On sequences of homomorphisms into measure algebras and the Efimov problem

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Joint work with Piotr Borodulin-Nadzieja.

Preliminaries

$\mathbb{A}, \mathbb{B}$ — Boolean algebras

$\mathcal{H}(\mathbb{A}, \mathbb{B})$ — the family of all Boolean homomorphisms $\mathbb{A} \to \mathbb{B}$

κ — a cardinal (finite or infinite)

$Bor(2^\kappa)$ — the Borel σ-field on 2^κ

λ_κ — the standard product measure on 2^κ

$\mathcal{N}(\lambda_\kappa) = \{A \in Bor(2^\kappa) \colon\ \lambda_\kappa(A) = 0\}$

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$\mathbb{M}_\kappa = Bor(2^\kappa)/\mathcal{N}(\lambda_\kappa)$

$\mathbb{M} = \mathbb{M}_\omega = Bor([0,1])/\mathcal{N}$

$\mathbb{M}_0 = \{0, 1\}$

Ultrafilters in $V^{\mathbb{M}_\kappa}$ induce homomorphisms in V

Let V be a ground model and $\mathbb{A} \in V$ a Boolean algebra.

$\mathbb{A}$ is a Boolean algebra in any $\mathbb{M}_\kappa$-generic extension $V[G]$.

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$$1 \Vdash_{\mathbb{M}_\kappa} \text{“}\dot{\mathcal{U}} \text{ is an ultrafilter on } \mathbb{A}\text{”}.$$

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In V, we define a Boolean homomorphism $\phi_{\dot{\mathcal{U}}} \colon \mathbb{A} \to \mathbb{M}_\kappa$:

$$\phi_{\dot{\mathcal{U}}}(A) = [\![A \in \dot{\mathcal{U}}]\!]$$

for every $A \in \mathbb{A}$.

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Define an $\mathbb{M}_\kappa$-name $\dot{\mathcal{U}}_\phi$:

$$\dot{\mathcal{U}}_\phi = \{(A, \phi(A))\colon\ A \in \mathbb{A}\}.$$

Then:

$$1 \Vdash_{\mathbb{M}_\kappa} \text{“}\dot{\mathcal{U}}_\phi \text{ is an ultrafilter on } \mathbb{A}\text{”}.$$

Duality between homomorphisms and ultrafilters

homomorphisms $\phi\colon \mathbb{A} \to \mathbb{M}_\kappa$

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Fact 1

If $\dot{\mathcal{U}}$ is an $\mathbb{M}_\kappa$-name for an ultrafilter on $\mathbb{A}$, then:

$$\Vdash_{\mathbb{M}_\kappa} \dot{\mathcal{U}}_{(\phi_{\dot{\mathcal{U}}})} = \dot{\mathcal{U}}.$$

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Fact 2

If $\phi\colon \mathbb{A} \to \mathbb{M}_\kappa$ is a homomorphism, then:

$$\phi_{(\dot{\mathcal{U}}_\phi)} = \phi.$$

But what about sequences?

Fix $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ and let $\dot{\mathcal{U}}_n = \dot{\mathcal{U}}_{\phi_n}$ and $\dot{\mathcal{U}} = \dot{\mathcal{U}}_\phi$.

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Question

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Proposition

The following conditions are equivalent:

1. $\Vdash_{\mathbb{M}_\kappa}$ "$(\dot{\mathcal{U}}_n)$ converges to $\dot{\mathcal{U}}$",
2. for every $A \in \mathbb{A}$ it holds:

$$\phi(A) = \bigwedge_n \bigvee_{m \geqslant n} \phi_m(A)$$

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If (2) holds, then we say that (ϕ_n) *converges algebraically to* ϕ.

Two topologies on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

Fréchet–Nikodym metric on $\mathbb{M}_\kappa$

For every $A, B \in \mathbb{M}_\kappa$ put:

$$d_\kappa(A, B) = \lambda_\kappa(A \triangle B).$$

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Pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

We may endow $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ with *the pointwise topology*:

$$V(\phi, A, \varepsilon) = \{\psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)\colon\ d_\kappa(\phi(A), \psi(A)) < \varepsilon\},$$

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where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa), A \in \mathbb{A}, \varepsilon > 0$.

$St(\mathbb{A})$ and $\mathcal{H}(\mathbb{A}, \mathbb{M}_0)$ with the pointwise topology are homeomorphic. In particular, $St(\mathbb{A})$ always embeds into $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$.

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Uniform metric on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

For every $\phi, \psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ put:

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d_{hom} is a metric bounded by 1.

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Fact

$\mathcal{H}(\mathbb{A}, \mathbb{M}_0)$ with the uniform topology is a discrete space.

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Convergence of sequences

1. (ϕ_n) *converges pointwise* to ϕ if it converges to ϕ in the pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$.
2. (ϕ_n) *converges uniformly* to ϕ if it converges to ϕ in the uniform topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$.

Algebraic/pointwise/uniform convergences

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If $\kappa = 0$, then algebraic convergence $\Leftrightarrow$ pointwise convergence.

Convergence of ultrafilters vs. convergence of homomorphisms

Proposition

The following conditions are equivalent:

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1. If $\Vdash_{\mathbb{M}_\kappa} \forall^\infty n \in \omega\colon\ \dot{\mathcal{U}}_n = \dot{\mathcal{U}}$, then (ϕ_n) converges uniformly to ϕ.

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Theorem

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2. If (ϕ_n) converges uniformly to ϕ, then for almost all $n \in \omega$ there is $p_n \in \mathbb{M}_\kappa$ such that $p_n \Vdash \dot{\mathcal{U}}_n = \dot{\mathcal{U}}$.

Interlude—distinguishing ultrafilters

Let $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ be $\mathbb{M}_\kappa$-names for ultrafilters on $\mathbb{A}$ st. $1 \Vdash \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$.

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If $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ are $\mathbb{M}_\kappa$-names for ultrafilters on $\mathbb{A}$ st. $1 \Vdash \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$, then for every $\varepsilon > 0$ there exists $p \in \mathbb{M}_\kappa$ and $C \in \mathbb{A}$ such that

- $\lambda_\kappa(p) > 1/4 - \varepsilon$, and
- $p \Vdash C \in \dot{\mathcal{U}} \triangle \dot{\mathcal{V}}$.

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Remark: $1/4$ is optimal!

The Efimov problem

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Consistent examples

1. CH (Fedorchuk, Dow–Pichardo-Mendoza, Talagrand)
2. $\diamondsuit$ (Fedorchuk, Kunen–Džamonja, de la Vega, S.–Zdomskyy)
3. $\mathfrak{s} = \omega_1$ & $\mathfrak{c} = 2^{\omega_1}$ (Fedorchuk)
4. $\mathrm{cof}([\mathfrak{s}]^\omega, \subseteq) = \mathfrak{s}$ & $2^\mathfrak{s} < 2^\mathfrak{c}$ (Dow)
5. $\mathfrak{b} = \mathfrak{c}$ (Dow–Shelah)
6. $\mathrm{cof}([\mathrm{cof}\,\mathcal{N}]^\omega, \subseteq) = \mathrm{cof}(\mathcal{N}) < \mathfrak{c}$ (S.)
7. and in many models obtained by forcing...

Efimov spaces in the random model

Theorem (Dow–Fremlin)

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If $\mathbb{A} \in V$ is a σ-complete Boolean algebra, then

$$\Vdash_{\mathbb{M}_\kappa} \text{“}St(\mathbb{A}) \text{ has no non-trivial convergent sequences.”}$$

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There is an Efimov space in the random model.

If $\mathbb{A} \in V$ is a σ-complete Boolean algebra, then

$$\Vdash_{\mathbb{M}_\kappa} \text{“}St(\mathbb{A}) \text{ has no non-trivial convergent sequences.”}$$

Theorem

If $\mathbb{A} \in V$, then TFAE:

1. $\Vdash_{\mathbb{M}_\kappa}$ “$St(\mathbb{A})$ has no non-trivial convergent sequences”;
2. every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ converges uniformly.

Efimov spaces in the random model

Corollary

If, in V, $\mathcal{A}$ is such a Boolean algebra that it has size $< \kappa$ and every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ converges uniformly, then, in $V^{\mathbb{M}_\kappa}$, the Stone space $St(\mathbb{A})$ is an Efimov space.

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Corollary

If $\mathbb{A}$ is such a Boolean algebra that $St(\mathbb{A})$ is an F-space, then every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ is uniformly convergent.

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Corollary

If $\mathbb{A}$ is such a Boolean algebra that $St(\mathbb{A})$ is an F-space, then every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ is uniformly convergent.
In particular, that holds for $\mathbb{A} = \wp(\omega)$ and $\mathbb{A} = \wp(\omega)/Fin$.

Two more topologies on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

Stone duality

Let $\mathbb{A}$ and $\mathbb{B}$ be Boolean algebras.

There is a one-to-one correspondence between homomorphisms $\mathbb{A} \to \mathbb{B}$ and continuous functions $St(\mathbb{B}) \to St(\mathbb{A})$:

$$\mathcal{H}(\mathbb{A}, \mathbb{B}) \ni \phi \mapsto f_\phi \in C(St(\mathbb{B}), St(\mathbb{A}))$$

$$f_\phi^{-1}[A] = \phi(A), \quad \forall A \in \mathbb{A}$$

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Borel Fréchet–Nikodym (pseudo)metric on $\mathbb{M}_\kappa$

For every $A, B \in Bor(St(\mathbb{M}_\kappa))$ put:

$$d_\kappa^{Bor}(A, B) = \widehat{\lambda}_\kappa(A \triangle B).$$

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d_κ^{Bor} is a pseudometric.

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Borel pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

Sub-basic open sets:

$$V(\phi, A, \varepsilon) = \{\psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa) : \; d_\kappa^{Bor}(f_\phi^{-1}[A], f_\psi^{-1}[A]) < \varepsilon\},$$

where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa), A \in Bor(St(\mathbb{A})), \varepsilon > 0$.

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(ϕ_n) *converges Borel pointwise* to ϕ if it converges to ϕ in the Borel pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$.

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$$d_{hom}^{Bor}(\phi, \psi) = \sup\{d_\kappa^{Bor}(f_\phi^{-1}[A], f_\psi^{-1}[A]) :\ A \in Bor(St(\mathbb{A}))\}.$$

d_{hom}^{Bor} is a metric bounded by 1.

Two (really?) more topologies on $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$

Fact

For every $\mathbb{A}$ and κ, $d_{hom} = d_{hom}^{Bor}$.

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Let $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$.

(ϕ_n) converges (Borel) uniformly to ϕ

$$\Downarrow$$

(ϕ_n) converges Borel pointwise to ϕ

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$$\Uparrow$$

(ϕ_n) converges algebraically to ϕ

Topologies on the space of probability measures

$P(\mathbb{A})$ — finitely additive probability measures on $\mathbb{A}$

Norm topology

$\mu, \nu \in P(\mathbb{A})$:

$$d_{var}(\mu, \nu) = \sup_{\substack{A,B \in \mathbb{A} \\ A \vee B = 0}} \left(|\mu(A) - \nu(A)| + |\mu(B) - \nu(B)| \right)$$

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Weak topology

$\mu \in P(\mathbb{A})$, $\varphi \in C(St(\mathbb{A}))^{**}$, $\varepsilon > 0$:

$$V(\mu; \varphi; \varepsilon) = \{\nu \in P(\mathbb{A}) \colon \ |\varphi(\mu) - \varphi(\nu)| < \varepsilon\}.$$

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Weak* topology

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Topologies on the space of probability measures

Weak* topology — equivalent definition

$\mu \in P(\mathbb{A})$, $A \in \mathbb{A}$, $\varepsilon > 0$:

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Topologies on the space of probability measures

Weak* topology — equivalent definition

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Fact

Let $\mu_n, \mu \in P(\mathbb{A})$. The sequence (μ_n) converges to μ with respect to the weak topology if and only if $(\widehat{\mu}_n(B))$ converges to $\widehat{\mu}(B)$ for every Borel set $B \subseteq St(\mathbb{A})$.

Homomorphisms and measures

$$F\colon\ \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa) \to P(\mathbb{A})$$

$$F(\phi) = \lambda_\kappa \circ \phi$$

$$F(\phi)(A) = \lambda_\kappa(\phi(A))$$

Homomorphisms and measures

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F is:

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Proposition

F is:

1. uniformly-norm continuous;
2. pointwise-weak* continuous.

Homomorphisms and measures

Corollary

$\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$. Then:

1. if (ϕ_n) converges uniformly to ϕ, then $(\lambda_\kappa \circ \phi_n)$ converges to $\lambda_\kappa \circ \phi$ in norm;

Homomorphisms and measures

Corollary

$\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$. Then:

1. if (ϕ_n) converges uniformly to ϕ, then $(\lambda_\kappa \circ \phi_n)$ converges to $\lambda_\kappa \circ \phi$ in norm;
2. if (ϕ_n) converges Borel pointwise to ϕ, then $(\lambda_\kappa \circ \phi_n)$ converges to $\lambda_\kappa \circ \phi$ weakly;

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Corollary

$\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$. Then:

1. if (ϕ_n) converges uniformly to ϕ, then $(\lambda_\kappa \circ \phi_n)$ converges to $\lambda_\kappa \circ \phi$ in norm;
2. if (ϕ_n) converges Borel pointwise to ϕ, then $(\lambda_\kappa \circ \phi_n)$ converges to $\lambda_\kappa \circ \phi$ weakly;
3. if (ϕ_n) converges pointwise to ϕ, then $(\lambda_\kappa \circ \phi_n)$ converges to $\lambda_\kappa \circ \phi$ weakly*.

Homomorphisms and measures

Uniform countable additivity

A sequence (μ_k) of Radon probability measures on a compact space K is *uniformly countably additive* if for every descending sequence (E_n) of Borel sets such that $\bigcap E_n = \emptyset$ and every $\varepsilon > 0$ there is $N \in \omega$ such that $\mu_k(E_n) < \varepsilon$ for every $n \geqslant N$ and $k \in \omega$.

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Nikodym Convergence Theorem

Every weakly convergent sequence of Radon measures on a compact space is uniformly countably additive.

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Nikodym Convergence Theorem

Every weakly convergent sequence of Radon measures on a compact space is uniformly countably additive.

Theorem

$\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$. If (ϕ_n) converges pointwise to ϕ and $(\widehat{\lambda}_\kappa \circ f_{\phi_n}^{-1})$ is uniformly countably additive, then (ϕ_n) converges Borel pointwise to ϕ.

Homomorphisms and measures

Grothendieck property

A Boolean algebra $\mathbb{A}$ has *the Grothendieck property* if every weakly* convergent sequence of measures on $St(\mathbb{A})$ is weakly convergent.

Examples: σ-complete Boolean algebras
Non-examples: countable Boolean algebras

Homomorphisms and measures

Grothendieck property

A Boolean algebra $\mathbb{A}$ has *the Grothendieck property* if every weakly* convergent sequence of measures on $St(\mathbb{A})$ is weakly convergent.

Examples: σ-complete Boolean algebras
Non-examples: countable Boolean algebras

Theorem

If $\mathbb{A}$ has the Grothendieck property, then every pointwise convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ is Borel pointwise convergent.

The end

Thank you for your attention!