On countably perfectly meager sets

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Countably perfectly meager sets

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X is a perfect Polish space.

Definition 1

A set $A \subseteq X$ is perfectly meager ($A \in \mathbf{PM}$), if for all perfect subsets P of X, the set $A \cap P$ is meager in P.

Definition 2

A set $A \subseteq X$ is universally meager ($A \in \mathbf{UM}$), if A is meager with respect to any perfect Polish topology τ ($A \in \mathcal{M}(X, \tau)$) on X giving the original Borel structure of X.

Equivalently, $A \in \mathbf{UM}$ if A does not contain any injective Borel image of a non-meager subset of any perfect Polish space.

Remark 1

$$\mathbf{UM} \subseteq \mathbf{PM}.$$

Proposition 1

Consistently (eg., under CH or MA):

$$\mathbf{UM} \subsetneq \mathbf{PM}.$$

Theorem 1 (Bartoszyński)

Consistently:

$$\mathbf{UM} = \mathbf{PM}.$$

Definition 3

We say that a set $A \subseteq X$ is countably perfectly meager in X ($A \in \mathbf{PM}_\sigma$), if for every sequence of perfect subsets $(P_n : n \in \mathbb{N})$ of X, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap P_n$ is meager in P_n for each n.

Proposition 2 (Bartoszyński)

$$\mathbf{PM}_\sigma \subseteq \mathbf{UM}.$$

Definition 4

A is an s_0-set if for every perfect set P there is a copy of the Cantor set $K \subseteq P$ with $K \cap A = \emptyset$.

Theorem 2

The following are equivalent for $A \subseteq X$:

1. $A \in \mathbf{PM}_\sigma$.
2. *A is an s_0-set and for every sequence $(K_n : n \in \mathbb{N})$ of pairwise disjoint and disjoint from A copies of the Cantor set in X there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $K_m \not\subseteq F_n$ for each $m, n \in \mathbb{N}$.*

Theorem 3

The product of two countably perfectly meager sets is countably perfectly meager in the sense that if A and B are $\mathbf{PM}_\sigma$-sets in perfect Polish spaces X and Y, respectively, then $A \times B$ is a $\mathbf{PM}_\sigma$-set in $X \times Y$.

Remark 2

If $\mathcal{I}$ is a σ-ideal of subsets of X, then by $\mathcal{I}^*$ we denote the σ-ideal generated by the closed subsets of X which belong to $\mathcal{I}$.

$A \in \mathbf{PM}_\sigma$ iff $A \in \mathcal{M}^*(X, \tau)$ for any perfect Polish topology τ on X giving the original Borel structure of X, i.e., for any such τ there are closed in the original topology, τ-meager sets $F_n \subseteq X$ with $A \subseteq \bigcup_n F_n$,

(whereas $A \in \mathbf{UM}$ iff $A \in \mathcal{M}(X, \tau)$ for any perfect Polish topology τ on X giving the original Borel structure of X).

Problem 1

$$\mathbf{PM}_\sigma = \mathbf{UM}\ ?$$

More precisely, is it true that every universally meager subset of X is countably perfectly meager in X?

Definition 5

- $A \subseteq X$ is a λ'-set in X if every countable set $D \subseteq X$ is relatively G_δ in $A \cup D$,
- $A \subseteq X$ is a λ-set if every countable set $D \subseteq A$ is relatively G_δ in A.

Proposition 3

λ'-sets in X are $\mathbf{PM}_\sigma$ in X. In particular, the following sets are $\mathbf{PM}_\sigma$ in the respective spaces:

1. *any $\mathfrak{b}$-scale in $\mathbb{N}^\mathbb{N}$ (Rothberger), i.e., a set of the form $\{f_\alpha : \alpha < \mathfrak{b}\}$ where*
   - *$\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,*
   - *for every $f \in \mathbb{N}^\mathbb{N}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$.*
2. *any Hausdorff (ω_1, ω_1^*)-gap in $\mathcal{P}(\mathbb{N})$.*

Definition 6

$A \subseteq X$ has the Hurewicz property, if every continuous image of A in $\mathbb{N}^\mathbb{N}$ is bounded in the ordering $\leq^*$ of eventual domination.

Proposition 4

Any set $A \subseteq X$ with the Hurewicz property and no perfect subsets is $\mathbf{PM}_\sigma$ in X. In particular, the following sets are $\mathbf{PM}_\sigma$ in the respective spaces:

1. *any subset of X of cardinality less than $\mathfrak{b}$,*
2. *any Sierpiński set in $2^\mathbb{N}$ (Fremlin and Miller),*
3. *γ-set in $2^\mathbb{N}$ (Galvin and Miller).*

By a theorem of Just, Miller, Scheepers and Szeptycki if $A \subseteq X$ has the Hurewicz property, then for every sequence $(K_n : n \in \mathbb{N})$ of copies of the Cantor set in X disjoint from A there are closed sets F_n in X such that $A \subseteq \bigcup_n F_n$ and $K_m \cap F_n = \emptyset$ for each $m, n \in \mathbb{N}$.

Definition 7

A subset A of $2^\mathbb{N}$ is perfectly meager in the transitive sense ($A \in \mathbf{AFC}'$) if for every perfect subset P of $2^\mathbb{N}$, there exists an F_σ-set F in X such that $A \subseteq F$ and $F \cap (P + t)$ is meager in $P + t$ for each $t \in 2^\mathbb{N}$.

Proposition 5 (Bartoszyński)

For $A \subseteq 2^\mathbb{N}$ the following are equivalent:

1. *$A \in \mathbf{PM}_\sigma$,*
2. *for every perfect subset P of $2^\mathbb{N}$, there exists an F_σ-set F in $2^\mathbb{N}$ such that $A \subseteq F$ and $F \cap (P + q)$ is meager in $P + q$ for every $q \in \mathbb{Q}$, where $\mathbb{Q}$ consists of all eventually zero binary sequences.*

Proposition 6

Any set $A \subseteq 2^\mathbb{N}$ perfectly meager in the transitive sense is $\mathbf{PM}_\sigma$ in $2^\mathbb{N}$. In particular, the following sets are $\mathbf{PM}_\sigma$ in $2^\mathbb{N}$:

1. *meager-additive sets,*
2. *γ-sets,*
3. *strongly meager sets,*
4. *Sierpiński sets.*

The following example, based on a result of Bartoszyński and Shelah and classical ideas of Rothberger shows that there exists (in ZFC) a countably perfectly meager set in $2^\mathbb{N}$ of cardinality $\mathfrak{b}$ that has neither the Hurewicz nor λ' property. It also shows that the Hurewicz and λ' properties are not the same.

Example 1

Inductively, one easily constructs a $\mathfrak{b}$-scale in $\mathbb{N}^\mathbb{N}$, i.e., a subset $\{f_\alpha : \alpha < \mathfrak{b}\}$ of $\mathbb{N}^\mathbb{N}$ with the following properties:

- f_α is strictly increasing,
- $\alpha < \beta < \mathfrak{b}$ implies $f_\alpha <^* f_\beta$,
- for every $f \in \mathbb{N}^\mathbb{N}$ there is $\alpha < \mathfrak{b}$ with $f_\alpha \not\leq^* f$.

By identifying each f_α with the characteristic function of its range, we obtain a homeomorphic copy A of $\{f_\alpha : \alpha < \mathfrak{b}\}$ in $2^\mathbb{N}$.

Let $B = A \cup \mathbb{Q}$, where $\mathbb{Q}$ consists of all eventually zero binary sequences. Then:

- $\{f_\alpha : \alpha < \mathfrak{b}\}$, being unbounded in $\mathbb{N}^\mathbb{N}$, does not have the Hurewicz property,
- $\{f_\alpha : \alpha < \mathfrak{b}\}$ is a λ'-set in $\mathbb{N}^\mathbb{N}$ (Rothberger),
- B has the Hurewicz property (first noted by Bartoszyński and Shelah) and has no perfect subsets, so B is countably perfectly meager in $2^\mathbb{N}$,
- A is countably perfectly meager in $2^\mathbb{N}$ as a subset of B,
- A does not the Hurewicz property as the homeomorphic image of $\{f_\alpha : \alpha < \mathfrak{b}\}$,
- if F is any F_σ-set in $2^\mathbb{N}$ such that $A \subseteq F$, then $F \cap \mathbb{Q} \neq \emptyset$ (since otherwise F viewed as a subset of $\mathbb{N}^\mathbb{N}$ is bounded, whereas A is unbounded). In particular, neither A nor B are λ'-sets in $2^\mathbb{N}$.

Proposition 7

If there is a λ'-set in $2^\mathbb{N}$ of cardinality of the continuum, then there is also one which is not perfectly meager in the transitive sense (but being a λ'-set in $2^\mathbb{N}$ it is $\mathbf{PM}_\sigma$-set in $2^\mathbb{N}$ as well).

Main results

Theorem 8

Let T be a subset of $2^\mathbb{N}$ of cardinality $2^{\aleph_0}$. There exist a set $H \subseteq T \times 2^\mathbb{N}$ intersecting each vertical section $\{t\} \times 2^\mathbb{N}$, $t \in T$, in a singleton and a homeomorphic copy E of H in $2^\mathbb{N}$ which is not a $\mathbf{PM}_\sigma$-set in $2^\mathbb{N}$. In particular, T is a continuous injective image of E.

Corollary 9

1. *If there exists a universally meager set in $2^\mathbb{N}$ of cardinality of the continuum, then there is also one which is not countably perfectly meager.*
   (Just take $T \in \mathbf{UM}$; then $E \in \mathbf{UM} \setminus \mathbf{PM}_\sigma$).
2. *If there exists a λ-set in $2^\mathbb{N}$ of cardinality of the continuum, then there is also one which is not countably perfectly meager.*
   (Just take $T \in \lambda$; then $E \in \lambda \setminus \mathbf{PM}_\sigma$).
3. *If there exists a λ'-set in $2^\mathbb{N}$ of cardinality of the continuum, then there is also one whose homeomorphic copy in $2^\mathbb{N}$ is not countably perfectly meager. In particular, the class $\mathbf{PM}_\sigma$ is not closed with respect to homeomorphic images.*
   (Just take T a λ' set in $2^\mathbb{N}$; then H is λ' in $2^\mathbb{N} \times 2^\mathbb{N}$ and E is its homeomorphic image in $2^\mathbb{N}$ which is not $\mathbf{PM}_\sigma$).

Sketch of proof of Theorem 8.

Let $C_0, C_1, \ldots$ be pairwise disjoint meager Cantor sets in $2^\mathbb{N}$ such that:

(1) each non-empty open set in $2^\mathbb{N}$ contains some C_n.

Let $P = 2^\mathbb{N} \setminus \bigcup_n C_n$.

Claim 1. There exists a set $H \subseteq T \times P$ intersecting each vertical section $\{t\} \times P$, $t \in T$, in a singleton, such that each F_σ-set in $2^\mathbb{N} \times 2^\mathbb{N}$ containing H contains also $\{t\} \times V$ for some $t \in T$ and a non-empty open set V in $2^\mathbb{N}$.

This is proved by a diagonalization argument.

Let $\{F_t : \ t \in T\}$ be a parametrization on T of all F_σ-sets in $2^\mathbb{N} \times 2^\mathbb{N}$.

For each $t \in T$, we pick $(t, \varphi(t)) \in (\{t\} \times P) \setminus F_t$, whenever this is possible, and we let $\varphi(t)$ be an arbitrary fixed element of P, otherwise.

Then the graph $H = \{(t, \varphi(t)) : t \in T\}$ has the required property.

Let F be an F_σ-set in $2^\mathbb{N} \times 2^\mathbb{N}$ containing H, and let $t \in T$ be such that $F = F_t$. Then $(t, \varphi(t)) \in F_t$, hence F_t contains $\{t\} \times P$. Consequently, P being a dense G_δ-set in $2^\mathbb{N}$, the Baire category theorem provides a non-empty open set V in $2^\mathbb{N}$ with $\{t\} \times V \subseteq F_t$, completing the proof of the claim.

For any $s \in 2^{<\mathbb{N}}$ let $N_s = \{x \in 2^\mathbb{N} : s \subseteq x\}$ be the standard basic open set in $2^\mathbb{N}$ determined by s.

Let $\sim$ be the equivalence relation on $2^\mathbb{N} \times 2^\mathbb{N}$, whose equivalence classes are given by:

$$[(x,y)]_\sim = \begin{cases} N_{x|n} \times \{y\}, & \text{if } y \in C_n, \\ \{(x,y)\}, & \text{if } y \in P \end{cases}$$

Let $\pi(x,y) = [(x,y)]_\sim$ be the quotient map onto the quotient space

$$K = (2^\mathbb{N} \times 2^\mathbb{N}) / \sim$$

(whose topology consists of sets $U \subseteq K$ such that $\pi^{-1}(U)$ is open in $2^\mathbb{N} \times 2^\mathbb{N}$).

Claim 2. The space K is homeomorphic to $2^\mathbb{N}$.

Indeed, K is compact, Hausdorff, second countable, zero-dimensional topological space without isolated points. This may be proved e.g. by defining explicitly a countable basis for K consisting of clopen sets.

Finally, let

$$E = \pi(H)$$

(cf. Claim 1).

Clearly, E is a homeomorphic copy of H in K and T is the injective image of E under the continuous function $\mathrm{proj}_1 \circ \pi^{-1}|E$, where proj_1 is the projection of $2^\mathbb{N} \times 2^\mathbb{N}$ onto the first axis.

We will complete the proof by showing that

E is not a $\mathbf{PM}_\sigma$-set in K.

More precisely, we will show that if F^* is an F_σ-set in K such that $E \subseteq F^*$, then F^* contains one of the members of a certain countable collection $\{P_s : \ s \in 2^{<\mathbb{N}}\}$ of perfect subsets of K.

To that end, let us consider an F_σ-set F^* in K such that $E \subseteq F^*$. Then

$F = \pi^{-1}(F^*)$

is an F_σ-set in $2^\mathbb{N} \times 2^\mathbb{N}$ containing H, so there are $t \in T$ and a non-empty open set V in $2^\mathbb{N}$ such $\{t\} \times V \subseteq F$, cf. Claim 1.

Let us fix $C_n \subseteq V$ (cf. (1)) and let $s = t|n$.

We have $\{t\} \times C_n \subseteq F$, so

(2) $\pi(\{t\} \times C_n) \subseteq F^*$.

Recall that for any $y \in C_n$

$$\pi(t, y) = N_s \times \{y\}.$$

It follows that:

(3) $\pi(\{t\} \times C_n) = \{N_s \times \{y\} : y \in C_n\} = \pi(N_s \times C_n)$.

Consequently, letting

$$P_s = \pi(N_s \times C_n),$$

we conclude that $P_s \subseteq F^*$ cf. (2) and (3).

It follows that any F_σ-set in K containing E also contains some of (countably many perfect sets in K of the form) P_s, which confirms that E is not a $\mathbf{PM}_\sigma$-set in K.