Nevanlinna Prize 2018:

How hard is it to compute Nash equilibrium?

Nevanlinna Prize 2018

- Awarded to **Constantinos Daskalakis** (MIT)

For transforming our understanding of the computational complexity of fundamental problems in markets, auctions, equilibria, and other economic structures. His work provides both efficient algorithms and limits on what can be performed efficiently in these domains.

Nevanlinna Prize

- Prize in mathematical aspects of information sciences
- Named in honor of Rolf Nevanlinna (1895-1980), president of the International Mathematical Union (1959-1963) and president of the International Congress of Mathematicians (1962)
- Awarded every 4 years since 1982
- Presented at International Congress of Mathematicians (along with Fields Medal)

Nevanlinna Prize

Complexity of Computing Nash Equilibrium

Rational choice (single player)

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- One **player**

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- Set of **strategies** S over which the choice is made
- **Utility function**: $u : S \to \mathbb{R}$
- Objective: maximise the value of u on S

Games

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- **Finite game:** There is finite number of players and each has finite set of strategies

Solution concepts

- Games specify possible choices of the players and their payoffs
- They do not specify the outcomes that will result from players' choices
- **Game solution** provides systematic description of what outcomes might emerge

Solution concepts

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C	3,3	-1,4
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Definition (Nash equilibrium)

Strategy profile such that no player can benefit from changing his strategy individually

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- **Problem:** Nash equilibrium may not exist
- **Extension:** allow players to choose probability distributions over set of strategies (**mixed strategies**)
- $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is stable: no one can benefit from changing his (mixed) strategy individually

Nash equilibrium

Theorem (Nash (1951))

Every finite game has (Nash) equilibrium in mixed strategies

Computation of Nash equilibria

- **Input** (the game):
  - Set of players, $N = \{1, \ldots, n\}$
  - Strategies of each player: $S_i = \{s_1, \ldots, s_{m_i}\}$ for $i \in N$
  - Utilities of players: $u_i^{\boldsymbol{s}}$, for each $\boldsymbol{s} \in \prod_{i \in N} S_i$ and $i \in N$
- **Output** (Nash equilibrium):
  - For each player i his mixed strategy: $\boldsymbol{x}_i = (x_i^{s_1}, \ldots, x_i^{s_{m_i}})$

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  - (Integer) $a > 0$
- **Output** ($1/a$-Nash equilibrium):
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Computation of Nash equilibria: what was known

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- **Bubelis (1979)**: efficient reduction of NE computation for k-player games ($k > 3$) to NE computation in 3-player games

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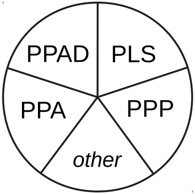

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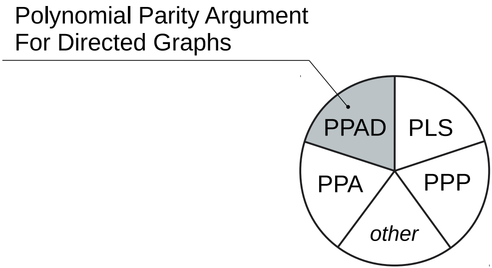

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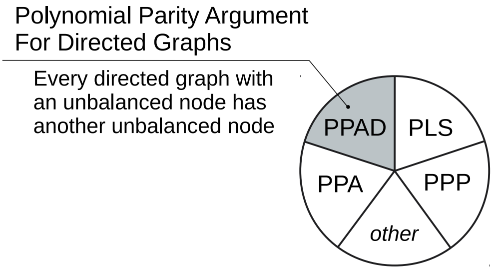
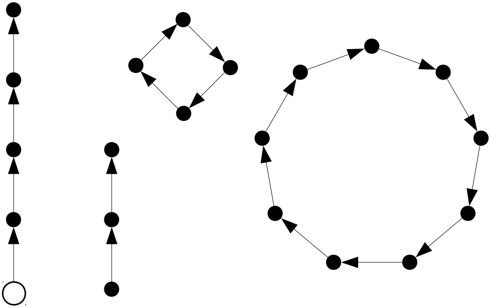


Class PPAD: completeness

- **End Of The Line Problem**: Given a directed graph over 2^n vertices with in-degree and out-degree ≤ 1 and a source vertex find a sink or another source



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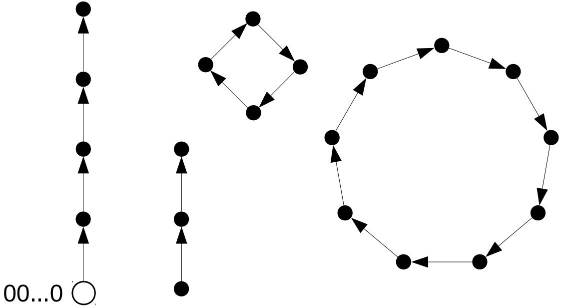

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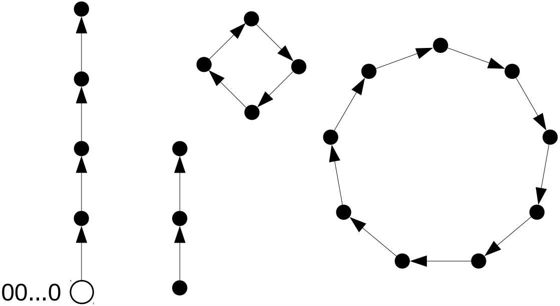
  - Vertices are 0-1 strings of length n
  - Edges are represented by two functions **S** and **P** encoded as boolean circuits of polynomial size



Class PPAD: completeness

Definition (Problem End Of The Line)

- **Input**: (graph of in- and out-degree at most 1)
  - Functions **S** and **P** representing edges of the graph over 2^n vertices such that $(0, \ldots, 0)$ is a source vertex
- **Output**
  - Source vertex different to $(0, \ldots, 0)$ or a sink vertex

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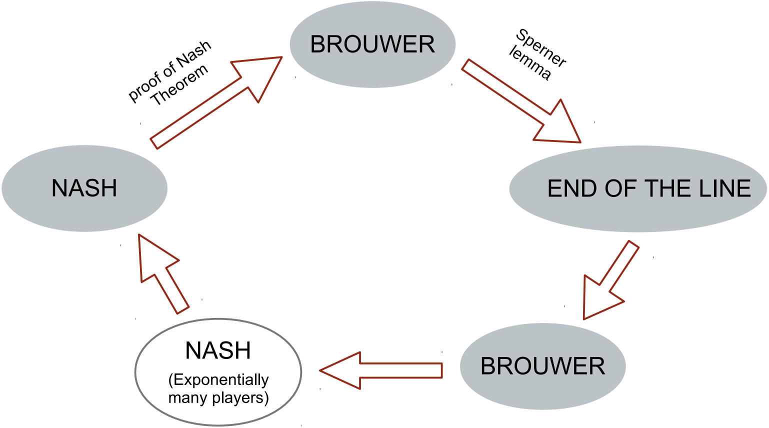
Complexity of NASH

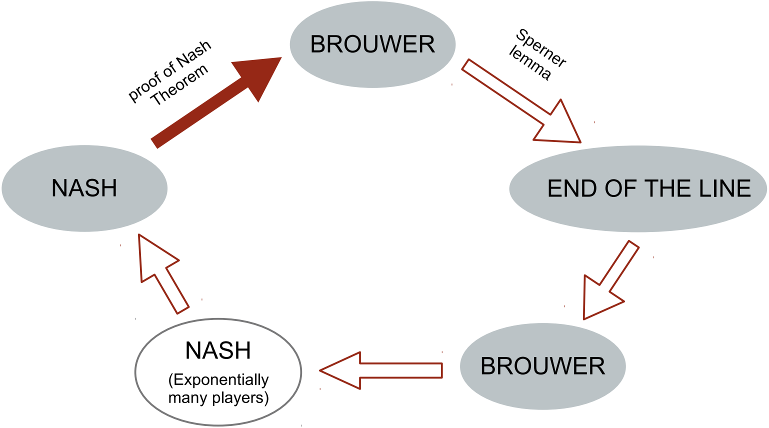
Theorem (Daskalakis, Goldberg, Papadimitriou (2006))

NASH *is PPAD-complete*

Proof


BROUWER
proof of Nash Theorem
Sperner lemma
NASH
END OF THE LINE
BROUWER
NASH
(Exponentially many players)

From NASH to BROUWER



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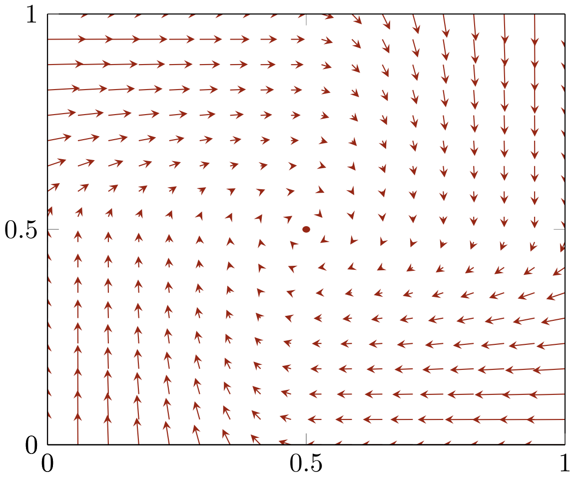

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- **BROUWER:** Given F find a fix point of F

From BROUWER to END OF THE LINE

- $F : [0,1]^m \to [0,1]^m$ satisfies Lipschitz condition with constant K

$$\text{For all } x_1, x_2 \in [0,1]^m,\ d(F(x_1), F(x_2)) \le K \cdot d(x_1, x_2)$$

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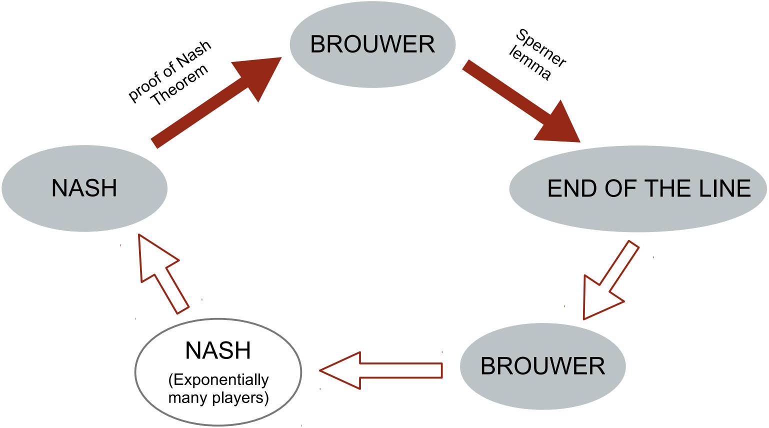
Definition (Problem BROUWER)

- **Input**: Efficient algorithm $\boldsymbol{\Pi_F}$ computing $F : [0,1]^m \to [0,1]^m$, Lipschitz constant $\boldsymbol{K}$ of F, and accuracy $\boldsymbol{a}$
- **Output:** $\boldsymbol{x}$ such that $d(F(x), x) \leq 1/a$

From BROUWER to END OF THE LINE



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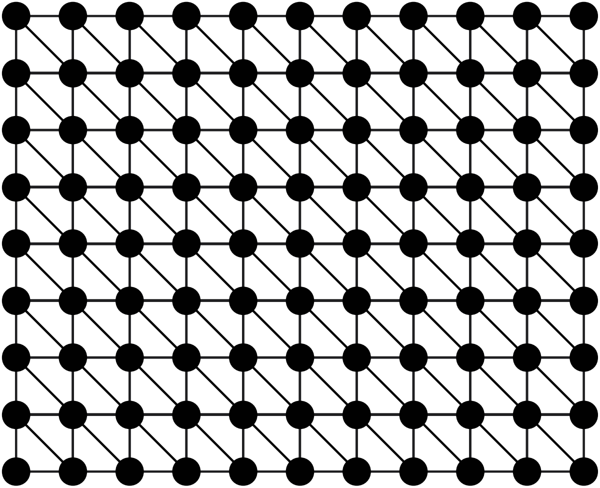
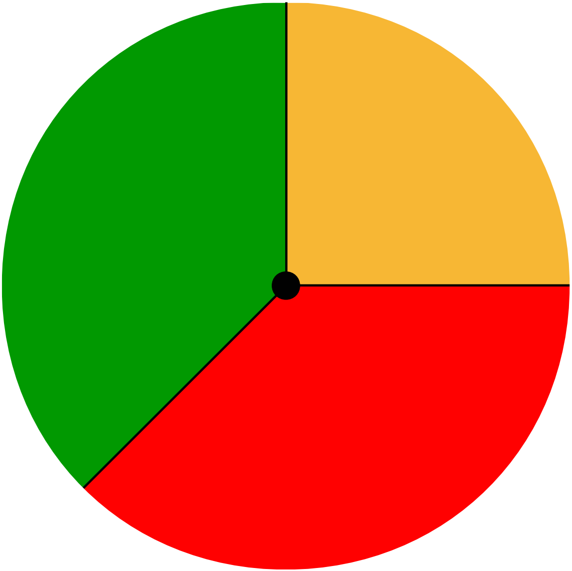


From Brouwer to End of The Line



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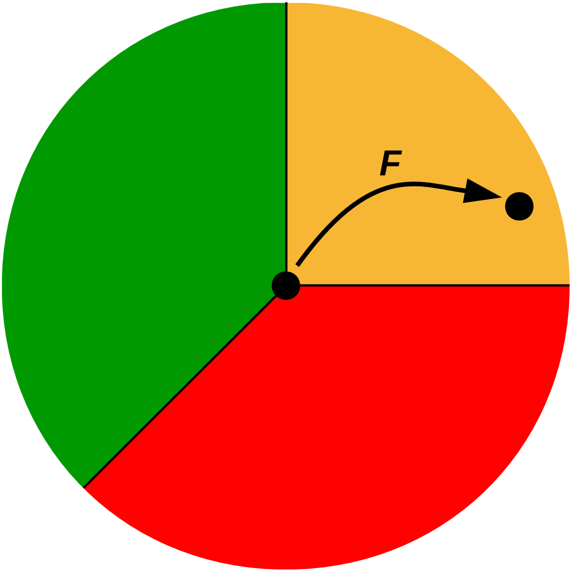

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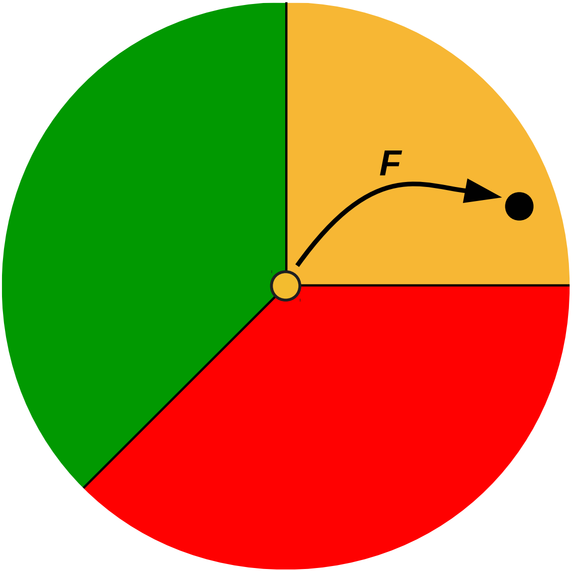

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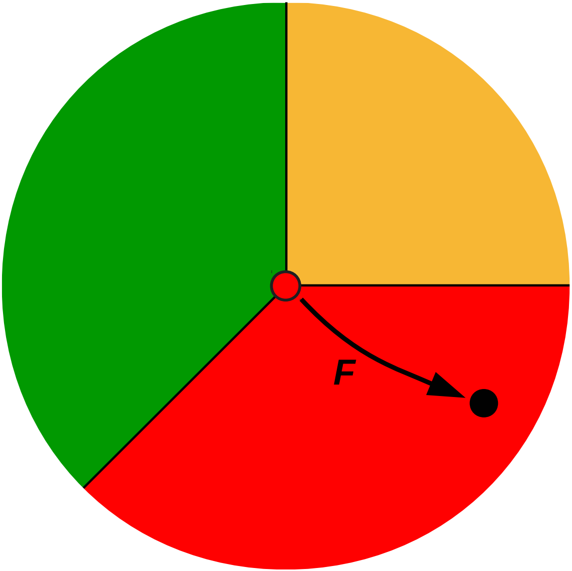

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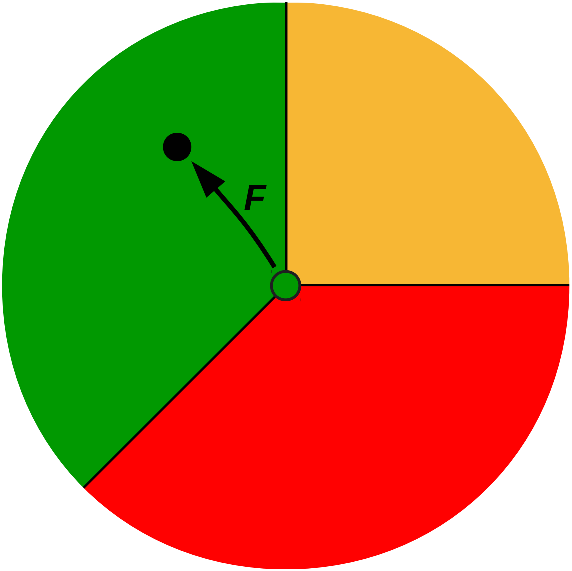

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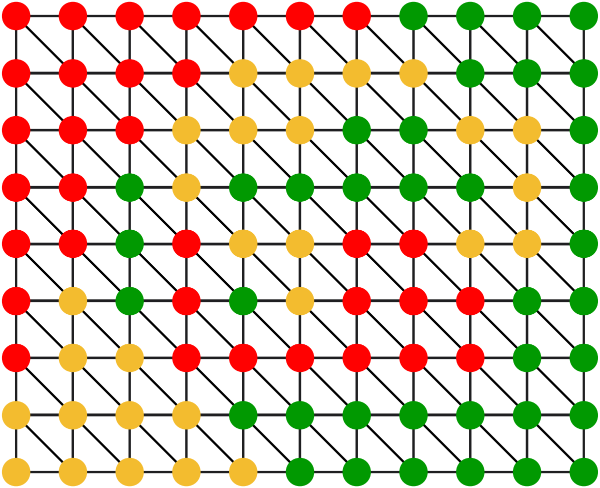

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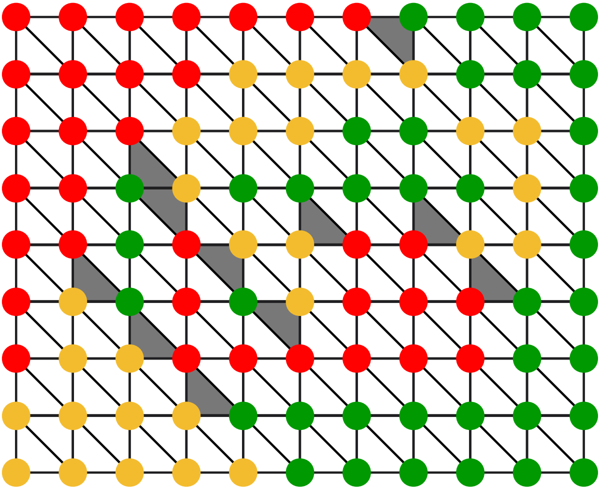

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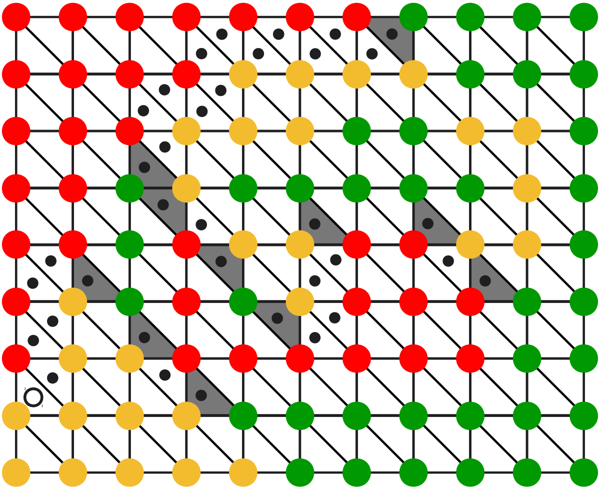

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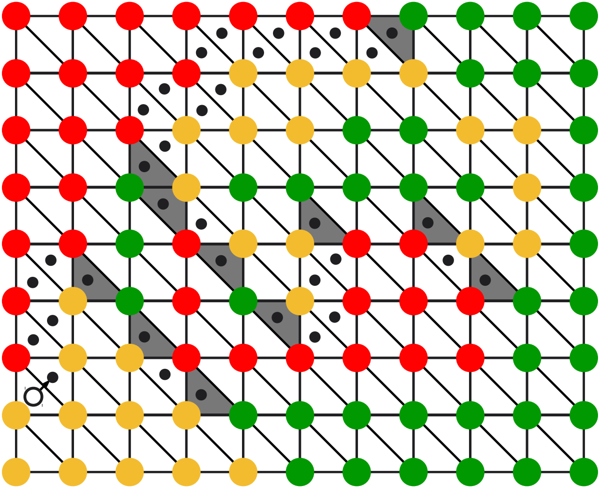

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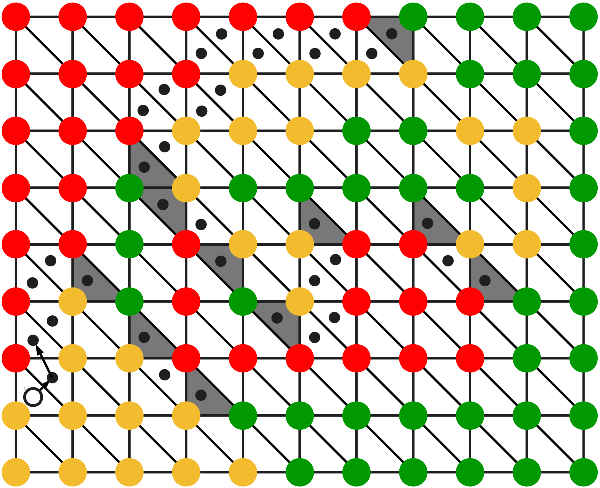

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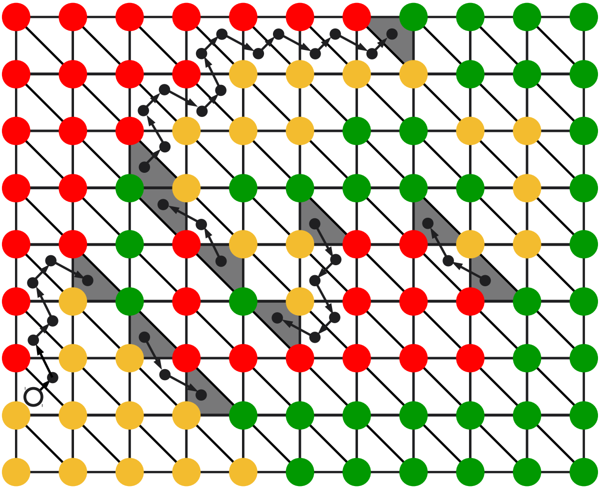
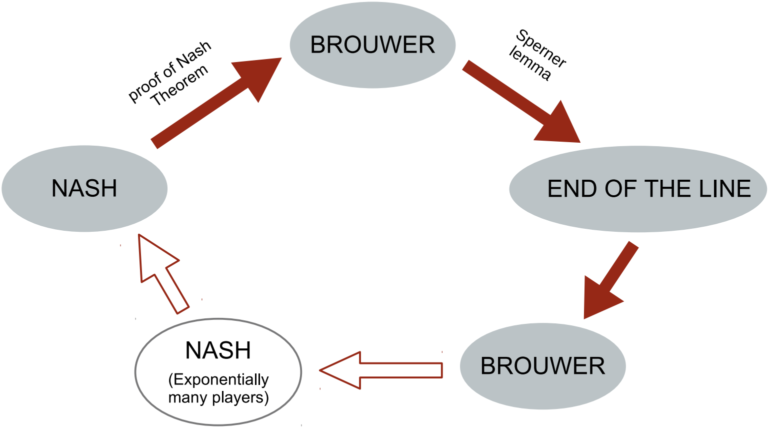
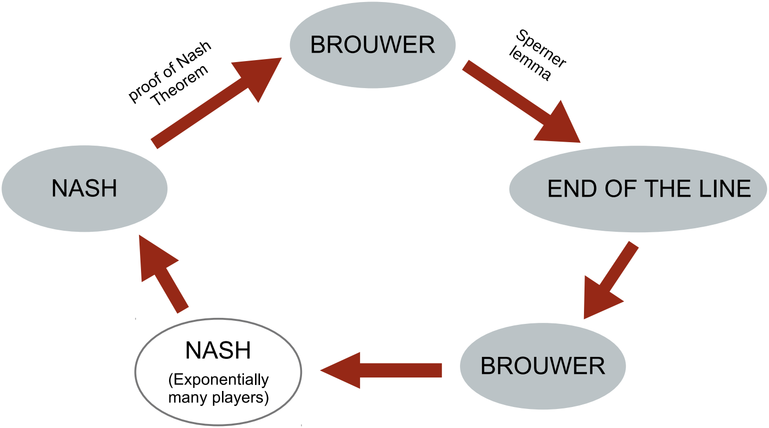


From End of The Line to Brouwer



From BROUWER to NASH



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- Few months after the result was published, Chen and Deng (2007) extended the result to 2-player games
- The problem of finding exact Nash equilibria (or approximating them) was studied by Etessami and Yannakakis (2007)
- It is at least as hard as finding ε-Nash equilibria