Quantitative Fatou Property and related notions

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Laplacian and harmonic functions

Let us begin with defining the Laplace operator. Let $\Omega \subset \mathbb{R}^n$ be an open set. For a function $u : \Omega \to \mathbb{R}$ laplacian of u is given by:

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u.$$

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We say that a function u is harmonic if

$$\Delta u = 0.$$

Classical Fatou theorem

Let us now state one of the versions of the Fatou theorem.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and let $u : \Omega \to \mathbb{R}$ be bounded and harmonic. Then for almost every point in the boundary of Ω there exists a nontangential limit of u.

Quantitative Fatou Property

To obtain the quantitative version of Fatou theorem let us introduce a counting function $\mathcal{N}$ of u at point $x \in \partial\Omega$. One may think about it as a way to count how much u oscillates when we approach point x.
Then Quantitative Fatou Property (QFP) for bounded functions reads:

$$\forall x \in \partial\Omega \sup_{0<r<\mathrm{diam}(\Omega)} \frac{1}{r^{n-1}} \int_{\partial\Omega \cap B(x,r)} \mathcal{N} u(y) \mathrm{d}\sigma(y) \leq C < \infty.$$

Quantitative Fatou Property

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Actually it is true that

Theorem (Bortz, Hofmann)

Let Ω be open and nontangentially accessible (NTA - a condition that ensures that Ω has nice topology). Let $u : \Omega \to \mathbb{R}$ be bounded harmonic. Then u satisfies QFP if and only if $\partial\Omega$ is uniformly rectifiable (geometric condition).

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1. for non-Euclidean spaces such as e.g. Riemannian manifolds or Carnot groups, in particular Heisenberg groups,
2. for wider classes of functions, i.e. satisfying more general PDEs or not even satisfying any PDE, but having "nice" growth or oscillation conditions.

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$$Nu(x) = \sup_{\Gamma(x)} |u(y)|, \quad (Su)^2(x) = \int_{\Gamma(x)} |\nabla u(y)|^2 (\mathrm{dist}(y, \partial\Omega))^{2-n} \mathrm{d}y,$$

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4. ε-approximability of functions from certain classes of functions.

Thank you for your attention!