p-adic Geometry: Peter Scholze's 2018 Fields Medal

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PETER SCHOLZE'S 2018 FIELDS MEDAL CITATION

«For transforming arithmetic algebraic geometry over p-adic fields through his introduction of perfectoid spaces, with application to Galois representations, and for the development of new cohomology theories.»

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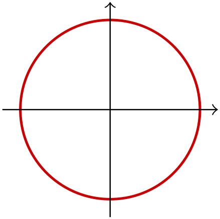
?	?	?	?
arithmetic algebraic geometry	p-adic numbers	Galois represen-tations	perfectoid spaces

Arithmetic Algebraic Geometry



$$x^2 + y^2 = 1$$

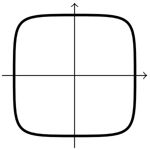
$x, y \in \mathbf{Q}$,

e.g. $(3/5, 4/5)$

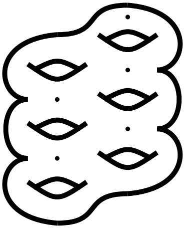
'geometry' 'algebraic' 'arithmetic'

Arithmetic Algebraic Geometry

Example. *Fermat curves* $X_n : x^n + y^n = 1$



$X_n(\mathbf{R})$



$X_n(\mathbf{C})$

Fermat's Last Theorem

$X_n(\mathbf{Q})$

$$|\overline{X}_3(\mathbf{F_p})| = p + 1 - a_p$$

where

$$q \prod (1-q^{3n})^2(1-q^{9n})^2 = \sum a_n q^n$$

$X_n(\mathbf{F_p})$

p-adic numbers

"numbers are like functions": compare

$$3x^2 - 7x + 2 = 5(x-2) + 3(x-2)^2 \quad \text{with} \quad 182 = 5 \cdot 7 + 3 \cdot 7^2$$

$$\tfrac{1}{3-x} = \textstyle\sum_{n \geq 0} (x-2)^n \quad \text{with} \quad -\tfrac{1}{6} = \textstyle\sum_{n \geq 0} 7^n \ ?$$

$$\text{power series in } (x-2)^n \quad \text{with} \quad \text{power series in } 7\ ??$$

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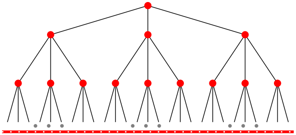
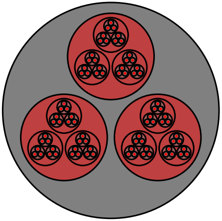


$$\mathbf{Z}_p = \varprojlim \left(\mathbf{Z}/p\mathbf{Z} \leftarrow \mathbf{Z}/p^2\mathbf{Z} \leftarrow \cdots \right)$$
$$= \text{sequences } a_n \in \mathbf{Z}/p^n\mathbf{Z} \text{ s.t } a_{n+1} \bmod p^n = a_n$$
$$= \text{'power series' } \sum_{n \geq 0} a_n p^n \text{ with } a_n \in \mathbf{F}_p$$



$$\mathbf{Q}_p = \mathbf{Z}_p[\tfrac{1}{p}] = \text{field of fractions of } \mathbf{Z}_p$$

***p*-adic metric:** $d_p(x, y) = p^{-k}$ if $x = y \bmod p^k$

Characteristic zero and characteristic p

The field $\mathbf{F}_p = \{0, \dots p-1\}$ has **characteristic p**, i.e. $\underbrace{1 + \cdots + 1}_{p} = 0$

Frobenius map:

$$\text{ring } R \supseteq \mathbf{F}_p \quad \rightsquigarrow \quad \varphi \colon R \to R \quad \varphi(x) = x^p$$

NEED TO CHECK:

$$\varphi(x+y) = (x+y)^p = x^p + \sum_{i=1}^{p-1} \underbrace{\frac{p!}{i!(p-i)!}}_{=0} x^i y^{p-i} + y^p = \varphi(x) + \varphi(y)$$

The Frobenius map makes many problems easier in characteristic p!

Characteristic zero and characteristic p

$$\begin{tikzcd}
\text{characteristic zero} \arrow[r, dotted, no head] \arrow[rrrr, bend right, "\text{perfectoid spaces}"'] & \mathbf{Q}_p & \mathbf{Z}_p \arrow[l] \arrow[r] & \mathbf{F}_p \arrow[r, dotted, no head] & \text{characteristic } p \arrow[loop below, "\text{Frobenius } \varphi"]
\end{tikzcd}$$

Fontaine–Wintenberger theorem

$$
\begin{tikzcd}
\mathbf{Q}_p(p^{1/p^\infty}) \arrow[r, equal] \arrow[d, no head] & {:K} \arrow[r, dotted, no head] & {K^\flat :} \arrow[r, equal] & \mathbf{F}_p((t))(t^{1/p^\infty}) \arrow[d, no head] \\
\cdots \arrow[d, no head] & & & \cdots \arrow[d, no head] \\
\mathbf{Q}_p(\sqrt[p]{p}) \arrow[d, no head] & & & \mathbf{F}_p((t))(\sqrt[p]{t}) \arrow[d, no head] \\
\mathbf{Q}_p \arrow[r, equal] & \{\sum a_n p^n,\ a_n \in \mathbf{F}_p\} \arrow[r, dotted, no head] & \{\sum a_n t^n,\ a_n \in \mathbf{F}_p\} \arrow[r, equal] & \mathbf{F}_p((t)).
\end{tikzcd}
$$

Theorem (Fontaine–Wintenberger 1979)

The fields K and $K^\flat$ have the same Galois groups. In other words, finite extensions of K correspond bijectively to finite extensions of $K^\flat$.

Prototype of a perfectoid space: the solenoid

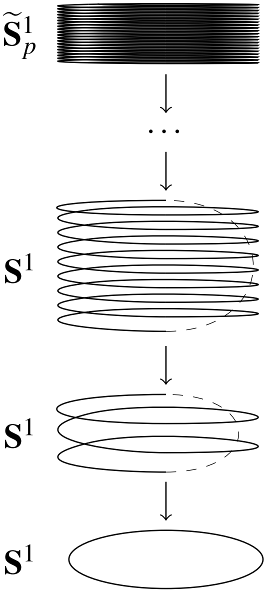
Circle solenoid

$$\widetilde{\mathbf{S}}^1_p = \varprojlim \left(\mathbf{S}^1 \xleftarrow{\varphi} \mathbf{S}^1 \xleftarrow{\varphi} \cdots \right)$$

$$= \text{sequences } z_n \in \mathbf{S}^1 \text{ s.t. } z_{n+1}^p = z_n$$

$\varphi(z) = z^p \colon \mathbf{S}^1 \to \mathbf{S}^1$ p-fold covering

$\mathbf{S}^1 = \{|z| = 1\}$ unit circle



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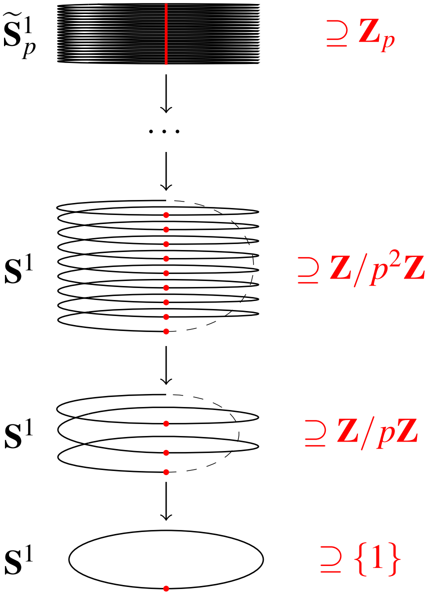

Perfectoid spaces

"Definition" (Scholze 2011)

A **perfectoid space** is a p-adic analytic space whose functions have 'enough p-power roots.'

Perfectoid spaces naturally appear as limits of towers of p-fold coverings over 'perfectoid fields' such as K and $K^\flat$.

Examples:

$$\text{“}\varprojlim\text{”}\left(\mathbf{P}^n_K \xleftarrow{\varphi} \mathbf{P}^n_K \xleftarrow{\varphi} \cdots\right) \qquad \varphi(x_0 : \ldots : x_n) = (x_0^p : \ldots : x_n^p)$$

$$\text{“}\varprojlim\text{”}\left(X_m \xleftarrow{\varphi} X_{pm} \xleftarrow{\varphi} X_{p^2m} \xleftarrow{\varphi} \cdots\right) \quad \text{tower of Fermat curves}$$

towers of abelian varieties ("tori"), Shimura varieties, ...

Tilting equivalence

Recall: $K = \mathbf{Q}_p(p^{1/p^\infty}) \sim K^\flat = \mathbf{F}_p((t))(t^{1/p^\infty})$ in the sense that

$$\left\{ \begin{matrix} \text{finite extensions} \\ \text{of } K \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite extensions} \\ \text{of } K^\flat \end{matrix} \right\}$$

Theorem (Tilting Equivalence, Scholze 2011)

$$\left\{ \begin{matrix} \text{perfectoid spaces} \\ \text{over } K \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{perfectoid spaces} \\ \text{over } K^\flat \end{matrix} \right\}$$

This equivalence preserves the ‘étale topology.’

“Theorem.” (Scholze 2012)

Every reasonable p-adic analytic space is ‘locally’ a perfectoid space.

Applications of perfectoid spaces

- cases of Deligne's weight-monodromy conjecture
- Shimura varieties, construction of Galois representations, Langlands program (with Caraiani and others)
- p-divisible groups (with Weinstein)
- p-adic Hodge theory (with Bhatt and Morrow)
- commutative algebra (André, Bhatt, Schwede–Ma)
- ...

Disclaimer. Avatars of perfectoid spaces appeared before Scholze in the work of Tate, Fontaine, Faltings, Kedlaya–Liu, ...

Illus. Dorota Budacz

Slogan:

After climbing an infinite tower, we arrive to the world of perfectoid spaces, whereupon we can jump over the fence separating characteristic zero and characteristic p geometry and eat the fruit of Frobenius' garden.

EXPANDED NOTES: `http://achinger.impan.pl/scholze.pdf` (in Polish)

CREDITS: 3-adic disk: A. T. Fomenko, featured in N. Koblitz *p-adic Numbers, p-adic Analysis, and Zeta Functions*, Springer 1984
P. S. photo: Bildarchiv des Mathematischen Forschungsinstituts Oberwolfach
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