On affinization

Tomasz Brzeziński

Swansea University & University of Białystok

MIMUW, May 2025

The affinization principle

- Affine spaces (algebraic structures) are defined with no reference to vector spaces (additive structures).
- Choosing points in affine spaces converts them into mutually isomorphic vector spaces – *tangent spaces* at these points.
- Any affine map is uniquely linearised to a linear map between tangent vector spaces – the *linearisation*.
- ‘Good’ algebraic structure on an affine space linearises to the corresponding linear structure on any tangent vector space.

Affinization of groups: heaps [Prüfer '24, Baer '29]

A **heap** is a nonempty set A together with a ternary operation

$$\langle -,-,-\rangle : A\times A\times A\to A,$$

such that for all $a_i\in A$, $i=1,\ldots,5$,

(a) $\big\langle\langle a_1,a_2,a_3\rangle,a_4,a_5\big\rangle=\big\langle a_1,a_2,\langle a_3,a_4,a_5\rangle\big\rangle$,

(b) $\langle a_1,a_2,a_2\rangle=a_1=\langle a_2,a_2,a_1\rangle$.

A heap A is **abelian** if $\langle a,b,c\rangle=\langle c,b,a\rangle$.

Homomorphism of heaps: a function $f:A\to B$ such that

$$f\left(\langle a_1,a_2,a_3\rangle\right)=\big\langle f(a_1),f(a_2),f(a_3)\big\rangle.$$

Heaps are ‘good’ affinizations of groups

- If $(A, +)$ is an (abelian) group, then A is an (abelian) heap with operation
$$\langle a, b, c\rangle = a - b + c.$$
- Let A be an (abelian) heap. For all $o \in A$, A is an (abelian) group (denoted by A_o) with addition and inverses
$$a + b := \langle a, o, b\rangle, \qquad -a = \langle o, a, o\rangle.$$
- A heap homomorphism $f : A \to B$ uniquely defines the group homomorphism:
$$\hat{f} : A_o \to B_p, \qquad \hat{f}(a) = f(a) - f(o).$$

Affine spaces defined intrinsically

[Breaz, TB, Rybołowicz, Saracco, 2023]

An affine space $\mathfrak{a}$ is an abelian heap with an $\mathbb{F}$-action $(\lambda, a, b) \mapsto \lambda \triangleright_a b$, such that

- $- \triangleright_a -$ is a bi-heap map,
- $(\alpha\beta) \triangleright_a b = \alpha \triangleright_a (\beta \triangleright_a b)$,
- $\lambda \triangleright_a b = \langle \lambda \triangleright_c b, \lambda \triangleright_c a, a \rangle$,
- $0 \triangleright_a b = a$, $1 \triangleright_a b = b$.

Intuition:

- $\langle a, b, c \rangle = a + \overrightarrow{bc}$;
- $\lambda \triangleright_a b := a + \lambda \overrightarrow{ab}$.

Affine spaces are ‘good’ affinizations of vector spaces

Fix an $o \in \mathfrak{a}$, then the abelian group $\mathfrak{a}_o = (\mathfrak{a}, +, o)$ with scalar multiplication:

$$\alpha\, a = \alpha \triangleright_o a$$

is a vector space called the **tangent space at** o or the **vector fibre at** o, denoted by $T_o\mathfrak{a}$.

$\mathfrak{a}$ is an affine space over $T_o\mathfrak{a}$:

- $\overrightarrow{ab} = \langle o, a, b \rangle = b - a$,
- $\alpha \triangleright_a b = a + \alpha\, \overrightarrow{ab} = (1-\alpha)a + \alpha b$.

An affine transformation $f : \mathfrak{a} \to \mathfrak{b}$ is a morphism of heaps such that

$$f(\lambda \triangleright_a b) = \lambda \triangleright_{f(a)} f(b).$$

The corresponding linear transformation $\hat{f} : T_o\mathfrak{a} \longrightarrow T_o\mathfrak{b}$,

$$\hat{f}(a) = \langle f(a), f(o), o \rangle = f(a) - f(o).$$

Affinization of associative algebras [TB '19]

Definition

An **associative affgebra** is an affine space $\mathfrak{a}$ together with an associative bi-affine map $\cdot : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$.

Bi-affine property implies that, for all $a, b, c, d \in \mathfrak{a}$,

$$a\langle b, c, d\rangle = \langle ab, ac, ad\rangle, \quad \langle b, c, d\rangle a = \langle ba, ca, da\rangle,$$

i.e. an associative affgebra is a **truss**.

Affgebras are 'good' affinizations of associative algebras [Andruszkiewicz-TB-Rybołowicz '22]

Theorem (RRA, TB & BR)

Let $\mathfrak{a}$ *be an associative affgebra and* $o \in \mathfrak{a}$.

(1) $T_o\mathfrak{a}$ *is an associative algebra with multiplication:*

$$a \bullet b = \langle ab, ao, o^2, ob, o \rangle = ab - ao - ob + o^2.$$

(2) $T_o\mathfrak{a} \oplus \mathbb{F}$ *is an associative algebra with product:*

$$(a, \alpha)(b, \beta) = (ab + (\beta - 1)ao + (\alpha - 1)ob + (\alpha - 1)(\beta - 1)o^2, \alpha\beta)$$

(3) $T_o\mathfrak{a} \cong \{(a, 1) \mid a \in \mathfrak{a}\} \subseteq T_o\mathfrak{a} \oplus \mathbb{F}$, *as associative affgebras.*

Lie algebras

Definition

A **Lie algebra** is a vector space $\mathfrak{g}$ together with a bilinear operation $[-,-]$ such that, for all $a, b, c \in \mathfrak{g}$,

$$[a, a] = 0,$$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0,$$

Note that

- If $\mathrm{char}\mathbb{F} \neq 2$, $[a, a] = 0$ iff $[a, b] = -[b, a]$;
- The Jacobi identity can be equivalently written as:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0;$$

- The Jacobi identity can be equivalently written as:

$$[[a, b], c] = [[a, c], b] + [a, [b, c]].$$

Lie affgebras [TB & Papworth '23], [Andruszkiewicz, TB, & Radziszewski '25]

Definition

A **(left) Lie bracket** on an affine space $\mathfrak{a}$ is a bi-affine map $\{-,-\} : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ such that, for all $a, b, c \in \mathfrak{a}$,

$$\big\langle \{a,b\}, \{a,a\}, \{b,a\} \big\rangle = \{b,b\},$$

$$\Big\langle \{a,\{b,c\}\}, \{a,\{a,a\}\}, \{b,\{c,a\}\}, \{b,\{b,b\}\}, \{c,\{a,b\}\} \Big\rangle = \{c,\{c,c\}\}.$$

An affine space with a Lie bracket is called a **Lie affgebra.**

In $T_o\mathfrak{a}$:

$$\{a,b\} - \{a,a\} + \{b,a\} = \{b,b\},$$

$$\{a,\{b,c\}\} - \{a,\{a,a\}\} + \{b,\{c,a\}\} - \{b,\{b,b\}\} + \{c,\{a,b\}\} = \{c,\{c,c\}\}.$$

Comments

- An affine map $f : \mathfrak{a} \to \mathfrak{b}$ is a homomorphism of Lie affgebras provided,

$$f(\{a, b\}) = \{f(a), f(b)\}.$$

- The intrinsic definition of Lie affgebras includes the Grabowska-Grabowski-Urbański '05 definition as a special case:

$$\{a, a\} = a,$$

for all $a \in \mathfrak{a}$.

Lie algebras are ‘good’ affinizations of Lie algebras

Theorem (TB-JP & RRA-TB-KR)

Let $(\mathfrak{a}, \{-,-\})$ *be a Lie affgebra.*

- *For all* $o \in \mathfrak{a}$*, the unique linearisation of* $\{-,-\}$*,*

$$\begin{aligned}[-,-] : & T_o\mathfrak{a} \times T_o\mathfrak{a} \longrightarrow T_o\mathfrak{a}, \\ & (a,b) \longmapsto \{a,b\} - \{a,o\} + \{o,o\} - \{o,b\},\end{aligned}$$

  is a Lie bracket on $T_o\mathfrak{a}$*.*
- *The linearisation of a Lie affgebra homomorphism* $f : \mathfrak{a} \to \mathfrak{b}$ *is a Lie algebra homomorphism.*
- $T_o\mathfrak{a} \cong T_p\mathfrak{a}$ *as Lie algebras.*

The Lie algebra on $T_o\mathfrak{a}$ is called the **tangent** Lie algebra to the Lie affgebra $\mathfrak{a}$.

Lie affgebras vs Lie algebras

Theorem (RRA-TB-KR)

Let $\mathfrak{g}$ be a Lie algebra. Let $\delta, \lambda, \in \mathrm{Lin}(\mathfrak{g})$ and $s \in \mathfrak{g}$.

$$\begin{aligned} \{-,-\} : & \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \\ & (a, b) \longmapsto [a, b] - \delta(a) + \lambda(b) + s. \end{aligned}$$

- $(\mathfrak{g}, \{-,-\})$ *is a Lie affgebra iff, for all* $a, b \in \mathfrak{g}$,

$$\lambda([a, b]) = [\delta(a), b] + [a, \lambda(b)].$$

  We write $\mathfrak{a}(\mathfrak{g}; \delta, \lambda, s)$.
- *For all* $o \in \mathfrak{g}$, $T_o\mathfrak{a}(\mathfrak{g}; \delta, \lambda, s) \cong \mathfrak{g}$.
- *If* $T_o\mathfrak{a} \cong \mathfrak{g}$, *then* $\mathfrak{a} \cong \mathfrak{a}(\mathfrak{g}; \delta, \lambda, s)$.

Comments

- In terminology of Leger and Luks (2000) the condition

$$\lambda([a,b]) = [\delta(a),b] + [a,\lambda(b)] \qquad (*)$$

  means that δ is a **generalised derivation** of $\mathfrak{g}$.
- (*) implies that $\kappa = \lambda - \delta$ is an element of the **quasicentroid** $\mathrm{QC}(\mathfrak{g})$ of $\mathfrak{g}$, i.e. $[a,\kappa(a)] = 0$, for all $a \in \mathfrak{g}$.
- δ is a derivation of $\mathfrak{g}$ iff κ is an element of the **centroid** of $\mathfrak{g}$, $\mathrm{C}(\mathfrak{g})$ (Jacobson (1962)), i.e.

$$\kappa([a,b]) = [a,\kappa(b)],$$

  for all $a,b \in \mathfrak{g}$.
- Note that there are no restrictions on s.

Isomorphisms

Theorem (RRA-TB-KR)

$\mathfrak{a}(\mathfrak{g};\delta,\lambda,s) \cong \mathfrak{a}(\mathfrak{g}';\delta',\lambda',s')$ *iff there exist a Lie algebra isomorphism* $\Psi : \mathfrak{g} \to \mathfrak{g}'$ *and* $q \in \mathfrak{g}$ *such that*

$$\delta' = \Psi(\delta - \mathrm{ad}_q)\Psi^{-1},$$
$$\lambda' = \Psi(\lambda - \mathrm{ad}_q)\Psi^{-1},$$
$$s' = \Psi(s + q - \lambda(q) + \delta(q)).$$

Lie affgebras with simple fibres

- We look at $\mathfrak{a}(\mathfrak{g};\delta,\lambda,s)$ with $\mathfrak{g}$ simple and finite dimensional over an algebraically closed field of characteristic 0.
- Leger and Luks (2000): $\mathrm{QC}(\mathfrak{g}) = \mathrm{C}(\mathfrak{g})$, hence δ is a derivation and $\kappa = \lambda - \delta \in C(\mathfrak{g})$.
- $\kappa \in \mathrm{C}(\mathfrak{g})$ means that $\kappa \circ \mathrm{ad} = \mathrm{ad} \circ \kappa$, so by Schur's lemma: $\kappa = \gamma\,\mathrm{id}$, $\gamma \in \mathbb{F}$.
- All derivations of $\mathfrak{g}$ are inner, so, up to isomorphism, $\delta = 0$.
- Up to isomorphism

$$\{a,b\} = [a,b] + \gamma b + s.$$

Towards classification

- Use classification of Lie algebras and determine allowed data δ, λ, s up to isomorphism.
- $\dim \mathfrak{g} = 1$, $[a, b] = 0$ and no conditions on δ and λ.
- $\mathrm{ad}_q = 0$, λ, δ are scalar multiples of identity, hence are not affected by any Ψ.
- Choosing suitable q and Ψ one obtains two types of isomorphism classes:

$$\{a, b\} = -\delta a + \lambda b,$$

$$\{a, b\} = (1 - \lambda)a + \lambda b + 1.$$

Classification of all $\mathfrak{a}$ with tangent $\mathfrak{sl}(2,\mathbb{C})$

- I use the Chevalley basis h, e, f:

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

- Any automorphism of $\mathfrak{sl}(2,\mathbb{C})$ is a conjugation by a special linear matrix.
- Isomorphism classes of $\mathfrak{a}$ such that $T_o\mathfrak{a} \cong \mathfrak{sl}(2,\mathbb{C})$:

$$\begin{aligned}
\mathfrak{a}(\mathfrak{sl}(2,\mathbb{C}); \gamma\,\mathrm{id}, \gamma\,\mathrm{id}, e) &: \quad \{x,y\} = [x,y] + \gamma y + e, \\
\mathfrak{a}(\mathfrak{sl}(2,\mathbb{C}); \gamma\,\mathrm{id}, \gamma\,\mathrm{id}, \sigma h) &: \quad \{x,y\} = [x,y] + \gamma y + \sigma h, \\
\mathfrak{a}(\mathfrak{sl}(2,\mathbb{C}); \gamma\,\mathrm{id}, \gamma\,\mathrm{id}, f) &: \quad \{x,y\} = [x,y] + \gamma y + f,
\end{aligned}$$

  where $\gamma, \sigma \in \mathbb{C}$.