Heterogeneous gradient flows

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Heterogeneous phenomena; mixtures, multiple species

Outline

1. Classical gradient flows - from ODEs to curves in Hilbert and Polish spaces.
2. Gradient flows in Wasserstein topology and continuity equation.
3. Heterogeneous continuity equation and fibered Wasserstein space.
4. Main result - heterogeneous gradient flows and their applications.
5. Further discussion.

Gradient flows in $\mathbb{R}^d$

$$\dot{x}(t) = -\nabla\mathcal{E}(x(t))$$

- $[0, T) \ni t \mapsto x(t) \in \mathbb{R}^d$ – a curve,
- $\mathcal{E} : \mathbb{R}^d \to [-\infty, +\infty]$ – potential functional (usually required to satisfy some regularity or convexity assumptions).

Position x evolves along the vector field $-\nabla\mathcal{E}$ evaluated at x "in an attempt" to minimize $\mathcal{E}(x)$.

Gradient flows in Hilbert spaces – Brezis, Pazy

$$\dot{x}(t) \in -\partial^{-}\mathcal{E}(x(t))$$

- $[0, T) \ni t \mapsto x(t) \in H$ – a curve with values in a Hilbert space
- $\mathcal{E}$ – λ-convex, *i.e.*

$$\mathcal{E}((1-t)x + ty) \leq (1-t)\mathcal{E}(x) + t\mathcal{E}(y) - \frac{\lambda}{2}t(1-t)|x-y|^2,$$
$$\partial^{-}\mathcal{E}(x) := \Big\{ v \in H : \ \forall y \in H$$
$$\mathcal{E}(x) + < v, y - x > + \frac{\lambda}{2}|x-y|^2 \leq \mathcal{E}(y) \Big\}$$

Equivalently - Evolutionary Variational Inequality

$$\forall y \in H \quad \frac{1}{2}\frac{d}{dt}|x(t)-y|^2 + \mathcal{E}(x(t)) + \frac{\lambda}{2}|x(t)-y|^2 \leq \mathcal{E}(y)$$

Gradient flows in Polish spaces

$$\forall y \in X \quad \frac{1}{2}\frac{d}{dt}\mathrm{dist}(x(t),y)^2 + \mathcal{E}(x(t)) + \frac{\lambda}{2}\mathrm{dist}(x(t),y)^2 \leq \mathcal{E}(y)$$

If $\mathcal{E} : X \to (-\infty, +\infty]$ is proper, coercive, lsc and λ-(geodesically) convex then the above EVI has only one absolutely continuous solution. The solution is exponentially stable

$$\mathrm{dist}(x_1(t), x_2(t)) \leq e^{-\lambda t}\mathrm{dist}(x_1(0), x_2(0)).$$

Example: $X = \mathcal{P}_2(\mathbb{R}^d)$, $\mathrm{dist}(\mu,\nu) = W_2(\mu,\nu)$, where

$$W_2^2(\mu,\nu) = \inf_{\gamma} \int_{\mathbb{R}^{2d}} |x - x'|^2 d\gamma(x,x'),$$

where γ are all probability measures with x-marginal μ and x'-marginal ν.

Weakly Riemannian structure of $(\mathcal{P}_2, W_2)$ – Otto, 2001

Wasserstein space $(\mathcal{P}_2, W_2)$ has a weakly Riemannian structure (tangent space at $x \in \mathcal{P}_2$ is isomorphic to L^2_x). Thus, theory of gradient flows known for Hilbert spaces applies. We again have

$$\dot{x}(t) \in -\partial_{W_2}\mathcal{E}(x(t))$$

Writing $x = \mu$ and interpreting $\dot{x}$ as local velocity $\boldsymbol{u}$ of the transport of μ. Then absolutely continuous curves satisfying the EVI

$$\forall \sigma \in \mathcal{P}_2 \quad \frac{1}{2}\frac{d}{dt}W_2^2(\mu(t),\sigma) + \mathcal{E}(\mu(t)) + \frac{\lambda}{2}W_2^2(\mu(t),\sigma) \le \mathcal{E}(\sigma)$$

are equivalent to distributional solutions to

$$\partial_t \mu + \mathrm{div}(\boldsymbol{u}\mu) = 0,$$
$$\boldsymbol{u}(t) \in \partial_{W_2}\mathcal{E}(\mu(t)).$$

Gradient flows in $\mathcal{P}_2$ and continuity equation

If $\mathcal{E} : \mathcal{P}_2 \to (-\infty, +\infty]$ is proper, coercive, lsc and λ-convex (along generalized geodesics) then there exists a unique gradient flow $\boldsymbol{\mu} \in AC(0, T; \mathcal{P}_2)$ issued at any $\mu_0 \in D(\mathcal{E})$. It satisfies continuity equation

$$\partial_t \boldsymbol{\mu} + \mathrm{div}(\boldsymbol{u}\boldsymbol{\mu}) = 0, \qquad u(t) \in -\partial_{W_2}\mathcal{E}(\mu(t))$$

in the sense of distributions.

We also get some more information for free *e.g.*

$$W_2(\mu_t^1, \mu_t^2) \lesssim e^{-\lambda t} W_2(\mu_0^1, \mu_0^2).$$

Applications

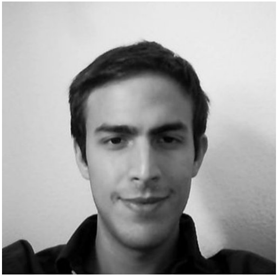
$$\partial_t \boldsymbol{\mu} + \mathrm{div}(\boldsymbol{u}\boldsymbol{\mu}) = 0, \qquad u(t) \in -\partial_{W_2}\mathcal{E}(\mu(t)).$$

- Absolutely continuous measure $\mu = g\mathcal{L}_1$.
- Entropy functional $\mathcal{E}(g) = \log g$.
- Subdifferential $\partial\mathcal{E}(g) = \frac{\nabla g}{g} = -u$.
- Continuity equation becomes the heat equation

$$\partial_t g - \Delta g = 0$$

- We further can consider such models as: Fokker-Planck, Vlasov, Keller–Segel, and many models of first-order collective dynamics (interaction potentials).

To give credit where credit is due

Left: **David Poyato**
Right: **Javier Morales**

Heterogeneous continuity equation

Consider $(x, \omega) \in \mathbb{R}^{d_1+d_2}$ and the continuity equation for $\mu_t = \mu_t(x, \omega)$

$$\partial_t \boldsymbol{\mu} + \mathrm{div}_x(\boldsymbol{u}\boldsymbol{\mu}) = 0$$

We treat $\boldsymbol{\mu} \in AC(0, T; \underbrace{(\mathcal{P}_{2,\nu}(\mathbb{R}^{d_1+d_2}), W_{2,\nu})}_{\mathcal{P}_{2,\nu}})$ as a curve

$t \in [0, T] \mapsto \mu_t \in \mathcal{P}_{2,\nu}$.

Equivalently

$$\partial_t \boldsymbol{\mu} + \mathrm{div}_{(x,\omega)}(\boldsymbol{v}\boldsymbol{\mu}) = 0, \quad \boldsymbol{v} = (\boldsymbol{u}, 0).$$

Fibered Wasserstein distance

Any probability measure $\mu = \mu(x, \omega)$ on $\mathbb{R}^{d_1+d_2}$ can be decomposed with respect to its marginal $\nu \in \mathcal{P}(\mathbb{R}^{d_2})$ as

$$\mu[\phi] = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \phi(x, \omega) d\mu^\omega(x) \right) d\nu(\omega)$$

for all Borel-measurable ϕ. We denote this representation as $\mu(x, \omega) = \mu^\omega(x) \otimes \nu(\omega)$

$$\mathcal{P}_{2,\nu} := \{\mu \in \mathcal{P} : \ (\pi_\omega)_\# \mu = \nu, \ \int_{\mathbb{R}^{d_1+d_2}} |x|^2 d\mu(x, \omega) < \infty\}$$

$$W_{2,\nu}^2(\mu, \sigma) := \int_{\mathbb{R}^{d_2}} W_2^2(\mu^\omega, \sigma^\omega) d\nu(\omega)$$

Heterogeneous gradient flow in $\mathcal{P}_{2,\nu}$

- First we prove that the fibered Wasserstein space $(\mathcal{P}_{2,\nu}, W_{2,\nu})$ is a Polish space, and we get the EVI gradient flows for free.

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- We develop a fibered subdifferential calculus with

$$\partial_{W_{2,\nu}}\mathcal{E}(\mu) = \Big\{ \boldsymbol{u} \in L^2_\mu : \mathcal{E}[\sigma] - \mathcal{E}[\mu] \geq \inf_{\gamma \in \Gamma_{o,\nu}(\mu,\sigma)} \int_{\mathbb{R}^{4d}} \boldsymbol{u}(x,\omega)\cdot(x'-x)\, d\gamma(x,x',\omega,\omega') + o(W_{2,\nu}(\mu,\sigma)) \Big\}.$$

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- First we prove that the fibered Wasserstein space $(\mathcal{P}_{2,\nu}, W_{2,\nu})$ is a Polish space, and we get the EVI gradient flows for free.
- Second, following Otto approach, we prove that $(\mathcal{P}_{2,\nu}, W_{2,\nu})$ is a Riemannian manifold.
- We develop a fibered subdifferential calculus.
- 'Assuming that $\mathcal{E} : \mathcal{P}_{2,\nu} \to (-\infty, +\infty]$ is proper, coercive, lsc and λ-convex (along fibered generalized geodesics) we prove that EVI is equivalent to the subdifferential notion of gradient flows

$$u_t \in -\partial_{W_{2,\nu}} \mathcal{E}(\mu_t)$$

where μ satisfies the parameterised continuity equation.

Main result

If $\mathcal{E} : \mathcal{P}_{2,\nu} \to (-\infty, +\infty]$ is proper, coercive, lsc and λ-convex (along fibered generalized geodesics), then there exists a unique gradient flow $\boldsymbol{\mu} \in AC(0, T; \mathcal{P}_{2,\nu})$ issued at any $\mu_0 \in D(\mathcal{E})$. It satisfies continuity equation

$$\partial_t \boldsymbol{\mu} + \mathrm{div}_x(\boldsymbol{u}\boldsymbol{\mu}) = 0, \qquad u_t \in -\partial_{W_{2,\nu}} \mathcal{E}(\mu_t)$$

in the sense of distributions.

We also get some more information for free *e.g.*

$$W_{2,\nu}(\mu_t^1, \mu_t^2) \lesssim e^{-\lambda t} W_{2,\nu}(\mu_0^1, \mu_0^2).$$

Applications

- Fokker Planck equations with various particles responding differently to the drag force and random motion.
- Vlasov equation for mixtures of different type of plasma.
- Multispecies first order collective dynamics.
- Kuramoto model of 1D synchronization.
- Lohe model of quantum coupled oscillators.
- Multispecies Keller-Segel model.

The fundamental example: Kuramoto-type equation

$$\mathcal{E}[\mu] = -\int_{\mathbb{R}^{2d}} \omega \cdot x d\mu + \int_{\mathbb{R}^{4d}} W(|x - x'|) d\mu d\mu',$$

$$W(|x - x'|) = \frac{1}{2-\alpha}\frac{1}{1-\alpha}|x - x'|^{2-\alpha}, \quad \alpha \in (0,1).$$

Then the fibered subdifferential is given by

$$\partial_{W_{2,\nu}} \mathcal{E}[\mu_t] = \nabla_x W * \mu - \omega$$

and the corresponding continuity equation reads

$$\partial_t \mu + \mathrm{div}_x[(\omega - \nabla_x W * \mu)\mu] = 0.$$

Two misconceptions

Ha! Aren't you just stacking a couple of (e.g. continuum) independent continuity equations one atop another? By disintegration:

$$\partial_t \mu^\omega(x) + \mathrm{div}_x(u^\omega(x)\mu^\omega(x)) = 0,$$
$$u_t^\omega \in -\partial_{W_2}\mathcal{E}^\omega(\mu_t^\omega)$$

then just integrate with respect to ν.

Ax explained – **NO**. It is generally not true that u^ω depends only on μ^ω!

$$\partial_t \mu + \mathrm{div}_x[(\omega - \nabla_x W * \mu)\mu] = 0.$$

After disintegration we have $u_t^\omega(x) = \omega - \nabla W * \mu_t(x)$ and the velocity field clearly depends on the whole μ even with fixed ω.

Two misconceptions

Anyway... aren't all of the proofs you need following the same idea? Step 1: disintegrate the energy $\mathcal{E}$. Step 2: solve the problem on each fiber using classical theory. Step 3: Integrate with respect to ν.

- Works 50 % of the time.
- In prticular classical approach usually involves narrow topology, which fails in fibered problems.
- Again, reversing the disintegration theorem usually requires some thought such as using Kuratowski–Ryll-Nardzewski measurable selection theorem.

Stability of optimality (classical)

Let $\mu_n^1 \to \mu^1$ and $\mu_n^2 \to \mu^2$ narrowly. Then any sequence of opitmal plans $\gamma_n \in \Gamma_{opt}(\mu_n^1, \mu_n^2)$ is narrowly relatively compact and any of its limit plans is belongs to $\Gamma_{opt}(\mu^1, \mu^2)$.

Sketch of the proof.

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  1. finitely valued costs (Ambrosio, Pratelli 2003)
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  1. finitely valued costs (Ambrosio, Pratelli 2003)
  2. infinitely valued but continuous costs (Pratelli 2008)
- Our cost function is neither finitely valued nor continuous. Back to the drawing board.

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- Thus for ν-a.a. ω, by Prokhorov, γ_n^ω is narrowly compact.
- Then we would like to use classical optimal stability for γ_n^ω. Done?
- No. The problem is we have a convergent subsequence fixed to match each ω (and there is continuum of ω).

Stability of optimality (fibered – attempt 3)

Let $\mu_n^1 \to \mu^1$ and $\mu_n^2 \to \mu^2$ narrowly. Then any sequence of opitmal plans $\gamma_n \in \Gamma_{opt}(\mu_n^1, \mu_n^2)$ is narrowly relatively compact and any of its limit plans is belongs to $\Gamma_{opt}(\mu^1, \mu^2)$.

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Let $\mu_n^1 \to \mu^1$ and $\mu_n^2 \to \mu^2$ ν-random narrowly. Then any sequence of opitmal plans $\gamma_n \in \Gamma_{opt,\nu}(\mu_n^1, \mu_n^2)$ is ν-random narrowly relatively compact and any of its limit plans is belongs to $\Gamma_{opt,\nu}(\mu^1, \mu^2)$.

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- The ν-random narrow convergence is tested with ν-random continuous functions ϕ:
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  2. For all ω the function $x \mapsto \phi(x, \omega)$ is bounded and continuous.
  3. We have
  $$\int_{\mathbb{R}^{d_2}} |\phi(\cdot, \omega)|_\infty d\nu(\omega) < \infty.$$

Stability of optimality (fibered – attempt 3)

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- Then the idea of attempt 2 works...?
- One still needs to proof a variant of "points in $\mathrm{supp}\gamma$ are limits of points in $\mathrm{supp}\gamma_n$" that works for the so called ν-random sets. Which we did.