On sequences of homomorphisms into measure algebras and the Efimov problem

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Joint work with Piotr Borodulin-Nadzieja.

Preliminaries

 \mathbb{A},\mathbb{B} — Boolean algebras

 $\mathcal{H}(\mathbb{A},\mathbb{B})$ — the family of all Boolean homomorphisms $\mathbb{A}\to\mathbb{B}$

 κ — a cardinal (finite or infinite)

 $Bor(2^{\kappa})$ — the Borel σ -field on 2^{κ}

 λ_{κ} — the standard product measure on 2^{κ}

 $\mathcal{N}(\lambda_{\kappa}) = \{A \in Bor(2^{\kappa}) \colon \ \lambda_{\kappa}(A) = 0\}$

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 $\mathbb{M}_\kappa = \textit{Bor}(2^\kappa)/\mathcal{N}(\lambda_\kappa)$

 $\mathbb{M}=\mathbb{M}_{\omega}=\textit{Bor}([0,1])/\mathcal{N}$

 $\mathbb{M}_0 = \{0,1\}$

Ultrafilters in $V^{\mathbb{M}_{\kappa}}$ induce homomorphisms in V

Let V be a ground model and $\mathbb{A} \in V$ a Boolean algebra.

A is a Boolean algebra in any \mathbb{M}_{κ} -generic extension V[G].

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Let $\dot{\mathcal{U}}$ be an \mathbb{M}_{κ} -name for an ultrafilter on \mathbb{A} , that is,

 $1 \Vdash_{\mathbb{M}_{\kappa}}$ " $\dot{\mathcal{U}}$ is an ultrafilter on \mathbb{A} ".

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In V, we define a Boolean homomorphism $\phi_{\dot{\mathcal{U}}} \colon \mathbb{A} \to \mathbb{M}_{\kappa}$:

$$\phi_{\dot{\mathcal{U}}}(A) = \llbracket A \in \dot{\mathcal{U}} \rrbracket$$

for every $A \in \mathbb{A}$.

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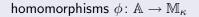
Define an \mathbb{M}_{κ} -name $\dot{\mathcal{U}}_{\phi}$:

$$\dot{\mathcal{U}}_{\phi} = \{ (A, \phi(A)) \colon A \in \mathbb{A} \}.$$

Then:

$$1 \Vdash_{\mathbb{M}_{\kappa}} ``\mathcal{U}_{\phi}$$
 is an ultrafilter on \mathbb{A} ".

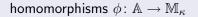
Duality between homomorphisms and ultrafilters



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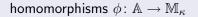
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Fact 1

If $\dot{\mathcal{U}}$ is an \mathbb{M}_{κ} -name for an ultrafilter on \mathbb{A} , then:

$$\Vdash_{\mathbb{M}_{\kappa}} \dot{\mathcal{U}}_{(\phi_{\dot{\mathcal{U}}})} = \dot{\mathcal{U}}.$$

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Fact 2

If $\phi \colon \mathbb{A} \to \mathbb{M}_{\kappa}$ is a homomorphism, then:

$$\phi_{(\dot{\mathcal{U}}_{\phi})} = \phi.$$

Fix $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_\kappa)$ and let $\dot{\mathcal{U}}_n = \dot{\mathcal{U}}_{\phi_n}$ and $\dot{\mathcal{U}} = \dot{\mathcal{U}}_{\phi}$.

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Question

What can we say about convergence properties of the sequence $(\dot{\mathcal{U}}_n)$ in $St(\mathbb{A}) \cap V^{\mathbb{M}_{\kappa}}$? When is it convergent to $\dot{\mathcal{U}}$?

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Proposition

The following conditions are equivalent:

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$$\Vdash_{\mathbb{M}_{\kappa}}$$
 " $(\dot{\mathcal{U}}_n)$ converges to $\dot{\mathcal{U}}$ "

2 for every $A \in \mathbb{A}$ it holds:

$$\phi(A) = \bigwedge_n \bigvee_{m \ge n} \phi_m(A)$$

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If (2) holds, then we say that (ϕ_n) converges algebraically to ϕ .

Two topologies on $\mathcal{H}(\mathbb{A},\overline{\mathbb{M}_{\kappa}})$

Fréchet–Nikodym metric on \mathbb{M}_{κ}

For every $A, B \in \mathbb{M}_{\kappa}$ put:

$$d_{\kappa}(A,B) = \lambda_{\kappa}(A \triangle B).$$

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Pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$

We may endow $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ with *the pointwise topology*:

$$V(\phi, A, \varepsilon) = \{ \psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa}) : d_{\kappa}(\phi(A), \psi(A)) < \varepsilon \},\$$

where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa}), A \in \mathbb{A}, \varepsilon > 0$.

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where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa}), A \in \mathbb{A}, \varepsilon > 0$.

 $St(\mathbb{A})$ and $\mathcal{H}(\mathbb{A}, \mathbb{M}_0)$ with the pointwise topology are homeomorphic. In particular, $St(\mathbb{A})$ always embeds into $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$.

Uniform metric on $\mathcal{H}(\mathbb{A},\mathbb{M}_{\kappa})$

For every $\phi, \psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ put:

$$d_{hom}(\phi,\psi) = \sup\{d_{\kappa}(\phi(A),\psi(A)): A \in \mathbb{A}\}.$$

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 d_{hom} is a metric bounded by 1.

Fact

 $\mathcal{H}(\mathbb{A}, \mathbb{M}_0)$ with the uniform topology is a discrete space.

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Fact

 $\mathcal{H}(\mathbb{A}, \mathbb{M}_0)$ with the uniform topology is a discrete space.

Convergence of sequences

- (φ_n) converges pointwise to φ if it converges to φ in the pointwise topology on H(A, M_κ).
- (φ_n) converges uniformly to φ if it converges to φ in the uniform topology on H(A, M_κ).

Let $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$.

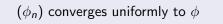
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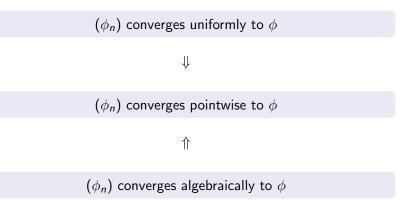
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∜

↑

 (ϕ_n) converges algebraically to ϕ

Let $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$.



If $\kappa = 0$, then algebraic convergence \Leftrightarrow pointwise convergence.

Proposition

The following conditions are equivalent:

- $\Vdash_{\mathbb{M}_{\kappa}}$ " $(\dot{\mathcal{U}}_n)$ converges to $\dot{\mathcal{U}}$ ",
- **2** (ϕ_n) converges algebraically to ϕ .

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Theorem

• If
$$\Vdash_{\mathbb{M}_{\kappa}} \forall^{\infty} n \in \omega$$
: $\dot{\mathcal{U}}_{n} = \dot{\mathcal{U}}$, then (ϕ_{n}) converges uniformly to ϕ .

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Theorem

- If $\Vdash_{\mathbb{M}_{\kappa}} \forall^{\infty} n \in \omega$: $\dot{\mathcal{U}}_{n} = \dot{\mathcal{U}}$, then (ϕ_{n}) converges uniformly to ϕ .
- ② If (ϕ_n) converges uniformly to ϕ , then for almost all $n \in \omega$ there is $p_n \in \mathbb{M}_{\kappa}$ such that $p_n \Vdash \dot{\mathcal{U}}_n = \dot{\mathcal{U}}$.

Let $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ be \mathbb{M}_{κ} -names for ultrafilters on \mathbb{A} st. $1 \Vdash \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$.

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Theorem

If $\dot{\mathcal{U}}$ and $\dot{\mathcal{V}}$ are \mathbb{M}_{κ} -names for ultrafilters on \mathbb{A} st. $1 \Vdash \dot{\mathcal{U}} \neq \dot{\mathcal{V}}$, then for every $\varepsilon > 0$ there exists $p \in \mathbb{M}_{\kappa}$ and $C \in \mathbb{A}$ such that

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$$\lambda_\kappa(p)>1/4-arepsilon$$
 , and

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 , and

• $p \Vdash C \in \dot{\mathcal{U}} \triangle \dot{\mathcal{V}}.$

Remark: 1/4 is optimal!

The Efimov problem

Problem

Does there exists an *Efimov space*, i.e. an infinite compact Hausdorff space with no non-trivial convergent sequences nor any copies of $\beta \omega$?

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Consistent examples

- O CH (Fedorchuk, Dow-Pichardo-Mendoza, Talagrand)
- ② ◊ (Fedorchuk, Kunen–Džamonja, de la Vega, S.–Zdomskyy)

$${f 3}\,\,{\mathfrak s}=\omega_1\,\,{\&}\,\,{\mathfrak c}=2^{\omega_1}$$
 (Fedorchuk)

- $\operatorname{cof}([\mathfrak{s}]^{\omega}, \subseteq) = \mathfrak{s} \& 2^{\mathfrak{s}} < 2^{\mathfrak{c}} (\operatorname{Dow})$
- **(**) $\mathfrak{b} = \mathfrak{c}$ (Dow-Shelah)
- $\operatorname{cof}([\operatorname{cof} \mathcal{N}]^{\omega}, \subseteq) = \operatorname{cof}(\mathcal{N}) < \mathfrak{c}(S.)$
- and in many models obtained by forcing...

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If $\mathbb{A} \in V$ is a $\sigma\text{-complete}$ Boolean algebra, then

 $\Vdash_{\mathbb{M}_{\kappa}}$ " $St(\mathbb{A})$ has no non-trivial convergent sequences."

Theorem

If $\mathbb{A} \in V$, then TFAE:

- $\Vdash_{\mathbb{M}_{\kappa}}$ "*St*(A) has no non-trivial convergent sequences";
- every algebraically convergent sequence in H(A, M_κ) converges uniformly.

If, in V, \mathcal{A} is such a Boolean algebra that it has size $< \kappa$ and every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ converges uniformly, then, in $V^{\mathbb{M}_{\kappa}}$, the Stone space $St(\mathbb{A})$ is an Efimov space.

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Corollary

If A is such a Boolean algebra that $St(\mathbb{A})$ is an F-space, then every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ is uniformly convergent.

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Corollary

If A is such a Boolean algebra that $St(\mathbb{A})$ is an F-space, then every algebraically convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ is uniformly convergent.

In particular, that holds for $\mathbb{A} = \wp(\omega)$ and $\mathbb{A} = \wp(\omega)/Fin$.

Stone duality

Let $\mathbb A$ and $\mathbb B$ be Boolean algebras.

There is a one-to-one correspondence between homomorphisms $\mathbb{A} \to \mathbb{B}$ and continuous functions $St(\mathbb{B}) \to St(\mathbb{A})$:

$$\mathcal{H}(\mathbb{A},\mathbb{B}) \ni \phi \mapsto f_{\phi} \in C(St(\mathbb{B}),St(\mathbb{A}))$$

$$f_{\phi}^{-1}[A] = \phi(A), \quad \forall A \in \mathbb{A}$$

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Borel Fréchet–Nikodym (pseudo)metric on \mathbb{M}_{κ}

For every $A, B \in Bor(St(\mathbb{M}_{\kappa}))$ put:

$$d_{\kappa}^{Bor}(A,B) = \widehat{\lambda}_{\kappa}(A \triangle B).$$

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 d_{κ}^{Bor} is a pseudometric.

Borel pointwise topology on $\mathcal{H}(\mathbb{A},\mathbb{M}_{\kappa})$

Sub-basic open sets:

$$V(\phi, A, \varepsilon) = \{ \psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa}) \colon \ d_{\kappa}^{Bor}(f_{\phi}^{-1}[A], f_{\psi}^{-1}[A]) < \varepsilon \},\$$

where $\phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa}), A \in Bor(St(\mathbb{A})), \varepsilon > 0$.

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Pointwise Borel convergence of sequences

 (ϕ_n) converges Borel pointwise to ϕ if it converges to ϕ in the Borel pointwise topology on $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$.

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Borel uniform metric on $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$

For every $\phi, \psi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ put:

$$d_{hom}^{\mathcal{B}or}(\phi,\psi) = \sup\{d_{\kappa}^{\mathcal{B}or}(f_{\phi}^{-1}[A],f_{\psi}^{-1}[A]): A \in \mathcal{B}or(St(\mathbb{A}))\}.$$

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 d_{hom}^{Bor} is a metric bounded by 1.

Fact

For every \mathbb{A} and κ , $d_{hom} = d_{hom}^{Bor}$.

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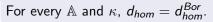
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 (ϕ_n) converges Borel pointwise to ϕ





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Fact For every A and κ , $d_{hom} = d_{hom}^{Bor}$. Let $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$. (ϕ_n) converges (Borel) uniformly to ϕ \downarrow (ϕ_n) converges Borel pointwise to ϕ 1 (ϕ_n) converges pointwise to ϕ ≙ (ϕ_n) converges algebraically to ϕ

Topologies on the space of probability measures

 $P(\mathbb{A})$ — finitely additive probability measures on \mathbb{A}

Norm topology

 $\mu, \nu \in P(\mathbb{A})$:

$$d_{\mathsf{var}}(\mu,\nu) = \sup_{\substack{A,B \in \mathbb{A} \\ A \lor B = 0}} (|\mu(A) - \nu(A)| + |\mu(B) - \nu(B)|)$$

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Weak topology

 $\mu \in \mathcal{P}(\mathbb{A}), \ arphi \in \mathcal{C}(\mathcal{S}t(\mathbb{A}))^{**}, \ arepsilon > 0$:

$$\mathscr{V}(\mu;arphi;arepsilon)=\{
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Weak topology

$$\mu \in P(\mathbb{A}), \ \varphi \in C(St(\mathbb{A}))^{**}, \ \varepsilon > 0$$
:

$$\mathcal{W}(\mu; \varphi; \varepsilon) = \{
u \in \mathcal{P}(\mathbb{A}) \colon |\varphi(\mu) - \varphi(\nu)| < \varepsilon \}.$$

Weak* topology

 $\mu \in P(\mathbb{A}), f \in C(St(\mathbb{A})), \varepsilon > 0$:

 $V(\mu; f; \varepsilon) = \{ \nu \in P(\mathbb{A}) : |\mu(f) - \nu(f)| < \varepsilon \}.$

Weak* topology — equivalent definition

 $\mu \in P(\mathbb{A}), \ A \in \mathbb{A}, \ \varepsilon > 0$:

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Fact

Let $\mu_n, \mu \in P(\mathbb{A})$. The sequence (μ_n) converges to μ with respect to the weak topology if and only if $(\hat{\mu}_n(B))$ converges to $\hat{\mu}(B)$ for every Borel set $B \subseteq St(\mathbb{A})$.

Homomorphisms and measures

Homomorphisms and measures

$$egin{aligned} F \colon & \mathcal{H}(\mathbb{A},\mathbb{M}_\kappa) o P(\mathbb{A}) \ & F(\phi) = \lambda_\kappa \circ \phi \ & F(\phi)(A) = \lambda_\kappa(\phi(A)) \end{aligned}$$

Proposition

F is:

uniformly-norm continuous;

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Proposition

F is:

- uniformly-norm continuous;
- 2 pointwise-weak* continuous.

- $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$. Then:
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Uniform countable additivity

A sequence (μ_k) of Radon probability measures on a compact space K is *uniformly countably additive* if for every descending sequence (E_n) of Borel sets such that $\bigcap E_n = \emptyset$ and every $\varepsilon > 0$ there is $N \in \omega$ such that $\mu_k(E_n) < \varepsilon$ for every $n \ge N$ and $k \in \omega$.

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Nikodym Convergence Theorem

Every weakly convergent sequence of Radon measures on a compact space is uniformly countably additive.

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Nikodym Convergence Theorem

Every weakly convergent sequence of Radon measures on a compact space is uniformly countably additive.

Theorem

 $\phi_n, \phi \in \mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$. If (ϕ_n) converges pointwise to ϕ and $(\widehat{\lambda}_{\kappa} \circ f_{\phi_n}^{-1})$ is uniformly countably additive, then (ϕ_n) converges Borel pointwise to ϕ .

Grothendieck property

A Boolean algebra \mathbb{A} has the Grothendieck property if every weakly* convergent sequence of measures on $St(\mathbb{A})$ is weakly convergent.

Examples: σ -complete Boolean algebras Non-examples: countable Boolean algebras

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Theorem

If \mathbb{A} has the Grothendieck property, then every pointwise convergent sequence in $\mathcal{H}(\mathbb{A}, \mathbb{M}_{\kappa})$ is Borel pointwise convergent.

Thank you for your attention!