Mathematical physics: on dirty stalls and liquid helium

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Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap. (...) In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic.

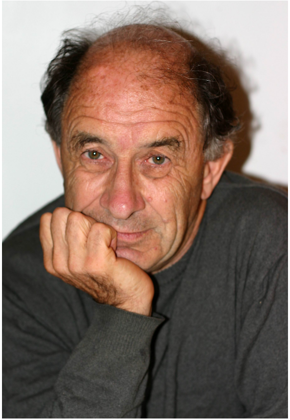


Figure: V.I. Arnold, 1938–2010

But these *cheap* experiments also take very long...

They ride the horses, we clean the stalls*

1. Physics is impossible without approximations and intuitive reasonings. Caring about every $\varepsilon > 0$ is, at initial stages, harmful. Taking care about the $\varepsilon > 0$'s is delegated to mathematical physicist, whose task is to verify the precise circumstances under which the approximations are correct. This is what known is as "cleaning the stalls".
2. But is is not just "cleaning the stalls". More often than not, the rigorous proof of intuitive contributes by itself to a deeper understanding of the physics involved (especially in those cases where the proof is of high mathematical quality). *When cleaning the stall, one gets to know the horse on a most intimate level...*

(* heard from a professor from the KMMF)

Example: helium atoms do not form a solid at low temperatures (and sufficiently small pressures).

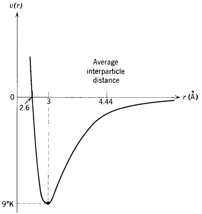
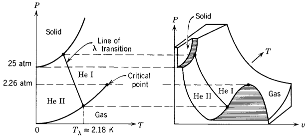


Fig. 13.1 Potential energy between two He atoms separated by distance r.

Consider a collection of He atoms at absolute zero under no external pressure. The most probable configuration of the atoms is determined by the ground state wave function, which, according to the variational principle, must be such as to minimize the total energy of the system, with no external constraint imposed. Hence energy consideration alone determines the most probable configuration. We can then make the following qualitative argument. If a He atom is to have a well-defined location, it must be confined to within a distance Δx that is small compared to the range of the potential, say $\Delta x \approx 0.5$ Å. By the uncertainty principle, we would then expect an uncertainty in energy (in units of Boltzmann's constant) of the order of

$$\Delta E \approx \frac{1}{2m}\left(\frac{\hbar}{\Delta x}\right)^2 \approx 10\ \mathrm{K} \qquad (13.2)$$

This is comparable to the depth of the potential well. Hence the localization is impossible. The fact that no other noble element can remain in liquid form down to very low temperatures is explained by their much greater masses. The fact that H_2, although lighter than He, solidifies at a finite temperature is explained by the strong molecular interactions between H_2 molecules. The argument we have given is independent of statistics and also explains why both He^4 and He^3 remain liquid down to absolute zero.



Mathematically, the problem is to determine whether there exists a function $\varphi : \mathbb{R}^3 \to \mathbb{C}$ with $\int_{\mathbb{R}^3} |\varphi(x)|^2 dx = 1$ such that

$$\mathcal{E}(\varphi) := \frac{\hbar^2}{2m} \int_{\mathbb{R}^3} |\nabla\varphi(x)|^2 + \int_{\mathbb{R}^3} V(x)|\varphi(x)|^2 dx < 0$$

in which case the helium atoms form a bound state and their collection solidifies. The "uncertainty principle" is better introduced via the *Sobolev inequality*

$$\int_{\mathbb{R}^3} |\nabla\varphi(x)|^2 \geq S_3 \left(\int_{\mathbb{R}^3} |\varphi(x)|^6 dx \right)^{1/3} \tag{1}$$

which combined with the *Hölder inequality*

$$\int_{x:V(x)<0} |V(x)||\varphi(x)|^2 dx \leq \left(\int_{x:V(x)<0} |V(x)|^{3/2} dx \right)^{2/3} \left(\int_{\mathbb{R}^3} |\varphi(x)|^6 dx \right)^{1/3} \tag{2}$$

yields $\mathcal{E}(\varphi) > 0$ for all admissible φ, provided

$$\left(\int_{x:V(x)<0} |V(x)|^{3/2} dx \right)^{2/3} < \frac{\hbar^2}{2m} S_3 \tag{3}$$

which inequality not only puts the intuitive reasoning on a rigorous basis, but it also quantifies it much better.

My stall

My area of interest: *many–body problems.*

- One atom of Au is does not shine like gold. One needs a great many of them, and only their mutual interactions as well as their interactions with the electromagnetic field produce the "emerging effect" One of the goals of many–body physics is to explain how does this happen.
- Example: bodies moving in liquid helium at low T do not experience any resistance resistance to the flow. Fundamentally, a large collection of helium atoms which interact among themselves and with the external object. How to deduce that such a phenomenon is possible?

Example: superfluidity in liquid helium

For instance, one of my works was essentially concerned with a rigorous justification of the highlighted part of the text below:

Therefore

$$|\Delta \mathbf{P}| \leq \sum_{\mathbf{k} \neq 0} |\mathbf{k}| n_{\mathbf{k}} \tag{13.28}$$

or

$$c|\Delta \mathbf{P}| \leq \Delta E \tag{13.29}$$

On the other hand, if the external object loses the amount of energy ΔE and momentum $\Delta \mathbf{P}$, we must have

$$\Delta E = \mathbf{v}_e \cdot \Delta \mathbf{P} \tag{13.30}$$

This is impossible unless $|\mathbf{v}_e| > c$. At low temperatures the transfer of energy and momentum through scattering of existing phonons can be neglected because as $T \to 0$ the number of phonons becomes zero. It is noted that the argument depends on the linearity of the phonon energy spectrum and does not apply to an ideal gas of bosons.

The foregoing argument is valid at absolute zero, but it breaks down at a higher temperature when there are many phonons present. The external object may now transfer energy and momentum by scattering the phonons. Experiments on the flow of He II past a wall at 1 K indicate that the critical velocity is orders of magnitude smaller than c.

Theorem (M-Seiringer 2020)

Let $\xi > 0$. Then for all eigenvalues $e_i(H_N)$ such that $e_i(H_N) - e_0(H_N) \leq \xi$, we have

$$|e_i(H_N) - E^{\mathrm{H}}(N) - e_i(\mathbb{H}^F)| \leq C_{v,w}\xi \left(\frac{\xi}{N}\right)^{1/2}$$

for some constant $C_{v,w} > 0$ independent of the parameters ξ and N.

Here, without coming into details:

- H_N is the object governing the exact microscopic dynamics of the system of N helium atoms + one additional object immersed in the flud
- $\mathbb{H}^F$ is the generator of the effective dynamics in terms of these *phonons*, in particular, it does not include scatterings between them
- ξ is the excitation energy of the system (low $T \iff$ low ξ)
- The non–excited energy $E^{\mathrm{H}}(N)$ is of order N, so that the bound is good as long as $\xi \ll N$.