Compressible fluids and related models

Nilasis Chaudhuri

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Uniwersytet Warszawski
MIM
Wydział
Matematyki, Informatyki i Mechaniki

Institute of Applied Mathematics and Mechanics

Real life: Examples

Cyclone

Traffic

Flock of Birds

These systems – fluids, vehicles, and animals – can all be described by Partial Differential Equations (PDEs) that express how density and velocity evolve in time.

[1] Source: wikipedia.org, phys.org

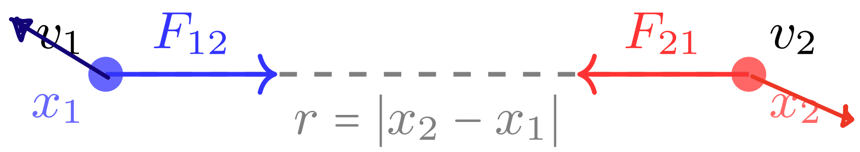
Different descriptions

Traffic (as Continuum)

Traffic (as Particle)

[2] Source: isarsoft.com, wikipedia.org

From Particles to the Continuum



Two particles Motivation: Newton's Law

Each particle has position $x_i(t) \in \mathbb{R}^d$ and velocity $v_i(t) \in \mathbb{R}^d$ for $i = 1, 2$:

$$\dot{x}_1 = v_1, \qquad \dot{x}_2 = v_2,$$

$$\dot{v}_1 = F_{12}(x_1, x_2, v_1, v_2), \qquad \dot{v}_2 = F_{21}(x_1, x_2, v_1, v_2),$$

where $\dot{X} \equiv \dfrac{dX}{dt}$ and $F_i(\cdot)$ represents the force encoding the interaction between the two particles. The space $\mathbb{R}^d$ denotes the *spatial domain* (or physical space).

From Particles to the Continuum

N particles Motivation: Newton's Law

Each particle has position $x_i(t) \in \mathbb{R}^d$ and velocity $v_i(t) \in \mathbb{R}^d$ for $i = 1, 2, \ldots, N$:

$$\dot{x}_i = v_i, \qquad \dot{v}_i = F_i(x_1, \ldots, x_N, v_1, \ldots, v_N), \quad i = 1, \ldots, N.$$

Here $F_i(\cdot)$ encodes the interaction among all particles.

"The continuum is the infinite limit of the discrete"

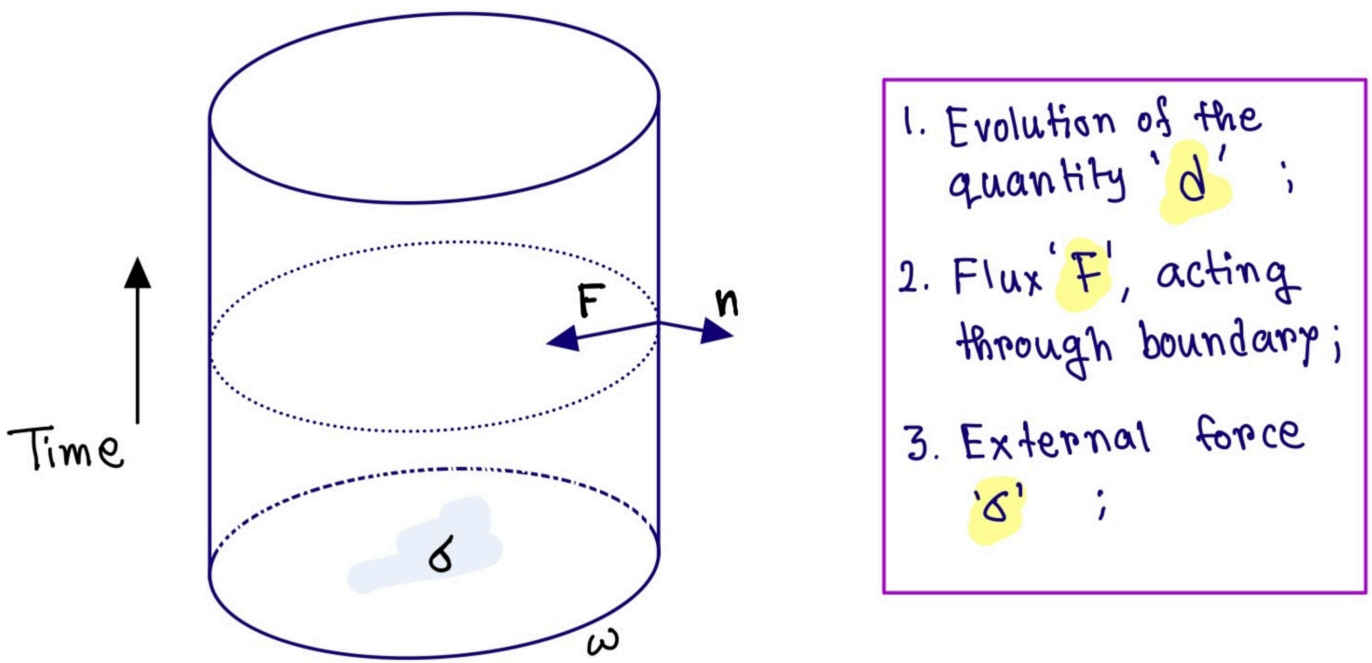
Number of particles N grow **very large** – in the ideal limit, $N \to \infty$.

Instead of describing each particle, the goal is to describe the *collective motion* of the medium.

$\rightsquigarrow$ System of ODEs $2dN$ equations $\longrightarrow$ System of PDEs $d + 1$ equations

$\rightsquigarrow$ PDEs of the conservation / balance law type.

A general conservation/balance law



Equation for macroscopic quantity d

$$\left[\int_\omega \mathsf{d}(t,x)\mathrm{d}x\right]_{t=t_1}^{t=t_2} = -\int_{t_1}^{t_2}\int_{\partial\omega} \mathbf{F}(t,x)\cdot\mathbf{n}\ \mathrm{dS}_x\mathrm{d}t + \int_{t_1}^{t_2}\int_\omega \sigma(t,x)\mathrm{d}x\mathrm{d}t,$$

An application of Gauss-Green theorem gives

$$\partial_t\mathsf{d} + \mathrm{div}\,\mathbf{F} = \sigma.$$

Compressible barotropic fluids

Continuity equation:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

Momentum equation:

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \operatorname{div} \mathbb{S}$$

- Time-space domain $(0,T) \times \Omega$, Ω: Bounded domain, $\mathbb{R}^d$ or $\mathbb{T}^d$;
- Density $\varrho = \varrho(t,x)$, Velocity $\mathbf{u} = \mathbf{u}(t,x)$;
- Pressure $p(\varrho)(\approx \varrho^\gamma$ with $\gamma > 1)$ and Viscous stress tensor $\mathbb{S}$;
- Initial and boundary condition;

- Navier–Stokes system : $\mathbb{S} \approx \nabla \mathbf{u}$;
- Euler system : $\mathbb{S} = 0$;

Compressible Navier–Stokes–Fourier system in $(0, T) \times \Omega$

Continuity equation:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

Momentum equation:

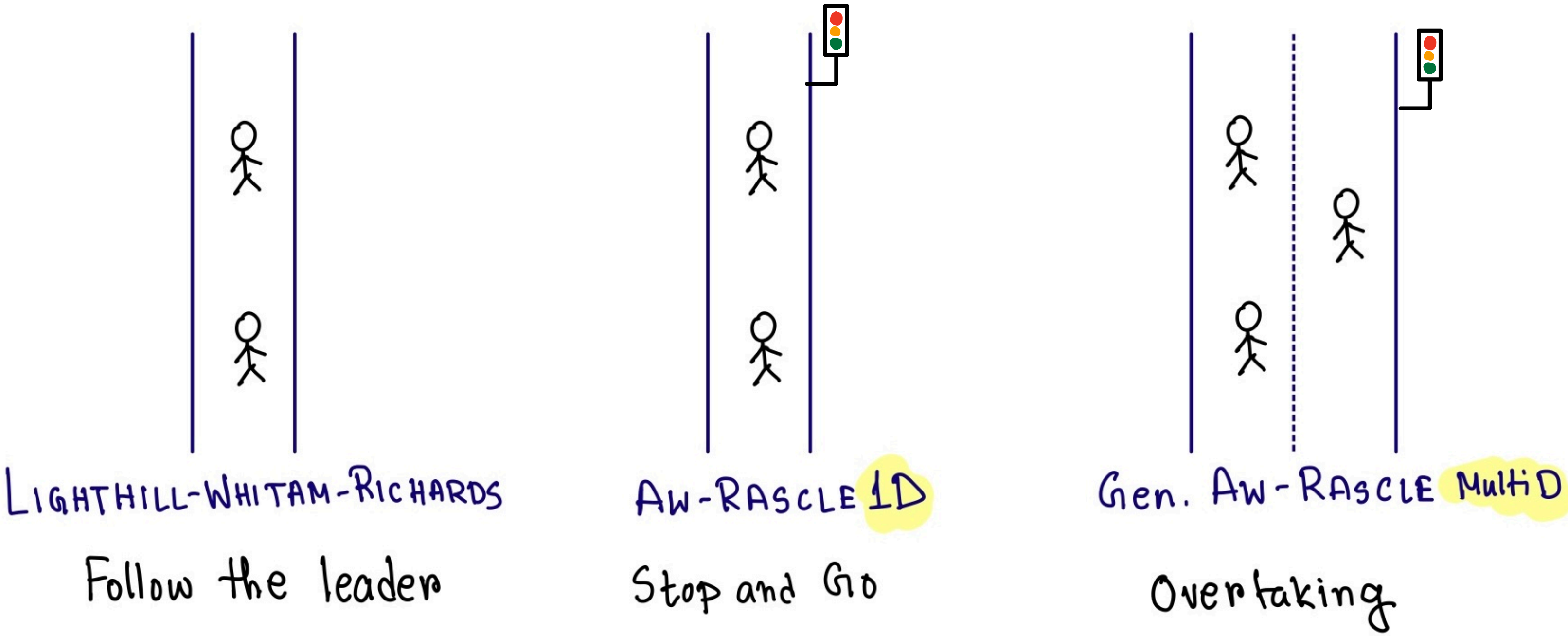
$$\partial_t (\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) = \operatorname{div} \mathbb{S}$$

Internal energy balance:

$$\partial_t (\varrho e(\varrho, \vartheta)) + \operatorname{div}(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div} \mathbf{q} = \mathbb{S} : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta) \operatorname{div} \mathbf{u}$$

- Absolute temperature $\vartheta = \vartheta(t, x)$;
- Pressure $p(\varrho, \vartheta)$, Internal energy $e(\varrho, \vartheta)$ and Heat flux $\mathbf{q}$;

Traffic/Pedestrian flows



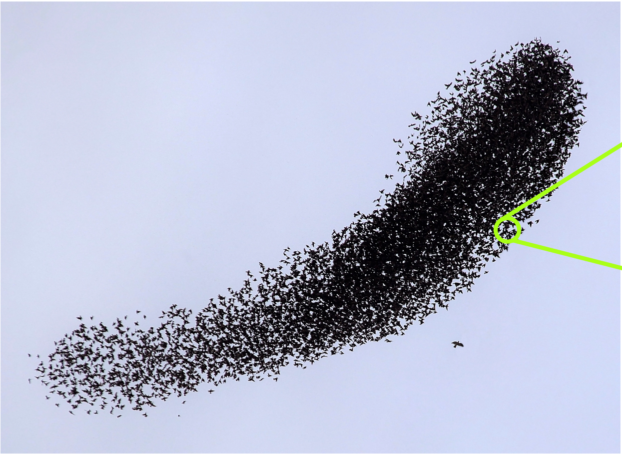
Generalized (Dissipative) Aw-Rascle model : DEGOND ET AL.

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,$$
$$\partial_t(\varrho \mathbf{w}) + \operatorname{div}(\varrho \mathbf{w} \otimes \mathbf{u}) = 0.$$

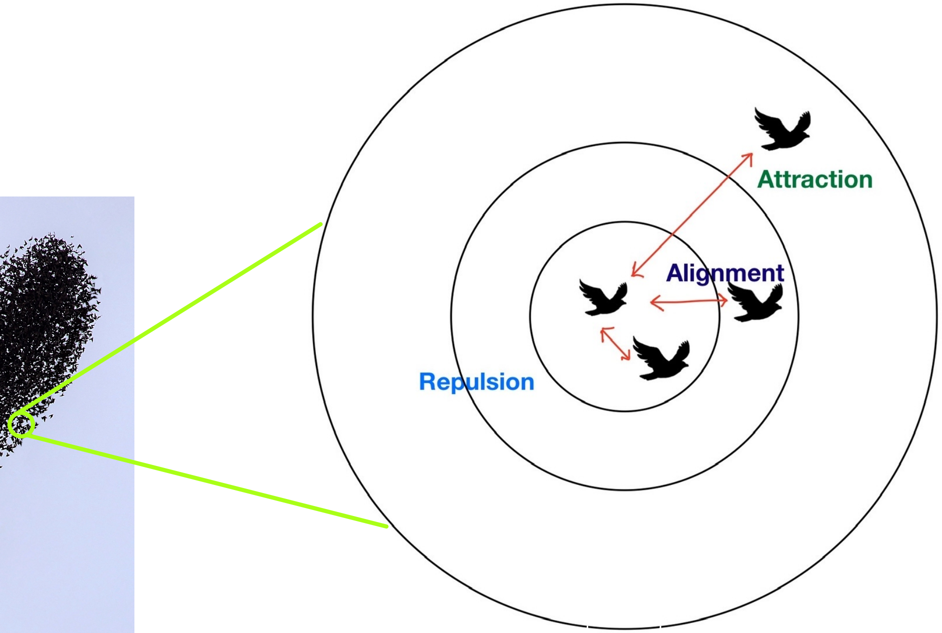
- ϱ is number of vehicles per unit length, i.e. density, $\mathbf{u}$ is the velocity for the cars, $\mathbf{w}$ is the preferred velocity,
- Additional relation $\mathbf{w} = \mathbf{u} + \nabla p(\varrho)$ with

$$p(\varrho) = a\varrho^\gamma \text{ with } a > 0,\ \gamma \geq 1 \text{ or } p(\varrho) = K * \varrho.$$

Swarming models



Macroscopic



Microscopic

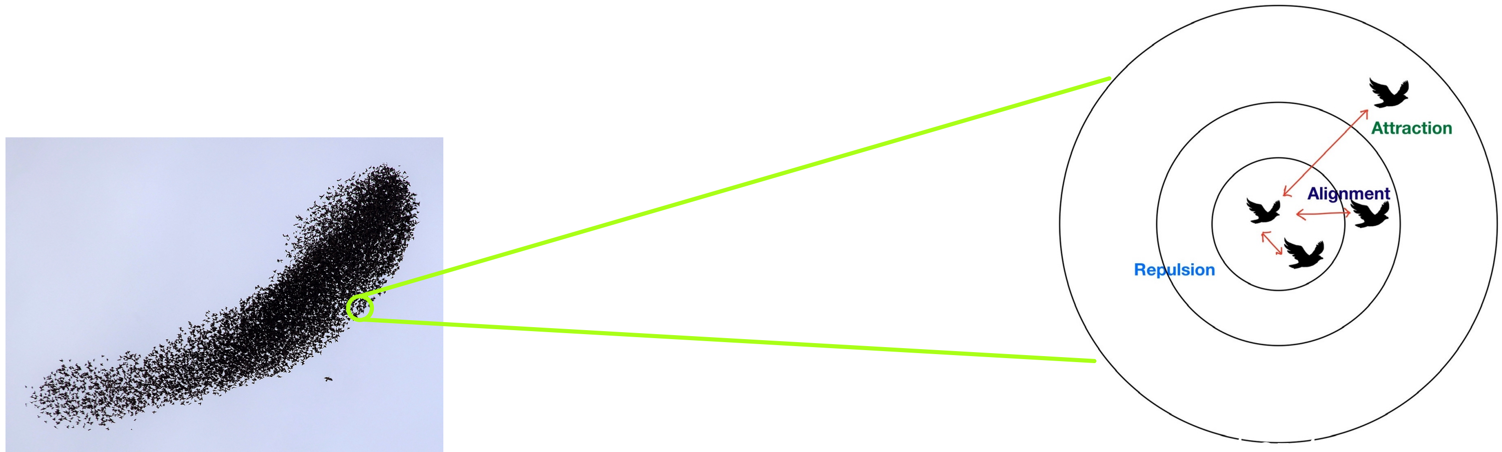
- Attraction-Repulsion interactions:

  $\rightsquigarrow$ Due to the presence of an individual in the sensitivity region ;

- Alignment interactions:

  $\rightsquigarrow$ Locally averaging the relative velocity with respect to neighbors with weights depending on the inter-particle distances;

Swarming models



Macroscopic

Microscopic

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0;$$
$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) = F(\rho, \mathbf{u});$$

$$\frac{d}{dt}\mathbf{x}_i = \mathbf{v}_i,$$
$$m_i \frac{d}{dt}\mathbf{v}_i = F_i$$

$N \to \infty$

Attraction / Repulsion force

$$F(\rho, \mathbf{u}) = -\rho \nabla W * \rho$$

Attraction / Repulsion force

$$F_i = -\tfrac{1}{N}\Sigma_{i \neq j} \nabla W(\mathbf{x}_j - \mathbf{x}_i)$$

Macroscopic Alignment

$$F(\rho, \mathbf{u}) = \rho(x) \int_{\mathbb{R}} \phi(x-y)(\mathbf{u}(y) - \mathbf{u}(x))\rho(y)\mathrm{d}y$$

Microscopic Alignment

$$F_i = \tfrac{1}{N}\Sigma_{i \neq j} \phi(\mathbf{x}_j - \mathbf{x}_i)(\mathbf{v}_j - \mathbf{v}_i)$$

Solution Concepts: A closer look into the comp. Navier–Stokes system

Continuity equation:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

- The uniqueness and strict positivity of solution depends on the regularity of $\mathbf{u}$; Di Perna–Lions' 1989

- $\varrho(t,x) > 0$ provided $\varrho_0 (= \varrho(0,x)) > 0$ and $\operatorname{div} \mathbf{u} \in L^1(0,T; L^\infty(\Omega))$.

Momentum equation: $\mathbb{S} \approx \nabla \mathbf{u}$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) = \Delta \mathbf{u}$$

$$\rightsquigarrow \boxed{\varrho \left(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u}\right) + \nabla p(\varrho) = \Delta \mathbf{u}}$$

- Semi-linear parabolic PDE, provided $\varrho > 0$;

An overview

$\rightsquigarrow$ Strong solution

A regular(smooth) pointwise solution (Hölder or Soblolev spaces) ;

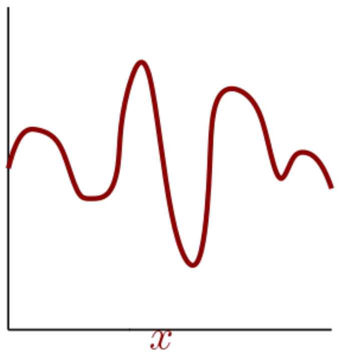
$\rightsquigarrow$ Weak Solution

- By weak solution we roughly replace the pointwise equations with the help of integral identities (Integration by parts) ;
- The main difficulty is the analysis of weak limit of a sequence and its non-linear composition – Concentration and Oscillation ;

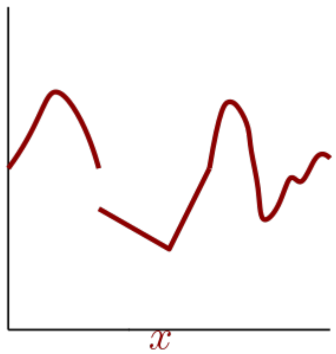
$\rightsquigarrow$ Measure valued solution

Analyse the lack of compactness due to non-linearity and allowing certain oscillations in the definition ;

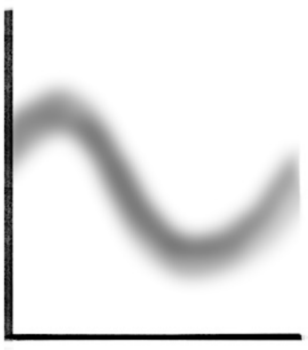
An overview



Strong soln.



Weak soln.



MV soln.

Generalized Solution

Generalized solutions

$\rightsquigarrow$ A Compatibility principle

A smooth generalized solution will be a strong solution.

$\rightsquigarrow$ A weak(measure-valued)-strong uniqueness principle

The principle asserts that a weak(measure–valued) solution and the strong solution emanating from the same initial data coincide as long as the strong solution exists.

$\rightsquigarrow$ Limit of numerical schemes

Comparatively easier to prove the convergence of numerical schemes to a generalized(measure-valued) solution.

Selected results

Compressible Navier–Stokes / Navier–Stokes–Fourier system

- Existence and Construction of weak solution : C. and Feireisl (App. Anal., 2022), C., Mucha and Zatorska (Math. Ann. 2024), C., Mucha and Pokorný (J. Differential Equations, 2025) ;
- Weak-Strong Uniqueness : C. (Non. Anal. RWA, 2019 ; J. Math. Fluid Mech., 2020 ; Nonlinearity 2024)

Generalized Aw-Rascle system

- Strong solution: C., Piasecki and Zatorska (Proc. Roy. Soc. Edinburgh Sect. A. 2025)
- Existence of Measure valued solution with weak(mv)–Strong uniqueness : C. Gwiazda and Zatorska (Comm. PDE., 2023)

Swarming models

- Existence of weak solution and long time asymptotics : C., Choi, Tse, Zatorska (J. Lond. Math. Soc. (2), 2025)

Details: https://sites.google.com/view/nilasischaudhuri/home

Thank you for your attention!