Non-locally compact Polish groups and non-essentially countable orbit equivalence relations

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Joint work with A.Kechris, A.Panagiotopoulos, and J.Zielinski

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On Polish groups admitting non-essentially countable actions,
to appear in Ergodic Theory and Dynamical Systems

Polish groups

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- additive groups of separable Banach spaces $(B, +)$;
- isometry groups of Polish metric spaces $\mathrm{Iso}(X)$;
- non-archimedeam groups $G \leq \mathrm{Sym}(\mathbb{N})$;

Orbit equivalence relations

Let α be a continuous action of a Polish group on a Polish space X (i.e., X is a Polish G-space.) The **orbit equivalence relation** E_α on X induced by α is defined by

$$x \, E_\alpha \, y \leftrightarrow \exists g \in G \; g.x = y.$$

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Examples:

- action by translation of a Polish(able) subgroup on a Polish group;
- evaluation actions of transformation groups;
- Bernoulli shifts for non-archimedean groups: $G \leq \mathrm{Sym}(\mathbb{N})$ acts on $\mathbb{R}^{\mathbb{N}}$ by permuting coordinates.

Complexity of equivalence relations

An equivalence relation E on a Polish space X is (Borel) **reducible** to an equivalence relation F on a Polish space Y (denoted by $E \leq_B F$) if there is a Borel mapping $f : X \to Y$ such that, for any $x, y \in X$,

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Important types of equivalence relations:

- **smooth relations**, e.g., relations reducible to the identity on a Polish space;
- **essentially countable relations**, e.g., relations reducible to a relation with countable classes.

Topological structure of G and Borel reducibility

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Theorem (Solecki)

Every non-compact Polish group induces a non-smooth orbit equivalence relation E_α (in fact, E_0 reduces to E_α.)
Thus, a Polish group G is non-compact iff G induces a non-smooth orbit equivalence relation.

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Orbit equivalence relations induced by locally compact Polish groups are essentially countable.

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Question: Does the converse to Kechris' theorem hold?

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The converse is true for every:

- (Solecki) additive group of a separable Banach space;
- (Thompson) Polish non-CLI group (i.e., not admitting a compatible complete left-invariant metric);
- (M.) abelian Polish group of isometries of a locally compact Polish metric space (in particular, every abelian non-archimedean group.)

Structure of abelian non-archimedean groups

Theorem (M.)

Let $G \leq \prod_n G_n$ be a non-locally compact Polish group, where G_n are countable discrete abelian groups. Then there exist infinite discrete groups K_n, $n \in \mathbb{N}$, and a closed $L \leq G$ such that

$$\prod_n K_n \leq G/L$$

(in particular, $E_0^{\mathbb{N}}$ reduces to E_α.)

Another partial answer

Theorem (Kechris, M., Panagiatopoulos, Zielinski)

Every non-locally compact Polish group of isometries of a locally compact Polish metric space (in particular, every non-locally compact non-archimedean group) induces a non-essentially countable orbit equivalence relation.

Main ingredients of the proof

Theorem (Feldman, Ramsey)

Let G be a non-locally compact Polish group which has a free Borel action α on a standard Borel space X admitting an invariant probability Borel measure. Then E_α is not essentially countable.

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Theorem (Kechris, M., Panagiatopoulos, Zielinski)

Let (X, μ) be standard non-atomic probability space and G a spatial closed subgroup of $\mathrm{Aut}(X, \mu)$ and denote by $g \cdot x$ the corresponding action of G on X. Consider the diagonal action of G on $X^{\mathbb{N}}$ given by $g \cdot (x_n) = (g \cdot x_n)$, which preserves the product measure $\mu^{\mathbb{N}}$. Then there is a G-invariant Borel set $Y \subseteq X^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(Y) = 1$ such that G acts freely on Y.

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Theorem (Kechris, M., Panagiatopoulos, Zielinski)

For each sequence (K_n) *of Polish locally compact groups, the group* $\prod_n (S_\infty \ltimes K_n^{\mathbb{N}})$ *can be embedded into* $\mathrm{Aut}(X, \mu)$.

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Theorem (Gao, Kechris)

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Theorem (Kwiatkowska, Solecki)

Every probability measure preserving Boolean action of the isometry group of a separable locally compact metric space has a spatial realization.

Games and non-reduction results

Recall that a Polish group is CLI if it admits a compatible left-invariant **and** complete metric.

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Let X be a Polish G-space, and let $x, y \in X$. In the Becker-embedding game $Emb(x, y)$, defined by Lupini and Panagiatopoulous, Odd and Eve play as follows:

- in the 1-st turn, Odd plays an open nbhd U_1 of y and an open nbhd V_1 of 1, and Eve responds with an element $g_0 \in G$;
- in the n-th turn, $n > 1$, Odd plays an open nbhd U_n of y and an open nbhd V_n of 1, and Eve responds with an element $g_{n-1} \in V_{n-1}$.

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Eve wins the game if $g_{n-1} \dots g_0.x \in U_n$ for $n > 1$. We say that x is *Becker embeddable* into y if Eve has a winning strategy for the game $Emb(x, y)$, and write $x \preceq_B y$.

Games and non-reduction results

Lemma (Lupini, Panagiatopoulous)

Let X be a Polish G-space, and let d be a compatible right-invariant metric on G. For any $x, y \in X$, the following statements are equivalent:

1. $x \preceq_B y$;
2. *there exists a right Cauchy sequence (h_n) in G such that* $h_n.x \to y$.

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Remark: If G is CLI, then 2. implies that $x \in [y]$.

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Theorem

Let G, H be Polish groups, X a Polish G-space, and Y a Polish H-space. Let $f : X \to Y$ be a Baire-measurable homomorphism of the involved orbit equivalence relations. Then there is an invariant comeager $C \subseteq X$ such that $c \preceq_B o$ implies $f(c) \preceq_B f(o)$ for $c, o \in C$.

Games and non-reduction result

Theorem (Lupini, Panagiatopoulous)

Suppose that X is a Polish G-space. If for any invariant comeager $C \subseteq X$ there are $c, o \in C$ such that $c \preceq_B o$ but $c \notin [o]$, then for any invariant comeager $C \subseteq X$, the orbit equivalence relation on X is not classifiable by an orbit equivalence relation induced by an action of a CLI group.

A game theoretic criterion for non-essential countability

Let X be a Polish G-space, and let $x, y \in X$. Let V be an open nbhd of 1 in G. We say that y admits a V-**approximation** from x if there is $g^* \in G$, and a sequence (g_n) in V so that $g_n g^*.x \to y$.

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Consider a game $\mathrm{Appr}_V(x, y)$, where Odd and Eve play as follows:

- in the 1-st turn, Odd plays an open nbhd U_1 of y, and Eve responds with an element $g^* \in G$;
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Eve wins the game if $g^*.x \in U_1$, and $g_{n-1} g^*.x \in U_n$ for $n > 1$. Clearly, Eve has a winning strategy in the game iff y admits a V-approximation from x. We write then $x \preceq^V y$.

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Remark: If V is compact, then $x \preceq^V y$ implies that $x \in [y]$.

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Let G, H be Polish groups, X a Polish G-space, and Y a Polish H-space. Let $f : X \to Y$ be a Baire-measurable homomorphism of the involved orbit equivalence relations. Then for every open nbhd W of 1 in H there is an open nbhd V of 1 in G, an invariant comeager $C \subseteq X$, and a non-meager $O \subseteq X$ such that $c \preceq^V o$ implies $f(c) \preceq^W f(o)$ for $c \in C$, $o \in O$. If $W = H$, one can put $V = G$, and O can be chosen to be comeager.

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Theorem (Kechris, M., Panagiatopoulos, Zielinski)

Let X be a Polish G-space. Assume that for every open nbhd V of 1, for every invariant comeager $C \subseteq X$, and for every non-meager $O \subseteq X$, there are $c \in C$ and $o \in O$ such that $c \preceq^V o$ but $c \notin [o]$. Then the induced orbit equivalence relation is not essentially countable.

Structure of non-archimedean grooups and Bernoulli shifts

Let G be a non-archimedean group.

Theorem

G is non-compact iff the Bernoulli shift for G has an invariant subspace that is generically ergodic with meager orbits (in particular, it is non-smooth).

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Theorem (Kechris, M., Panagiatopoulos, Zielinski)

G is non-locally compact iff the Bernoulli shift for G satisfies the assumptions of the above theorem (in particular, its orbit equivalence relation is non-essentially countable).

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Theorem

G is non-cli iff for the Bernoulli shift for G, and every invariant comeager subset C of $\mathbb{R}^{\mathbb{N}}$ there are $c, o \in C$ with different orbits so that $c \preceq_B o$ (i.e., o is Becker-embeddable in c.)

Actions of Banach spaces

Theorem (Solecki)

Let Z be an infinite dimensional separable Banach space viewed as a group under addition. Then there exists a Polish Z-space X whose associated orbit equivalence relation is not essentially countable.

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Theorem (Kechris, M., Panagiatopoulos, Zielinski)

Let Z be an infinite dimensional separable Banach space viewed as a group under addition. Then there exists a Polish Z-space X whose associated orbit equivalence relation is Borel and not essentially countable.