Set-theoretic methods in functional analysis

Damian Głodkowski

MIM Faculty Colloquium

14 XI 2024

Functional analysis:

- Banach spaces and algebras
- Spaces of continuous functions
- Operator algebras

Functional analysis:

- Banach spaces and algebras
- Spaces of continuous functions
- Operator algebras

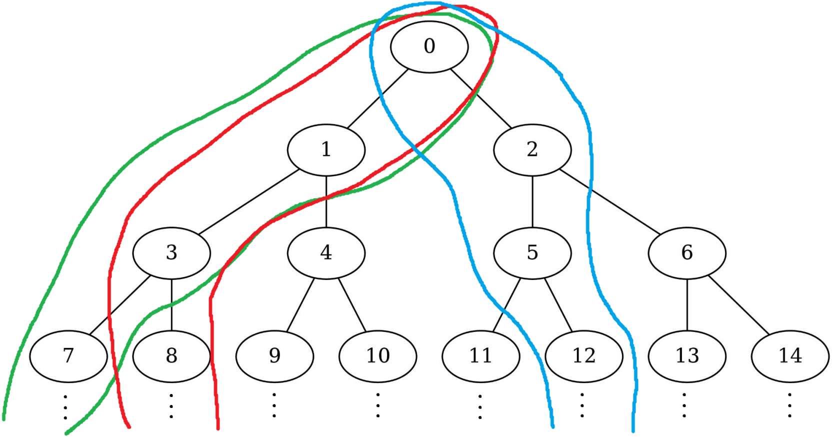
Tools from set theory:

- Families of sets with special combinatorial properties
- Methods of proving unprovability

Example: almost disjoint families

Fact: there is an uncountable family $(A_i)_{i \in I}$ of infinite subsets of $\mathbb{N}$ such that $A_i \cap A_j$ is finite for $i \neq j$ (so called **almost disjoint family**).

Fact: there is an uncountable family $(A_i)_{i \in I}$ of infinite subsets of $\mathbb{N}$ such that $A_i \cap A_j$ is finite for $i \neq j$ (so called **almost disjoint family**).



Example: almost disjoint families

Fact: there is an uncountable family $(A_i)_{i \in I}$ of infinite subsets of $\mathbb{N}$ such that $A_i \cap A_j$ is finite for $i \neq j$ (so called **almost disjoint family**).

Moreover, every countable almost disjoint family may be extended to a bigger one.

Example: almost disjoint families

Fact: there is an uncountable family $(A_i)_{i \in I}$ of infinite subsets of $\mathbb{N}$ such that $A_i \cap A_j$ is finite for $i \neq j$ (so called **almost disjoint family**).

Moreover, every countable almost disjoint family may be extended to a bigger one.

Almost disjoint families may be used in constructions of some infinite-dimensional spaces with some special properties.

For example, Johnson and Lindenstrauss used them to show that there is a Banach space X that is not weakly compactly generated, but X^* is weakly compactly generated.

Infinite matrices

H - infinite-dimensional complex vector (Hilbert) space.

$T\colon H \to H$ - a linear operator.

T may be represented as an infinite matrix

$$T = \begin{bmatrix} 1 & 0 & -1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 1 & 2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Projections

Definition

We say that a matrix P is a **projection**, if $P = P^2 = P^*$, where P^* denotes the Hermitian conjugate of P.

Projections

Definition

We say that a matrix P is a **projection**, if $P = P^2 = P^*$, where P^* denotes the Hermitian conjugate of P.

Projections P and Q are **orthogonal**, if $PQ = 0$.

Projections

Definition

We say that a matrix P is a **projection**, if $P = P^2 = P^*$, where P^* denotes the Hermitian conjugate of P.

Projections P and Q are **orthogonal**, if $PQ = 0$.

Fact: every family of pairwise orthogonal projections is countable.

Projections

Definition

We say that a matrix P is a **projection**, if $P = P^2 = P^*$, where P^* denotes the Hermitian conjugate of P.

Projections P and Q are **orthogonal**, if $PQ = 0$.

Fact: every family of pairwise orthogonal projections is countable.

Definition

Projections P and Q are **almost orthogonal**, if PQ is compact, that is, it can be approximated by finite matrices.

Projections

Definition

We say that a matrix P is a **projection**, if $P = P^2 = P^*$, where P^* denotes the Hermitian conjugate of P.

Projections P and Q are **orthogonal**, if $PQ = 0$.

Fact: every family of pairwise orthogonal projections is countable.

Definition

Projections P and Q are **almost orthogonal**, if PQ is compact, that is, it can be approximated by finite matrices.

Fact: There are uncountable families of pairwise almost orthogonal projections. Moreover, every countable family of pairwise almost orthogonal projections may be extended to a bigger one.

The Calkin algebra

- $\mathcal{B}$ - the algebra of matrices that define continuous linear operators
- $\mathcal{K}$ - the algebra of matrices that may be approximated by finite matrices
- $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ - the Calkin algebra

The Calkin algebra

- $\mathcal{B}$ - the algebra of matrices that define continuous linear operators
- $\mathcal{K}$ - the algebra of matrices that may be approximated by finite matrices
- $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ - the Calkin algebra

Definition

An automorphism $T \colon \mathcal{Q} \to \mathcal{Q}$ is **inner** if it is of the form $T(X) = AXA^*$ for some $A \in \mathcal{Q}$.
An automorphism that is not inner is called **outer**.

The Calkin algebra

- $\mathcal{B}$ - the algebra of matrices that define continuous linear operators
- $\mathcal{K}$ - the algebra of matrices that may be approximated by finite matrices
- $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ - the Calkin algebra

Definition

An automorphism $T \colon \mathcal{Q} \to \mathcal{Q}$ is **inner** if it is of the form $T(X) = AXA^*$ for some $A \in \mathcal{Q}$.
An automorphism that is not inner is called **outer**.

(Phillips & Weaver, 2007) "The Calkin algebra has outer automorphisms"

The Calkin algebra

- $\mathcal{B}$ - the algebra of matrices that define continuous linear operators
- $\mathcal{K}$ - the algebra of matrices that may be approximated by finite matrices
- $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ - the Calkin algebra

Definition

An automorphism $T : \mathcal{Q} \to \mathcal{Q}$ is **inner** if it is of the form $T(X) = AXA^*$ for some $A \in \mathcal{Q}$.
An automorphism that is not inner is called **outer**.

(Phillips & Weaver, 2007) “The Calkin algebra has outer automorphisms”

(Farah, 2011) “All automorphisms of the Calkin algebra are inner”

Thank you for your attention!