Multiplicative structure of matrices; from classical to exotic

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Plan of the talk

1) a motivation: representation theory

2) full matrix monoid $M_n(K)$ over a field K
   - structure of $M_n(K)$
   - Rees matrix semigroups
   - an application: semisimplicity of the algebra $\mathbb{C}[M_n(\mathbb{F}_q)]$

3) monoid of tropical matrices $M_n(\mathbb{T})$ over the tropical semiring $\mathbb{T}$
   - how $M_n(\mathbb{T})$ differs from $M_n(K)$ and why it might be useful
   - an application: representing plactic monoids

1. Classical representation theory

Groups G (more generally semigroups S) are studied via homomorphisms

$$\phi : S \longrightarrow M_n(K),$$

called linear representations.

ϕ is faithful if it is injective.
ϕ is irreducible if K^n has no $\phi(S)$-invariant subspaces.

If S is a semigroup with operation written multiplicatively, then the semigroup algebra $K[S]$ of S we mean the K-algebra with basis S and with multiplication (uniquely) extending the operation in S.

Every ϕ extends to a homomorphism of the semigroup algebra

$$\overline{\phi} : K[S] \longrightarrow M_n(K)$$

and ϕ is irreducible iff $\overline{\phi}$ is onto, provided that K is algebraically closed.

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Significance of faithful representations

Two examples: celebrated theorems on group representations

1. The class of linear groups does not allow pathological properties. For example:

'Tits alternative' (J.Tits, 1972):

if $G \subseteq GL_n(K)$ is a finitely generated subgroup, then either G is almost solvable (has a solvable subgroup of finite index) or G contains a free noncommutative subgroup.

2. Concrete important classes of groups: 'braid groups B_n are linear':

$$B_n \hookrightarrow GL_{n(n-1)/2}(\mathbb{C})$$

(Bigelow, 2001, J. AMS; Krammer; 2002, Annals of Math.)

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Significance of irreducible representations

Two fundamental results (on finite groups).

Definition

A finite dimensional algebra A is semisimple if it has no nonzero nilpotent ideals (ideals I such that $I^k = 0$ for some k).

Theorem (Wedderburn)

A (finite dimensional) algebra A over $\mathbb{C}$ is semisimple if and only if $A \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_k}(\mathbb{C})$ for some k and some $n_1, \ldots, n_k$.

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Conclusion:

in semisimple algebras one can apply a rich blend of methods of linear algebra, group theory, ring theory, topology (including Zariski topology), analysis, geometry and algebraic number theory (by replacing $\mathbb{C}[G]$ by $K[G]$, where K is an appropriately chosen finite field extension of $\mathbb{Q}$).

2. Structure of the monoid $M_n(K)$; K - an arbitrary field

Let $M_j = \{a \in M_n(K) \mid \text{rank}(a) \leq j\}$, $j = 0, 1, \ldots, n$. Then

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M_n(K)$$

and these are the only ideals of the monoid $M_n(K)$.

Let Y_j= the set of matrices of rank j that are in the reduced row echelon form, and let $X_j = Y_j^t$ (the transpose).
Let (in the block form)

$$G_j = \left\{ \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \mid z \in GL_j(K) \right\}$$

Then the elements $a \in M_j \setminus M_{j-1}$ can be written uniquely in the from

$$a = xgy, \quad \text{where} \quad x \in X_j, g \in G_j, y \in Y_j$$

(use elementary row- and column- reductions).

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In block matrix form such elements multiply as follows:

$xgy \cdot x'g'y' =$

$$\underbrace{\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}}_{x}\underbrace{\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}}_{g}\underbrace{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}}_{yx'}\underbrace{\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}}_{g'}\underbrace{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}}_{y'}$$

if $\mathrm{rank}(yx') = j$ then the entire product is of rank j;

so that the product $xgy \cdot x'g'y'$ lies in $M_j \setminus M_{j-1}$.

Let $P_j = (p_{yx})$ be the $Y_j \times X_j$ - matrix with coefficients in $G_j \cup \{0\}$, where

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Then $M_j/M_{j-1} := (M_j \setminus M_{j-1}) \cup \{0\} \cong \mathcal{M}(G_j, X_j, Y_j; P_j)$
(called a Rees matrix semigroup),

where the nonzero elements are the triples (x, g, y), with multiplication

$$(x, g, y) \circ (x', g', y') = (x, g p_{yx'} g', y'),$$

while all triples $(x, 0, y)$ are identified with the zero element of $\mathcal{M}(G_j, X_j, Y_j; P_j)$.

Identifying (x, g, y) with the $X_j \times Y_j$ - matrix with the only nonzero entry g in position (x, y), we see that the operation in $\mathcal{M} = \mathcal{M}(G_j, X_j, Y_j; P_j)$ takes the form

$$\mathcal{A} \circ \mathcal{B} = \mathcal{A} P_j \mathcal{B}.$$

A classical observation (in case $\mathcal{M} = \mathcal{M}(G, X, Y; P)$ is finite) : the algebra $\mathbb{C}[\mathcal{M}]$ is semisimple if and only if $|X| = |Y|$ and P is invertible as a matrix in the algebra $M_{|X|}(\mathbb{C}[G])$.

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An example of applying this strategy

Conjecture:

(D.K.Faddeev, 1976): For every finite field $\mathbb{F}_q$ the algebra $\mathbb{C}[M_n(\mathbb{F}_q)]$ is semisimple.
(Here $q = p^k$ for a prime p and some $k \geq 1$; and $|\mathbb{F}_q| = q$.)

In fact, Faddeev claimed that this is true, but later he admitted that he never had a correct proof.

An approach: groups and monoids of Lie type

Groups of Lie type form a class of axiomatically defined groups.
A model example: $G = GL_n(K)$.

Reminder: Classification of finite simple groups

cyclic groups C_p, p - a prime,
alternating groups A_n, $n \geq 5$,
16 infinite families of groups of Lie type,
26 'sporadic' simple groups.

There are two parallel worlds:
- certain algebraic groups (groups admitting a structure of an algebraic variety; for example Zariski closed subgroups of $GL_n(\mathbb{C})$)
- and their finite counterparts, such as $GL_n(\mathbb{F}_q)$.

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Monoids of Lie type

They are defined axiomatically. There are three groups of axioms:

1. there is a group of Lie type G such that

$$G = \text{ the group of invertible elements in the monoid } M,$$

2. $M = \bigcup_{e \in E} GeG$, where $E \subseteq M$ is a set of idempotents of M,

3. (roughly speaking) the properties of M reflect the structure of the group G
(relations between idempotents in M and certain 'parabolic subgroups' of the group G are imposed).

The most prominent example: $G = GL_n(K)$ and $M = M_n(K)$.

Note: again there are two parallel worlds, namely:

- algebraic monoids of Lie type (closed in Zariski topology); such as $M_n(\mathbb{C})$
- and their finite counterparts; such as $M_n(\mathbb{F}_q)$.

A class of such monoids that arises in a natural way:

Theorem (M.Putcha)

If $G \subseteq GL_n(\mathbb{C})$ is a reductive algebraic group and M is the Zariski closure of G in $M_n(\mathbb{C})$, then M is a monoid of Lie type.

Theorem (M.Putcha, JO; 1991)

If M is a finite monoid of Lie type then the algebra $\mathbb{C}[M]$ is semisimple. In particular, all algebras $\mathbb{C}[M_n(\mathbb{F}_q)]$ are semisimple.

The proof uses an induction of certain type ('parabolic induction'), based on the notion of 'cuspidal representations' (Harish-Chandra, Lusztig) for reductive groups.

This naturally requires considering a much wider class of monoids (those of Lie type), since this induction is not possible within $M_n(\mathbb{F}_q)$.

The main difficulty in the proof is to show that all matrices P_j are invertible as matrices in $M_{n_j}(K[GL_j(\mathbb{F}_q)])$, where $n_j = |X_j|$, $j = 1, \ldots, n$. One proves this more generally for similar matrices arising from M.

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3. Tropical matrices

By the tropical semiring we mean the set $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ considered with the operations of maximum and summation, namely

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \odot b = a + b \quad \text{for } a, b \in \mathbb{T}.$$

Then $(\mathbb{T}, \oplus, \odot)$ is a (commutative) semiring; meaning that $(\mathbb{T}, \oplus)$ is a semigroup (as opposed to 'a group' in case of a ring).
Note that $-\infty$ is the zero element and 0 is the unity of $\mathbb{T}$.

For every nonnegative integer n the set $M_n(\mathbb{T})$ of all $n \times n$ matrices over $\mathbb{T}$ is considered as a multiplicative semigroup, with the operation defined in the standard way in terms of the operations $\oplus$ and $\odot$.

It is then natural to refer to the set

$$U_n(\mathbb{T}) = \{(a_{ij}) \in M_n(\mathbb{T}) \mid a_{ij} = -\infty \text{ for } i > j\}$$

as the subsemigroup of all upper triangular matrices in $M_n(\mathbb{T})$.

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So, how do we multiply tropical matrices?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} \max\{a+a', b+c'\} & \max\{a+b', b+d'\} \\ \max\{c+a', d+c'\} & \max\{c+b', d+d'\} \end{pmatrix}$$

in $\in M_2(\mathbb{T})$.

Tropical matrices (introduced in the 1970') have been used in a variety of contexts: combinatorics, algebraic geometry, phylogenetics, convex geometry, optimization, control theory, formal language and automata theory, ...

Structure of $M_n(\mathbb{T})$ has been studied in the recent 10 years, but is not yet completely understood. Several aspects of 'tropical algebra' over $\mathbb{T}$ (analogous to linear algebra) have also been studied.

Note: potential applications may involve tropical matrices over semirings other than $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$.

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Note: potential applications may involve tropical matrices over semirings other than $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$.

$M_n(K)$ versus $M_n(\mathbb{T})$

Let $A = \begin{pmatrix} -1 & 1 \\ -\infty & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ -\infty & -1 \end{pmatrix} \in M_2(\mathbb{T})$. Then we get

$$E = AB = \begin{pmatrix} 0 & 0 \\ -\infty & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & 2 \\ -\infty & 0 \end{pmatrix},$$

and

$$EA = A = AE, \quad EB = B = BE, \quad E^2 = E.$$

Conclusion

In the monoid $EM_n(\mathbb{T})E \subseteq M_n(\mathbb{T})$ (with identity E) we have elements A, B such that $AB = E, BA \neq E$.

(In this case, one says that $M_n(\mathbb{T})$ is not von Neumann finite.)

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Theorem (Y.Shitov; 2012)

Every subgroup of $M_n(\mathbb{T})$ embeds into the semidirect product

$$(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}) \rtimes S_n \quad (S_n \text{ acting on } \mathbb{R}^n \text{ by permutation }).$$

Hence, H has an abelian normal subgroup of finite index $\leq n!$.

(One says that H is abelian-by-finite.)

A few aspects of the comparison

$M_n(K)$	$M_n(\mathbb{T})$
von Neumann finite ($AB = 1 \Rightarrow BA = 1$)	NOT von Neumann finite
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Conclusion: perhaps tropical representation theory should be developed, as the properties of tropical matrices differ dramatically from the properties of matrices in $M_n(K)$.

So, tropical representations might be useful for testing/applying other properties than those detected by standard representations.

We will present a supporting example.

Young tableaux

Informally speaking, a (semistandard) Young tableau is a planar object with decreasing columns and non-decreasing rows. An example:

5					
3	4	4			
2	3	3	3		
1	1	2	2	2	3

The so called ‘column reading’ of this tableau yields the word

$$w = x_5x_3x_2x_1 \cdot x_4x_3x_1 \cdot x_4x_3x_2 \cdot x_3x_2 \cdot x_2 \cdot x_3$$

in the free monoid $F_n = \langle x_1, \ldots, x_n \rangle$ (if the entries are taken from the set $\{1, \ldots, n\}$).

Such tableaux can be multiplied (via the so called ‘insertion algorithm’), and the product yields another tableaux (involving the same set of generators $x_1, \ldots, x_n$). And this product is associative.

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Key algebraic tool - plactic monoids

The set of semistandard Young tableaux on generators $x_1, \ldots, x_n$, with this operation is called the plactic monoid P_n of rank n.

For an integer $n \geq 1$ we consider the finitely generated monoid $M_n = \langle a_1, \ldots, a_n \rangle$ defined by the relations

$$a_i a_k a_j = a_k a_i a_j \qquad \text{for } i \leq j < k,$$
$$a_j a_i a_k = a_j a_k a_i \qquad \text{for } i < j \leq k.$$

The crucial result is that the elements of M_n can be written in a canonical form, and these canonical forms are in a one-to-one correspondence with semistandard Young tableaux. Even more:

Theorem (D.Knuth)

The monoids P_n and M_n are isomorphic (via $x_i \mapsto a_i$, $i = 1, \ldots, n$).

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The crucial result is that the elements of M_n can be written in a canonical form, and these canonical forms are in a one-to-one correspondence with semistandard Young tableaux. Even more:

Theorem (D.Knuth)

The monoids P_n and M_n are isomorphic (via $x_i \mapsto a_i$, $i = 1, \ldots, n$).

These monoids originated from certain problems in combinatorics (Schensted).

Because of deep connections to Young tableaux, the plactic monoids have proved to be a very powerful tool in several aspects of representation theory, algebraic combinatorics, geometry, quantum groups, statistical mechanics, ...

For example: the Littlewood-Richardson rule, which yields the formula for the decomposition of the tensor product of representations of the full linear groups into irreducible components, is proved (and can be expressed) in terms of the plactic monoid.

Two observations:
- all finite dimensional irreducible representations of P_n are 1-dimensional,
- if $n > 1$ then P_n does not admit any faithful representation in $M_t(K)$.

So, ordinary (over a field) finite dimensional representations are not useful in the context of P_n.

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So, ordinary (over a field) finite dimensional representations are not useful in the context of P_n.

Conjecture

The plactic monoid P_n satisfies a nontrivial identity $v_n \equiv w_n$, where v_n, w_n are (distinct) words in the free monoid $\langle x, y \rangle$.

Theorem (Ł.Kubat, JO; 2014)

If $n \leq 3$ then P_n satisfies an identity.

This was a byproduct of the description of all irreducible (infinite dimensional) representations of P_3. Such a description is not known for $n \geq 4$.

Theorem (Z.Izhakian; 2014/2016 and JO; 2015)

$U_n(\mathbb{T})$ satisfies a nontrivial identity (depending on n).

Theorem (M.Johnson, M.Kambites; 2020)

There is an embedding $P_n \longrightarrow U_{2^n}(\mathbb{T})$. So the conjecture holds.

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