Partial differential equations with linear growth

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Calculus of variations

The main goal of calculus of variations is to find critical points of functions defined over infinite-dimensional objects and study their properties.

In this lecture, we only consider *minimisation problems*, i.e., given a set X and a function $I : X \to \mathbb{R} \cup \{+\infty\}$, we look for solutions of

$$\min \left\{ I(u) : \ u \in X \right\}.$$

This type of problems appears frequently in relation to partial differential equations, via a formalism called the *Euler-Lagrange equations*.

Example: brachistochrone problem

Proposed by Johann Bernoulli in 1696, solved independently by himself and Newton in 1697.

Brachistochrone problem

Find the curve along which a point mass will move from point A to B in the shortest time.

What is the functional to minimise?

$$mgy = \frac{1}{2}mv^2 \implies v = \sqrt{2gy}$$

Since $dt = ds/v$,

$$I[y] = \int_{x_A}^{x_B} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\,dx.$$

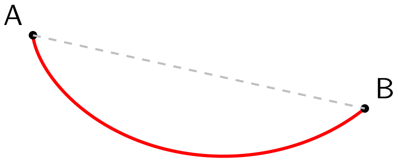

Figure: Shape of the path of quickest descent (brachistochrone).

Euler-Lagrange equations

For an integral functional of the form

$$I[y] = \int_{x_1}^{x_2} L(x, y, y')\, dx$$

where L is called the *Lagrangian*, any function $y(x)$ that minimises or maximises $I[y]$ satisfies the following differential equation:

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

For the brachistochrone problem, setting

$$L(x, y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}$$

one obtains the equation of (a part of) the (inverted) cycloid.

Euler-Lagrange equations

This concept can be applied to more general functionals: for a function $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ and the Lagrangian

$$L := L(x, u, u_{x_1}, ..., u_{x_N})$$

the Euler-Lagrange equation for $I = \int_\Omega L$ becomes

$$\frac{\partial L}{\partial u} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial u_{x_i}} \right) = 0$$

If the functional I is strictly convex, this becomes a 1-to-1 correspondence between the minimiser and the solution of the Euler-Lagrange equation.

This now becomes a *partial differential equation* (PDE).

Example: Laplace equation

Minimising the Dirichlet energy

$$I[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx = \frac{1}{2}\int_\Omega \left(\sum_{i=1}^N \left(\frac{\partial u}{\partial x_i} \right)^2 \right) \, dx$$

corresponds to the Euler-Lagrange equation

$$-\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} = 0$$

or equivalently

$$-\Delta u := -\operatorname{div}(\nabla u) = 0,$$

called the *Laplace equation*. It is very common in physics, e.g. in electromagnetism or heat transfer.

Weak solutions: Laplace equation

Consider the Laplace equations with *Dirichlet boundary conditions*

$$\begin{cases} -\mathrm{div}\,(\nabla u) = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

for sufficiently regular h. Solutions of the PDE are known to be smooth inside Ω; can we directly prove existence of smooth solutions?

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for sufficiently regular h. Solutions of the PDE are known to be smooth inside Ω; can we directly prove existence of smooth solutions?

Using the Euler-Lagrange equation, we may equivalently find a solution to the minimisation problem

$$\min\left\{ \int_\Omega |\nabla u|^2 \, dx : \ \ u|_{\partial\Omega} = h \right\}.$$

Direct method of calculus of variations

Let X be a complete metric space. Given a functional $I : X \to \mathbb{R} \cup \{+\infty\}$, our goal is to find a solution of the minimisation problem

$$\min\{I(u) : \, u \in X\}.$$

Assuming that the following two assumptions hold:

(Coe) Coercivity: for all $t \in \mathbb{R}$, every sequence $(u_n) \subset X$ with $I(u_n) \leq t$ has a convergent subsequence in X.

(Lsc) Lower semicontinuity: for every sequence $(u_n) \subset X$ with $u_n \to u$ in X, it holds that

$$I(u) \leq \liminf_{n \to \infty} I(u_n).$$

there exists at least one solution to the minimisation problem.

Weak solutions: Laplace equation

Applying the direct method of calculus of variations in $C^\infty(\Omega) \cap C(\overline{\Omega})$, we do not obtain a minimiser:

Step 1. Find a minimising sequence $u_k \in C^\infty(\Omega) \cap C(\overline{\Omega})$ with $u = h$ on $\partial\Omega$, i.e., $\int_\Omega |\nabla u|^2 \, dx \to \inf$.

Step 2. Thus, the minimising sequence $u_k \in C^\infty(\Omega) \cap C(\overline{\Omega})$ has uniformly bounded energy, i.e., $\int_\Omega |\nabla u|^2 \, dx \leq M$.

Step 3 - failure. This is not enough to conclude that there exists a limit function $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$; even in 1D the limit may fail to be $C^1(\Omega)$.

To obtain existence of solutions, we need a larger function space.

Failure of Step 3

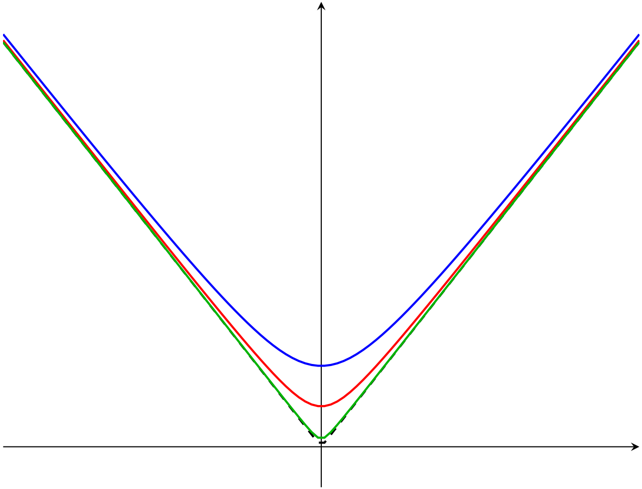

Figure: Smooth approximations of the modulus.

Weak solutions

We separately consider existence and regularity of solutions.

Step 1. Use the variational formulation to prove existence of a solution in a large enough class;

Step 2. Use a different set of techniques to conclude that this solution lies in a smaller class with better properties.

Sobolev spaces

The correct choice is the *Sobolev space* $W^{1,p}(\Omega)$ with $p = 2$, i.e.,

$$u \in W^{1,p}(\Omega) \Leftrightarrow u \in L^p(\Omega) \text{ and its weak derivative } \nabla u \in L^p(\Omega; \mathbb{R}^N),$$

where ∇u is the unique function *defined* via integration by parts, so that

$$\int_\Omega u \, \nabla \varphi \, dx = - \int_\Omega \varphi \, \nabla u \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We set $\|u\|_{W^{1,p}(\Omega)} = (\|u\|_p^p + \|\nabla u\|_p^p)^{1/p}$.

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Crucially for this argument, Sobolev spaces have three key properties:

(a) The embedding $id : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact;

(b) For $p \in (1, \infty)$, the Sobolev spaces are reflexive;

(c) The trace (boundary values) of every Sobolev function is well-defined.

Weak solutions: Laplace equation

We apply the direct method again.

Step 1. Find a minimising sequence $u_k \in W^{1,2}(\Omega)$ with $u = h$ on $\partial\Omega$, i.e., $\int_\Omega |\nabla u|^2 \, dx \to \inf$.

Step 2. Thus, the minimising sequence $u_k \in W^{1,2}(\Omega)$ has uniformly bounded energy, i.e., $\int_\Omega |\nabla u|^2 \, dx \leq M$.

Step 3 - success! There exists a limit function $u \in W^{1,2}(\Omega)$. Indeed, by the Poincaré inequality estimating the norm $\|u\|_2$ by $\|\nabla u\|_2$, reflexivity and the compact embedding, there exists $u \in W^{1,2}(\Omega)$ such that

$$u_n \to u \quad \text{in } L^2(\Omega) \qquad \text{and} \qquad \nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^N).$$

(We call this weak convergence in $W^{1,2}(\Omega)$.)

Weak solutions: Laplace equation

Step 4. u is a minimiser: by the lower semicontinuity of the Dirichlet energy with respect to weak convergence in $W^{1,2}(\Omega)$,

$$\inf \leq \int_\Omega |\nabla u|^2 \, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 \, dx \to \inf .$$

Step 5. u satisfies the boundary condition; one can show that the subspace

$$W^{1,2}_h(\Omega) := \Big\{ u \in W^{1,2}(\Omega) : \quad u|_{\partial\Omega} = h \Big\}$$

is weakly closed, so if $u_n \in W^{1,2}_h(\Omega)$, we also have $u \in W^{1,2}_h(\Omega)$. We thus have a solution to the minimisation problem $u \in W^{1,2}(\Omega)$.

One separately shows that it is smooth inside Ω: one possible approach is to prove that it lies in $C^1(\Omega)$, use linearity of the equation, and notice that every partial derivative $\frac{\partial u}{\partial x_i}$ also solves the Laplace equation.

Nonlinear PDEs

A generalisation of the above is the *p-Laplace equation*

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

for $p \in (1, \infty)$. Using the Euler-Lagrange equation, we may equivalently find a solution to the minimisation problem

$$\min \bigg\{ \int_\Omega |\nabla u|^p \, dx : \;\; u|_{\partial\Omega} = h \bigg\}.$$

A similar scheme produces solutions in $W^{1,p}(\Omega)$ for any admissible h. Solutions to this PDE are of class $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, but in general not better; since the equation is not linear, $\frac{\partial u}{\partial x_i}$ does not satisfy the same equation.

Linear growth functionals

In the formal limit $p \to 1$, we get the 1-*Laplace equation*

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Formally, it is the Euler-Lagrange equation of the *least gradient problem*

$$\min \left\{ \int_\Omega |\nabla u| \, dx : \ \ u \in W^{1,1}(\Omega), \ \ u|_{\partial\Omega} = h \right\}.$$

Is this problem well-posed?

Weak solutions: 1-Laplace equation

Let us once more try the direct method.

Step 1. Find a minimising sequence $u_k \in W^{1,1}(\Omega)$ with $u = h$ on $\partial\Omega$, i.e., $\int_\Omega |\nabla u| \, dx \to \inf$.

Step 2. Thus, the minimising sequence $u_k \in W^{1,1}(\Omega)$ has uniformly bounded energy, i.e., $\int_\Omega |\nabla u| \, dx \leq M$.

Step 3 - failure. There might be no limit function in $W^{1,1}(\Omega)$.

Since $W^{1,1}(\Omega)$ is not reflexive, the limiting sequence converges in $L^1(\Omega)$, but the gradients do not necessarily converge weakly in $L^1(\Omega; \mathbb{R}^N)$.

We again need a larger function space.

Failure of Step 3

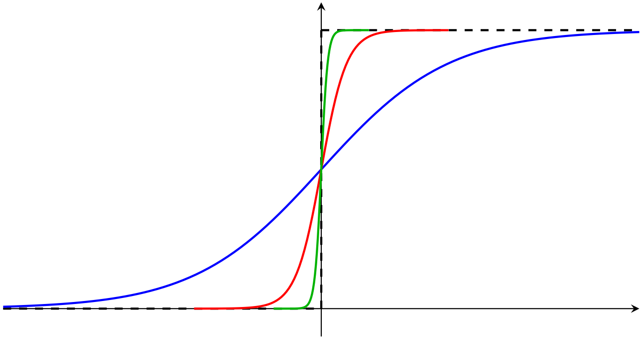

Figure: Smooth approximations of a step function.

BV spaces

The correct choice is the space of functions of *bounded variation* $BV(\Omega)$, i.e.,

$u \in BV(\Omega) \Leftrightarrow u \in L^1(\Omega)$ and its distributional derivative $Du \in \mathcal{M}(\Omega; \mathbb{R}^N)$,

where Du is the unique measure *defined* via integration by parts, so that

$$\int_\Omega u \, \nabla\varphi \, dx = - \int_\Omega \varphi \, dDu \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We set $\|u\|_{BV(\Omega)} = \|u\|_1 + \|Du\|_{\mathcal{M}}$.

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We set $\|u\|_{BV(\Omega)} = \|u\|_1 + \|Du\|_{\mathcal{M}}$.

Similarly to the Sobolev case, it holds that

(a) The embedding $id : BV(\Omega) \to L^1(\Omega)$ is compact;

(b) The trace (boundary values) of every BV function is well-defined.

Weak solutions: 1-Laplace equation

... and again the direct method.

Step 1. Find a minimising sequence $u_k \in W^{1,1}(\Omega)$ with $u = h$ on $\partial\Omega$, i.e., $\int_\Omega |\nabla u| \, dx \to \inf$.

Step 2. Thus, the minimising sequence $u_k \in W^{1,1}(\Omega)$ has uniformly bounded energy, i.e., $\int_\Omega |\nabla u| \, dx \leq M$.

Step 3 - success! There exists a limit function in $BV(\Omega)$. Indeed, by the Poincaré inequality estimating the norm $\|u_n\|_1$ by $\|\nabla u_n\|_1$, the compact embedding and lower semicontinuity of the total variation, there exists $u \in BV(\Omega)$ such that

$$u_n \to u \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \nabla u_n \rightharpoonup Du \quad \text{weakly* in } \mathcal{M}(\Omega; \mathbb{R}^N).$$

(We call this weak* convergence in $BV(\Omega)$.)

Weak solutions: 1-Laplace equation

Step 4. u is a minimiser: by the lower semicontinuity of the total variation with respect to convergence in $L^1(\Omega)$,

$$\inf \leq \int_\Omega |Du| \leq \liminf_{n\to\infty} \int_\Omega |Du_n| \to \inf .$$

Step 5 - failure. u does not necessarily satisfy the boundary condition; the subspace

$$BV_h(\Omega) := \left\{ u \in BV(\Omega) : \quad u|_{\partial\Omega} = h \right\}$$

is **not** weakly* closed, so if $u_n \in BV_h(\Omega)$, we may have that $u|_{\partial\Omega} \neq h$.

It turns out that for linear-growth PDEs attainment of boundary values depends on the geometry of the domain and the boundary data.

Failure of Step 5

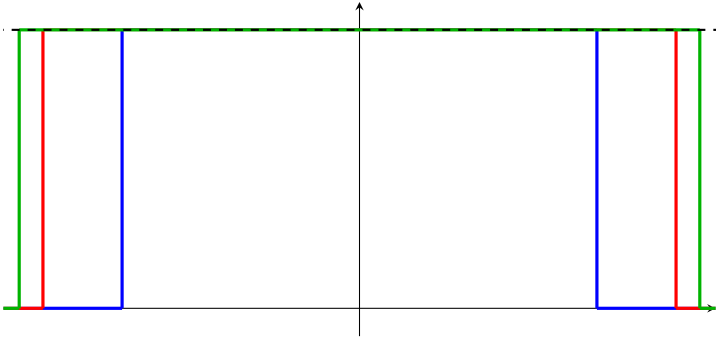


Figure: We take characteristing functions of an increasing family of intervals $\chi_{[-a,a]}(x)$ with $a \to 1^-$. The boundary values of the limit function is not equal to the limit of boundary values of the approximating sequence.

The least gradient problem

Thus, the correct formulation of the least gradient problem is

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}$$

which can be equivalently described as the 1-Laplace equation

$$\begin{cases} -\mathrm{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Note that the object $\frac{Du}{|Du|}$ has to be carefully defined and proving this equivalence is non-trivial.

- R.V. Kohn, G. Strang, Comm. Pure Appl. Math. **39** (1986).
- J.M. Mazón, J. Rossi, S. Segura de León, Indiana Univ. Math. J. (2014).

General linear growth functionals

More generally, consider minimisation of a linear growth integral functional

$$\min \left\{ \int_\Omega g(x, Du) : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\},$$

where

$$c_1|\xi| - c_2 \leq |g(x,\xi)| \leq c_3(1 + |\xi|).$$

Formally, the Euler-Lagrange equation for such a problem is

$$-\mathrm{div}(\nabla_\xi g(x, Du)) = 0.$$

- F. Andreu, V. Caselles, J.M. Mazón, Birkhäuser (2004).
- L. Beck, T. Schmidt, J. Funct. Anal. (2015).
- W. Górny, J.M. Mazón, J. Funct. Anal. (2022).
- W. Górny, J.M. Mazón, Publ. Mat. (2025).

Geometric viewpoint

Let us formally look at the equation

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = 0.$$

One can show that:

(a) If u is a solution, then $\chi_{\{u>t\}}$ is also a solution;

(b) For $u = \chi_E$ with ∂E smooth enough, the left-hand side is the (minus) mean curvature of ∂E;

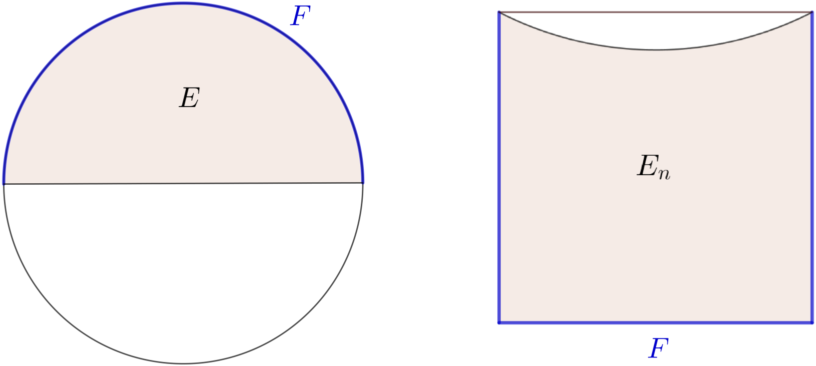
(c) Locally, after choosing the right coordinates the level sets even minimise the area functional $\int_B \sqrt{1 + |Du|^2}$ and thus the level sets are quite regular.

Geometric viewpoint

Setting $u = \chi_E$ and $h = \chi_F$ in

$$\min\left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}$$

the least gradient problem has a simple geometric meaning.



Existence and properties of solutions depend on the shape of the domain!

Classical results

Let $N \geq 2$. To tackle the question of boundary values of solutions to

$$\min \left\{ \int_\Omega |Du| : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}$$

one directly estimates the values of u at the boundary using geometric measure theory techniques.

If Ω is strictly convex, then

$$h \in C(\partial\Omega) \Rightarrow \exists! \text{ a solution } u \in BV(\Omega)$$

and $u \in C(\overline{\Omega})$.

If Ω is uniformly convex, then

$$h \in C^{0,\alpha}(\partial\Omega) \Rightarrow u \in C^{0,\alpha/2}(\overline{\Omega}).$$

P. Sternberg, G. Williams, W. Ziemer, J. Reine Angew. Math. (1992).

Modern research directions

The *anisotropic least gradient problem* is

$$\min \left\{ \int_\Omega \phi(x, Du) : \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}$$

where $\phi(x, \cdot)$ is a uniformly bounded family of norms.

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Existence of solutions: discontinuous boundary data

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(a') If Ω is strictly convex and h is continuous a.e. on $\partial\Omega$, there exist solutions for every norm ϕ;

(b) If we allow for less regular h, for every two different norms ϕ_1 and ϕ_2 there exists h which is admissible in the anisotropic least gradient problem for only one of them.

W. Górny, Indiana Univ. Math. J. (2021).

W. Górny, Math. Ann. (2023).

Existence of solutions: positive result

Theorem (WG, Indiana Univ. Math. J. (2021))

Let $\Omega \subset \mathbb{R}^N$ be strictly convex and suppose that $h \in L^1(\partial\Omega)$ is continuous $\mathcal{H}^{N-1}$-a.e. Then, there exists a solution $u \in BV(\Omega)$ to

$$\min \left\{ \int_\Omega \phi(Du) : \; u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}$$

for every norm ϕ and $u(y) \xrightarrow{y \to x_0} h(x_0)$ at each continuity point x_0 of h.

Proof:

1. Show existence of a generalised solution u;
2. Approximate h by continuous functions h_n which satisfy a series of key inequalities around a given continuity point x_0;
3. Modify each level set of u to construct a competitor for minimality which locally around x_0 satisfies a similar series of inequalities;
4. Verify that this competitor has lower energy unless the boundary datum is attained at x_0.

Existence of solutions: negative result

Theorem (WG, Math. Ann. (2023))

Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Suppose that ϕ_1 and ϕ_2 are two strictly convex norms of class C^2. Unless $\phi_1 = c\phi_2$ for some $c > 0$, there exists a function $h \in L^\infty(\partial\Omega) \setminus BV(\partial\Omega)$ such that there exists a solution to

$$\min \left\{ \int_\Omega \phi_1(Du) : \; u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}$$

but there is no solution to

$$\min \left\{ \int_\Omega \phi_2(Du) : \; u \in BV(\Omega), \quad u|_{\partial\Omega} = h \right\}.$$

Proof: explicit construction of a fat Cantor set C on $\partial\Omega$ such that $h = \chi_C$ has the desired properties.

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Thank you for your attention!