On forcing names for ultrafilters

Piotr Borodulin-Nadzieja

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seminar in Warsaw, 2021

Based on . . .

The talk is based on two preprints:

- PBN, Damian Sobota, *On sequences of homomorphisms into measure algebras and the Efimov Problem* (arxiv)
- PBN, Katarzyna Cegiełka, *On measures induced by forcing names for ultrafilters*

Names for ultrafilters

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- reals are ultrafilters on the Cantor algebra,
- Stone spaces of *old* Boolean algebras may provide interesting examples of topological spaces.

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Consider $\varphi\colon \mathbb{A} \to \mathbb{P}$ defined by

$$\varphi(A) = \|A \in \dot{u}\|.$$

Then φ is a Boolean homomorphism.

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Consider a $\mathbb{P}$-name $\dot{u}$ defined by

$$\dot{\varphi} = \{\langle A, \varphi(A)\rangle \colon A \in \mathbb{A}\}.$$

Then $\dot{\varphi}$ is a name for an ultrafilter on $\mathbb{A}$.

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Remark. Ultrafilters = homomorphisms to $\{0, 1\}$.

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- $\mathbb{M}_\kappa$ supports the standard Haar measure λ_κ,
- $\mathbb{M}_1 = \{0,1\}$,
- $\mathbb{M}_\omega = \mathrm{Bor}[0,1]_{/\mathcal{N}}$,
- Forcing with $\mathbb{M}_\kappa$ = adding κ random reals (for $\kappa > \omega$).

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Remark. By a "measure" we mean here a *finitely additive* measure.

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- If μ is purely atomic, then

$$1 \Vdash \dot{\varphi} \in V.$$

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Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{C}$ (a real). Let μ be the measure induced by φ.

- if μ is the standard Lebesgue measure, then $\dot{\varphi}$ is a "random" real.
- if μ is non-atomic, then $\dot{\varphi}$ is a "new" real.

Example

Suppose that α is a property of subsets of ω such that

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- Let $\varphi \colon \mathbb{C} \to \mathbb{M}_\kappa$ be a homomorphism such that $\mu = \lambda_\kappa \circ \varphi$.
- Then $\dot{\varphi}$ is a name for a new subset and $1 \Vdash \alpha(\dot{\varphi})$.

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Under MA *each well-ordered chain in $\mathcal{P}(\omega)/Fin$ of size $< \mathfrak{c}$ can be extended.*

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Proposition

Assume GCH. *Suppose* $(\varphi_\alpha)_{\alpha<\omega_2}$ *is a sequence of homomorphisms* $\varphi_\alpha\colon \mathbb{C} \to \mathbb{M}_{\omega_2}$. *Then there are* $\alpha \neq \beta \in \omega_2$ *and an automorphism* $\Phi\colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ *such that* $\Phi \circ \varphi_\alpha = \varphi_\beta$ *and* $\Phi \circ \varphi_\beta = \varphi_\alpha$.

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- But this means that

$$\mathbb{M}_{\omega_2} \Vdash \dot{\varphi}_\beta \subseteq^* \dot{\varphi}_\alpha.$$

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- Fix an independent family $\{C_n \colon n \in \omega\} \subseteq \mathbb{C}$ generating $\mathbb{C}$.
- We say that $A \in \mathbb{C}$ is *a chunk* if it is of the form

$$A = (C_{i_0} \wedge \cdots \wedge C_{i_k}) \wedge (C^c_{j_0} \wedge \cdots \wedge C^c_{j_l})$$

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- We say that α and β are *symmetric* if for each chunks A, B

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

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Claim. There are $\alpha < \beta < \omega_2$ which are symmetric, i.e.

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- GCH + Erdös-Rado implies that there is an uncountable monochromatic Λ (WLOG $= \omega_1$) with color $\langle A, B \rangle$.
- "Then" for each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.

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- For each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.
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- Then $D \wedge \varphi_\gamma(B) = 0$.
- But $\varphi_{\gamma+1}(A) \wedge \varphi_\gamma(B) \neq 0$ and $\varphi_{\gamma+1}(A) \leq D$. A contradiction.

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Claim. There are α and $\beta < \omega_2$ which are symmetric, i.e.

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So, by Sikorski's Extension Lemma, there is an automorphism $\Phi\colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ such that

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and we are done.

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Kamburelis' theorem

Strictly positive measures

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$\mathbb{A}$ - σ-centered $\implies$ $\mathbb{A}$ supports a measure $\implies$ $\mathbb{A}$ is ccc.

Kamburelis' theorem

Theorem (Kamburelis)

Let $\mathbb{A}$ be a Boolean algebra. TFAE

- $\mathbb{A}$ *supports a measure,*
- *there is κ such that $\Vdash_{\mathbb{M}_\kappa} \mathbb{A}$ is σ-centered.*

Ergodic automorphisms

Definition

A measure preserving automorphism $\varphi\colon (\mathbb{A},\mu) \to (\mathbb{A},\mu)$ is *ergodic* if for every non-null $N, M \in \mathbb{A}$ there is $n \in \omega$ such that $\varphi^n(N) \cap M \neq 0$.

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For $\kappa = \omega$ consider $f \colon S^1 \to S^1$ defined by

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for an irrational α.

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Let $\varphi(A) = f^{-1}[A]$.

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- So T induces an ergodic automorphism of $\mathrm{Bor}(X)_{/\mu=0}$.

Ergodic automorphisms of measure algebras

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- Let $\varphi = \psi \circ T^{-1} \circ \psi^{-1}$.

Kamburelis' theorem

Theorem (Kamburelis)

Let $\mathbb{A}$ be a Boolean algebra. TFAE

- $\mathbb{A}$ *supports a measure,*
- *there is* κ *such that* $\Vdash_{\mathbb{M}_\kappa} \mathbb{A}$ *is* σ*-centered.*

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- $Q \Vdash A \in \dot{\varphi}_n$.

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- Thus $\mu_n(A) > 0$ and so $\mu(A) > 0$.

Frechet-Nikodym metric

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If $\mathbb{M}$ is a measure algebra, then $d_\lambda \colon \mathbb{M} \to [0, \infty)$ defined by

$$d_\lambda(A, B) = \mu(A \triangle B)$$

is a metric (called *Frechet-Nikodym metric*).

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Remark. Stone topology = pointwise convergence topology.

Pointwise convergence and convergence in the extension

Proposition (PBN, Sobota)

Suppose that

$$\mathbb{M} \Vdash (\dot{\varphi}_n) \text{ converges to } \dot{\varphi}.$$

Then (φ_n) *converges metrically pointwise to* φ.

Uniform convergence

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We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges *uniformly* to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

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If $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges non-trivially to $\dot{\varphi}$, then (φ_n) does not converge to φ uniformly.

Thank you and greetings from Wrocław