On uniformly continuous surjections between C_p-spaces over metrizable spaces

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New results in my talk are published in joint papers [1], [2], [3], [4].

[1] Ali Emre Eysen, Arkady Leiderman and Vesko Valov,
On uniformly continuous surjections between C_p-spaces over metrizable spaces, 2024 (submitted for publication).
https://arxiv.org/pdf/2408.01870

[2] Jerzy Kąkol, Ondřej Kurka and Arkady Leiderman,
Some classes of topological spaces extending the class of Δ-spaces, Proc. Amer. Math. Soc. **152** (2024), 883–899.

[3] Jerzy Kąkol and Arkady Leiderman,
Basic properties of X for which the space $C_p(X)$ is distinguished, Proc. Amer. Math. Soc., series B, **8** (2021), 267–280.

[4] Jerzy Kąkol and Arkady Leiderman,
On linear continuous operators between distinguished spaces $C_p(X)$, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM (2021) 115:199.

1. Relations of ℓ- and u-dominance. Some known results.

For a Tychonoff space X, by $C(X)$ we denote the linear space of all continuous real-valued functions on X. $C^*(X)$ is a subspace of $C(X)$ consisting of the bounded functions.
We write $C_p(X)$ (resp., $C_p^*(X)$) if $C(X)$ (resp., $C^*(X)$) is endowed with the pointwise convergence topology.
By $D(X)$ we denote either $C^*(X)$ or $C(X)$.

Following A. Arkhangel'skii, we say that a space Y is **ℓ-dominated** (**u-dominated**) by a space X if there exists a linear (uniform, respectively) continuous operator onto $T : C_p(X) \to C_p(Y)$.

There are many topological properties which are invariant under defined above relations, and there are many which are not.

By dimension we mean the *covering dimension* dim.

- Let X and Y be metrizable compact spaces. If free topological groups $F(X)$ and $F(Y)$ are isomorphic then $\dim X = \dim Y$ (M.I. Graev (1940-s)).

Let $C_p(X)$ and $C_p(Y)$ be linearly homeomorphic.

- Assuming that X and Y are metrizable compact spaces, D. Pavlovskii (1980) proved that $\dim X = \dim Y$.
- Assuming that X and Y are compact spaces, A. Arkhangel'skii (1980) proved that $\dim X = \dim Y$.
- For any Tychonoff spaces X and Y, V. Pestov (1982) proved that $\dim X = \dim Y$.

- Let X and Y be Tychonoff spaces. If $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic, then $\dim X = \dim Y$ (S. Gul'ko (1987, in English 1992)).

Results about ℓ-dominance

A. Arhangel'skii in 1990 posed a problem for metrizable compacta X and Y, whether $\dim Y \leq \dim X$ if there is continuous (and open) linear surjection from $C_p(X)$ onto $C_p(Y)$, i.e. if Y is ℓ-dominated by X.
These questions were answered negatively.

- For every finite-dimensional metrizable compact space Y there exists a continuous linear surjection $T : C_p([0,1]) \to C_p(Y)$ (A.L., S. Morris, V. Pestov (1992, published in 1997)).
- For every natural $n > 1$ there exist n-dimensional metrizable compact space Y and one-dimensional metrizable compact space X such that $C_p(X)$ admits a continuous open linear surjection onto $C_p(Y)$ (A.L., M. Levin, V. Pestov (1997)).
- Later, M. Levin (2011) showed that for every finite-dimensional metrizable compact space Y there exists a continuous open linear surjection $T : C_p([0,1]) \to C_p(Y)$.

- It is easy to show that if there exists a continuous linear surjection $T : C_p([0,1]) \to C_p(Y)$, then Y must be a strongly countable-dimensional metrizable compact space.
- If there exists a continuous linear surjection $T : C_p([0,1]) \to C_p(Y)$, then Y does not have to be finite-dimensional (P. Gartside, Z. Feng (2017)).
- The problem of a characterization of those strongly countable-dimensional metrizable compact spaces Y which admit a continuous linear surjection $T : C_p([0,1]) \to C_p(Y)$, is still open.

So, the dimension can be increased by ℓ-dominance. However, it turned out that zero-dimensional case is an exception.

- If there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$ for compact metrizable spaces, then $\dim X = 0$ implies that $\dim Y = 0$ (A.L., M. Levin, V. Pestov (1997)).
- If there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$ for compact spaces, then $\dim X = 0$ implies that $\dim Y = 0$ (A.L., K. Kawamura (2017)).

The natural question arose whether the same statement is true without assumption of compactness of X and Y. Very recently, this difficult question was answered positively.

- Let X and Y be Tychonoff spaces. If there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$, then $\dim X = 0$ implies that $\dim Y = 0$ (A. Eysen, V. Valov (2024)).

Results about u-dominance

The following question was posed by R. Górak, M. Krupski and W. Marciszewski).

Open Problem

Let X be a compact metrizable strongly countable-dimensional [zero-dimensional] space. Suppose that Y is u-dominated by X. Is Y necessarily strongly countable-dimensional [zero-dimensional]?

- A map $T : D_p(X) \to D_p(Y)$ is called *uniformly continuous* if for every neighborhood U of the zero function in $D_p(Y)$ there is a neighborhood V of the zero function in $D_p(X)$ such that $f, g \in D_p(X)$ and $f - g \in V$ implies $T(f) - T(g) \in U$.
- For every bounded function $f \in C(X)$ by $||f||$ we denote its *supremum*-norm. A map $T : D(X) \to D(Y)$ is called *c-good* if for every bounded function $g \in C(Y)$ there exists a bounded function $f \in C(X)$ such that $T(f) = g$ and $||f|| \leq c||g||$.

Theorem 1.1

(R. Górak, M. Krupski and W. Marciszewski (2019)) Let X be a compact metrizable space. Suppose that there is a uniformly continuous surjection $T : C_p(X) \to C_p(Y)$ which is c-good for some $c > 0$. Then

(a) If X is zero-dimensional, then so is Y.

(b) If X is strongly countable-dimensional, then so is Y.

Theorem 1.2

(R. Górak, M. Krupski and W. Marciszewski (2019)) Every strongly countable-dimensional metrizable compact space K is u-dominated by the unit interval $[0, 1]$.

Theorem 1.1 was recently generalized for σ-compact metrizable spaces.

Theorem 1.3

(A. Eysen, V. Valov (2024)) Let X and Y be σ-compact metrizable spaces. Suppose that there is a uniformly continuous surjection $T : C_p(X) \to C_p(Y)$ which is c-good for some $c > 0$. Then

(a) If X is zero-dimensional, then so is Y.

(b) If X is strongly countable-dimensional, then so is Y.

We strengthen the last result in two directions: we prove this statement for all (not necessarily σ-compact) metrizable spaces, and assuming a weaker condition that T is a uniformly continuous inversely bounded surjection.

A map $T : D(X) \to D(Y)$ is called *inversely bounded* if for every norm bounded sequence $\{g_n\} \subset C^*(Y)$ there is a norm bounded sequence $\{f_n\} \subset C^*(X)$ with $T(f_n) = g_n$ for each $n \in \mathbb{N}$.

Evidently, every linear continuous map between $D_p(X)$ and $D_p(Y)$ is uniformly continuous and every c-good map is inversely bounded. Also, every linear continuous surjection $T : C_p^*(X) \to C_p^*(Y)$, where X and Y are arbitrary Tychonoff spaces, is inversely bounded.

In fact we develop a general scheme for the proof as follows. For the brevity, we write $X \in \mathcal{P}$ if X has the property $\mathcal{P}$.

We consider the properties $\mathcal{P}$ of metrizable spaces such that:

(a) if $X \in \mathcal{P}$ and $F \subset X$ is closed, then $F \in \mathcal{P}$;

(b) $\mathcal{P}$ is closed under finite products;

(c) if X is a countable union of closed subsets each having the property $\mathcal{P}$, then $X \in \mathcal{P}$;

(d) if $f : X \to Y$ is a closed continuous map with finite fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$.

From the classical results of dimension theory it follows that *zero-dimensionality*, *countable-dimensionality* and *strongly countable-dimensionality* satisfy conditions $(a) - (d)$ above.

Theorem 2.1

Let X be a metrizable space and Y be a perfectly normal topological space. Suppose that $T : D_p(X) \to D_p(Y)$ is a uniformly continuous inversely bounded surjection. For any topological property $\mathcal{P}$ satisfying conditions $(a) - (d)$ above, if $X \in \mathcal{P}$ then $Y \in \mathcal{P}$.

Corollary 2.2

Let X be a metrizable space and Y be a perfectly normal topological space. Suppose that $T : D_p(X) \to D_p(Y)$ is a uniformly continuous inversely bounded surjection.

(i) If X is either countable-dimensional or strongly countable-dimensional, then so is Y.

(ii) If X is zero-dimensional, then so is Y.

Note that item (ii) was established in Theorem 1.3 for arbitrary Tychonoff spaces X, Y and c-good surjections T. However, we don't know whether every uniformly continuous inversely bounded map is c-good for some $c > 0$.

A linear continuous version of Theorem 2.1 is also true.

Theorem 2.3

Let X be a metrizable space and Y be a perfectly normal topological space. Suppose that $T : D_p(X) \to D_p(Y)$ is a linear continuous surjection. For any topological property $\mathcal{P}$ satisfying conditions $(a) - (d)$ above, if $X \in \mathcal{P}$ then $Y \in \mathcal{P}$.

3. Scattered-like properties $\mathcal{P}$

Definitions

① A topological space X is called *scattered* if every closed subset $A \subseteq X$ has an isolated (in A) point.

② If X is a countable union of closed scattered subspaces then X is called *strongly σ-scattered*.

③ A Tychonoff space X is called pseudocompact if every continuous function $f : X \to \mathbb{R}$ is bounded.

Open Problem

Let X and Y be Tychonoff spaces.

(1) Suppose that $T : D_p(X) \to D_p(Y)$ is a continuous linear surjection (continuous linear isomorphism). Is Y scattered provided X is scattered?

(2) Suppose that $T : D_p(X) \to D_p(Y)$ is a uniformly continuous inversely bounded surjection. Is Y scattered provided X is scattered?

Quick result

Let X and Y be compact spaces. Suppose that $T : C_p(X) \to C_p(Y)$ is a continuous linear surjection. If X is scattered then Y is also scattered.

Proof.

Consider T as an operator between Banach spaces. Then T remains continuous. Banach spaces $C(X)$ is Asplund if and only if X is scattered. Then $C(Y)$ also is Asplund, i.e. Y is scattered.

④ A topological space X is said to be a *Δ-space* (*Δ_1-space*) if for every decreasing sequence $\{D_n : n \in \omega\}$ of subsets of X (countable subsets of X, respectively) with empty intersection, there is a decreasing sequence $\{V_n : n \in \omega\}$ consisting of open subsets of X, also with empty intersection, and such that $D_n \subseteq V_n$ for every $n \in \omega$.

Equivalently, X is a Δ-space (Δ_1-space) iff every countable sequence of disjoint sets (distinct points) admits a point-finite open expansion.

⑤ A Δ-space $X \subset \mathbb{R}$ is said to be a *Δ-set*.

Δ-sets $X \subset \mathbb{R}$ are defined by Reed and van Douwen (1980).

⑥ A topological space X is said to be a *Q-space* if every subset of X is a G_δ-set (equivalently, every subset of X is a F_σ-set).

Q-sets $X \subset \mathbb{R}$ are defined by Hausdorff (1933).

⑦ $X \subset \mathbb{R}$ is called a *λ-set* if each countable $A \subset X$ is G_δ in X.

λ-sets $X \subset \mathbb{R}$ are defined by Kuratowski (1933).

- Q-set $\Rightarrow$ Δ-set $\Rightarrow$ λ-set;
  similarly, Q-space $\Rightarrow$ Δ-space $\Rightarrow$ Δ_1-space.
- Existence of uncountable Q- and Δ-sets depends on additional axioms of ZFC; there are uncountable λ-sets in ZFC.

Theorem 3.0

(A.L., J. Kąkol)

- X is a Δ-space $\Leftrightarrow$ $C_p(X)$ is a distinguished locally convex space.
- A Corson compact space X is a Δ-space $\Leftrightarrow$ X is a scattered Eberlein compact space.
- $[0, \omega_1]$ is an example of a scattered compact but not Δ-space.

(A.L., J. Kąkol, O. Kurka)

- Let X be Čech-complete. Then X is a Δ_1-space $\Leftrightarrow$ X is scattered.
- Let X be pseudocompact. Then X is a Δ_1-space $\Leftrightarrow$ every countable subset of X is scattered.

Theorem 3.1

(A.L., J. Kąkol) Let Y be ℓ-dominated by X.

(a) If X is σ-scattered (σ-discrete), then Y is σ-scattered (σ-discrete, respectively).

(b) If X is a scattered Eberlein compact space, then Y also is a scattered Eberlein compact space.

(c) If X is a Δ-space, then Y also is a Δ-space.

(d) Let X and Y be normal spaces (for instance, let both be metrizable spaces or both be subsets of the real line $\mathbb{R}$). If X is a Q-space, then Y also is a Q-space.

(e) Let X and Y be metrizable spaces. If X is scattered, then Y also is scattered.

Theorem 3.1

(A.L., V. Valov)

(f) Let X and Y be metrizable spaces. Suppose that $T : C_p^*(X) \to C_p^*(Y)$ is a linear continuous surjection. If X is scattered, then Y also is scattered.

Theorem 3.1 (e), (f) strengthen the following result of J. Baars:

Let X and Y be metrizable spaces.

(a) Suppose that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is scattered if and only if Y is scattered.

(b) Suppose that $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic. Then X is scattered if and only if Y is scattered.

Sketch of the proof of Theorem 3.1 (e)

Let Y be ℓ-dominated by X. Let X and Y be metrizable spaces. If X is scattered, then Y also is scattered.

If X is metrizable and scattered, then X homeomorphically embeds into a scattered Eberlein compact space (T. Banakh & A.L.), hence X is a Δ-space. So, by Theorem 3.1 (c) the space Y is also a Δ-space. From another hand, every metrizable and scattered space is completely metrizable. We have that a metrizable space Y is ℓ-dominated by a completely mertizable space X, therefore Y is completely metrizable due to the known result of J. Baars, J. de Groot and J. Pelant. Finally, Y is a Čech-complete Δ-space, and therefore Y is scattered.

Theorem 3.2

(A.L., J. Kąkol, O. Kurka) Let Y be ℓ-dominated by X.

(a) If X is a Δ_1-space, then Y also is a Δ_1-space.

(b) Let X and Y be metrizable spaces. If X is a λ-space, then Y also is a λ-space.

(c) If X is pseudocompact and every countable set in X is scattered, then Y has the same properties.

(d) If X is a compact scattered space, then Y is a pseudocompact space such that its Stone–Čech compactification βY is scattered.

Sketch of the proof of Theorem 3.2 (b)

Let Y be ℓ-dominated by X. Let X and Y be metrizable spaces. If X is a λ-space, then Y also is a λ-space.

Metrizable space is a Δ_1-space iff every countable subset of X is G_δ. So, X is a Δ_1-space, then applying Theorem 3.2 (a), Y is also a Δ_1-space. Finally, every countable subset of Y is G_δ, i.e. Y also is a λ-space.

Proposition

Let X and Y be metrizable spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is a linear continuous surjection. If X is strongly σ-scattered, then so is Y.

We don't know whether analogues of Theorem 3.1 (e), (f) above are valid under a weaker assumption: $T : D_p(X) \to D_p(Y)$ is a uniformly continuous surjection.

Question: Let X and Y be (separable) metrizable spaces and let $T : D_p(X) \to D_p(Y)$ be a uniformly continuous surjection. Is Y scattered provided X is scattered?

This is because the following major question posed by W. Marciszewski and J. Pelant is open.

Open Problem

Let X and Y be (separable) metrizable spaces and let $T : D_p(X) \to D_p(Y)$ be a uniformly continuous surjection (uniform homeomorphism). Let X be completely metrizable. Is Y also completely metrizable?

Moreover, the next problem is also open:

Open Problem

Let X and Y be (separable) metrizable spaces and let $T : D_p(X) \to D_p(Y)$ be an inversely bounded uniformly continuous surjection. Let X be completely metrizable. Is Y also completely metrizable?

Theorem 3.3

(A.L., V. Valov) Let X and Y be metrizable spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is an inversely bounded uniformly continuous surjection. If X is strongly σ-scattered, then Y also is strongly σ-scattered.

Proof.

Any product of finitely many scattered (resp., strongly σ-scattered) spaces is scattered (resp., strongly σ-scattered). Evidently, any closed subset of a strongly σ-scattered space is strongly σ-scattered. It is also true that the preimage of a strongly σ-scattered space under a continuous map with finite fibers is strongly σ-scattered. Hence, all properties (a)-(d) from Theorem 2.1 are satisfied and we complete the proof.

Theorem 3.4

(A.L., V. Valov) Let X and Y be metrizable spaces. Suppose that $T : D_p(X) \to D_p(Y)$ is an inversely bounded uniformly continuous surjection. If X is a Δ_1-space then Y also is a Δ_1-space.

Proof.

All properties (a)-(d) from Theorem 2.1 are satisfied.

Theorem 3.5

(A.L., J. Kąkol) Let α be a fixed infinite countable ordinal. Then for a Tychonoff space Y the following are equivalent.

(1) There exists a linear continuous surjection $T : C_p([1,\alpha]) \to C_p(Y)$.

(2) Y is homeomorphic to $[1,\beta]$, where β is a countable ordinal such that either $\beta < \alpha$, or $\alpha \leq \beta < \alpha^\omega$.

Sketch of the proof of Theorem 3.5

Assumption (1) implies that Y has to be a countable compact space, i.e Y is homeomorphic to $[1, \beta]$, where β is a countable ordinal. If $\beta < \alpha$ there is nothing to prove. So, let us assume that $\alpha \leq \beta$. Applying the Closed Graph Theorem we consider T as a linear continuous operator from the Banach space $C([1, \alpha])$ onto the Banach space $C([1, \beta])$.
Recall that the Szlenk index of a Banach space E, denoted $\mathrm{Sz}(E)$, is an ordinal number, which is invariant under linear isomorphisms. The key tool is the following precise result of Samuel.

Sketch of the proof of Theorem 3.5

Fact A. For any $0 \leq \gamma < \omega_1$

$$\mathrm{Sz}(C([1, \omega^{\omega^\gamma}])) = \omega^{\gamma+1}.$$

We need also

Fact B. Let E_1 and E_2 be given Banach spaces with norm-separable duals. Assume that E_2 is isomorphic to a subspace of a quotient space of E_1. Then $\mathrm{Sz}(E_2) \leq \mathrm{Sz}(E_1)$.

In order to finish the proof of (1) $\Rightarrow$ (2) suppose the contrary: $\beta \geq \alpha^\omega$. Then by Fact A, $\mathrm{Sz}(C([1, \beta])) > \mathrm{Sz}(C([1, \alpha]))$ which contradicts Fact B.
(2) $\Rightarrow$ (1) is known.

Thank you!