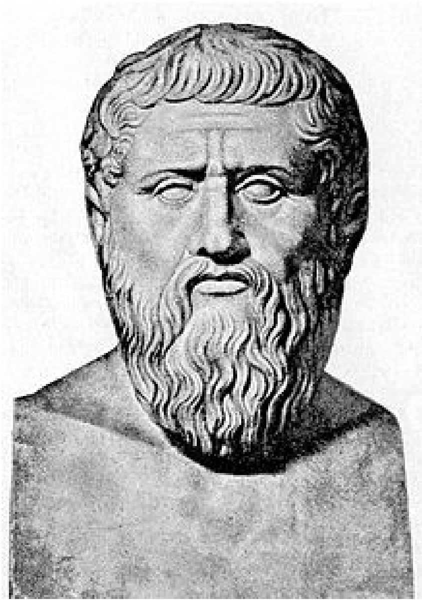
Kolokwium Wydziału MIM UW

14 czerwca 2018

Quasicrystals - global order from local rules

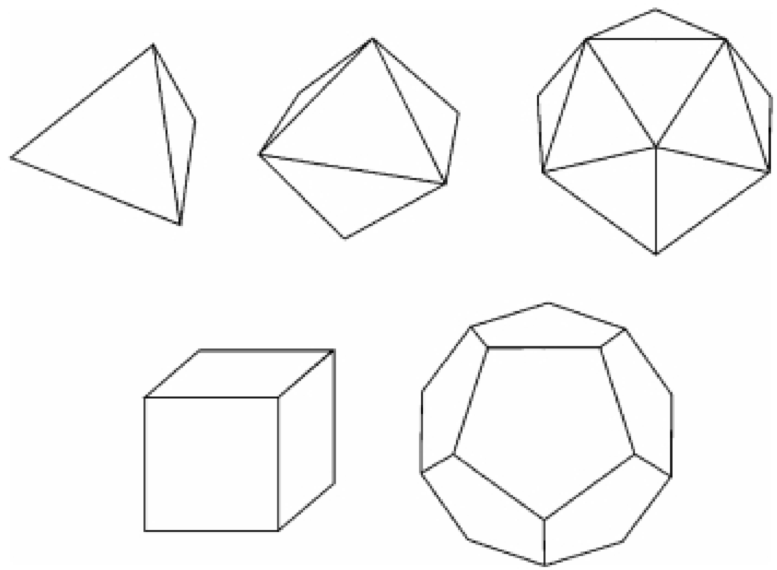
Jacek Miękisz
IMPAN and MIMUW



Plato 427 BC - 347 BC

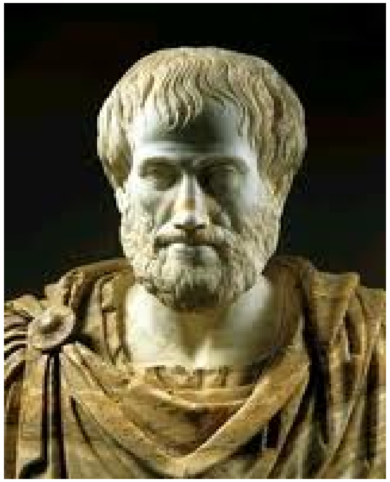
tetraedr oktaedr ikosaedr

fire air water



earth cosmos

heksaedr dodekaedr



Aristotle 384 BC - 322 BC

He criticizes Plato

It is improper to ascribe natural forms to platonic solids because they cannot cover the whole space.

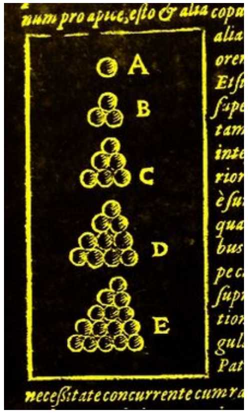
Sir Walter Raleigh (1554 – 1618)
an English gentleman, writer, poet, soldier, politician

Thomas Harriot (1560 – 1621)
an English astronomer, mathematician, ethnographer, translator

expedition to Roanoke Island, 1585

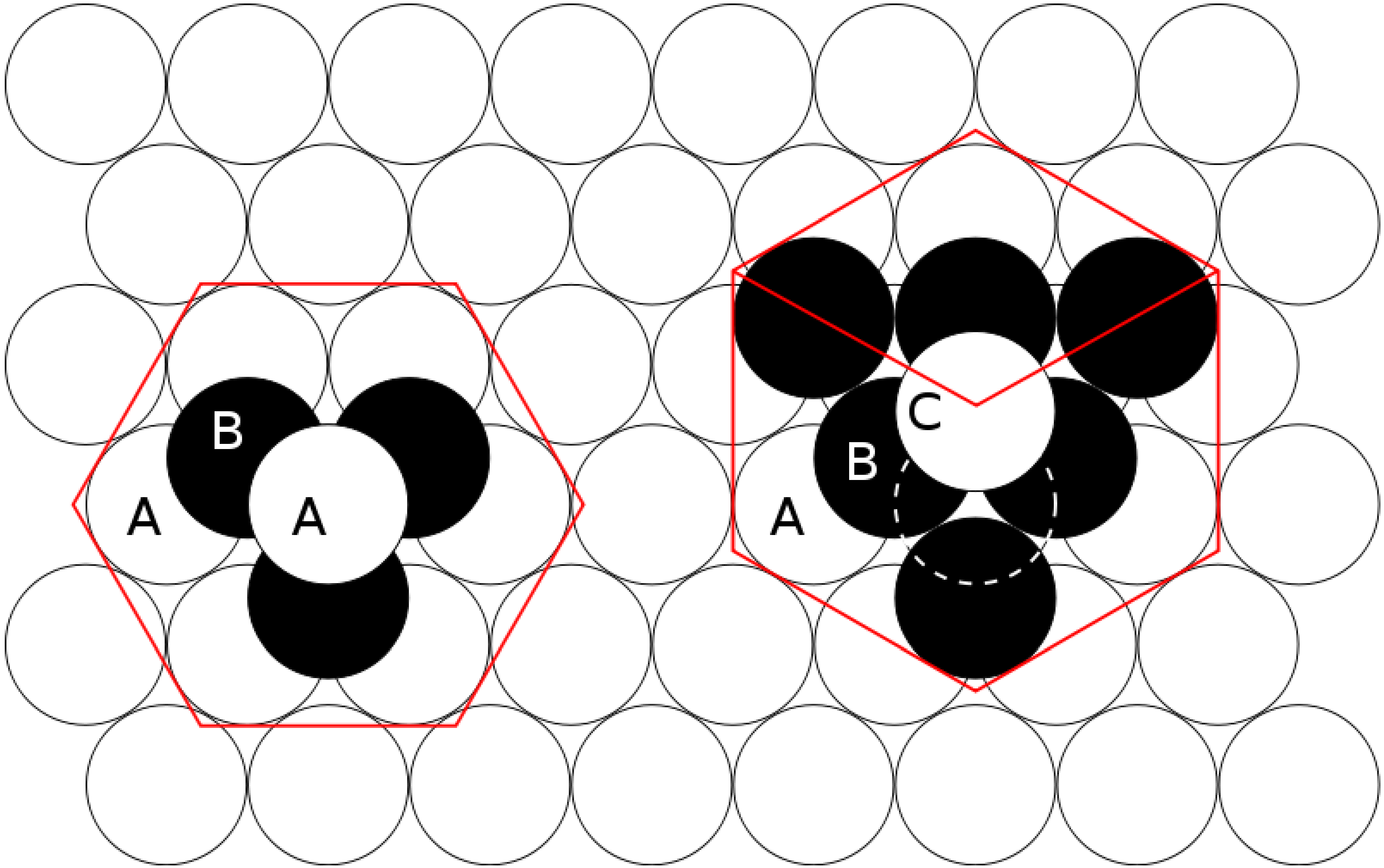
Johannes Kepler 1571 - 1630

Kepler Cannonball Conjecture,
Strena Seu de Nive Sexangula, 1611



Densest packing of spheres is realized on orange fruit stands.

spheres cover 74% of the space $$\frac{\pi}{\sqrt{18}} \simeq 0.74048.$$


B
A
A
C
B
A

Carl Friedrich Gauss 1777 - 1855

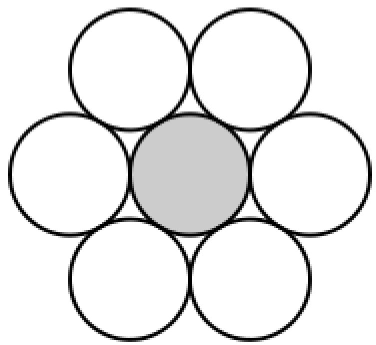
Partial proof of Kepler’s Cannonball Conjecture, 1831

Orange pyramid is the densest packing of spheres
if you consider packings in which centers of spheres form a periodic lattice.

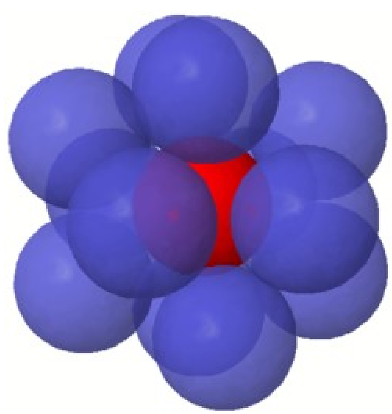
Kissing number

How many d-dimensional unit spheres
can simultaneously touch a unit sphere?

d=2 6



d=3 12



Newton's hypothesis
Proof, 1953

d=4	24,	Musin 2003
d=8	240	
d=24	196 560	

Thomas Hales 1958 -

Proved Kepler Cannonball Conjecture

Announced, 1998

Annals of Mathematics, 2005

computer-assisted proof

12 referees read 300 pages for 7 years

Final paper, Forum of Mathematics, 2017

David Hilbert 1862 - 1943

23 problems, 1900

Problem 18 Part II

Does there exist a polyhedron which can cover the space
but only in a nonperiodic way ?

Hao Wang 1921 - 1995

Hilbert problem for domino players

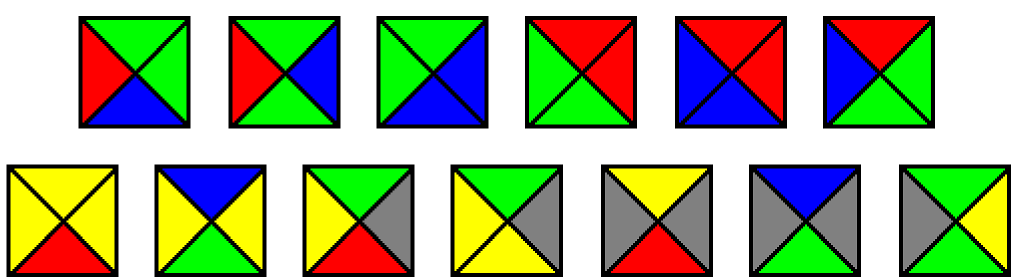
Wang tiles ---- squares with colored sides ---- square dominoes

Wang Hypothesis 1961

Each finite set of dominoes which covers the plane,
may also cover it in a periodic way.

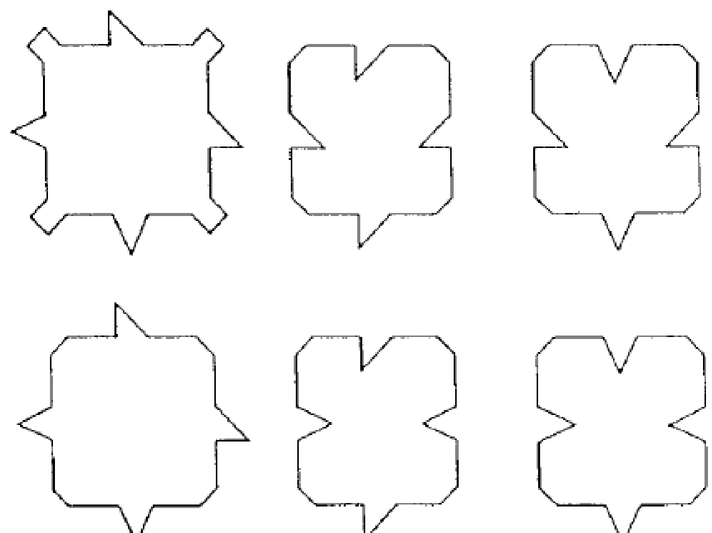
A brief history of the world record

Robert Berger,	20426 tiles,	1966
Raphael Robinson,	56 tiles,	1971
Robert Ammann,	16 tiles,	1977
Jarkko Kari,	14 tiles,	1996
Karel Culik II,	13 tiles,	1996
Emmanuel Jeandel and Michael Rao	11 tiles,	2015

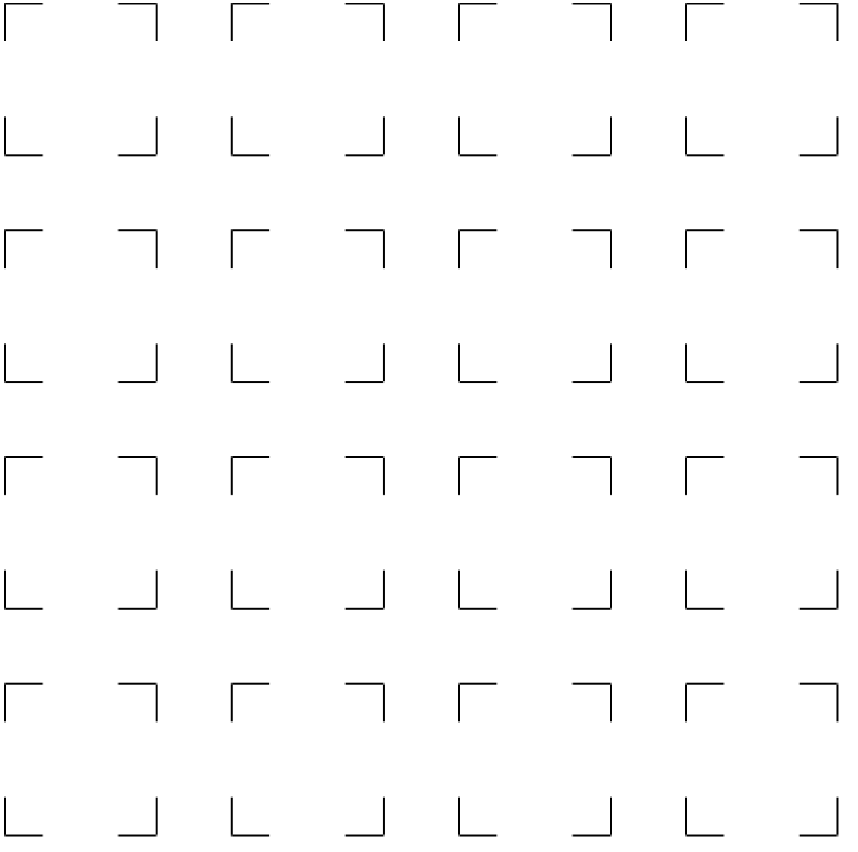


Raphael Robinson 1911 - 1995

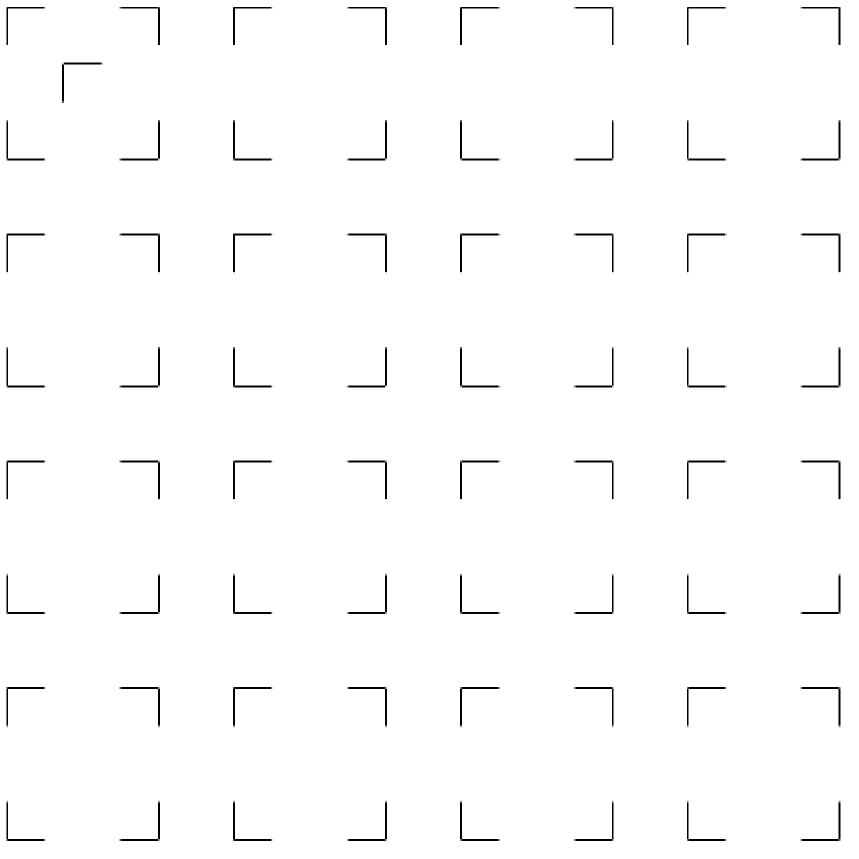
6 (56) tiles which cover planes but only in a non-periodic way, 1971



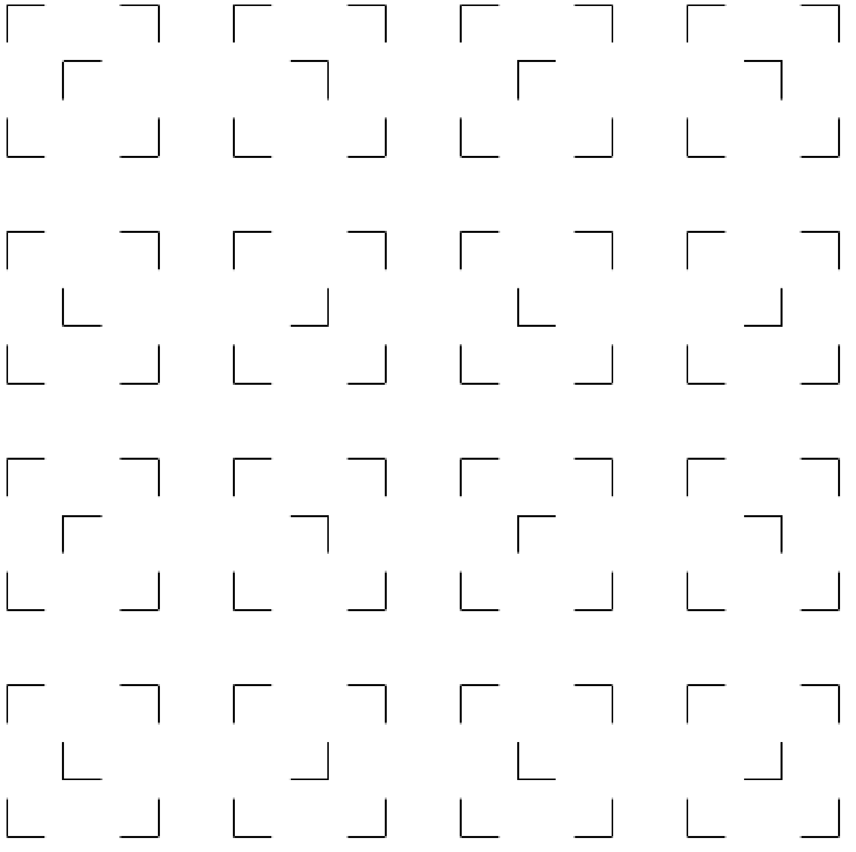
Structure of an infinite tiling



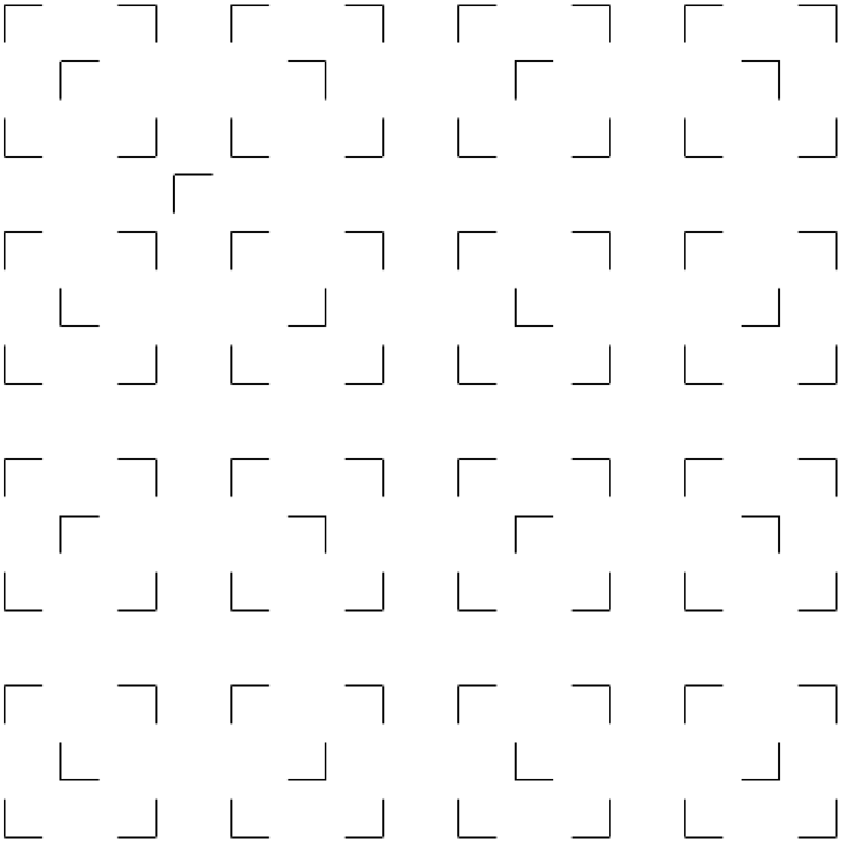
Structure of an infinite tiling



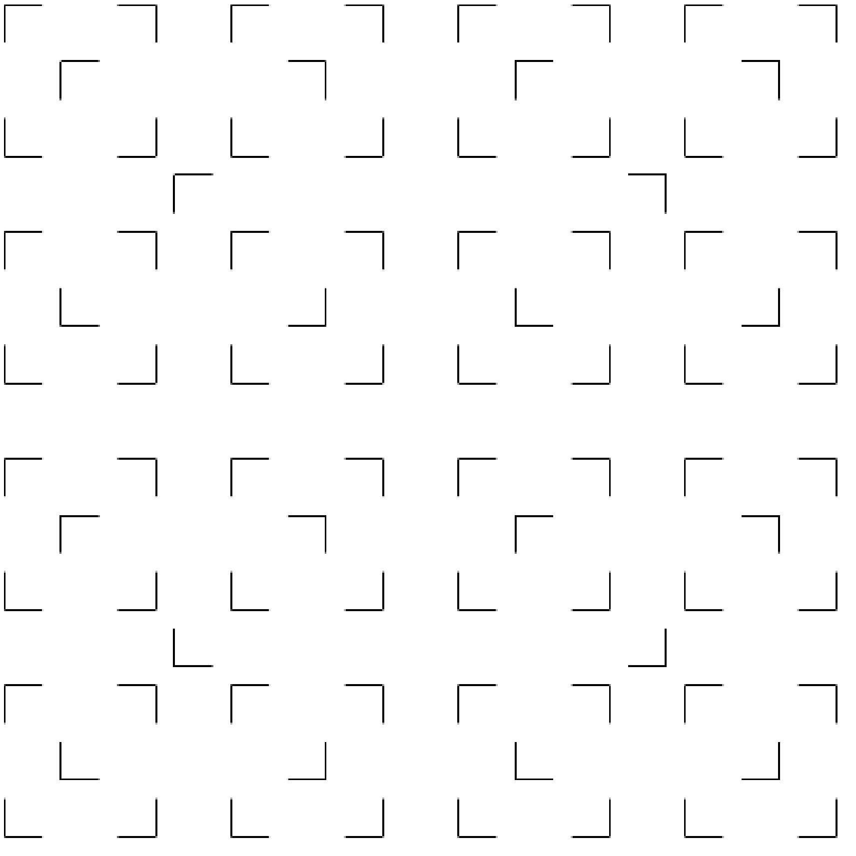
Structure of an infinite tiling



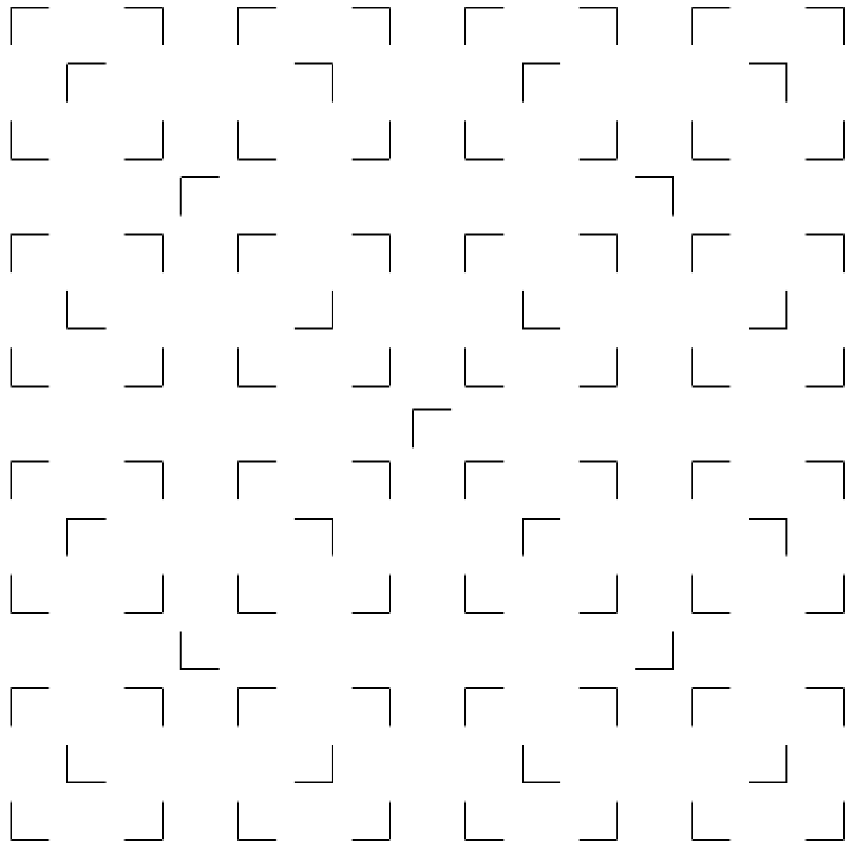
Structure of an infinite tiling



Structure of an infinite tiling



Structure of an infinite tiling



Configurations with period 2^{n+1} on sublattices $2^n Z^2$ $n \geq 1$

Global order from local rules

Systems of finite type

$$\Omega = \{1, ..., 56\}^{Z^2}$$

Let X be a Robinson's tiling, $X \in \Omega$

$$R = closure(\{T_a X, a \in Z^2\})$$

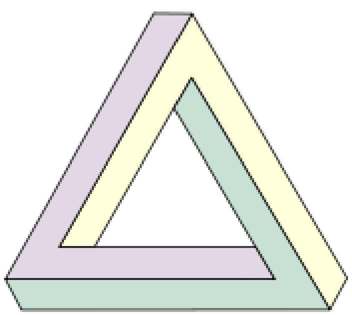
$$\mu_R = lim_{\Lambda \to Z^2} \frac{1}{|\Lambda|} \Sigma_{a \in \Lambda} \delta_{T_a} X$$

R is defined by a finite number of forbidden patterns –
two neighboring tiles that do not match

(R, T, μ_R) is a dynamical system of finite type

Roger Penrose 1931 -

In 1954 Penrose was at the exhibition of M. C. Escher

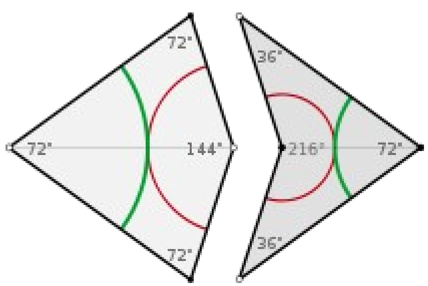


Waterfall, M. C. Escher, 1961, Wikipedia

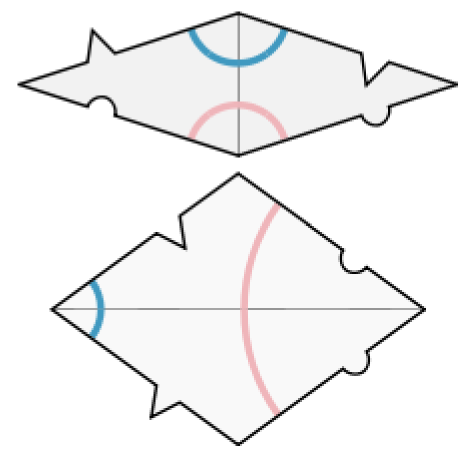
Roger Penrose 1931 -

Two tiles which cover the plane but only in non-periodic way, 1974

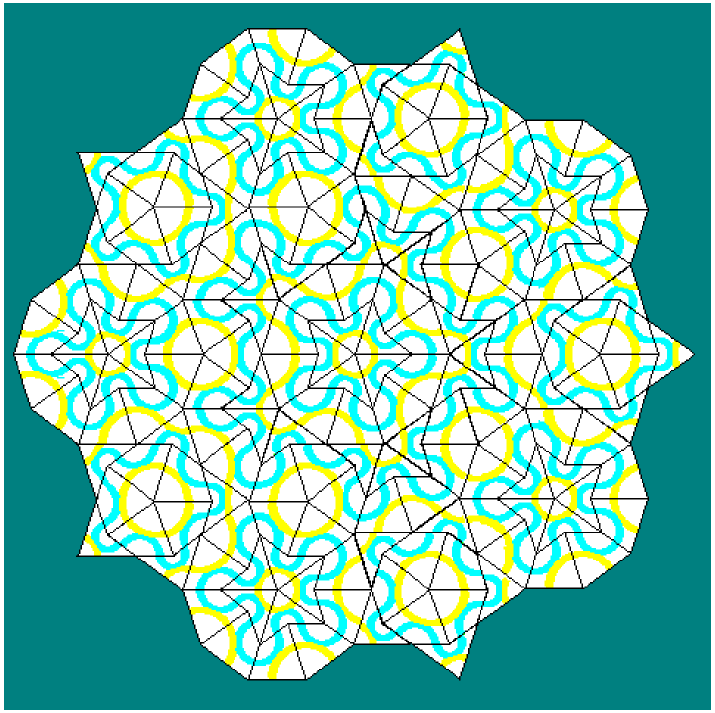
dart and kite

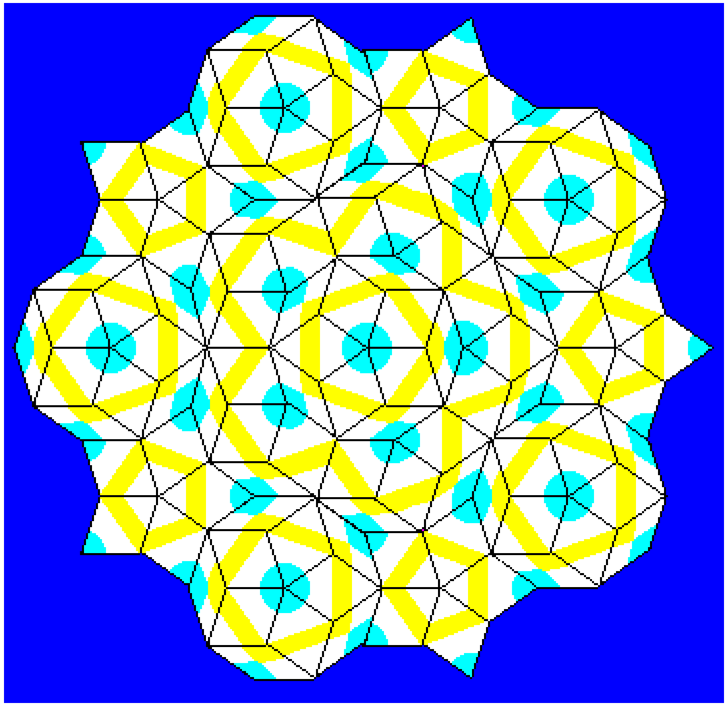


rhombs



Penrose tilings

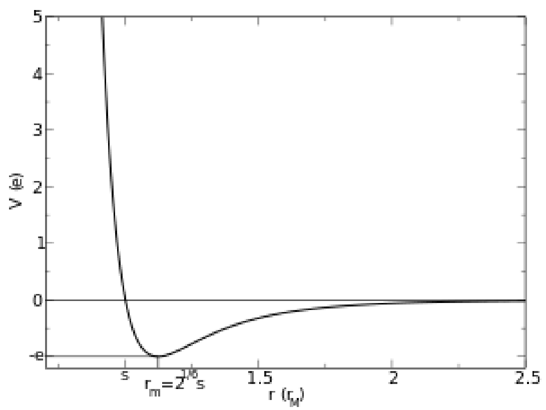


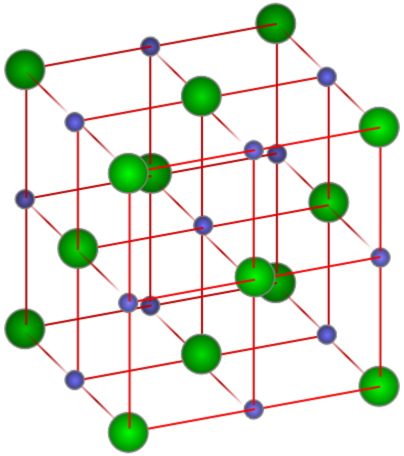


The Crystal Problem

Equilibrium state of many interacting particles
minimizes free energy $F = E - TS$
(or energy E in temperature $T = 0$)

To prove that minimization of the energy of realistic particle interactions, for example Lennard-Jones, leads to periodic crystal lattices.



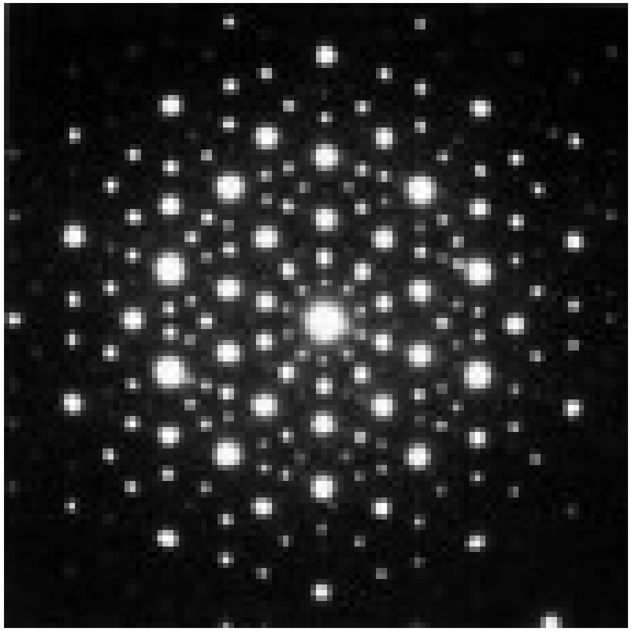


Dan Shechtman 1941 -

Technion Israel Institute of Technology
Iowa State University

8 April 1982 in National Bureau of Standards in Washington

Dan Shechtman was observing rapidly solidified aluminum transition metal alloys



Nobel Prize in Chemistry, 2011

Philip W. Anderson

Basic notions of condensed matter physics, 1984

Benjamin/Cummings Pub. Co.

„Proved” that every interaction has at least one periodic ground-state configuration

Classical lattice-gas models based on tilings

$$\text{tiles} \to \text{particles}$$

$$\text{matching rules} \to \text{interactions}$$

If two tiles do not match, then the energy of interaction between corresponding particles is positive, say 1, otherwise the energy is zero

ground-state configurations – configurations which minimize the energy

forbidden patterns have positive energy

$$\text{tilings} \to \text{ground-state configurations}$$

ground-state configurations have zero energy systems of finite type ---> lattice-gas models with finite-range interactions

ergodic measure ----- ground-state measure

However

Lattice gas models with finite-range interactions
are not necessarily dynamical systems of finite type

JM, Journal of Statistical Physics, 1999

Low-temperature stability, Gibbs measures

$$X \in \{1, ..., n\}^{\Lambda}, \;\; \Lambda \subset Z^d, \;\; \Lambda \;\; finite$$

$$\rho_{\Lambda}(X) = \frac{e^{-\frac{H(X)}{T}}}{Z_{\Lambda}}$$

$$\rho_{\Lambda} \to \{Gibbs \;\; Measures\} \;\; as \;\; \Lambda \to Z^d$$

Open Problem

Does there exist a lattice-gas model with translation-invariant finite-range interactions without periodic ground-state configurations and non-periodic equilibrium state at positive temperature ?

so far

Theorem (JM, 1990)

There is a decreasing sequence of temperatures, T_n, such that if $T < T_n$, then there exists a Gibbs state with a period at least 2×6^n in both directions.

Thue-Morse sequences

substitutions

$0 \rightarrow 01$
$1 \rightarrow 10$

0

01

0110

01101001

0110100110010110

Let X be a Thue-Morse sequence, $X \in \{0,1\}^Z$

$$TM = closure(\{T_a X, a \in Z\})$$

$$\mu_{TM} = lim_{\Lambda \to Z} \frac{1}{|\Lambda|} \Sigma_{a \in \Lambda} \delta_{T_a} X$$

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} f(T^i(X)) = \int f d\mu_{TM}$$

(TM, T, μ_{TM}) is a uniquely ergodic dynamical system, Michael Keane, 1968

Characterizations of Thue-Morse sequences

Goal: looking for the minimal set of forbidden patterns

Gottschalk and Hedlung, 1964

TM is uniquely characterized by the absence of BBb,

where B is any word and b is its first character, 0 or 1

TM is uniquely characterized by the absence of infinitely many 4-point patterns
(Gardner, Radin, Miękisz, van Enter, 1989)

4-body Hamiltonian with Thue-Morse sequences as unique ground states

Fibonacci sequences

substitutions

$0 \rightarrow 01$
$1 \rightarrow 0$

0	1
01	2
010	3
01001	5
01001010	8
0100101001001	13

$(F, T, \mu F)$ *is a uniquely ergodic system*

density of 0's = $\dfrac{2}{1+\sqrt{5}} = \gamma$

Another construction of Fibonacci sequences

Let $0 \leq \phi \leq 2\pi$ and let T_γ be a rotation by $2\pi\gamma$ on a unit circle.

If $T_\gamma^n(\phi) \in [0, 2\pi\gamma)$ *mod* 2π then let $a(n) = 0$, otherwise $a(n) = 1$.

Ergodic optimization in dynamical systems

Survey article in Ergodic Theory and Dynamical systems, O. Jenkinson, 2018

$X \quad compact\ space$

$T : X \to X$

$M_T \quad T-invariant\ probability\ measures\ on\ X$

$f : X \to R, \ \ f\ \ continuous$

$$\int f\, dm = \max_{\mu \in \mathcal{M}_T} \int f\, d\mu$$

m is called a **maximizing measure** for *f*

In our examples:

$X = \{1, ..., n\}^{Z^d}, \ \ T\ \ is\ \ a\ \ translation\ \ operator,\ \ f\ \ is\ \ -energy\ \ per\ \ lattice\ \ site$

Goals of ergodic optimization

Properties of maximizing measures for particular f

non-periodicity, mixing properties,

spectrum of T in $L^2(X,m)$

Properties for generic f

Example

In the second Baire category set of interactions in a suitable Banach space, the unique ergodic ground-state mea sure is non-noperiodic and non-mixing or just concentrated on one uniform configuration.

JM, 1988; JM and Charles Radin, 1989

Special thanks to

Grant NCN Harmonia

Mathematical models of quasicrystals

We need young people to solve fundamental problems

NARODOWE
CENTRUM
NAUKI

9–13 July 2018
Będlewo, Poland

BANACH CENTER SCHOOL:

Mathematical Physics of Non–periodic Structures

- **Quasicrystals**
- **Non-periodic tilings**
- **Gibbs States**
- **Spin Glasses**

Organizers:
Jacek Miękisz (IMPAN and MIMUW, Warsaw)
Daniel Stein (Courant, New York City)
Jan Wehr (University of Arizona, Tucson)

HP REVEAL: KWAZI

To be continued