On cardinalities of Lindelöf first countable spaces

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A question of Alexandrov 192?

Is it true that every first-countable Hausdorff compact space has cardinality at most c?

Arkhangel'skii 1969: Yes

Theorem (Arkhangel'skii 1969)

If X is a Hausdorff space, then $|X| \leq 2^{L(X)\chi(X)}$.

Definition. Let $\langle X, \tau \rangle$ be a T_1 topological space and $x \in X$.

- $\chi(X; x) = \min\{|\mathcal{B}| : \mathcal{B}$ is a local base at $x\}$;
- $\chi(X) = \sup\{\chi(X; x) : x \in X\}$;
- $\psi(X; x) = \min\{|\mu| : \mu \subset \tau, \{x\} = \cap \mu\}$;
- $\psi(X) = \sup\{\psi(X; x) : x \in X\}$; and
- $L(X) = \min\{\kappa :$ every open cover $\mathcal{U}$ of X has a subcover $\mathcal{V}$ of cardinality $\leq \kappa\}$.

What about T_1-spaces?

Frank Tall in his 2011 survey paper on Lindelöf spaces attributes the next question to Arhangel'skii:

Question

Is it true that $|X| \leq \mathfrak{c}$ for every Lindelöf first-countable T_1 space?

Theorem (Gryzlov 1980)

$|X| \leq \mathfrak{c}$ *for every compact first-countable T_1 space X.*

Theorem (LLL(L) 2022)

Suppose that GCH holds in V and κ is a regular cardinal. Then there exists a countably closed cardinal preserving poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there exists a Lindelöf first-countable T_1 space X ($L(X) = \chi(X) = \omega$) of cardinality κ.

What we do not know

The idea of the proof is taken from the following 30 years old result of I. Gorelic:

Theorem

Suppose that GCH holds in V and κ is a regular cardinal. Then there exists a countably closed cardinal preserving poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there exists a Lindelöf subspace X of 2^κ of cardinality κ such that $\psi(X) = \omega$. □

Informally speaking, we were able to modify his proof in such a way that we gain the first countability, and (necessarily) lose the Hausdorff property. This method does not allow to go beyond 2^{ω_1}, which is $2^{\mathfrak{c}}$ in this setting.

Question. Is it true (consistent) that $|X| \leq 2^{\omega_1}$ [$|X| \leq 2^{\mathfrak{c}}$] for every

(Gor): Lindelöf Hausdorff (resp. regular, zero-dimensional) space X with $\psi(X) = \omega$?

(LLL): Lindelöf T_1 space X with $\chi(X) = \omega$?

The poset: Notation

- Let κ be regular, $\kappa = \sqcup_{\alpha\in\kappa} A_\alpha$, $|A_\alpha| = \omega$, $A_\alpha = \{a^\alpha_n : n \in \omega\}$;
- $\{B_n : n \in \omega\} \subset [\omega]^\omega$ is a.d. and such that *for every $s \in 2^{<\omega}$ and $n \in \omega$ there exists $m > n$ such that $s \subset \chi_{B_m}$ and $B_m \cap B_i \cap (\omega \setminus |s|) = \emptyset$ for all $i \leq n$*;
- $A_{\alpha,n} := \{a^\alpha_k : k \in B_n\}$;
- For each finite $s \subset \kappa$ set $U_s = \{x : x \text{ is a function into } 2,\ s \subset \mathrm{dom}(x) \subset \kappa, \text{ and } x \restriction s \equiv 1\}$; For $A \subset \kappa$ we set $\mathcal{B}_A := \{U_s : s \in [A]^{<\omega}\}$;
- Given $F \subset 2^A$, $s \in [A]^{<\omega}$, and $\xi \in A$, we say that s *requires* ξ (in written $s \vDash \xi$) if for every $x \in F$, if $x \restriction s \equiv 1$, then $x(\xi) = 1$. I.e., $U_s \cap F \subset U_{\{\xi\}} \cap F$. (Depends on F!)

The poset: Conditions

$\mathbb{P}$ consists of conditions
$p = \langle I, A, F, \mathsf{U}, \ell, r \rangle = \langle I^p, A^p, F^p, \mathsf{U}^p, \ell^p, r^p \rangle$ such that

1. $I \in [\kappa]^\omega$ and $A = \bigcup_{\alpha \in I} A_\alpha$;
2. $F = \{x_{\alpha,n} \in 2^A : \alpha \in I, n \in \omega\}$ is such that
   - (i) $x_{\alpha,n} \upharpoonright A_\alpha = \chi_{A_{\alpha,n}}$, i.e., $x_{\alpha,n}(a_k^\alpha) = 1$ iff $k \in B_n$;
   - (ii) $x_{\alpha,n} \upharpoonright (A \setminus A_\alpha) = x_{\alpha,m} \upharpoonright (A \setminus A_\alpha)$ for all $\alpha \in I$ and $n, m \in \omega$;
   - (iii) If $\alpha, \beta \in I$, $\alpha \neq \beta$, then for any $n, m \in \omega$ we have $|x_{\alpha,m}^{-1}(1) \cap A_{\beta,n}| < \omega$;
3. $\mathsf{U} \subset \mathcal{P}(\mathcal{B}_A)$ is a countable family of covers of F;
4. $r : A \times \omega \to [\omega]^{<\omega}$ is such that $r(\alpha, n)$ equals $\{j \in B_n : \exists \beta \in I \setminus \{\alpha\} \, \exists m \in \omega \, (x_{\alpha,n}^{-1}(1) \cap A_{\beta,m} \vDash a_j^\alpha)\}$;
5. ℓ is a function with domain consisting of $\langle \alpha, n, \xi \rangle$ such that $\xi \in A \setminus A_\alpha$ and $x_{\alpha,n}(\xi) = 1$, $\ell(\alpha, n, \xi) \in \omega$, such that

$$S^{\alpha,n}_{\ell(\alpha,n,\xi)} \vDash \xi, \quad \text{where } S_k^{\alpha,n} = \{a_j^\alpha : j \in B_n, j \leq k\}.$$

The poset: Order relation

$q \leq p$ iff $I^q \supset I^p$, $A^q \supset A^p$, $\mathsf{U}^q \supset \mathsf{U}^p$, $\ell^q \supset \ell^p$, $r^q \supset r^p$, and $x^q_{\alpha,n} \restriction A^p = x^p_{\alpha,n}$ for all $\alpha \in I^p$ and $n \in \omega$. Obviously, $\mathbb{P}$ is countably closed.

Lemma

- *Let $p \in \mathbb{P}$ and $\gamma \notin I^p$, then there exists $q \leq p$ such that $\gamma \in I^q$;*
- *If GCH holds in V, then $\mathbb{P}$ is ω_2-c.c.* □

If G is $\mathbb{P}$-generic, then for every $\alpha \in \kappa$ and $n \in \omega$ set

$$x_{\alpha,n} = \bigcup \{x^p_{\alpha,n} : p \in G, \alpha \in I^p\} \text{ and } X = \{x_{\alpha,n} : \alpha \in \kappa, n \in \omega\}.$$

$X \subset 2^\kappa$ by Lemma above

Theorem

X with the topology generated by $\mathcal{B}_\kappa \restriction X = \{U_s \cap X : s \in [\kappa]^{<\omega}\}$ is a Lindelöf, T_1, and first-countable space.

Sketch of the proof: Lindelöf

Let $p_0 \in \mathbb{P}$ and $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name such that p_0 forces $\dot{\mathcal{U}} \subset \mathcal{B}_\kappa$ and $\dot{X} \subset \cup\dot{\mathcal{U}}$. Let $p_1 \leq p_0$ and $\mathcal{U}_1 \in [\mathcal{B}_\kappa]^\omega$ be such that

$$p_1 \Vdash \{\dot{x}_{\alpha,n} : \alpha \in \check{I}^{p_0}, n \in \omega\} \subset \cup\check{\mathcal{U}}_1,$$

and $\mathcal{U}_1 \subset \mathcal{B}_{A^{p_1}}$. Let $p_2 \leq p_1$ and $\mathcal{U}_2 \in [\mathcal{B}_\kappa]^\omega$ be such that

$$p_2 \Vdash \{\dot{x}_{\alpha,n} : \alpha \in \check{I}^{p_1}, n \in \omega\} \subset \cup\check{\mathcal{U}}_2,$$

and $\mathcal{U}_2 \subset \mathcal{B}_{A^{p_2}}$. And so on...

Set p_ω be the "union" of all the p_n's, with one extra elements $\mathcal{U}_\omega := \bigcup_{n\in\omega} \mathcal{U}_n$ included in addition into U^{p_ω}. Then $p_\omega \Vdash$ "$\mathcal{U}_\omega \subset \dot{\mathcal{U}}$ and $\mathcal{U}_\omega$ is a cover of $\dot{X}$".

Indeed, pick $G \ni p_\omega$, $\gamma \notin I^{p_\omega}$ and n, and $q \leq p_\omega$ with $q \in G$ and $\gamma \in I^q$. Then $\mathcal{U}_\omega \in \mathsf{U}^q$, and hence $x^q_{\gamma,n} \in \cup\mathcal{U}_\omega$, which implies $x_{\gamma,n} \in \cup\mathcal{U}_\omega$.

Sketch of the proof: First-countable

Given α, n, we claim that

$$\{U_{S_k^{\alpha,n}} : k \in \omega\}$$

is a base at $x_{\alpha,n}$. Recall that $S_k^{\alpha,n} = \{a_j^\alpha : j \in B_n, j \leq k\}$. Indeed, pick $\xi \in \kappa$ such that $x_{\alpha,n}(\xi) = 1$, and find $p \in G$ such that $\alpha \in I^p$ and $\xi \in A^p$. By the definition of a condition, we have $S_{\ell^p(\alpha,n,\xi)}^{\alpha,n} \vDash \xi$, which means

$$U_{S_{\ell^p(\alpha,n,\xi)}^{\alpha,n}} \cap F^p \subset U_\xi.$$

For stronger q, $\ell^q(\alpha, n, \xi) = \ell^p(\alpha, n, \xi)$, and hence $S_{\ell^q(\alpha,n,\xi)}^{\alpha,n} \vDash \xi$, which means

$$U_{S_{\ell^q(\alpha,n,\xi)}^{\alpha,n}} \cap F^q \subset U_\xi.$$

Thus

$$U_{S_{\ell^p(\alpha,n,\xi)}^{\alpha,n}} \cap X \subset U_\xi.$$

The proof of Gryzlov's theorem

Pick $M \prec V$, $|M| = 2^\omega$, $M^\omega \subset M$, and $X, \tau \in M$. Enough to prove that $X \subset M$.

Step 1. $X \cap M$ is countably compact as a subspace of X.

Proof. Let $Y \in [X \cap M]^\omega$. Then

$$V \models \text{“}Y \text{ has an accumulation point”}.$$

Since $Y \in M$,

$$M \models \text{“}Y \text{ has an accumulation point”},$$

i.e., Y has an accumulation point in M (and hence in $M \cap X$). □

The proof of Gryzlov's theorem, continuation

Step 2. $X \cap M$ is compact.

Proof. Suppose not and fix a maximal family $\mathcal{F}$ of closed subsets of $M \cap X$ closed under finite intersection, and such that $\bigcap \mathcal{F} = \emptyset$. Pick $x \in \bigcap\{cl_X(F) : F \in \mathcal{F}\}$ and note that $x \notin X \cap M$. Let $\{U_n : n \in \omega\}$ be a decreasing base at x. $x \notin cl_X(X \setminus U_n)$ implies $(M \cap X) \setminus U_n \notin \mathcal{F}$, and hence there exists $F_n \in \mathcal{F}$ such that $F_n \cap (X \setminus U_n) = \emptyset$, i.e., $F_n \subset U_n$. Thus $\bigcap_{n\in\omega} F_n = \emptyset$, a contradiction. □

The proof of Gryzlov's theorem, continuation

Step 3. $X \subset M$.

Proof. Suppose not and fix $z \in X \setminus M$. For every $x \in X \cap M$ find a neighborhood $U(x) \not\ni z$, $U(x) \in M$. Let $\{U(x_i) : i < n\} \in M$ be a finite cover of $X \cap M$. Then

$$M \vDash \bigcup_{i<n} U(x_i) = X,$$

and hence

$$V \vDash \bigcup_{i<n} U(x_i) = X,$$

i.e., $\bigcup_{i<n} U(x_i) = X$, a contradiction because z is not covered. $\square$

The last slide

Thank you for your attention!

Dziękuję Polsce za wsparcie dla Ukrainy!