To be a $C(K)$-space is not a three-space property

Alberto Salguero-Alarcón

(joint work with Grzegorz Plebanek)

Universidad de Extremadura, Badajoz, Spain

Seminar on Topology and Set Theory
University of Warsaw, November 24th, 2021

This work has been supported by project IB20038 (Junta de Extremadura) and by grants FPU18/00990 and EST19/00738 (Ministerio de Ciencia, Innovación y Universidades).

FEDER
Fondo Europeo de Desarrollo Regional
UNIÓN EUROPEA
"Una manera de hacer Europa"

GOBIERNO DE ESPAÑA
MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES

JUNTA DE EXTREMADURA

Banach spaces

Let K be a compact Hausdorff space.

Let K be a compact Hausdorff space.

1. $C(K) = \{f : K \to \mathbb{R} \text{ continuous}\}$ is a Banach space with the norm $\|f\| = \sup_{x\in K} |f(x)|$.

Banach spaces

Let K be a compact Hausdorff space.

1. $C(K) = \{f : K \to \mathbb{R} \text{ continuous}\}$ is a Banach space with the norm $\|f\| = \sup_{x \in K} |f(x)|$.

2. $C(K)^* = M(K)$ the space of (bounded, regular, Borel, signed) measures, with the norm

$$\|\mu\| = |\mu|(K)$$

Banach spaces

Let K be a compact Hausdorff space.

1. $C(K) = \{f : K \to \mathbb{R} \text{ continuous}\}$ is a Banach space with the norm $\|f\| = \sup_{x\in K} |f(x)|$.

2. $C(K)^* = M(K)$ the space of (bounded, regular, Borel, signed) measures, with the norm

$$\|\mu\| = |\mu|(K)$$

3. We say that X is a $\mathcal{C}$-space if there is some K so that X is isomorphic to $C(K)$.

Twisted sum of Banach spaces

A *twisted sum* of two Banach spaces Y and X is another space Z so that

Twisted sum of Banach spaces

A *twisted sum* of two Banach spaces Y and X is another space Z so that

1. Z contains Y as a closed subspace;

Twisted sum of Banach spaces

A *twisted sum* of two Banach spaces Y and X is another space Z so that

1. Z contains Y as a closed subspace;
2. Z/Y is isomorphic to X.

A *twisted sum* of two Banach spaces Y and X is another space Z so that

1. Z contains Y as a closed subspace;
2. Z/Y is isomorphic to X.

Exact sequences:

$$0 \longrightarrow Y \xrightarrow{\ i\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

Twisted sum of Banach spaces

A *twisted sum* of two Banach spaces Y and X is another space Z so that

1. Z contains Y as a closed subspace;
2. Z/Y is isomorphic to X.

Exact sequences:

$$0 \longrightarrow Y \xrightarrow{\ i\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

We say Z is *trivial* if one of the following equivalent conditions hold:

Twisted sum of Banach spaces

A *twisted sum* of two Banach spaces Y and X is another space Z so that

1. Z contains Y as a closed subspace;
2. Z/Y is isomorphic to X.

Exact sequences:

$$0 \longrightarrow Y \xrightarrow{\;i\;} Z \xrightarrow{\;q\;} X \longrightarrow 0$$

We say Z is *trivial* if one of the following equivalent conditions hold:

- There is a "projection" (left-inverse) $P : Z \to Y$ so that $Pi = \mathrm{Id}_Y$.

Twisted sum of Banach spaces

A *twisted sum* of two Banach spaces Y and X is another space Z so that

1. Z contains Y as a closed subspace;
2. Z/Y is isomorphic to X.

Exact sequences:

$$0 \longrightarrow Y \xrightarrow{\ i\ } Z \xrightarrow{\ q\ } X \longrightarrow 0$$

We say Z is *trivial* if one of the following equivalent conditions hold:

- There is a "projection" (left-inverse) $P : Z \to Y$ so that $Pi = \mathrm{Id}_Y$.
- There is a "selection" (right-inverse) $S : Z \to Y$ so that $qS = \mathrm{Id}_X$.

Simple examples

1. *Sobczyk's theorem*: if X is separable, every twisted sum

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is trivial.

Simple examples

1. *Sobczyk's theorem*: if X is separable, every twisted sum

$$0 \longrightarrow c_0 \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is trivial.

2. A non-trivial example:

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \longrightarrow \ell_\infty / c_0 \longrightarrow 0$$

Twisted sums of c_0 and $C(K)$

Definition

A *countable discrete extension* of a compact space K is another compact space $L = K \cup \omega$ contaning a countable set ω of isolated points and a copy of K.

Twisted sums of c_0 and $C(K)$

Definition

A *countable discrete extension* of a compact space K is another compact space $L = K \cup \omega$ contaning a countable set ω of isolated points and a copy of K.

Every countable discrete extension of K gives rise to a twisted sum:

$$0 \longrightarrow c_0 \xrightarrow{\ i\ } C(K \cup \omega) \xrightarrow{\ q\ } C(K) \longrightarrow 0$$

Twisted sums of c_0 and $C(K)$

Definition

A *countable discrete extension* of a compact space K is another compact space $L = K \cup \omega$ contaning a countable set ω of isolated points and a copy of K.

Every countable discrete extension of K gives rise to a twisted sum:

$$0 \longrightarrow c_0 \xrightarrow{i} C(K \cup \omega) \xrightarrow{q} C(K) \longrightarrow 0$$

...but it can be trivial.

Three-space properties

Definition

A property $\mathcal{P}$ of Banach spaces is said to be a *three-space property* (3SP) if it is stable under twisted sums.

Three-space properties

Definition

A property $\mathcal{P}$ of Banach spaces is said to be a *three-space property* (3SP) if it is stable under twisted sums.

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

Three-space properties

Definition

A property $\mathcal{P}$ of Banach spaces is said to be a *three-space property* (3SP) if it is stable under twisted sums.

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

That is, if a Banach space X has a subspace Y so that both Y and X/Y have $\mathcal{P}$, then X also has $\mathcal{P}$.

Three-space properties

Definition

A property $\mathcal{P}$ of Banach spaces is said to be a *three-space property* (3SP) if it is stable under twisted sums.

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

That is, if a Banach space X has a subspace Y so that both Y and X/Y have $\mathcal{P}$, then X also has $\mathcal{P}$.

Examples:

Three-space properties

Definition

A property $\mathcal{P}$ of Banach spaces is said to be a *three-space property* (3SP) if it is stable under twisted sums.

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

That is, if a Banach space X has a subspace Y so that both Y and X/Y have $\mathcal{P}$, then X also has $\mathcal{P}$.

Examples:

1. Being finite-dimensional is a 3SP.

Three-space properties

Definition

A property $\mathcal{P}$ of Banach spaces is said to be a *three-space property* (3SP) if it is stable under twisted sums.

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

That is, if a Banach space X has a subspace Y so that both Y and X/Y have $\mathcal{P}$, then X also has $\mathcal{P}$.

Examples:

1. Being finite-dimensional is a 3SP.
2. Separability is a 3SP.

To be a $\mathcal{C}$-space...

...is **not** a three-space property.

To be a $\mathcal{C}$-space...

...is **not** a three-space property.

(1) Benyamini-Cabello-Castillo:

To be a $\mathcal{C}$-space...

...is **not** a three-space property.

(1) Benyamini-Cabello-Castillo:

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\ell_\infty/c_0) \longrightarrow 0$$

To be a $\mathcal{C}$-space...

...is **not** a three-space property.

(1) Benyamini-Cabello-Castillo:

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\ell_\infty/c_0) \longrightarrow 0$$

(2) Cabello-Castillo-Kalton-Yost:

$$0 \longrightarrow C(\omega^\omega) \longrightarrow X \longrightarrow c_0 \longrightarrow 0$$

$$0 \longrightarrow C[0,1] \longrightarrow X \longrightarrow c_0 \longrightarrow 0$$

To be a $\mathcal{C}$-space...

...is **not** a three-space property *even for the "simplest"* $\mathcal{C}$*-spaces.*

To be a $\mathcal{C}$-space...

...is **not** a three-space property *even for the "simplest" $\mathcal{C}$-spaces.*

Theorem 1

$[\mathfrak{p} = \mathfrak{c}]$ *There is a twisted sum of* c_0 *and* $c_0(\mathfrak{c})$ *which is not a* $\mathcal{C}$*-space.*

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.

First step

How to produce a “complicated” twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.
2. Consider $L = M_1(K) \cup \omega$ and

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.
2. Consider $L = M_1(K) \cup \omega$ and

$$0 \longrightarrow c_0 \longrightarrow C(M_1(K) \cup \omega) \longrightarrow C(M_1(K)) \longrightarrow 0$$

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.
2. Consider $L = M_1(K) \cup \omega$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(M_1(K) \cup \omega) & \longrightarrow & C(M_1(K)) & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & Z & & & & \end{array}$$

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.
2. Consider $L = M_1(K) \cup \omega$ and

$$0 \longrightarrow c_0 \longrightarrow C(M_1(K) \cup \omega) \longrightarrow C(M_1(K)) \longrightarrow 0$$

$$\uparrow$$

$$Z$$

$$Z = \{f \in C(M_1(K) \cup \omega) : \exists g \in C(K) : f|_{M_1(K)} = \widehat{g}\}$$

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.

2. Consider $L = M_1(K) \cup \omega$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(M_1(K) \cup \omega) & \longrightarrow & C(M_1(K)) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & Z & \longrightarrow & C(K) & \longrightarrow & 0 \end{array}$$

$$Z = \{f \in C(M_1(K) \cup \omega) : \exists g \in C(K) : f|_{M_1(K)} = \widehat{g}\}$$

First step

How to produce a "complicated" twisted sum of c_0 and $C(K)$?

1. Every function $g \in C(K)$ can be extended to a function $\widehat{g} \in C(M_1(K))$.
2. Consider $L = M_1(K) \cup \omega$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(M_1(K) \cup \omega) & \longrightarrow & C(M_1(K)) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & Z & \longrightarrow & C(K) & \longrightarrow & 0 \end{array}$$

$$Z = \{f \in C(M_1(K) \cup \omega) : \exists g \in C(K) : f|_{M_1(K)} = \widehat{g}\}$$

☞ How to produce $L = M_1(K) \cup \omega$?

Second step

How to produce Z so that it is not a $\mathcal{C}$-space?

Second step

How to produce Z so that it is not a $\mathcal{C}$-space?

Let $F \subseteq B_{X^*}$ be weak*-compact, and $0 < c \leq 1$.

Second step

How to produce Z so that it is not a $\mathcal{C}$-space?

Let $F \subseteq B_{X^*}$ be weak*-compact, and $0 < c \leq 1$.

Definition

We say that F is c-norming if

$$c \cdot \|x\| \leq \sup_{x^* \in F} |x^*(x)|$$

Second step

How to produce Z so that it is not a $\mathcal{C}$-space?

Let $F \subseteq B_{X^*}$ be weak*-compact, and $0 < c \leq 1$.

Definition

We say that F is c-norming if

$$c \cdot \|x\| \leq \sup_{x^* \in F} |x^*(x)|$$

In other words, the "evaluation map":

$$e : X \to C(F) \quad , \quad e(x)(x^*) = x^*(x)$$

is an isomorphic embedding with $\|e^{-1}\| \leq \frac{1}{c}$.

Second step

How to produce Z so that it is not a $\mathcal{C}$-space?

Let $F \subseteq B_{X^*}$ be weak*-compact, and $0 < c \leq 1$.

Definition

We say that F is c-norming if

$$c \cdot \|x\| \leq \sup_{x^* \in F} |x^*(x)|$$

In other words, the "evaluation map":

$$e : X \to C(F) \quad , \quad e(x)(x^*) = x^*(x)$$

is an isomorphic embedding with $\|e^{-1}\| \leq \frac{1}{c}$.

Definition

We say that F is free if e is onto.

Lemma 2

For a Banach space X, TFAE:

Lemma 2

For a Banach space X, TFAE:

1. X *is a* $\mathcal{C}$*-space.*

Lemma 2

For a Banach space X, TFAE:

1. *X is a $\mathcal{C}$-space.*
2. *There is a weak*-compact set $F \subseteq B_{X^*}$ which is c-norming for some $0 < c \leq 1$ and free.*

Third step

How to recognize sets that are not free?

How to recognize sets that are not free?

Lemma 3

Let F be a c-norming set.

Third step

How to recognize sets that are not free?

Lemma 3

Let F be a c-norming set. If there are different points $x_0^*, x_1^*, x_2^* \in F$ such that

$$\|x_0^* - \frac{1}{2}(x_1^* + x_2^*)\| < c$$

Third step

How to recognize sets that are not free?

Lemma 3

Let F be a c-norming set. If there are different points $x_0^*, x_1^*, x_2^* \in F$ such that

$$\|x_0^* - \frac{1}{2}(x_1^* + x_2^*)\| < c$$

then F cannot be free.

Third step

How to recognize sets that are not free?

Lemma 3

Let F be a c-norming set. If there are different points $x_0^*, x_1^*, x_2^* \in F$ such that

$$\|x_0^* - \tfrac{1}{2}(x_1^* + x_2^*)\| < c$$

then F cannot be free.

We call the points x_0^*, x_1^*, x_2^* a *c-forbidden triple.*

Idea of the construction

Idea of the construction

For an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

we know:

Idea of the construction

For an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

we know:

1. X^* is isomorphic to $\ell_1 \times \ell_1(\mathfrak{c})$.

Idea of the construction

For an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

we know:

1. X^* is isomorphic to $\ell_1 \times \ell_1(\mathfrak{c})$.
2. Every c-norming set $F \subset B_{X^*}$ contains a sequence $(\varphi_n)_{n=1}^{\infty}$ so that,

Idea of the construction

For an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

we know:

1. X^* is isomorphic to $\ell_1 \times \ell_1(\mathfrak{c})$.
2. Every c-norming set $F \subset B_{X^*}$ contains a sequence $(\varphi_n)_{n=1}^{\infty}$ so that, writing

$$\varphi_n = \mu_n + \lambda_n \quad , \quad \mu_n \in \ell_1,\ \lambda_n \in \ell_1(\mathfrak{c})$$

we have $|\mu_n\{n\}| \geq c \ \forall n \in \omega$.

Idea of the construction

For an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

we know:

1. X^* is isomorphic to $\ell_1 \times \ell_1(\mathfrak{c})$.
2. Every c-norming set $F \subset B_{X^*}$ contains a sequence $(\varphi_n)_{n=1}^{\infty}$ so that, writing

$$\varphi_n = \mu_n + \lambda_n \quad , \quad \mu_n \in \ell_1, \ \lambda_n \in \ell_1(\mathfrak{c})$$

we have $|\mu_n\{n\}| \geq c \ \forall n \in \omega$.
Let us call such a sequence $(\mu_n)_{n=1}^{\infty}$ *c-admissible.*

Idea of the construction

For an exact sequence

$$0 \longrightarrow c_0 \longrightarrow X \longrightarrow c_0(\mathfrak{c}) \longrightarrow 0$$

we know:

1. X^* is isomorphic to $\ell_1 \times \ell_1(\mathfrak{c})$.
2. Every c-norming set $F \subset B_{X^*}$ contains a sequence $(\varphi_n)_{n=1}^{\infty}$ so that, writing

$$\varphi_n = \mu_n + \lambda_n \quad , \quad \mu_n \in \ell_1,\ \lambda_n \in \ell_1(\mathfrak{c})$$

   we have $|\mu_n\{n\}| \geq c \ \forall n \in \omega$.
   Let us call such a sequence $(\mu_n)_{n=1}^{\infty}$ *c-admissible.*

☞ *What we do:* we "destroy" the possibility that every admissible sequence lies inside a norming free set.

The core

The core

Theorem 4

[$\mathfrak{p} = \mathfrak{c}$] *Fix a collection* $\{\mathcal{D}_\alpha : \alpha < \mathfrak{c}\}$ *of dense families. There is an almost disjoint family* $\mathcal{A}$ *so that the attached space*

$$L = \frac{K_{\mathcal{A}} \sqcup B_{\ell_1(\mathfrak{c})}}{\equiv}$$

has the following property:

Theorem 4

[$\mathfrak{p} = \mathfrak{c}$] *Fix a collection* $\{\mathcal{D}_\alpha : \alpha < \mathfrak{c}\}$ *of dense families.*
There is an almost disjoint family $\mathcal{A}$ *so that the attached space*

$$L = \frac{K_{\mathcal{A}} \sqcup B_{\ell_1(\mathfrak{c})}}{\equiv}$$

has the following property:
For every c-admissible sequence $(\mu_n)_{n=1}^{\infty}$ *and every* $\alpha < \mathfrak{c}$ *there is:*

Theorem 4

$[\mathfrak{p} = \mathfrak{c}]$ *Fix a collection* $\{\mathcal{D}_\alpha : \alpha < \mathfrak{c}\}$ *of dense families.*
There is an almost disjoint family $\mathcal{A}$ *so that the attached space*

$$L = \frac{K_{\mathcal{A}} \sqcup B_{\ell_1(\mathfrak{c})}}{\equiv}$$

has the following property:
For every c-admissible sequence $(\mu_n)_{n=1}^{\infty}$ *and every* $\alpha < \mathfrak{c}$ *there is:*

- $D \in \mathcal{D}_\alpha$.

Theorem 4

[$\mathfrak{p} = \mathfrak{c}$] *Fix a collection* $\{\mathcal{D}_\alpha : \alpha < \mathfrak{c}\}$ *of dense families.*
There is an almost disjoint family $\mathcal{A}$ *so that the attached space*

$$L = \frac{K_{\mathcal{A}} \sqcup B_{\ell_1(\mathfrak{c})}}{\equiv}$$

has the following property:
For every c-admissible sequence $(\mu_n)_{n=1}^{\infty}$ *and every* $\alpha < \mathfrak{c}$ *there is:*

- $D \in \mathcal{D}_\alpha$.
- *a forbidden c-triple in the closure of* $\{\mu_n : n \in D\}$ *inside* $M(L)$.

Finally...

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent } \}$$

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent }\}$$

2. Suppose F is a c-norming set.

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent }\}$$

2. Suppose F is a c-norming set.

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent }\}$$

2. Suppose F is a c-norming set. Then F contains a c-admissible sequence

$$\varphi_n = \mu_n + \lambda_n^\alpha$$

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent }\}$$

2. Suppose F is a c-norming set. Then F contains a c-admissible sequence

$$\varphi_n = \mu_n + \lambda_n^\alpha$$

   - $\{x_i^*, i = 0, 1, 2\}$ is a c-forbidden triple in $\overline{\{\mu_n : n \in D\}}$.

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent } \}$$

2. Suppose F is a c-norming set. Then F contains a c-admissible sequence

$$\varphi_n = \mu_n + \lambda_n^\alpha$$

   - $\{x_i^*, i = 0, 1, 2\}$ is a c-forbidden triple in $\overline{\{\mu_n : n \in D\}}$.
   - $\lim_{n\in D} \lambda_n^\alpha = \lambda$.

Finally...

$$Z = \{f \in C(L) : \exists g \in c_0(\mathfrak{c}) : f|_{B_{\ell_1(\mathfrak{c})}} = \widehat{g}\}$$

cannot be a $\mathcal{C}$-space.

1. Enumerate all the sequences in $B_{\ell_1(\mathfrak{c})}$ as $\{(\lambda_n^\alpha)_{n\in\omega} : \alpha < \mathfrak{c}\}$, and apply theorem 4 with

$$\mathcal{D}_\alpha = \{D : (\lambda_n^\alpha)_{n\in D} \text{ is convergent }\}$$

2. Suppose F is a c-norming set. Then F contains a c-admissible sequence

$$\varphi_n = \mu_n + \lambda_n^\alpha$$

   - $\{x_i^*, i = 0, 1, 2\}$ is a c-forbidden triple in $\overline{\{\mu_n : n \in D\}}$.
   - $\lim_{n\in D} \lambda_n^\alpha = \lambda$.

3. $\{x_i^* + \lambda : i = 0, 1, 2\}$ is a forbidden c-triple inside F.

THANK YOU
FOR YOUR ATTENTION