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Evaluations of Polynomials in a noncommutative setting

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This talk is essentially extracted from a few joint works with

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CONTENT

1 I) One Variable.

Ia) **Classical case.**

Let k be a field, $f(x), g(x) \in R = k[x]$, $a \in k$.
a) $f(a) = 0$ iff $x - a$ divides $f(x)$ iff $\overline{f(x)} = \overline{0}$ in $R/(x - a)$.
b) $(f \cdot g)(a) = f(a)g(a)$.
c) $|\{a \in k \mid f(a) = 0\}| \leq deg(f)$.
Remark that c) is not valid for rings with zero divisors:
$(x - 2)(x - 3) \in (\mathbb{Z}/6\mathbb{Z})[x]$ has 4 zeroes in $\mathbb{Z}/6\mathbb{Z}$.
If a ring R is such that every nonzero polynomial has a finite number of zeros then R is a domain (Fuchs, Maxson, Pilz).

Ib) **Noncommutative case.**

Examples 1.1. [1] $ix = xi \in \mathbb{H}[x]$. So,

$$(ix)(j) = \begin{cases} ij, & \text{right evaluation,} \\ ji & \text{left evaluation.} \end{cases} \tag{1.I}$$

[2] $x^2 + 1 \in \mathbb{H}[x]$ & $\forall\ a \in \mathbb{H}^*, (aia^{-1})^2 + 1 = 0$. So, $x^2 + 1$ has an infinite number of roots. In fact $V(x^2 + 1) = \{aia^{-1} \mid a \in \mathbb{H}^*\}$
[3] $\mathbb{H}[x] = (x - j)(x - i)$ only one right root: i.
[4] If K is a finite dimensional division algebra over its center F and $f(x) = (x - d)g(x) \in K[x]$, then a conjugate d' of d is a right root of $f(x)$.

Theorem 1.2 (Niven)**.** *The quaternion algebra $\mathbb{H}$ defined over the real numbers is algebraically closed, meaning that every polynomial $p(y) \in \mathbb{H}[y]$ possesses at least one root in $\mathbb{H}$.*

For a noncommutative coefficient ring R, $R_G[y]$ is the general polynomials with coefficients from R. The elements are sums of monomials structured as $a_1 y a_2 y \cdots y a_n$, where $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in R$.

$$a_1 y a_2 y a_3 y + b_1 y b_2 y b_3 y b_4 y b_5 y b_6 + \cdots$$

The product of two monomials is obtained by concatenation. The evaluation at some element $a \in R$ of such a polynomial is obtained by replacing y by a.

Theorem 1.3 (Fuchs, Maxson, Pilz). *Each nonzero polynomial in $R_G[y]$ possesses a finite number of zeros if and only if the ring R is finite.*

We note the following Theorem due to P.M. Cohn.

Theorem 1.4 (P.M. Cohn). *Let K be an arbitrary division ring. For any polynomial equation of degree $n > 0$,*

$$y^n + a_1 y^{n-1} + \ldots + a_n = 0 \quad (a_i \in K)$$

there exists a right root in some extension of the division ring K.

Looking at polynomials with noncommutative variables we can consider polynomials in the free algebra over a field K, i.e. $K < x_1, \ldots, x_n >$. The evaluation of such polynomials over any K algebra makes sense and leads to a very wide area, that includes polynomial identities.

Ic) **Skew polynomial rings**

A a ring, $S \in End(A)$, D a S-derivation:

$$D \in End(A, +) \qquad D(ab) = S(a)D(b) + D(a)b, \forall a, b \in A.$$

For $a \in A$, L_a left multiplication by a.

In $End(A,+)$, we then have : $D \circ L_a = L_{S(a)} \circ D + L_{D(a)}$.

Define a ring $R := A[t; S, D]$; Polynomials $f(t) = \sum_{i=0}^{n} a_i t^i \in R$.

Degree and addition are defined as usual, the product is based on:

$$\forall a \in A, \quad ta = S(a)t + D(a).$$

Examples 1.5. 1) If $S = id.$ and $D = 0$ we get back the usual polynomial ring $A[x]$.

2) $R = \mathbb{C}[t; S]$ where S is the complex conjugation. If $x \in \mathbb{C}$ is such that $S(x)x = 1$ then

$$t^2 - 1 = (t + S(x))(t - x).$$

$t^2 + 1$ is central and irreducible in R, $R/(t^2 + 1) \cong \mathbb{H}$

3) K a field, $q \in K \setminus \{0\}$ and $S \in End_K(K[x])$ defined by $S(x) = qx$. $R = K[x][y; S]$. Commutation rule: $yx = qxy$.

4) $t^n a = \sum_{i=0}^{n} f_i^n(a) t^i$ where f_i^n is the sum of all words in S and D of length n having i letters S and $n - i$ letters D.

Facts Let K be a division ring.

a) Ore (1933): $R = K[t; S, D]$ is a left principal ideal domain.

b) Ore (1933): $R = K[t; S, D]$ is a unique factorization domain:

If $f(t) = p_1(t) \dots p_n(t) = q_1(t) \dots q_m(t)$, $p_i(t), q_i(t)$ irreducible then $m = n$ and there exists $\sigma \in \mathcal{S}_n$ such that,

$$\text{For } 1 \le i \le n, \qquad \frac{R}{Rq_i} \cong \frac{R}{Rp_{\sigma(i)}}$$

Id) Evaluations and roots

Definition 1.6. Let $f(t) \in R = A[t; S, D]$ and $a \in A$, there exists a unique $b \in A$ such that $b + R(t-a) = f(t) + R(t-a)$. b is called the right evaluation of $f(t)$ at a and is denoted $f(a)$

To express $f(a)$ we introduce, for $a \in A$, the following maps:

$$N_0(a) = 1, N_1(a) = a \quad \text{and} \quad N_{i+1}(a) = S(N_i(a))a + D(N_i(a))$$

For $f(t) = \sum_{i=0}^{n} a_i t^i$ and $a \in R$ we have

$$f(a) = \sum_{i=0}^{n} a_i N_i(a)$$

Examples 1.7. 1. $N_2(a) = S(a)a + D(a)$,

2. $N_3(a) = S(N_2(a))a + D(N_2(a)) = ...$
3. If $D = 0$ $N_n(a) = S^{n-1}(a)S^{n-2}(a) \cdots S(a)a$.

Question: How to compute $fg(a)$ for $f, g \in R = A[t; S, D]$?
If $A = K$ is a division ring we have the "product formula"

$$(1.II) \qquad (f.g)(a) := \begin{cases} 0 & ; g(a) = 0, \\ f(a^{g(a)}).g(a) & ; g(a) \neq 0. \end{cases}$$

where for $c \in K/\{0\}$, $a^c = \sigma(c)ac^{-1} + \delta(c)c^{-1}$.

Remark 1.8. The above formula needs to be applied even in the "classical case" (i.e. $S = Id.$ and $D = 0$).

Question: and if A is not a division ring?

Ie) PLT

Definition 1.9. Let A be a ring, S an endomorphism of A and D a S-derivation of A. Let also V stand for a left A-module. An additive map $T : V \longrightarrow V$ such that, for $\alpha \in A$ and $v \in V$,

$$T(\alpha v) = S(\alpha)T(v) + D(\alpha)v.$$

is called an (S, D) pseudo-linear transformation (or a (S, D)-PLT, for short).

Example 1.10. For $a \in A$, $T_a \in End(A, +)$ is defined by

$$T_a(x) = S(x)a + D(x) \quad \forall x \in A.$$

Examples: $T_0 = D,\ T_1 = S + D$.

Proposition 1.11. *Let A be a ring $S \in End(A)$ and D a S-derivation of A. For an additive group $(V, +)$ the following conditions are equivalent:*

(i) V is a left $R = A[t; S, D]$-module;

(ii) V is a left A-module and there exists an (S, D) pseudo-linear transformation $T : V \longrightarrow V$;

(iii) There exists a ring homomorphism $\Lambda : R \longrightarrow End(V, +)$.

Corollaire 1.12. *For any $f, g \in R = A[t; S, D]$ and any pseudo-linear transformation T we have: $(fg)(T) = f(T)g(T)$.*

Theorem 1.13. *(a) $f(T_a)(1) = f(a)$.*

(b) For $f, g \in R$, $fg(a) = f(T_a)(g(a))$.

(c) For $a, b \in A$ with $b \in U(A)$, we have $(t - c)b = S(b)(t - a)$ where $c := S(b)ab^{-1} + D(b)b^{-1}$. This will be denoted $c = a^b$).

(d) For $b \in U(A)$, $(f(t)b)(a) = f(a^b)b$.

(e) For $b \in U(A)$, $f(T_a)(b) = f(a^b)b$.

(f) If $g(a) \in U'A)$, we have $fg(a) = f(a^{g(a)})g(a)$.

We define

$$E(f,a) := \ker f(T_a)$$

If $A = K$ is a division ring we have

$$E(f,a) = \{0 \neq b \in K \mid f(a^b) = 0\} \cup \{0\}$$

Recent news about Ore extensions with finite order automorphisms and zero derivations (L., Lopez and K. S. Lee).
Let $\sigma, \tau \in Aut(R)$ be a of finite orders m, l. There is $1-1$ ring homomorphism φ_m from $R[x;\sigma]$ into $\oplus^m R[t;\pi_m]$ such that $\varphi_m(r) = (r, \sigma(r), \ldots, \sigma^{m-1}(r))$ and $\varphi_m(x) = t^m$ and $\pi_m(a_0, \ldots, a_{m_1}) = (a_1, a_2, \ldots, a_{m-1}, a_0)$. We have $R[x;\sigma] \cong S_\sigma[t;\pi]$, where $S_\sigma = \{(a, \sigma(a), \ldots, \sigma^{m-1}(a)) \mid a \in R\}$.
We denote $R^\sigma = \{a \in R \mid \sigma(a) = a\}$. Suppose that $\sigma, \tau \in Aut(R)$ has order m and l respectively. We denote by s the least common multiple of m and r. we have the following injective ring homomorphisms :

$$R^\sigma[x^m] \longrightarrow R[x^m] \longrightarrow R[x,\sigma] \cong S_\sigma[t,\pi_m] \longrightarrow \oplus^m R[t_m,\pi_m] \xrightarrow{\gamma_{m,s}} \oplus^s R[t_s,\pi_s]$$

$$R^\tau[x^l] \longrightarrow R[x^l] \longrightarrow R[x,\tau] \cong S_\tau[t,\pi_l] \longrightarrow \oplus^l R[t_l,\pi_l] \xrightarrow{\gamma_{l,s}} \oplus^s R[t_s,\pi_s]$$

The map $\gamma_{m,s}$ is defined as follows. Write $s = mr$

We define the embedding $\gamma_{m,s} : \oplus^m R[t_m, \pi_m] \longrightarrow \oplus^s R[t_s, \pi_s]$ by $\gamma_{m,s}(t_m) = t_s^r$ and $\gamma_{m,s}((a_1, \ldots, a_m)) = (b_1, \ldots, b_s)$, where $b_i = 0$ if $i \notin m\mathbb{N}$ and $b_{ml} = a_l$ for $1 \leq l \leq r$.

One particular feature of this diagram is that, by embedding different $R[x; \sigma]$ and $R[x; \tau]$ we in an adequate $\oplus R^m[t; \pi]$, we can consider the elemnts of these Ore extensions in a single one and hence compute the product....

I f) Counting the number of roots.

K a division ring.

Facts and notations

$a \in K$, K a division ring, $R = K[t; S, D]$.

1) $\Delta(a) := \{a^c = S(c)ac^{-1} + D(c)c^{-1} \mid 0 \neq c \in K\}$.

2) T_a defines a left R-module structure on K via $f(t).x = f(T_a)(x)$.

3) In fact, ${}_RK \cong R/R(t-a)$ as left R-module.

4) ${}_RK_S$ where $S = End_R({}_RK) \cong End_R(R/R(t-a))$, a division ring. isomorphic to the division ring $C(a) := \{0 \neq x \in K \mid a^x = a\} \cup \{0\}$.

5) For any $a \in K$ and $f(t) \in R = K[t; S, D]$, $\ker f(T_a)$ is a right vector space on the division ring $C(a)$.

Theorem 1.14. *Let $f(t) \in R = K[t; S, D]$ be of degree n. We have*

(a) The roots of $f(t)$ belong to at most n conjugacy classes, say $\Delta(a_1), \ldots, \Delta(a_r)$; $r \leq n$ (Gordon Motzkin in "classical" case).

(b) $\sum_{i=1}^r dim_{C_i} \ker f(T_{a_i}) \leq n$.

For any $f(t) \in R = K[t; S, D]$ we thus "compute" the number of roots by adding the dimensions of the vector spaces consisting of "exponents" of roots in the different conjugacy classes...

Theorem 1.15. *let p be a prime number, $\mathbb{F}_q$ a finite field with $q = p^n$ elements, θ the Frobenius automorphism ($\theta(x) = x^p$). Then:*

a) There are p distinct classes of θ-conjugation in $\mathbb{F}_q$.

b) $0 \neq a \in \mathbb{F}_q$ we have $C^\theta(a) = \mathbb{F}_p$ and $C^\theta(0) = \mathbb{F}_q$.

c) $R = \mathbb{F}_q[t; \theta]$, $t - a$ for $a \in \mathbb{F}_q$ is

$$G(t) := [t - a \,|\, a \in \mathbb{F}_q]_l = t^{(p-1)n+1} - t.$$

We have $RG(t) = G(t)R$.

The polynomial $G(t)$ in the above theorem is a Wedderburn polynomial...
Conjecture (Werner) R a finite ring. Then the set $\{f(x) \in R[x] \mid f(R) = 0\}$ is a two-sided ideal.

Lemme 1.16. *Every nonzero element in a finite ring is a sum of finite number of units.*

Theorem 1.17. *Let R be a ring such that its elements are sums of a finite number of units. Then the set*
$K := \{f(t) \in R[t; \sigma, \delta] \mid f(R) = 0\}$ is a two sided ideal of $R[t; \sigma, \delta]$.

Proof. $f(t) \in K$ and $g(t) \in R[t; \sigma, \delta]$. Clear: $g(t)f(t) \in K$.
For $r \in R$ $(f(t)g(t))(r) = f(T_r)(g(r))$. There exists units $g(r) = u_1 + \cdots u_s$. Since $f(T_r)(u_i) = (f.u_i)(r) = f(r^{u_i})u_i = 0$ This gives $f(T_r)(u_1 + \cdots u_s) = 0$ □

d) Wedderburn polynomials and their factorizations

Definitions 1.18. 1. (a) A monic polynomial $p(t) \in R = K[t; S, D]$ (K a division ring) is a Wedderburn polynomial if we have equality in the "counting roots formula"(cf Theorem 1.14).

(b) For $a_1, \ldots, a_n \in K$ the matrix

$$V_n^{S,D}(a_1, \ldots, a_n) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ T_{a_1}(1) & T_{a_2}(1) & \ldots & T_{a_n}(1) \\ \ldots & \ldots & \ldots & \ldots \\ T_{a_1}^{n-1}(1) & T_{a_1}^{n-1}(1) & \ldots & T_{a_1}^{n-1}(1) \end{pmatrix}$$

Theorem 1.19. *Let $f(t) \in R = K[t; S, D]$ be a monic polynomial of degree n. The following are equivalent:*

(a) $f(t)$ is a Wedderburn polynomial.

(b) There exist n elements $a_1, \ldots, a_n \in K$ such that $f(t) = [t - a_1, \ldots, t - a_n]_l$ where $[g, h]_l$ stands for LLCM of g, h.

(c) There exist n elements $a_1, \ldots, a_n \in K$ such that

$$S(V)C_f V^{-1} + D(V)V^{-1} = Diag(a_1, \ldots, a_n)$$

Where C_f is the companion matrix of f and $V = V(a_1, \ldots, a_n)$

(d) Every quadratic factor of f is a Wedderburn polynomial.

Example 1.20. Construction of Wedderburn polynomials: Let $a, b \in K$ be two different elements in K.

$$f(t) := [t - a, t - b]_l = (t - b^{b-a})(t - a) = (t - a^{a-b})(t - b).$$

Assume now that $c \in K$ is such that $f(c) \neq 0$ then:

$$g(t) := [t - a, t - b, t - c]_l = (t - c^{f(c)})f(t).$$

Remarks 1.21.

(b) **Question**: Is every left V-domain a right V-domain?
Can we use $R = K[t; S, D]$ to construct such an example?
One necessary condition for R to be a right V domain is that every monic polynomial is Wedderburn... (-,T.Y.Lam, S.K.Jain)

(c) Matrices $A \in M_n(K)$ that are (S, D)-diagonalizable can be characterized by Wedderburn polynomials ($S \in Aut(K)$.)

How can we build all the linear factorizations of a Wedderburn polynomial?

Theorem 1.22. *Let $f \in R$ be a Wedderburn polynomial and $V(f)$ the set of its right roots.*

(a) Assume that $V(f) \subseteq \Delta(a)$, then the linear factorizations are in bijection with the complete flags of right $C(a)$-vector spaces in $E(f, a)$.

(b) Assume that $V(f) \subseteq \bigcup_{i=1}^{r} \Delta(a_i)$ then the linear factorizations of f are in bijection with the "shuffle complete flags" of $\bigcup_{i=1}^{r} E(f, a_i)$.

Since a polynomial which is linearly factorizable is a product of Wedderburn polynomials we can use the above factorizations to get factorizations of such polynomials.

Example 1.23. Let us describe all the factorizations of $f = [t - a^x, t - a]_l$. These factorizations are in bijection with the complete flags in the two dimensional vector space $E(f, a) = C + xC$ where $C := C^{S,D}(a)$. The flags are of the form $0 \neq yC \subset E(f, a)$. Apart from the flag $0 \subset xC \subset E(f, a)$, they are given by

$0 \subset (1+x\beta)C \subset E(f,a)$, where $\beta \in C^{S,D}(a)$. Hence we get the following factorizations $f = (t - a^{a-a^x})(t - a^x)$ and
$(t - a^{a-\gamma})(t - a^{1+x\beta})$, where $\gamma = a - a^{1+x\beta}$.

Pursuing the theme of the above example we can develop a theory of symmetric functions.

c)**Factorizations in $\mathbb{F}_q[t;\theta]$.**

Aim: reduce factorization in $\mathbb{F}_q[t;\theta]$ to factorisation in $\mathbb{F}_q[x]$

Definitions 1.24. p a prime number,
(a)$i \geq 1$, put $[i] := \frac{p^i-1}{p-1} = p^{i-1} + p^{i-2} + \cdots + 1$ and put $[0] = 0$.
(b) $q = p^n$. define $\mathbb{F}_q[x^{[]}] \subset \mathbb{F}_q[x]$ by:

$$\mathbb{F}_q[x^{[]}] := \{\sum_{i\geq 0} \alpha_i x^{[i]} \in \mathbb{F}_q[x]\}$$

Elements of $\mathbb{F}_q[x^{[]}]$ are called $[p]$-polynomials.

Extend θ to $F_q[x]$ via $\theta(x) = x^p$ i.e. $\theta(g) = g^p$ for $g \in F_q[x]$.

Let us consider $R := F_q[t;\theta] \subset S := F_q[x][t;\theta]$.

For $f \in R := \mathbb{F}_q[t;\theta] \subset \mathbb{F}_q[x][t;\theta]$
We may evaluate f in x.

Theorem 1.25. *Let*
$f(t) = \sum_{i=0}^{n} a_i t^i \in R := \mathbb{F}_q[t;\theta] \subset S := \mathbb{F}_q[x][t;\theta]$. *We have:*

1) *for every* $b \in \mathbb{F}_q$, $f(b) = \sum_{i=0}^{n} a_i b^{[i]}$.

2) $f^{[]}(x) = \sum_{i=0}^{n} a_i x^{[i]} \in \mathbb{F}_q[x^{[]}]$.

3) $\{f^{[]} | f \in R = \mathbb{F}_q[t;\theta]\} = \mathbb{F}_q[x^{[]}]$.

4) *For* $i \geq 0$ *and* $h(x) \in \mathbb{F}_q[x]$ *we have* $T_x^i(h) = h^{p^i} x^{[i]}$.

5) If $g(t) \in S = F_q[x][t;\theta]$ et $h(x) \in \mathbb{F}_q[x]$ $g(T_x)(h(x)) \in \mathbb{F}_q[x]h(x)$.

6) For $h(t) \in R = \mathbb{F}_q[t;\theta]$, $f(t) \in Rh(t)$ iff $f^{[]}(x) \in \mathbb{F}_q[x]h^{[]}(x)$.

Corollaire 1.26. *$f(t) \in \mathbb{F}_q[t;\theta]$ is irréducible iff the corresponding p-polynomial $f^{[]}$ does not have non trivial factors in $\mathbb{F}_q[x^{[]}]$.*

Ih) Roots and coefficients: twisted symmetric functions

Example 2.2. Suppose $x_1 \neq x_2$ are elements in K. We have :

$$[t-x_1, t-x_2]_l = \left(t - x_1^{x_1-x_2}\right)(t-x_2) = \left(t - x_2^{x_2-x_1}\right)(t-x_1)$$

Comparing coefficients of degree 0 and 1

$$\Lambda_1(x_1,x_2) = x_1^{x_1-x_2} + S(x_2) = x_2^{x_2-x_1} + S(x_1)$$

$$\Lambda_2(x_1,x_2) = x_1^{x_1-x_2} \cdot x_2 - D(x_2) = x_2^{x_2-x_1} \cdot x_1 - D(x_1)$$

Put $p_j = [t - x_i \mid i \leq j]_l$ for $j = 1, \ldots, n$. It is useful to also define $p_0 := 1$

Lemme 1.27. *$\{x_1, \ldots, x_n\} \subseteq K$ is P-independent if and only if for any $i \in \{1, \ldots, n-1\}, p_i(x_{i+1}) \neq 0$*

we put

$$y_i = x_i^{p_{i-1}(x_i)} \text{ for } i \in \{1, \ldots, n\}$$

We then have:

$$[t - x_j \mid j = 1, \ldots, i]_l = p_i(t) = (t-y_i)\ldots(t-y_1) \text{ for } i \in \{1, \ldots, n\}$$

2 II) n Variables.

II a) Iterated Ore extensions.

In what follows, we consider an iterated Ore extension

$$R = A[t_1; \sigma_1, \delta_1][t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n].$$

We will always assume that, for any $1 \leq i \leq n$, we have $\sigma_i(A) \subseteq A$ and $\delta_i(A) \subseteq A$.

Definitions 2.1. For $(a_1, \ldots, a_n) \in A^n$, we define

$$I = R(t_1 - a_1) + R(t_2 - a_2) + \cdots R(t_n - a_n)$$

$$I_n(a_1, \ldots, a_n) = R_1(t_1 - a_1) + \cdots + R_{n-1}(t_{n-1} - a_{n-1}) + R(t_n - a_n),$$

where, for each $1 \leq i \leq n$, R_i stands for $R_i = A[t_1; \sigma_1, \delta_1] \cdots [t_i; \sigma_i, \delta_i]$. the evaluation of $f(t_1, t_2, \ldots, t_n)$ at a point $(a_1, \ldots, a_n)$, denoted by $f(a_1, \ldots, a_n)$, as the representative in A of $f(t_1, t_2, \ldots, t_n)$ modulo I_n

Remark 2.2. 1) If $n = 1$, we obviously have $I = I_1$ and all points are good.
2) In general, $I_n \subset I$ are different.
3) The left R-module I can be the entire ring. This the case in the Weyl algebra $R = A_1(K) = K[t_1][t_2; id, \frac{d}{dt_1}]$ for the point $(0, 0)$ since we then have $t_2t_1 - t_1t_2 = 1$ and hence $Rt_1 + Rt_2 = R$.
3) Consider the Ore extension $R = K[t_1; \sigma_1][t_2; \sigma_2]$ where K is a field and σ_2 is an endomorphism of $K[t_1; \sigma_1]$ such that $\sigma_2(t_1) = t_1$. It is easy to check that for any $(a_1, a_2) \in K^2$ we have
$(t_2 - \sigma_1(a_2))(t_1 - a_1) + (-t_1 + \sigma_2(a_1))(t_2 - a_2) = \sigma_1(a_2)a_1 - \sigma_2(a_1)a_2$.
So that if $\sigma_1(a_2)a_1 - \sigma_2(a_1)a_2 \neq 0$, then the left ideal $I(a_1, a_2) = R$.

Definition 2.3. A point $(a_1, \dots, a_n) \in K^n$ will be called a good point if the two ways of evaluating a polynomial *in* $K[t_1; \sigma_1, \delta_1][t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n]$ at $(a_1, \dots, a_2)$ coincide i.e. if $I_n(a_1, \dots, a_n) = I$.

II b) Iterated PLT or MLT.

Definition 2.4. Let A a ring, $R = A[t_1; \sigma_1, \delta_1][t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n]$ be an iterated Ore extension, ${}_AV$ a left A-module, and $(T_1, \dots, T_n)$ be a sequence of maps in $\mathrm{End}(V, +)$ such that for each $1 \le i \le n$, T_i is a (σ_i, δ_i)-PLT of ${}_AV$. This sequence $(T_1, \dots, T_n)$ is called *good* if $({}_AV, T_1)$ gives a ${}_{R_1}V$ structure on V, and T_2 is a (σ_2, δ_2)-PLT on ${}_{R_1}V$ so that $({}_{R_1}V, T_2)$ defines an ${}_{R_2}V$ structure on V and inductively, for any $1 \le i < n$, T_{i+1} is a (σ_i, δ_i)-PLT on ${}_{R_i}V$-structure which leads to an R_{i+1} module structure on V.

Example 2.5. Let $a = (a_1, \dots, a_n) \in A^n$ and consider the following iterated Ore extension

$$R = A[t_1; \sigma_1, \delta_1][t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n].$$

For $1 \le i \le n$, we define the map $T_i : A \to A$ given by $T_i(x) = \sigma_i(x)a_i + \delta_i(x)$ for all $x \in A$. This sequence of PLT's defined on A corresponds to a left R-module structure on A.

Theorem 2.6. *Let A be a ring and $R = A[t_1; \sigma_1, \delta_1][t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n]$ an iterated Ore extension on A. For $a = (a_1, \dots, a_n) \in A^n$, we let $T_i = T_{a_i}$ be the PLT on A defined in the above example and $f = f(t_1, \dots, t_n) \in R$. Then the following statements hold.*

1. *For any $x \in A$, we have $(fx)(a_1, \dots, a_n) = f(T_{a_1}, \dots, T_{a_n})(x)$.*

2. We have $f(a_1,\ldots,a_n)=f(T_{a_1},\ldots,T_{a_n})(1)$*.*

3. For any $x\in U(A)$*, we have*

$$f(T_{a_1},\ldots,T_{a_n})(x)=(fx)(a_1,\ldots,a_n)=f(a_1^x,\ldots,a_n^x)x,$$

where for each $i\in\{1,\ldots,n\}$*,* $a_i^x=\sigma_i(x)a_ix^{-1}+\delta_i(x)x^{-1}$*.*

Example 2.7. The statement (3) above can be used to obtain a closed formula for the evaluation of $f(t_1,\ldots,t_n)=\sum\alpha_{l_1,\ldots,l_n}t_1^{l_1}t_2^{l_2}\cdots t_n^{l_n}$ at the point $(a_1,\ldots,a_n)\in K^n$. For instance, in the case in which $n=2$ and $(a,b)\in K^2$, we consider the evaluation of $f(t_1,t_2)=\sum_{i=0,j=0}^{l_1,l_2}a_{i,j}t_1^it_2^j$ at (a,b) and, assuming $x_j:=N_j^{\sigma_2,\delta_2}(b)\neq 0$ for $0\leq j\leq l_2$, we deduce that

$$f(a,b)=\sum_{i,j}a_{i,j}N_i^{\sigma_1,\delta_1}(a^{x_j})x_j.$$

Theorem 2.8. *Let* A *be a ring and* $R=A[t_1;\sigma_1,\delta_1][t_2;\sigma_2,\delta_2]\cdots[t_n;\sigma_n,\delta_n]$*. We consider* $(a_1,a_2,\ldots,a_n)\in A^n$ *and put*

$$I=R(t_1-a_1)+R(t_2-a_2)+\cdots+R(t_n-a_n),$$

and

$$I_n=R_1(t_1-a_1)+\cdots+R_{n-1}(t_{n-1}-a_{n-1})+R(t_n-a_n),$$

where, for each $1\leq i\leq n$*,* $R_i=A[t_1;\sigma_1,\delta_1]\cdots[t_i;\sigma_i,\delta_i]$*. With these notations, the following statements are equivalent:*

1. $I_n=I$*;*

2. $R(t_i-a_i)\subseteq I_n$*;*

3. $I\neq R$*;*

4. For $1 \leq i < j \leq n$*, we have* $t_j(t_i - a_i) \in I_n$*;*

5. For $1 \leq i < j \leq n$*, we have* $\sigma_j(t_i - a_i)a_j + \delta_j(t_i - a_i) \in I_n$*;*

6. For $1 \leq i < j \leq n$*, we have* $(t_j t_i)(a_1, \ldots, a_n) = \sigma_j(a_i)a_j + \delta_j(a_i)$*;*

7. For all $f, g \in R$*, we have*

$$(fg)(a_1, a_2, \ldots, a_n) = (f(T_{a_1}, T_{a_2}, \ldots, T_{a_n}) \circ g(T_{a_1}, T_{a_2}, \ldots, T_{a_n}))(1);$$

8. The sequence $(T_{a_1}, T_{a_2}, \ldots, T_{a_n})$ *of PLT on* A *is good;*

9. The map $\psi : R = A[t_1; \sigma_1, \delta_1][t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n] \longrightarrow End(A, +)$ *defined by* $\psi(f(t_1, \ldots, t_n)) = f(T_{a_1}, \ldots, T_{a_n})$ *is a ring homomorphism.*

II c) Other kind of skew multivariables polynomials.

Other kind of multivariable Ore-like extensions are used
a) (E. Wexkler-Kreindler) Let A be a ring and consider S the commutative semigroup generated by $t_1, \ldots, t_n$. Let R be the left free A-module with the element of S as a basis.
Let also $\sigma : A \to M_n(A)$ be a ring homomorphism and, for $1 \leq i \leq n$ let $\delta_i : A \to A$ be additive mappings. Commutation rules

$$1 \leq i \leq n, \forall a \in A \quad t_i a = \sum_{j=1}^{n} \sigma(a)_j t_j + \delta_i(a)$$

b) (Martinez Peñas)The same construction but we don't impose any commutation rule between the variables. (used in coding theory).

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Thank you !!