Modal logic for topologists

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- *All spaces are Tychonoff*
- What is modal logic?
- *You do not really need to know!*
- It suffices to have good friends to tell you what it is and to keep you on track!
- I will try to explain that there are interesting problems in modal logic that have a purely topological translation and consequently can be attacked by topologists fairly effectively.
- An S4-*algebra* is a pair $\mathfrak{U} = (B, \Box)$, where B is a Boolean algebra and $\Box\colon B \to B$ satisfies Kuratowski's axioms for interior:
  1. $\Box(a \wedge b) = \Box a \wedge \Box b$,
  2. $\Box 1 = 1$,
  3. $\Box a \leq a$,
  4. $\Box a \leq \Box\Box a$.

  (Observe that by (4) and (3), $\Box a \leq \Box(\Box a) \leq \Box a$.)

- These algebras were introduced by McKinsey and Tarski in 1944 under the name of *closure algebras*, Rasiowa and Sikorski in 1963 call them *topological Boolean algebras*, and Blok in 1976 calls them *interior algebras*.
- Typical examples come from topology.
- If X is a topological space, then $\mathfrak{U}_X := (\mathcal{P}(X), {}^o)$ is an S4-algebra, where $\mathcal{P}(X)$ is the powerset of X and ^{o} is the interior operator of X.
- By the McKinsey–Tarski Representation Theorem, each S4-algebra is isomorphic to a subalgebra of $\mathfrak{U}_X$ for some topological space X.
- S4 is a certain set of formulas in the basic propositional modal language (with $\Box$).
- In fact, S4 is the logic of the class of all topological spaces.
- A formula φ is *valid* in in the S4-algebra $\mathfrak{U}$, written $\mathfrak{U} \models \varphi$, provided it evaluates to 1 under all interpretations.

- S4 $\vdash \varphi$ iff φ is valid in every S4-algebra.
- In words: φ is a theorem of S4 iff φ is valid in every S4-algebra.

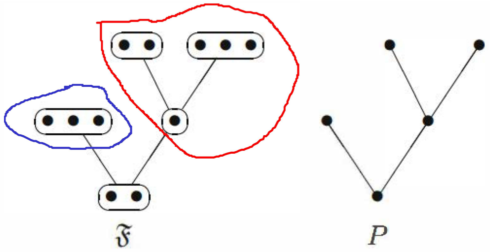
Theorem (McKinsey-Tarski, 1944)

If X is any dense-in-itself metrizable space, then S4 $\vdash \varphi$ *iff* $\mathfrak{U}_X \models \varphi$.

- S4 is the logic of any dense-in-itself metrizable space.
- How to prove the McKinsey–Tarski Theorem?
- An S4-frame is a pair $\mathfrak{F} = (W, R)$, where W is a set and R is a reflexive and transitive binary relation on W.
- That is: xRx and $aRb \wedge bRc \longrightarrow aRc$.
- $R[A] = \{v \in W : (\exists w \in A)(wRv)\}$,
  $R^{-1}[A] = \{v \in W : (\exists w \in A)(vRw)\}$.

- $A \subseteq W$ is called an *R-cone* if $A = R[A]$.
- The collection of all R-cones is a topology on W with closure operator R^{-1}. Moreover, for each $w \in W$, $R[w]$ is the least open neighborhood of w.
- *Hence $\mathfrak{F}$ has very, very bad separation properties.*
- We call $\mathfrak{F}$ rooted if there is an $r \in W$ such that $R[r] = W$.
- For the proof of the McKinsey-Tarski Theorem, all one needs to do is show that every finite rooted S4 frame is an *open* and *continuous* image of every dense-in-itself metrizable space.
- (What allows this is Kripke completeness of S4 with respect to finite rooted S4-frames. This is not trivial.)
- Such a map is called *interior* in the literature on modal logic.
- One can restrict the class of finite rooted S4-frames, as follows.

- A *cluster* is an equivalence class of the equivalence relation $\{(w, v) : wRv \wedge vRw\}$.
- A *quasi-chain* is a subset Q of W such that wRv or vRw for $w, v \in Q$.
- We call $\mathfrak{F}$ a *quasi-tree* if $\mathfrak{F}$ is rooted and $R^{-1}[w]$ is a quasi-chain for each $w \in W$.
- In fact, all one needs to do for McKinsey-Tarski is show that every finite quasi-tree is an *open* and *continuous* image of every dense-in-itself metrizable space.



- So we are in a purely topological situation now.
- The quasi-trees also have very bad separation properties of course.
- In the recent literature many simplified proofs of the McKinsey–Tarski Theorem have been produced for specific dense-in-itself metrizable spaces. Usually based on computations with metrics.
- For a recent proof of it which is based on the Bing Metrization Theorem, see below. No metrics!
- (Whether this is good or bad, I do not know.)

A New Proof of the McKinsey–Tarski Theorem

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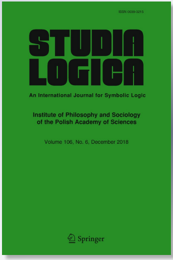

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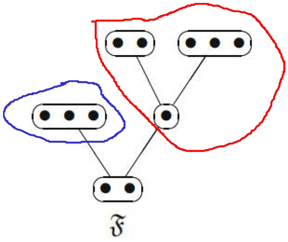
- Let us try to analyze what it means that we can map a dense-in-itself metrizable space X onto this quasi-tree by an open continuous map.



- There are several related results.
- $\mathsf{S4.2} = \mathsf{S4} + \Diamond\Box p \to \Box\Diamond p$.
- A space is *extremally disconnected* (ED) if the closure of every open subset is open.
- S4.2 is the logic of the absolute (= projective cover) of the closed unit interval (which is ED); G. Bezhanishvili and J. Harding, 2012.
- $\mathsf{S4.3} = \mathsf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$.
- S4.3 is the logic of some countable subspace of the absolute of the closed unit interval (which is hereditarily ED); G. Bezhanishvili, N. Bezhanishvili, J. Lucero-Bryan and J. van Mill, 2015.
- In fact, we have a complete characterization of the logics arising from hereditarily extremally disconnected Tychonoff spaces; G Bezhanishvili, N Bezhanishvili, J Lucero-Bryan, J van Mill, 2018.

- In the proofs a new dimension function for topological spaces became important.
- The *modal Krull dimension* of a topological space X is the Krull dimension of the S4-algebra of the powerset of X.
- It can be defined recursively, and relates to the so-called *nodec* spaces of van Douwen. Here *nodec* stands for Nowhere Dense Closed.
- Suppose X is nonempty.
  1. X is 0-nodec iff X is discrete.
  2. X is 1-nodec iff X is nodec.
  3. For $n \geq 1$, X is n-nodec iff every nowhere dense subset of X is $(n-1)$-nodec.

Theorem

Let X be a nonempty T_1-space and $n \in \omega$. Then X has modal Krull dimension $\leq n$ iff X is n-nodec.

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KRULL DIMENSION IN MODAL LOGIC

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Abstract. We develop the theory of Krull dimension for **S4**-algebras and Heyting algebras. This leads to the concept of modal Krull dimension for topological spaces. We compare modal Krull dimension to other well-known dimension functions, and show that it can detect differences between topological spaces that Krull dimension is unable to detect. We prove that for a T_1-space to have a finite modal Krull dimension can be described by an appropriate generalization of the well-known concept of a nodec space. This, in turn, can be described by modal formulas zem_n which generalize the well-known Zeman formula zem. We show that the modal logic $\mathbf{S4.Z}_n := \mathbf{S4} + \mathsf{zem}_n$ is the basic modal logic of T_1-spaces of modal Krull dimension $\leq n$, and we construct a countable dense-in-itself ω-resolvable Tychonoff space Z_n of modal Krull dimension n such that $\mathbf{S4.Z}_n$ is complete with respect to Z_n. This yields a version of the McKinsey-Tarski theorem for $\mathbf{S4.Z}_n$. We also show that no logic in the interval $[\mathbf{S4}_{n+1}, \mathbf{S4.Z}_n)$ is complete with respect to any class of T_1-spaces.

§1. Introduction. Topological semantics of modal logic was pioneered by Tsao-Chen [45], McKinsey [36], and McKinsey and Tarski [37]. The celebrated McKinsey–Tarski theorem states that if we interpret modal diamond as closure and hence modal box as interior, then **S4** is the modal logic of any dense-in-itself separable metric space. Rasiowa and Sikorski [42] showed that separability can be dropped from the assumptions of the theorem. However, dropping the dense-in-itself assumption may result in logics strictly stronger than **S4**. A complete description of when a modal logic is the logic of a metric space was given in [5], where it was shown that such logics form the chain:

$$\mathbf{S4} \subset \mathbf{S4.1} \subset \mathbf{S4.Grz} \subset \cdots \subset \mathbf{S4.Grz}_n \subset \cdots \subset \mathbf{S4.Grz}_1.$$

Here $\mathbf{S4.1} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$ is the McKinsey logic, $\mathbf{S4.Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$ is the Grzegorczyk logic, and $\mathbf{S4.Grz}_n = \mathbf{S4.Grz} + \mathsf{bd}_n$, where

$$\mathsf{bd}_1 = \Diamond\Box p_1 \to p_1,$$
$$\mathsf{bd}_{n+1} = \Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1}.$$

An important generalization of the class of metric spaces is the class of Tychonoff spaces. It is a classic result of Tychonoff that these are exactly the spaces that up to homeomorphism are subspaces of compact Hausdorff spaces (see, e.g., [19, Theorem 3.2.6]). Because of this important feature, the class of Tychonoff spaces is

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βY such that each point in $\beta\omega$ is a remote point of Y. Consider the quotient space Q_i of βY obtained by the equivalence relation whose only nontrivial equivalence classes are the fibers of g, namely $g^{-1}(x)$ for each $x \in \beta X_i$. By [19, Theorem 2.4.13] the quotient mapping of βY onto Q_i is closed. Intuitively, Q_i is obtained from βY by replacing the copy of $\beta\omega$ that 'is remote from Y' by βX_i. We identify Y, βX_i, and X_i with their respective images in Q_i, see Figure 6. For a nowhere dense subset N of Y, we have $\mathbf{C}_{\beta Y}(N) \cap \beta\omega = \varnothing$, so $\mathbf{C}_{\beta Y}(N)$ is saturated, and hence $\mathbf{C}_{Q_i}(N) \cap \beta X_i = \varnothing$.

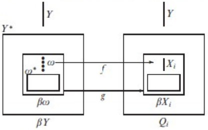


FIGURE 6. Identifying Y, βX_i, and X_i in the quotient Q_i of βY.

Viewing $Y \cup X_i$ as a subspace of Q_i, the subsets Y and X_i are complements of each other, Y is dense, and X_i is closed and nowhere dense. Let A_i be the adjunction space of ω copies of $Y \cup X_i$ glued through the identity map on the copies of X_i. That is, up to homeomorphism, A_i is the quotient of the topological sum $\bigoplus_{m\in\omega}(Y \cup X_i) \times \{m\}$ under the equivalence relation whose nontrivial equivalence classes are $\{(x,m) \mid m \in \omega\}$ for each $x \in X_i$; see Figure 7.

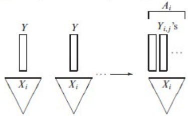


FIGURE 7. The adjunction space A_i obtained by gluing ω copies of $Y \cup X_i$ through X_i.

To facilitate defining $\alpha_{n+1} : Z_{n+1} \to \mathfrak{Z}_{n+2}$ we denote the ω copies of Y in A_i by $Y_{i,j}$ where $j \in \omega$. We also identify X_i with its homeomorphic copy in A_i. The quotient mapping from $\bigoplus_{j\in\omega}(Y_{i,j} \cup X_i)$ onto A_i is closed. Thus, in A_i we have that $\bigcup_{j\in\omega} Y_{i,j}$ and X_i are complements of each other, $\bigcup_{j\in\omega} Y_{i,j}$ is dense, and X_i is closed and nowhere dense.

We define Z_{n+1} as the adjunction space of the A_i for $i \in \omega$ through the following gluing. For each A_i consider the inclusion mapping $I_i : X_i \to Z_n$. Glue through

- Consider the so-called *Kripke frame* $\mathfrak{D}$:

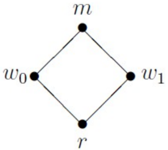


FIGURE 1. The Kripke frame $\mathfrak{D} = (D, \leq)$ where $D = \{r, w_0, w_1, m\}$.

- We ran across the modal logics S4.2 and S4.3 on the previous slide.
- The simplest modal logic above S4.2 that is not above S4.3 is the logic $\mathsf{L} = \mathsf{L}(\mathfrak{D})$ of $\mathfrak{D}$.

Theorem

There exists a measurable cardinal iff there exists a normal space Z such that $\mathsf{L}(Z) = \mathsf{L}$.

- Recall that an uncountable cardinal κ is *measurable* if there exists a κ-complete free uf on κ.
- In purely topological language, the theorem from the last slide is equivalent to:

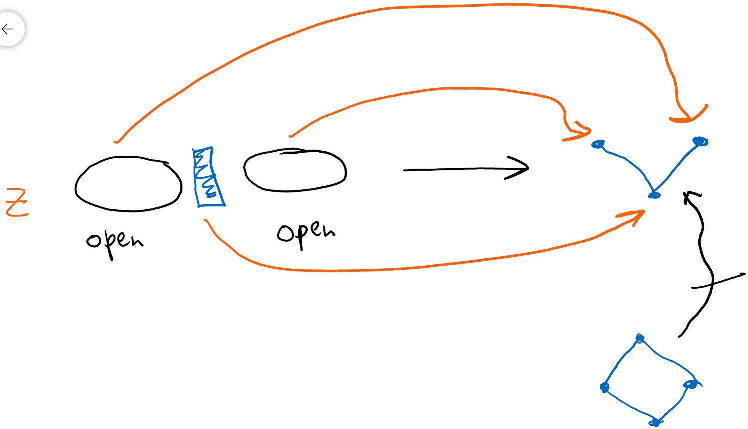
Theorem

The following statements are equivalent:

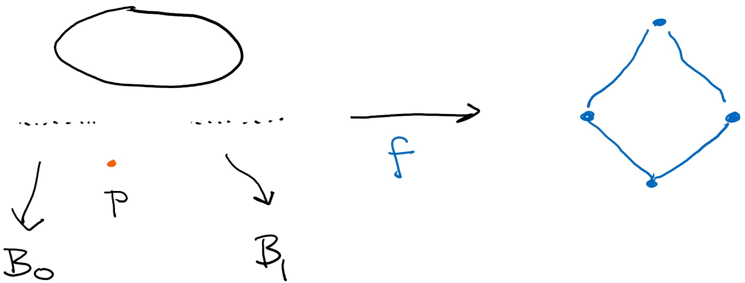
1. *There is a measurable cardinal.*
2. *There is a normal space Z which has the following properties:*
    1. *Z admits an open and continuous map onto the Kripke frame $\mathfrak{D}$,*
    2. *if Z admits an open continuous map onto some finite rooted S4-frame $\mathfrak{F}$, then $\mathfrak{F}$ is an open continuous image of $\mathfrak{D}$.*

- Now we can do business!

- (1) $\Rightarrow$ (2).
- Z is *extremally disconnected*.
- If not, then there are two disjoint nonempty open sets which have nonempty closures.
- In the space, we then see the following picture:

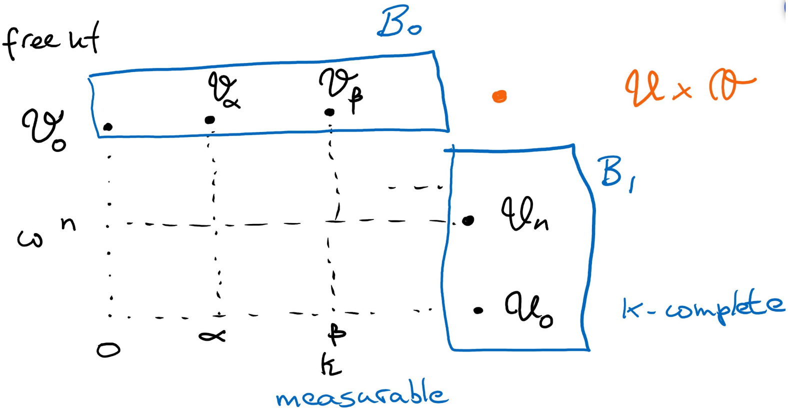


- Z is not *hereditarily extremally disconnected*, hence it is uncountable.
- Why a measurable cardinal?

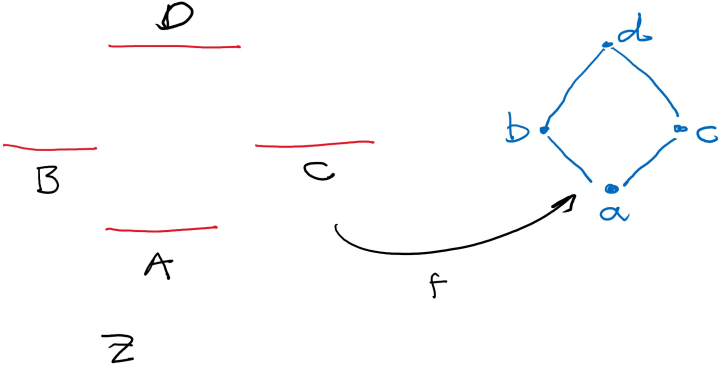


- Since Z is extremally disconnected, the point p must be inaccessible by a countable set, either in B_0 or in B_1.

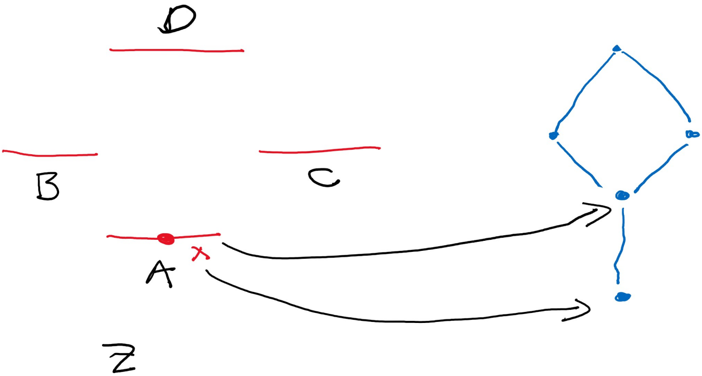
- So with a little luck, we indeed run into measurable cardinals (which would be cool)!
- Staring at the Kripke frame, it is now not a problem to construct Z from a measurable cardinal.



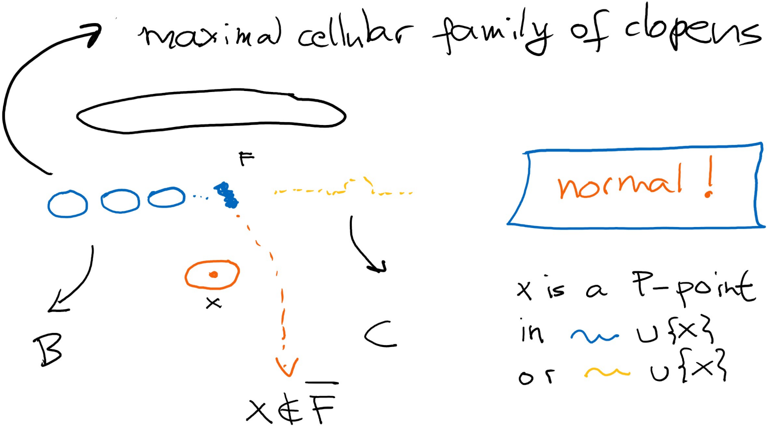
- (2) $\Rightarrow$ (1).
- Let Z be as in the theorem.



- D is dense open and infinite, Z being infinite (we observed that Z has to be uncountable).
- A is discrete. If not, let x be a non-isolated point of A.



- If F is a nowhere dense subset of $B \cup C$, then x is not in the closure of F.



- How about *Tychonoff*?

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See also Esakia, Shehtman, Goldblatt, Gabelaia.

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THANK YOU!