Isoperimetric problems in convex geometry

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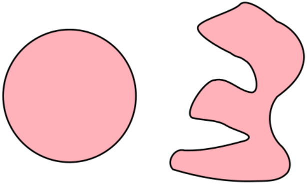
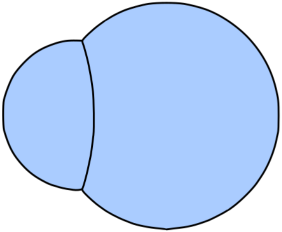
21 May, Warsaw

$$|A| = |\text{ball}| \quad \Longrightarrow \quad |\partial A| \geq |\partial \text{ ball}|$$

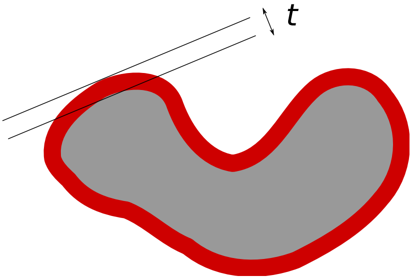




Cheap definition of surface area measure:

$$|\partial A| \quad = \quad \liminf_{t \to 0^+} \quad \frac{|A_t| - |A|}{t}$$

where $A_t = \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, A) \leq t\}$.



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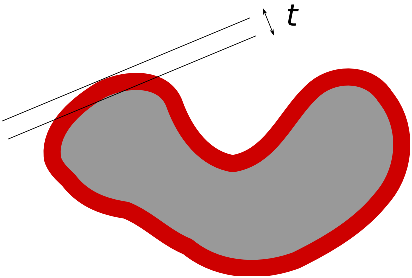

Isoperimetric inequality: if B is a ball then

$$|A| = |B| \quad \Longrightarrow \quad |A_t| \geq |B_t|, \qquad t \geq 0.$$

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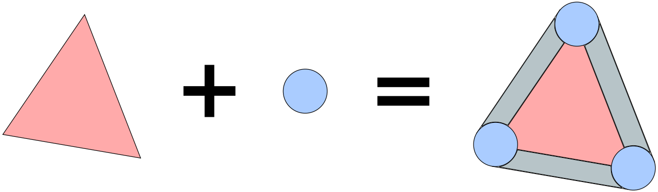

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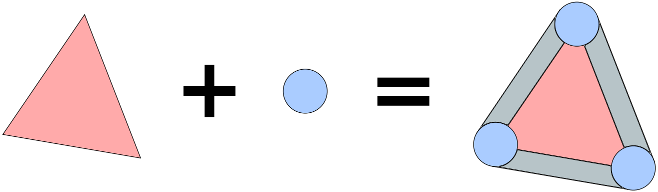

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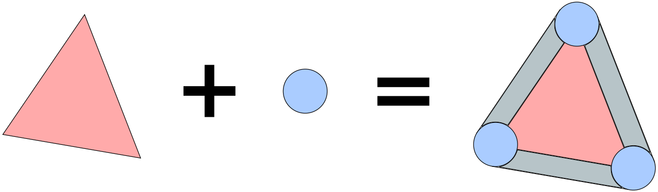

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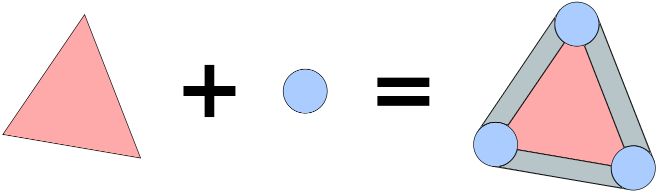

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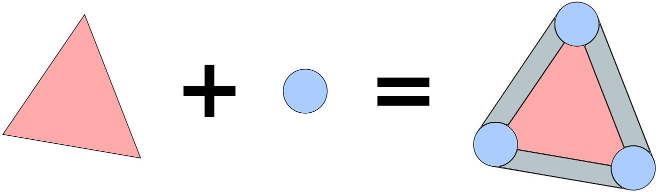

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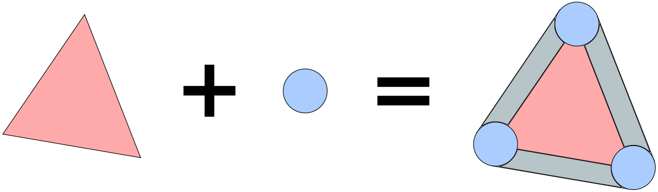

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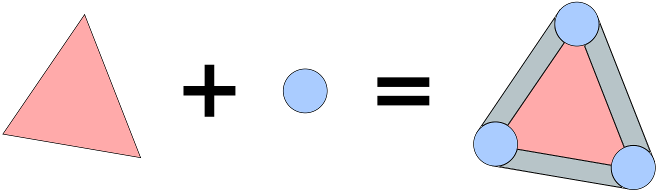

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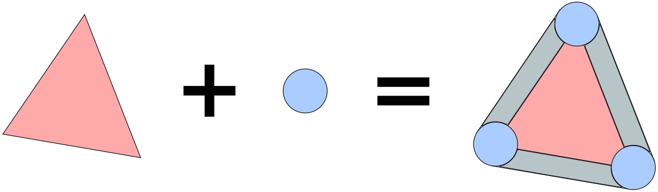

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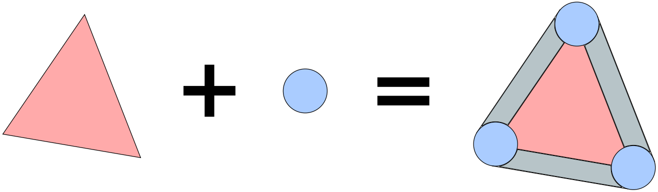

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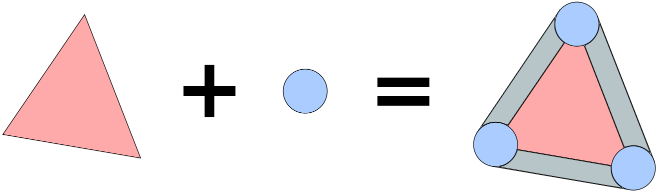

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&= \frac{\sqrt[n]{a_1 \cdot \ldots \cdot a_n} + \sqrt[n]{b_1 \cdot \ldots \cdot b_n}}{\sqrt[n]{(a_1+b_1) \cdot \ldots \cdot (a_n+b_n)}}
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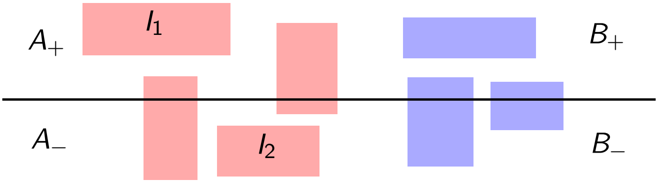
General case (induction on the number of boxes in $A \cup B$):



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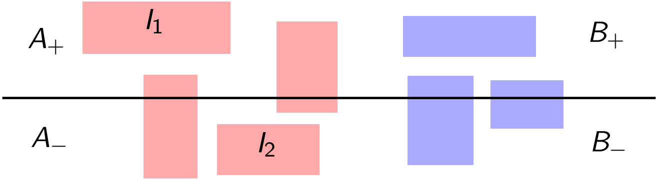

Shift A so that $\{x_i = 0\}$ separates boxes I_1 and I_2 in A. Then shift B to obtain

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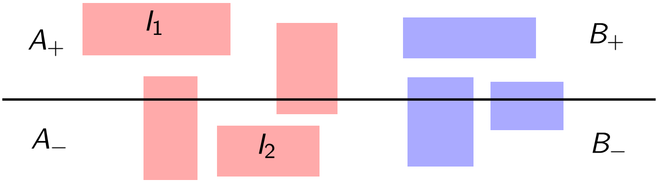

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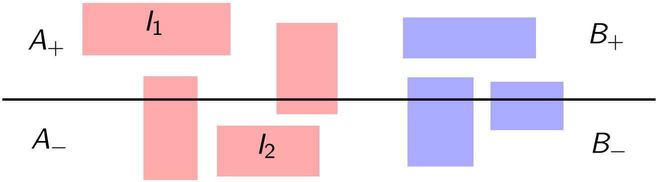

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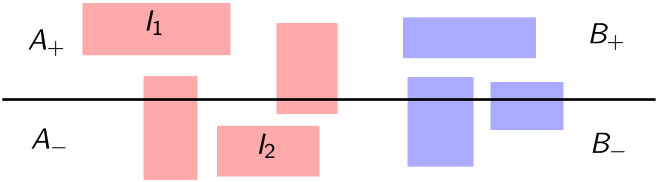

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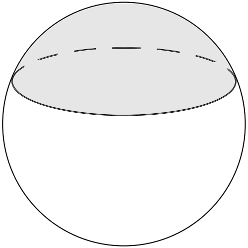
For general metric measure space we can define

$$\mu^+(\partial A) = \liminf_{t \to 0^+} = \frac{\mu(A_t) - \mu(A)}{t}.$$



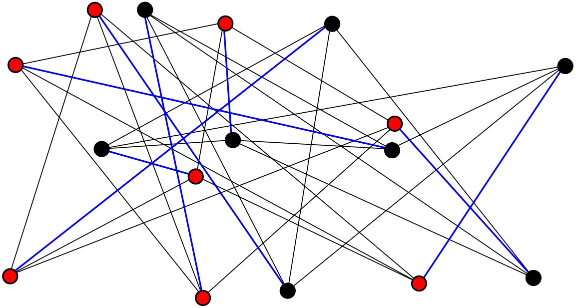
P. Levý (1951)

$$\mathrm{d}\mu(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|x|^2} \mathrm{d}x$$

Borell (1975)
Sudakov-Tsirelson (1974)

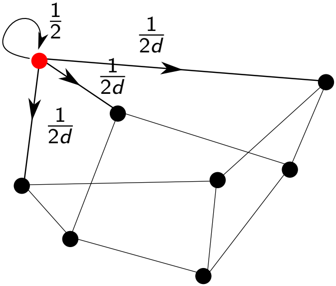
Isoperimetric problem on graphs: find

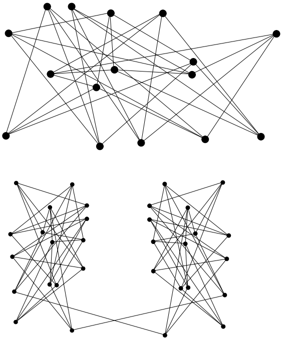
$$\alpha_k(G) = \min\{E(A, A^c) : |A| = k\}.$$



For this graph we have $\alpha_8(G) = 8$.

Symmetric random walk on d-regular graph





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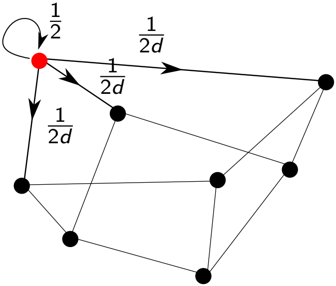
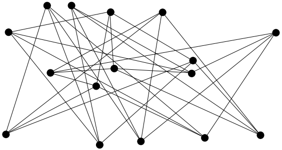
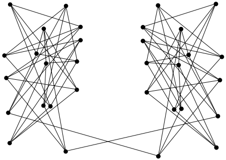






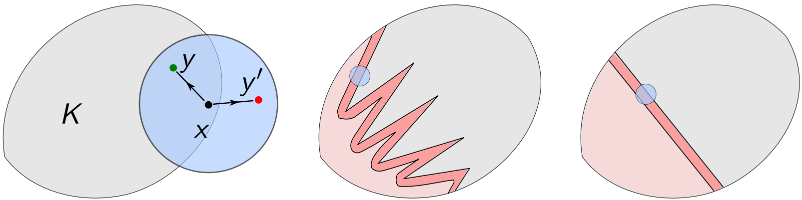
Conductance:

$$\Phi = \min_{|S| \leq \frac{|G|}{2}} \frac{|E(S, S^c)|}{d|S|} \qquad \max_{u,v \in G} \left| P^n(u \to v) - \frac{1}{|G|} \right| \leq \left(1 - \frac{1}{8}\Phi^2\right)^n$$

Large conductance $\implies$ easy to escape small sets

Ball walk: at a current point x

1. Pick a random point y from $B(x, \delta)$, where δ is fixed.
2. If $y \in K$ go to y, otherwise stay at x.



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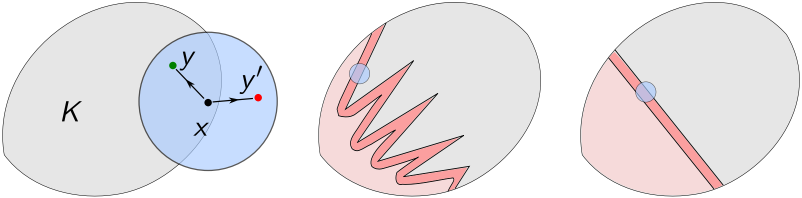

Conductance (Cheeger constant) of a convex set K:

$$\Phi(K) = \inf_{S \subseteq K:\ |S| \leq \frac{1}{2}|K|} \frac{|\partial_K S|}{|S|}$$

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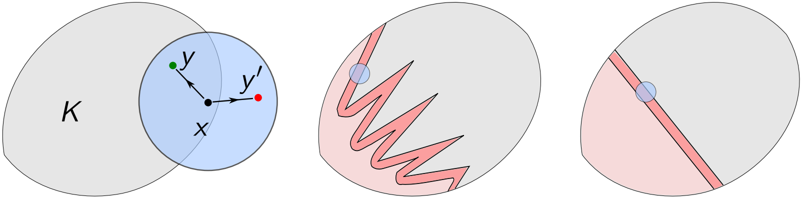

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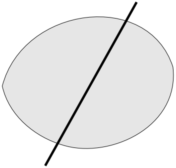
KLS Conjecture: $\Phi(K)$ is achieved, up to a universal multiplicative constant, for $S = K \cap H$, where H is a halfspace.

Some reformulations and closely related conjectures:

1. Up to a constant, the cheapest way of cutting K into two parts of equal volume is to use a hyperplane.



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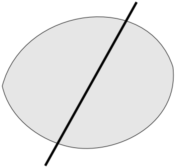

1. Up to a constant, the cheapest way of cutting K into two parts of equal volume is to use a hyperplane.

2. Under the normalization $\frac{1}{|K|}\int_K x_i \mathrm{d}x = 0$ and $\mathrm{Cov}_{ij}(K) = \frac{1}{|K|}\int_K x_i x_j \mathrm{d}x = \delta_{ij}$ (isotropic position) we have

$$|\partial_K S| \geq c|S|, \qquad S \subseteq K,\ |S| \leq \frac{1}{2}|K|, \qquad \text{that is } \Phi(K) \geq c.$$

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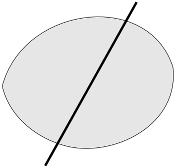

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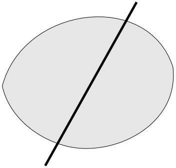

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3. **Slicing conjecture:** if $|K| = 1$ then there exists a hyperplane such that $|K \cap H| \geq c$, where c is a universal constant.
4. **Thin shell conjecture:** if K is isotropic then $\frac{1}{|K|}\int_K (|x| - \sqrt{n})^2 \mathrm{d}x \leq C$. Mass concentrated in a thin shell around sphere of radius $\sqrt{n}$.

KLS Conjecture $\iff$ Poincaré inequality

$$\mathrm{Var}_K(f) := \frac{1}{|K|}\int_K f^2 - \left(\frac{1}{|K|}\int_K f\right)^2 \leq C(K) \cdot \frac{1}{|K|}\int_K |\nabla f|^2$$

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The best constant $C(K)$ is equal to λ_1^{-1}, where λ_1 is the smallest non-zero eigenvalue of $-\Delta$ with Neumann boundary conditions.

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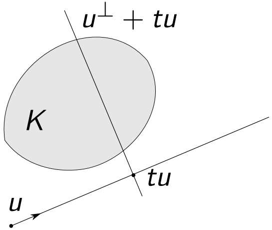
KLS conjecture is equivalent to the following statement:

$$\lambda_1 = \inf_{\int_K f = 0} \frac{\int_K |\nabla f|^2}{\int_K f^2} \qquad \text{is } \approx \text{ saturated for linear } f.$$

Geometric tomography:

Fix some direction $u \in S^{n-1}$ and define

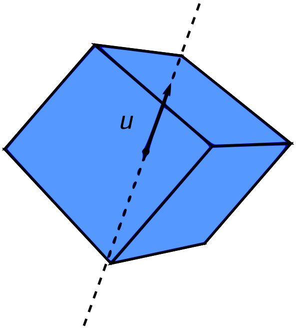
$$f(t) = |K \cap (u^{\perp} + tu)| = |K_t|$$



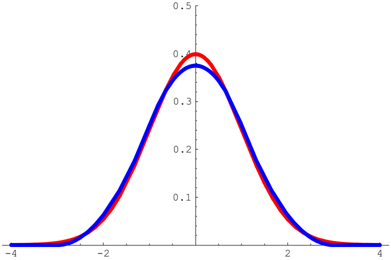
From Brunn-Minkowski $t \mapsto f(t)^{\frac{1}{n-1}}$ is concave on its support.

Affine for cones!

Central limit theorem for convex sets (B. Klartag)



$u = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



$f_u(x)$ vs. $(2\pi)^{-\frac{1}{2}} e^{-x^2/2}$

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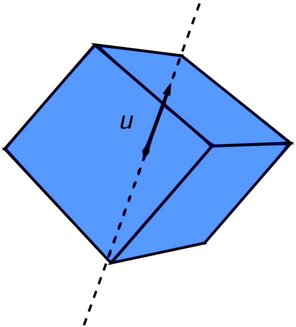
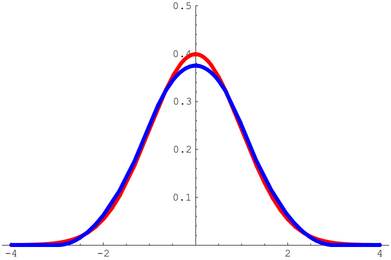




$u = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

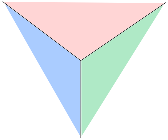
$f_u(x)$ vs. $(2\pi)^{-\frac{1}{2}} e^{-x^2/2}$

For any K satisfying $\mathrm{Cov}(K) = I$ for $1 - e^{-\sqrt{n}}$ fraction of directions u we have

$$\sup_{t \in \mathbb{R}} \left| \int_{-\infty}^{t} f_u(x) \mathrm{d}x - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} \right| \leq Cn^{-\frac{1}{16}}$$

Two recent remarkable results around isoperimetric inequality and Brunn-Minkowski inequality in Gauss space:

$$\gamma_n(A) = \int_A (2\pi)^{-\frac{n}{2}} e^{-|x|^2/2} \mathrm{d}x$$



1. Double-bubble in Gauss space (Milman-Neeman, 2018), minimizing boundary of division of $\mathbb{R}^n$ into three parts: minimizers are standard tripods.

2. Brunn-Minkowski inequality in Gauss space (Eskenazis-Moschidis, 2020), for A, B convex origin-symmetric

$$\gamma_n(\lambda A + (1-\lambda)B)^{\frac{1}{n}} \geq \lambda \gamma_n(A)^{\frac{1}{n}} + (1-\lambda)\gamma_n(B)^{\frac{1}{n}}.$$

Thank you!