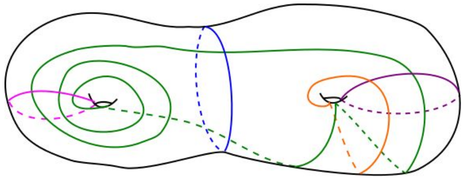
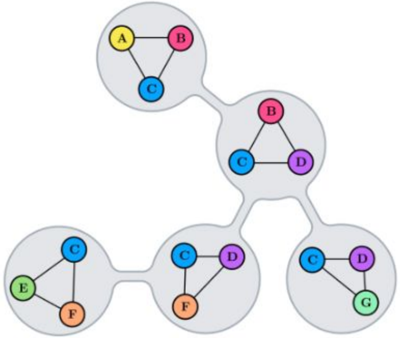
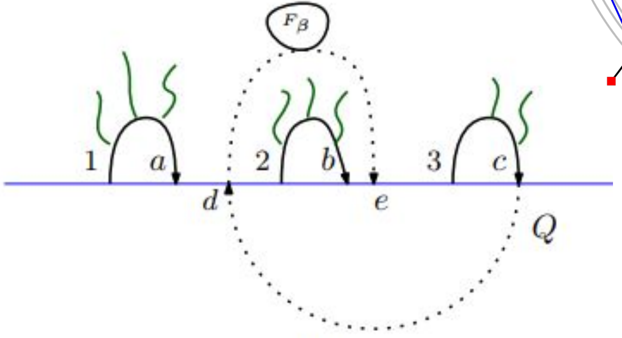
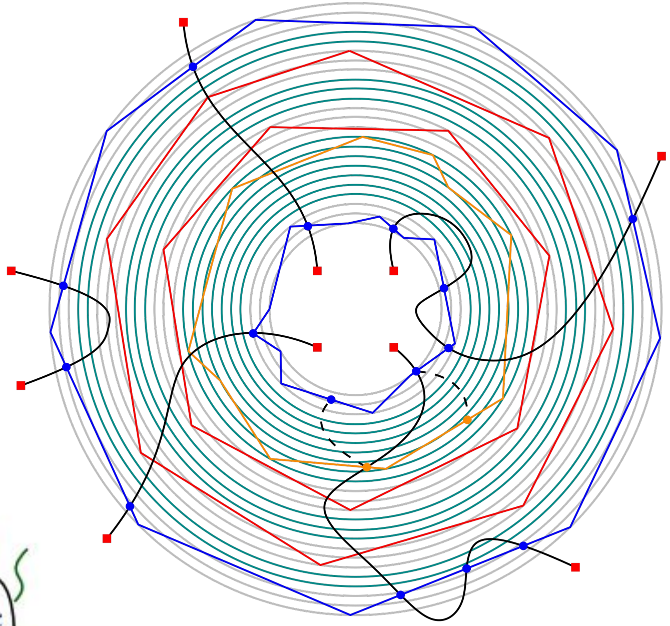
[Parameterized Complexity] x [Topological Graph Theory]

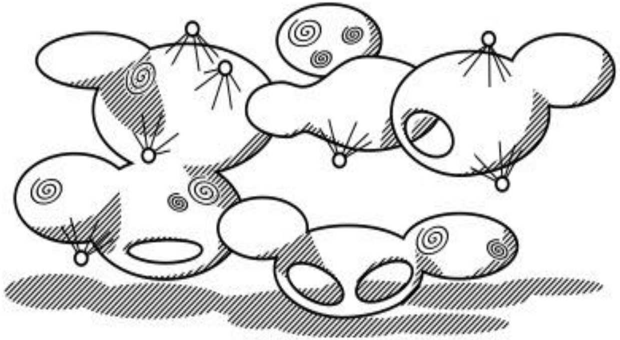
Michał Włodarczyk
MIMUW Colloquium 13/04/25

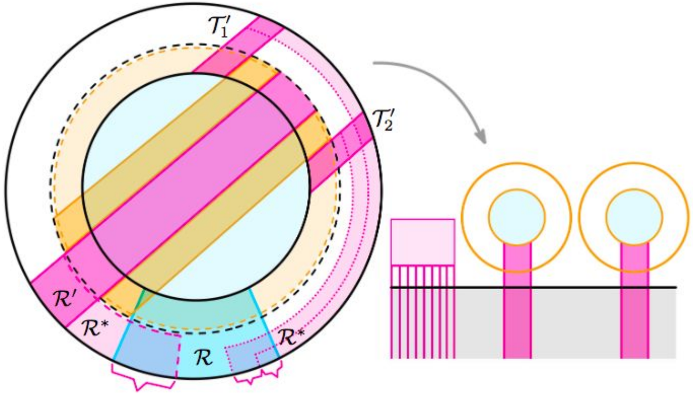








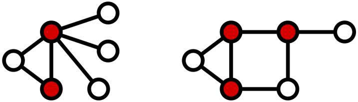




Parameterized Complexity

VERTEX COVER

Given: n-vertex graph G, integer k
Question: is there $X \subseteq V(G)$ of size k such that every edge touches a vertex from X?



- NP-hard so no hope for even $\mathcal{O}(n^{100})$-time algorithm
- There is a 2-approximation algorithm
- VERTEX COVER can be solved exactly in time $\mathcal{O}(2^k \cdot n^2)$
- Running time $f(k) \cdot \text{poly}(n)$: *fixed parameter tractable* (FPT)
- General question: which problems are FPT?
- Next question: what is the best running time in terms of the function f ?
- VERTEX COVER is solvable in time $\mathcal{O}(1.26^k \cdot n)$

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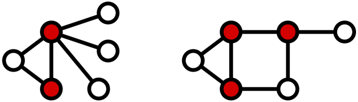

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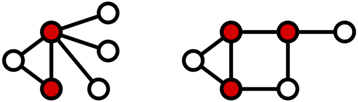

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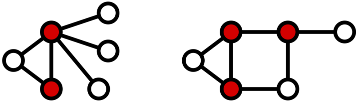

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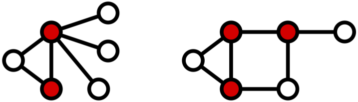

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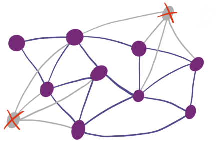
Def: a graph is called *planar* if it can be drawn on the plane without edge crossings.

VERTEX PLANARIZATION

Given: n-vertex graph G, integer k
Question: is there $X \subseteq V(G)$ of size k such that $G - X$ is planar?



- At least as hard as VERTEX COVER
- Constant-factor approximation is a major open problem
- VERTEX PLANARIZATION is FPT: solvable in time $2^{\mathcal{O}(k \log k)} \cdot n$
- Shaving off $(\log k)$ in another big open problem
- Another variant: find a drawing of G with the smallest number of crossings; also not well understood

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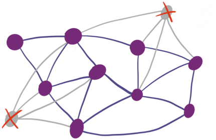

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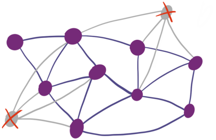

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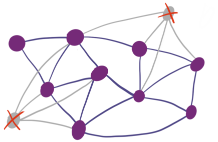

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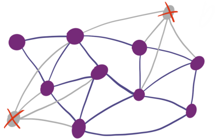

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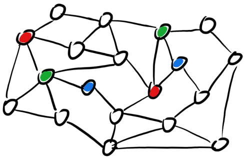
Disjoint Paths

Disjoint Paths

Given: n-vertex graph G, collection $\mathcal{T}$ of k pairs $(s_i, t_i) \in V(G) \times V(G)$
Question: is there a family $(P_1, \ldots, P_k)$ of vertex-disjoint paths s.t. P_i is a (s_i, t_i)-path?



- Famously FPT due to the Graph Minors Project
- Running time $f(k) \cdot \text{poly}(n)$ but the function f is “galactic”
- When G is planar, solvable in time $2^{\mathcal{O}(k^2)} \cdot n$
- The Disjoint Shortest Paths problem is unlikely to be FPT
- Our recent theorem [G. Stamoulis, Mi. Pilipczuk, W.]: Disjoint Shortest Paths is FPT when G is planar

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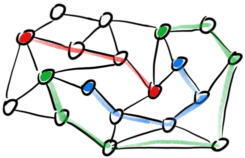

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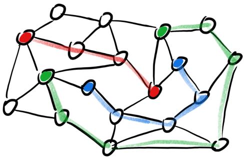

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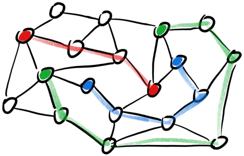

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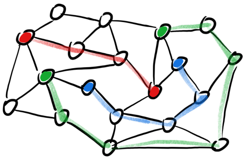

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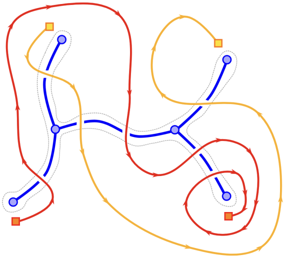
- Consider a plane embedding of G
- The problem is polynomial solvable when we fix the *homotopy class* of a solution
- A homotopy class can be encoded using the free non-abelian group F_k
- A priori the number of homotopy classes is $n^{\mathcal{O}(k)}$
- Main challenge: find a solution by inspecting only $f(k)$ homotopy classes



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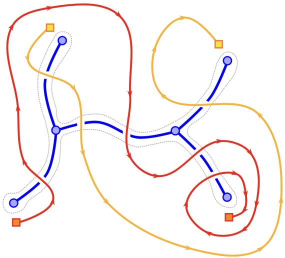

Homotopy

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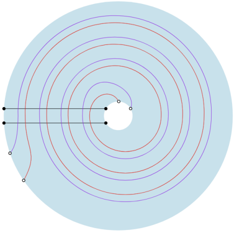
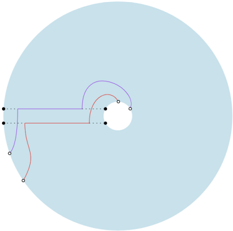
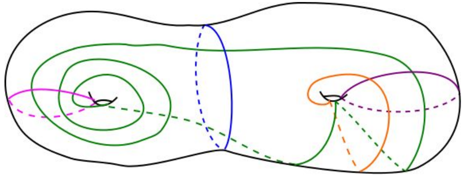
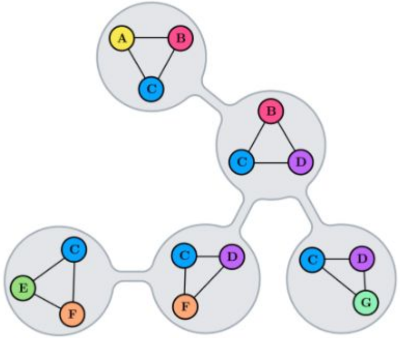




The End

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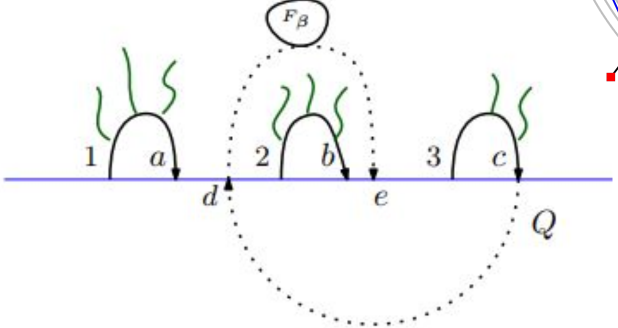
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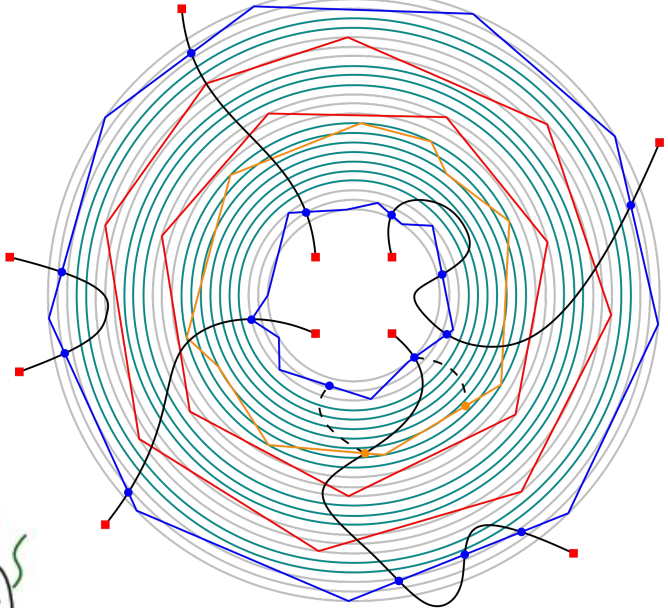

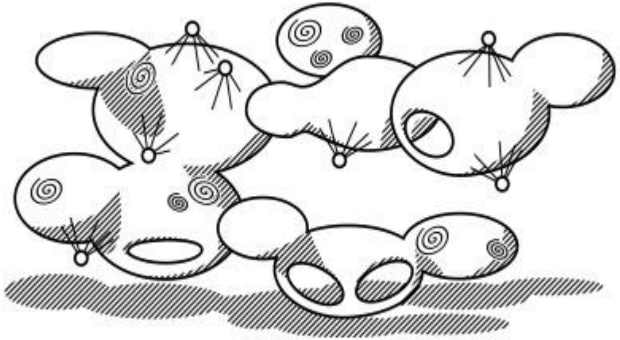

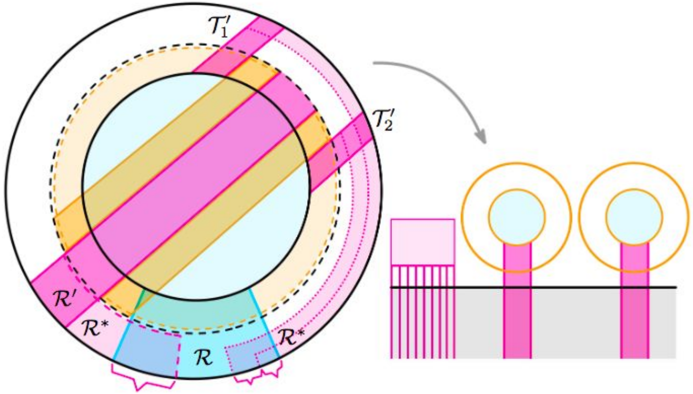
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