A Topological Ramsey Theorem

Paul J. Szeptycki

Department of Mathematics and Statistics
York University
Toronto Canada
and
Czech Acadamy of Sciences
szeptyck@yorku.ca

Introduction

Joint work with Wiesław Kubiś

Ramsey's Theorem

For any $r, n \in \omega$ and any $f : [\omega]^r \to n$ there is $H \subseteq \omega$ and $i < n$ such that

$$f(x) = i \text{ for all } x \in [H]^r$$

H is homogeneous for f

I.e., $\omega \to (\omega)^r_n$

Given $f : [\omega]^r \to K$ where K is compact,
in what sense can we assert that there is an H homogeneous for f?

A generalized notion of a convergent sequence

Definition

Let $r \in \omega \setminus \{0\}$, X a space, $S \subseteq \omega$ infinite and $f : [S]^r \to X$, *f converges to $p \in X$* if for every neighborhood U of p there is a finite set F such that $f''[S \setminus F]^r \subseteq U$.

1. If $r = 1$, then $f : [S]^1 \to X$ is a sequence and this notion is the same as usual.
2. If $(x_n : n \in \omega) \to p$ and we define f on $[\omega]^r$ by $f(s) = x_{\min(s)}$, then f converges to p.
3. If $f : [S]^r \to X$ converges to p and $\{s_i : i \in \omega\}$ is pairwise disjoint, then $(f(s_i) : i \in \omega) \to p$.

Definition

Given $r \in \omega$, a space X is said to be r-Ramsey if, for each $f : [\omega]^r \to X$, there is $S \subseteq \omega$ infinite such that $f \restriction [S]^r$ converges. X has the Ramsey property if it is r-Ramsey for all $r \in \omega$.

1. 1-Ramsey $\iff$ sequentially compact.
2. $r + 1$-Ramsey $\Rightarrow$ r-Ramsey
3. Ramsey's Theorem can be restated as *every finite space has the Ramsey property*.

Compact metrizable spaces

Theorem

If X is compact metrizable then it has the Ramsey property.

Observations:

1. Applying the theorem to finite X, we obtain Ramsey's classical theorem as a corollary.

$$\forall r, n \in \omega \big(\omega \to (\omega)^r_n\big)$$

2. $r = 1$: Compact metrizable spaces are sequentially compact.
3. $r = 2$: Due to M. Bojańczyk, E. Kopczyński, S. Toruńczyk. Applied to obtain idempotents in compact metrizable semigroups as limits of some particular functions on $[\omega]^2$.

Proof.

Theorem

If X is compact metrizable then it is r-Ramsey for all $r \in \omega$.

Proof: For each n fix a finite cover $\mathcal{U}_n$ by $1/2^n$ balls and let $f : [\omega]^r \to X$.
f and $\mathcal{U}_n$ induce a finite coloring of $[\omega]^r$.
Using Ramsey's Theorem, let

$$S_0 \supseteq S_1 \supseteq \ldots S_n \supseteq \ldots \text{ so that for all } n$$

1. $S_n \subseteq \omega$ is infinite.
2. the diameter of $F_n = f''[S_n]^r$ is less than $1/2^n$

If $p \in \bigcap\{\overline{F_n} : n \in \omega\}$ and $S \subseteq^* S_n$ for all n then $f \restriction [S]^r$ converges to p. $\dashv$

Corollaries

Corollary

If X is compact and the closure of every countable set is first countable, then X has the Ramsey property.

1. Any 1-point compactification of a discrete space is Ramsey
2. and so is any Corson compact,
3. and any compact linearly ordered space.

This can be improved a bit:

Theorem

Sequentially compact spaces of character $< \mathfrak{b}$ have the Ramsey property.

Examples

Let $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ be almost disjoint, $\Psi(\mathcal{A})$ its Isbell-Mrówka space and $K(\mathcal{A})$ its one-point compactification.

Example

If $\mathcal{A}$ is a maximal almost disjoint family, then $K(\mathcal{A})$ is not 2-Ramsey (but is sequentially compact).

Proof: $K(\mathcal{A})$ is r-Ramsey if and only if it is r-Ramsey with respect to $f : [\omega]^r \to \omega$.

$f : [S]^r \to \omega$ converges to $a \in \mathcal{A}$ if and only if there is n such that

$$f''[S \setminus n]^r \subseteq a$$

$f : [S]^r \to \omega$ converges to ∞ if and only if for every $a \in \mathcal{A}$ there is n such that

$$f''[S \setminus n]^r \cap a = \emptyset$$

Proof continued

We may assume $\mathcal{A} \subseteq [\omega \times \omega]^{\aleph_0}$ and $\{n\} \times \omega \in \mathcal{A}$ for all n.

Define $f : [\omega]^2 \to K(\mathcal{A})$ by $f(\{k, n\}) = (k, n)$ $(k < n)$

Then, for any infinite $S \subseteq \omega$, and any n

$$f''[S \setminus n]^2 \in I^+(\mathcal{A})$$

Lemma (Mathias)

For $\mathcal{A}$ mad, for any decreasing sequence $B_n \in I^+(\mathcal{A})$ there is $B \in I^+(\mathcal{A})$ such that $B \subseteq^* B_n$ for all n.

So, for any S, there is $A \in \mathcal{A}$ such that for all n

$$f''[S \setminus n]^2 \cap A \text{ is infinite}$$

So, no $f \restriction [S]^2$ can be convergent. $\dashv$

Preservation under operations

Theorem

The r-Ramsey property is preserved under

1. *Closed subspaces*
2. *Continuous images*
3. *Countable products and* Σ*-products*

Theorem (van Douwen)

The minimal cardinal κ such that 2^κ is not sequentially compact is the splitting number $\mathfrak{s}$

$\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq 2^\omega$ is splitting. I.e., for no A is $f \restriction A$ constant mod finite for all $f \in \mathcal{F}$.

2^κ may be sequentially compact and not Ramsey

Definition (Blass)

(1) A is almost homogeneous for a family of functions $\mathcal{F} \subseteq 2^{[\omega]^r}$ if for each $f \in \mathcal{F}$ there is n such that f is constant on $[A \setminus n]^r$.
(2) $\mathfrak{par}_r$ is the minimal cardinality of a family of functions $[\omega]^r \to 2$ with no almost homogeneous set.

Theorem (Blass)

For each $r \geq 2$, $\mathfrak{par}_r = \mathfrak{par}_2 = \min\{\mathfrak{b}, \mathfrak{s}\}$

Analogous to van Douwen's characterization of $\mathfrak{s}$, we have

Theorem

$\mathfrak{par}_2$ *is the minimal cardinal* κ *such that* 2^κ *is not* r*-Ramsey.*

And so,
$\mathfrak{b} < \mathfrak{s}$ implies that $2^\mathfrak{b}$ is sequentially compact not 2-Ramsey

r-Ramsey not $(r+1)$-Ramsey

Theorem

Assuming CH ($\mathfrak{b} = \mathfrak{c}$ should suffice). For each r there is an almost disjoint family $\mathcal{A}$ on ω such that $K(\mathcal{A})$ is r-Ramsey and not $(r+1)$-Ramsey.

Proof. Build $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ on ω^{r+1} starting with

$$\{a_n : n \in \omega\} = \{\{s\} \times \omega : s \in \omega^r\}$$

Not $(r+1)$-Ramsey will be witnessed by G defined by

$$G(\{k_0, k_1, ..., k_r\}_<) = (k_0, ..., k_r)$$

$(B_\alpha)_\alpha$ enumerate $[\omega]^{\aleph_0}$ and $(f_\alpha)_\alpha$ enumerate all $f : [\omega]^r \to \omega^{r+1}$

To make the construction work, we need to fix S_α convergent for f_α and add a new a_α witnessing $G \restriction [B_\alpha]^{r+1}$ is not convergent.

FIN^n

Definition

FIN is the ideal of finite subsets of ω.
FIN^n is the Fubini product of FIN: defined recursively by
$X \in \mathrm{FIN}^{n+1}$ if

$$\{s \in \omega^n : \{k : s^\frown k \in X\} \notin \mathrm{FIN}\} \in \mathrm{FIN}^n$$

1. $\{a_n : n \in \omega\} = \{\{s\} \times \omega : s \in \omega^r\} \subseteq \mathrm{FIN}^{r+1}$,
2. and $a \in \mathrm{FIN}^{r+1}$ whenever a is a.d. from all a_n
3. $G''[B]^{r+1} \notin \mathrm{FIN}^{r+1}$ for any B

Lemma

For every $f : [\omega]^r \to \omega^{r+1}$, there is $S \subseteq \omega$ such that

$$f''[S]^r \in FIN^{r+1}$$

More on products

1. (P. Simon): The productivity number for sequential compactness is $\mathfrak{h}$

   $\mathfrak{h}$ is the minimal number of mad families needed to split every infinite subset of ω.

   If $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ witness, then

   $$\prod_{\alpha<\mathfrak{h}} K(\mathcal{A}_\alpha)$$

   is not sequentially compact.
2. The productivity number for the Ramsey property is $\geq \mathfrak{h}$

Question

Are there $\mathfrak{h}$ many 2-Ramsey spaces whose product is not 2-Ramsey?