MIM UW Faculty Colloquium

Banach space properties of Sobolev Spaces

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My interest revolves around the Banach space properties of spaces of smooth functions. In particular, I am interested in the properties of Sobolev spaces:

$$W_1^1(\Omega) = \{f \in L^1(\Omega) : D^\alpha f \in L^1(\Omega) \text{ for } |\alpha| = 1\},$$

and BV space

$$BV(\Omega) = \{f \in L^1(\Omega) : D^\alpha f \text{ is a bounded measure for } |\alpha| = 1\},$$

where $D^\alpha f$ is a weak derivative of f i.e. it satisfies "integration by parts" for any $\phi \in C_0^\infty(\Omega)$:

$$\int_\Omega f \; D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega \phi \; D^\alpha f.$$

Definitions

We denote by X^* a dual space of a Banach space X i.e.

$$X^* = \{\phi : X \to \mathbb{R} : \phi \text{ is linear and continuous}\}$$

We say that a sequence $(x^{(n)})$ is weakly convergent to x if for every $\phi \in X^*$ we have

$$\lim_{n\to\infty} \phi(x^{(n)}) = \phi(x).$$

We write

$$x^{(n)} \xrightarrow{w} x \text{ in } X.$$

Dunford-Pettis property (DPP)

We say that space X has DPP

$$\left\{\begin{array}{ll} x^{(n)} \xrightarrow{w} 0 & \text{in } X, \\ \phi^{(n)} \xrightarrow{w} 0 & \text{in } X^*. \end{array}\right. \Rightarrow \lim_{n\to\infty} \phi^{(n)}\left(x^{(n)}\right) = 0$$

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- $L^1(\Omega)$, $C(\Omega)$, $L^\infty(\Omega)$, $C^1(\Omega)$ have DPP.

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Does W_1^1 or BV have DPP?
We don't know but...

Decomposition of gradient measure

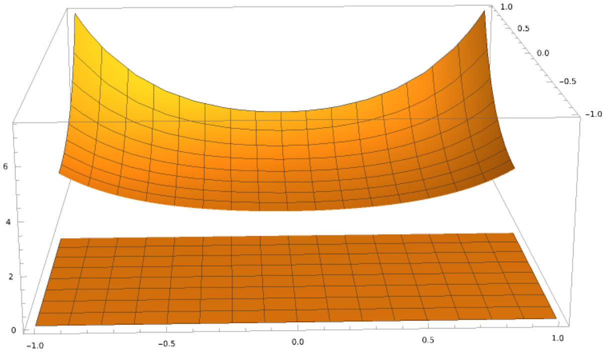
The gradient of a function u from BV has the following canonical decomposition:

$$Du = D^a u + D^c u + D^j u.$$

- $D^a u$ is a Lebesgue measurable part of the gradient measure Du.
- $D^j u$ is jump part of the gradient measure Du.
- $D^c u$ is a Cantor part of a gradient measure Du.

Decomposition of gradient measure

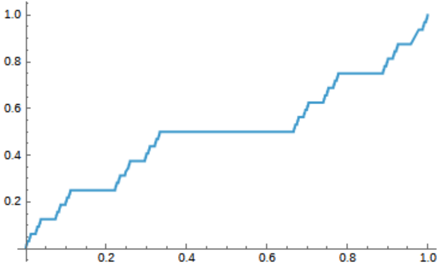
- $D^j u$ is jump part of the gradient measure Du. It is connected to the discontinuities of u.



In the picture measure $|D^j u| = |u^+(x,0) - u^-(x,0)|dx = |u^+(x,0)|dx$

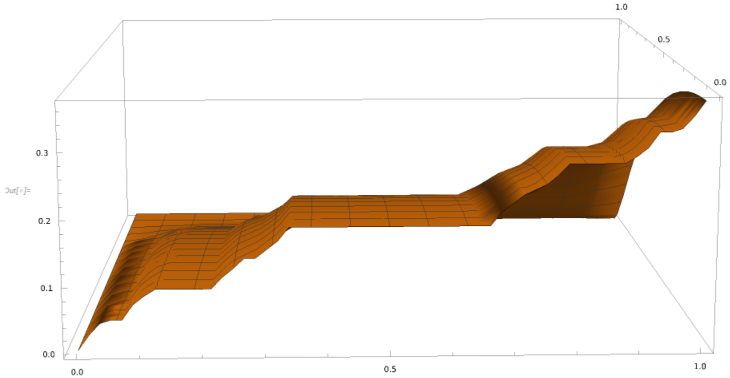
Decomposition of gradient measure

The Cantor's staircase function is a function that is continuous and has a derivative equal to zero almost everywhere.



Decomposition of gradient measure

- $D^c u$ appears if we have a function with a similar behavior to the Cantor's staircase function.



In the picture for the fixed y, the function on $f(\cdot, y)$ is a dilated Cantor's staircase function.

KK, A. Tselishchev, M. Wojciechowski

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in BV. If (f_n) converges weakly in $BV(\mathbb{R}^d)$ to a function $f \in BV$ then

$$\lim_{n\to\infty} \| \left(D^j(f - f_n)\right) \|_M = 0,$$

where $\| \cdot \|_M$ is a total variation of a measure.

If, as in the picture, the functions $f^{(n)}$ have jumps only on the line $\mathbb{R} \times \{0\}$ then the theorem states that

$$\int_{\mathbb{R}} |f^{(n),+}(x,0) - f^{(n),-}(x,0) - \left(f^{+}(x,0) - f^{-}(x,0)\right)| dx \overset{n\to\infty}{\to} 0.$$