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Gyárfás' path

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22.04.2021

Difficult graph questions

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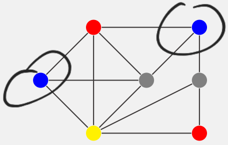
Graph theory

Algorithms

Difficult graph questions

Graph theory



$\chi(G)$: chromatic number

Algorithms

Difficult graph questions

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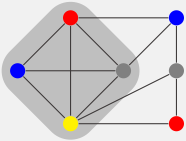

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Algorithms

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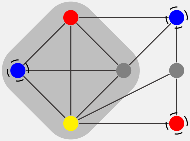
Graph theory



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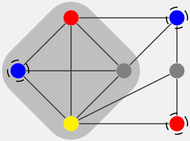

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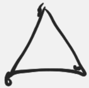
$\omega(G), \frac{|V(G)|}{\alpha(G)} \leq \chi(G)$.

Maybe $\omega(G) = 2$ but large $\chi(G)$

Algorithms



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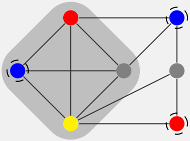

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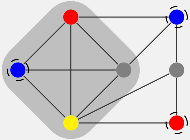

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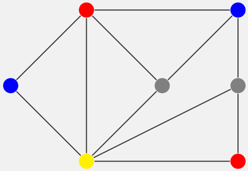
$\alpha(G)$ and $\omega(G)$ hard to $|V(G)|^{0.999}$-approximate.

n vertex

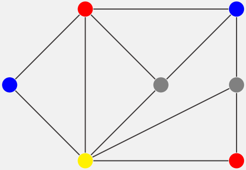
$\alpha = n^{0.01}$ $\quad$ $\alpha \geq n^{0.55}$

Planar graphs



Restricting graph class

Planar graphs



$\chi(G) \leq 4$

Restricting graph class

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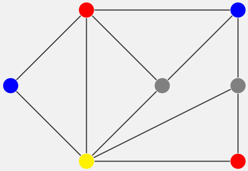

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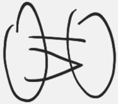

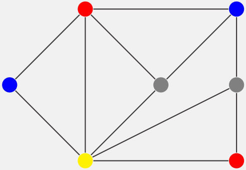

$\chi(G) \leq 4$
$\omega(G) = 2 \Rightarrow \chi(G) \leq 3$

Checking $\chi(G) = 3$ NP-hard

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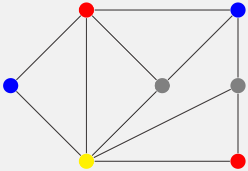
Planar graphs



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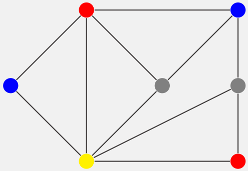

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Checking $\chi(G) = 3$ NP-hard

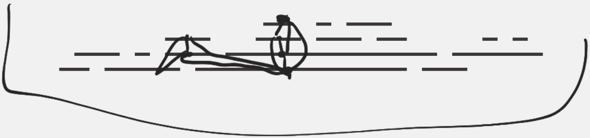
PTAS for $\alpha(G)$.

$\alpha(G) \leq k$ or $\alpha(G) \geq k/(1+\varepsilon)$

$\forall \varepsilon > 0$

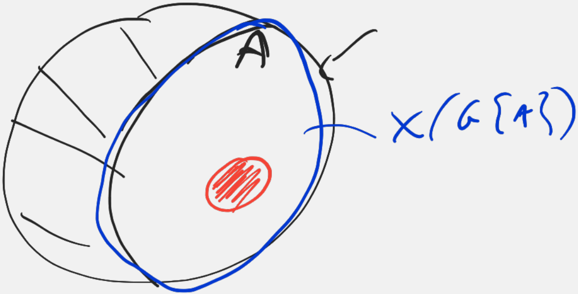
$n^{0.01}$ — $n^{0.99}$

Perfect graphs



Perfect graphs

$$\forall_{A \subseteq V(G)} \omega(G[A]) = \chi(G[A])$$



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[redacted]

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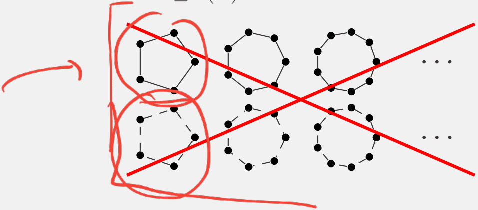

$\omega = 2$

$\chi = 3$

Restricting graph class

Perfect graphs

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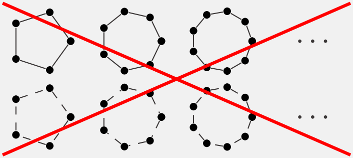

$$\chi(G) = \omega(G)$$

Restricting graph class

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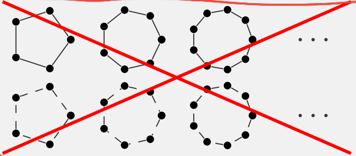

$\chi(G) = \omega(G)$

closed under taking complements

Restricting graph class

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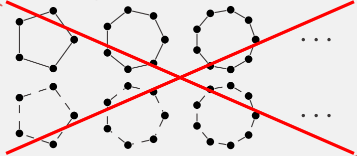
[redacted]

$\forall_{A \subset V(G)} \omega(G[A]) = \chi(G[A])$

$\omega \leq \chi$



$\chi(G) = \omega(G)$

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$\alpha(G), \chi(G)$ poly-time computable

Tractability boundaries

- $\mathcal{G}$ — graph class
  - closed under vertex deletion (*hereditary*);
  - $\mathcal{H}$-free graphs;
  - $\mathcal{H} = \{H\} \longrightarrow H$-free.

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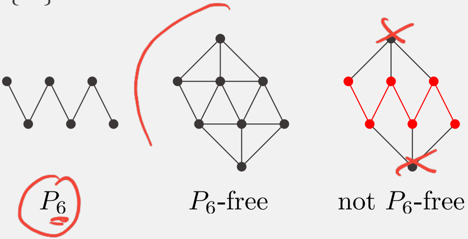

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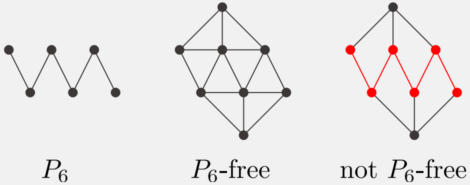

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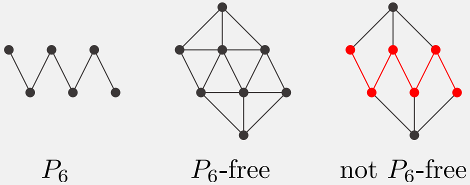

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For which $\mathcal{H}$, $\mathcal{H}$-free graphs have similar properties?

χ-boundedness

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- $\{C_b, C_{a+b}, C_{2a+b}, C_{3a+b}, \ldots\}$ -free χ-bounded.

χ-boundedness of P_t-free graphs

Gyárfás 1987: P_t-free are χ-bounded.

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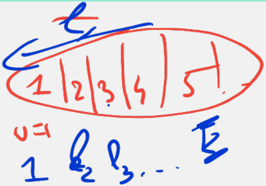
Induction over $\omega(G)$: $\omega(G') < k \Longrightarrow \chi(G') \leq \ell$.
We have G, $\omega(G) = k$, we bound $\chi(G)$.



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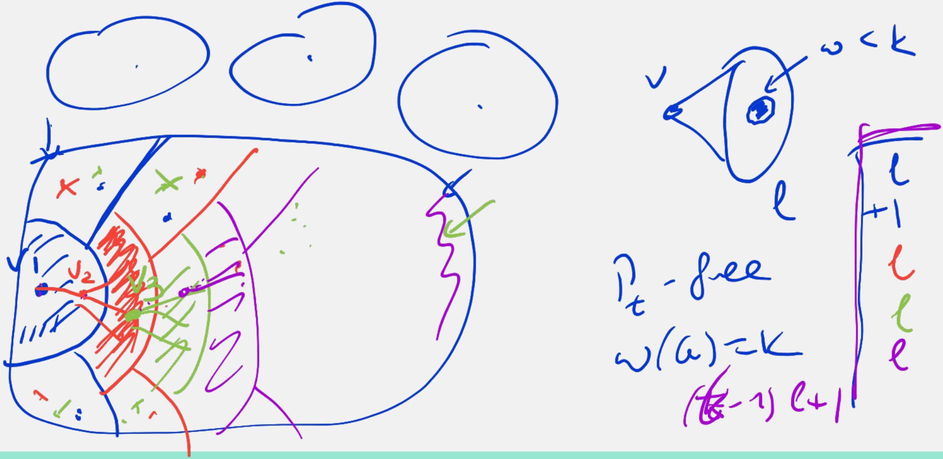
We have G, $\omega(G) = k$, we bound $\chi(G)$. $\forall_{v \in V(G)} \chi(G[N(v)]) \leq \ell$.

$\chi(G) \leq 1 + (t-1)\ell$

$\chi(G) \leq t^{\omega(G)}$



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- $\{H, \overline{H}\}$-free for caterpillars H, (Liebenau, P.) + (Seymour, Spirkl), 2017.

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  (Liebenau, P.) + (Seymour, Spirkl), 2017.
- $\{H, \overline{H}\}$-free for trees H,
  (Chudnovsky, Scott, Seymour, Spirkl), 2018.

Pure pairs

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Theorem (CSSS 2018)

For every tree H there exists $\delta > 0$, such that in every H-free G there exist disjoint $A, B \subseteq V(G)$, $|A|, |B| \geq \delta |V(G)|$ so that there are none or all edges between A and B.

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*Called **pure pair** (A, B).*

Not yet published:

Detecting a long even hole (with Linda Cook).

Pure pairs. VIII. Excluding a sparse graph (with Alex Scott and Sophie Spirkl).

Induced subgraphs of bounded treewidth and the container method (with Tara Abrishami, Maria Chudnovsky, Marcin Pilipczuk, Paweł Rzążewski).

Holes with hats and Erdős-Hajnal (with Maria Chudnovsky).

Pure pairs. VI. Excluding an ordered tree (with Alex Scott and Sophie Spirkl).

Finding an induced path that is not a shortest path (with Eli Berger, Sophie Spirkl).

Pure pairs. IV. Trees in bipartite graphs (with Alex Scott and Sophie Spirkl).

Finding a shortest odd hole (with Maria Chudnovsky, Alex Scott).

Pure pairs. V. Excluding some long subdivision (with Alex Scott and Sophie Spirkl).

Subdivided claws and the clique-stable set separation property (with Maria Chudnovsky).

Pure pairs. VII. Homogeneous submatrices in 0/1-matrices with a forbidden submatrix (with Alex Scott).

Even-hole-free graphs still have bisimplicial vertices (with Maria Chudnovsky).

Small families under subdivision (with Maria Chudnovsky and Martin Loebl).

Detecting a long odd hole (with Maria Chudnovsky and Alex Scott).

Concatenating bipartite graphs (with Maria Chudnovsky, Patrick Hompe, Alex Scott and Sophie Spirkl).

A survey of chi-boundedness (with Alex Scott).

Pure pairs. III. Sparse graphs with no polynomial-sized anticomplete pairs (with Maria Chudnovsky, Jacob Fox, Alex Scott and Sophie Spirkl).

Pure pairs. I. Trees and linear anticomplete pairs (with Maria Chudnovsky, Alex Scott and Sophie Spirkl).

Pure pairs. II. Excluding all subdivisions of a graph (with Maria Chudnovsky, Alex Scott and Sophie Spirkl).

Clustered colouring in minor-closed classes (with Sergey Norin, Alex Scott and David Wood).

Induced subgraphs of graphs with large chromatic number. V. Chandeliers and strings (with Maria Chudnovsky, Alex Scott).

Pure pairs

Theorem (CSSS 2018)

For every tree H there exists $\delta > 0$, such that in every H-free G there exist disjoint $A, B \subseteq V(G)$, $|A|, |B| \geq \delta|V(G)|$ so that there are none or all edges between A and B.
*Called **pure pair** (A, B).*

For $H = P_t$; regularization $\Longrightarrow$ max degree $o(|V(G)|)$.

Separators in H-free graphs

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Theorem

Every graph G contains an induced path Q such that every connected component of $G - N[Q]$ contains at most $|V(G)|/2$ vertices.

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Remark: $C_{>t}$-free $\Longrightarrow$ can also get $|V(Q)| \leq t$.

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- NP-hard, hard to $|V(G)|^{0.999}$-approximate.

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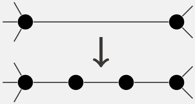

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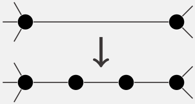
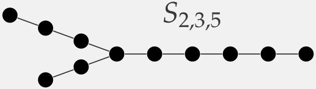


- NP-hard, APX-hard in H-free if one connected component of H is not a path nor a subdivision of $K_{1,3}$.

P_6

$S_{2,3,5}$



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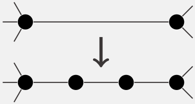

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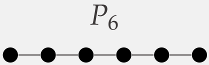
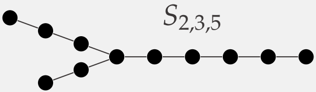
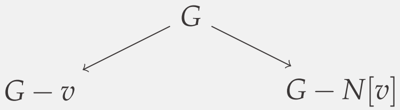




- Polynomial-time algorithms:
  - P_4-free (bounded cliquewidth)
  - P_5-free (Lokshtanov, Vatshelle, Villanger, 2014)
  - P_6-free (Grzesik, Klimosová, P., Pilipczuk, 2018)
  - $K_{1,3}$-free (Sbihi, Minty, 1980)
  - $S_{1,1,2}$-free (Lozin, Milanic, 2006)
  - $C_{>4}$-free (Abrishami, Chudnovsky, P., Rzążewski, Seymour, 2020)

QP for $\alpha(G)$ in P_t-free

function MIS(G)
  if $V(G) = \emptyset$ **then**
    return 0
  if G disconnected **then**
    return $\sum_C$ MIS(C)
  pick $v \in V(G)$
  $k_1 \leftarrow$ MIS($G - v$)
  $k_2 \leftarrow 1+$ MIS($G - N[v]$)
  return $\max(k_1, k_2)$



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```
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    k_1 ← MIS(G − v)
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    return max(k_1, k_2)
```

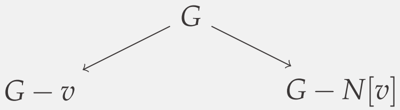

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Theorem (Gartland, Lokshtanov + P., Pilipczuk, Rzążewski, 2020)

In P_t-free graphs there is a way to choose the pivot that guarantees that the recursion tree has size $\exp(\mathcal{O}(t^3 \log^3 |V(G)|))$.

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Generalization

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Theorem (Gartland, Lokshtanov, P., Pilipczuk, Rzążewski, STOC 2021)

For every d and t and MSO_2 formula ϕ, given a P_t-free graphf G with vertex weights, one can in time $\exp(\mathcal{O}_{d,t,\phi}(\log^4 |V(G)|))$ find an induced subgraph H of G of maximum possible weight among all induced subgraphs that are d-degenerate and are models for ϕ.

Open questions

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Thanks!