EXTENSION OPERATORS AND TWISTED SUMS OF c_0 AND $C(K)$ SPACES

WITOLD MARCISZEWSKI AND GRZEGORZ PLEBANEK

Abstract. We investigate the following problem posed by Cabello Sanchéz, Castillo, Kalton, and Yost:

Let K be a nonmetrizable compact space. Does there exist a nontrivial twisted sum of c_0 and $C(K)$, i.e., does there exist a Banach space X containing a non-complemented copy Z of c_0 such that the quotient space X/Z is isomorphic to $C(K)$?

Using additional set-theoretic assumptions we give the first examples of compact spaces K providing a negative answer to this question. We show that under Martin's axiom and the negation of the continuum hypothesis, if either K is the Cantor cube 2^{ω_1} or K is a separable scattered compact space of height 3 and weight ω_1, then every twisted sum of c_0 and $C(K)$ is trivial.

We also construct nontrivial twisted sums of c_0 and $C(K)$ for K belonging to several classes of compacta. Our main tool is an investigation of pairs of compact spaces $K \subseteq L$ which do not admit an extension operator $C(K) \to C(L)$.

1. Introduction

A *twisted sum* of Banach spaces Z and Y is a short exact sequence

$$0 \to Z \to X \to Y \to 0,$$

where X is a Banach space and the maps are bounded linear operators. Such a twisted sum is called *trivial* if the exact sequence splits, i.e. if the map $Z \to X$ admits a left inverse (in other words, if the map $X \to Y$ admits a right inverse). This is equivalent to saying that the range of the map $Z \to X$ is complemented in X; in this case, $X \cong Y \oplus Z$. We can, informally, say that X is a nontrivial twisted sum of Z and Y if Z can be isomorphically embedded onto an uncomplemented copy Z' of X so that X/Z' is isomorphic to Y. Twisted sums of Banach spaces and their connection with injectivity-like properties are discussed in a recent monograph [3].

The classical Sobczyk theorem asserts that every isomorphic copy of c_0 is complemented in every separable superspace. This implies that $Z = c_0$ admits a nontrivial twisted sum with no separable Banach space Y. In particular, there is no nontrivial twisted sum of c_0

Date: March 8, 2017.

2010 *Mathematics Subject Classification.* Primary 46B25, 46B26, 46E15; Secondary 03E35, 54C55, 54D40.

The first author was partially supported by the Polish National Science Center research grant DEC-2012/07/B/ST1/03363. The second author was partially supported by the Polish National Science Center research grant NCN 2013/11/B/ST1/03596 (2014-2017).

and $C(K)$, whenever K is a compact metric space. Castillo [7] and Correa and Tausk [9] investigated the following problem originated in [5] and [6].

Problem 1.1. *Given a nonmetrizable compact space K, does there exist a nontrivial twisted sum of c_0 and $C(K)$?*

There are several classes of nonmetrizable compacta for which Problem 1.1 has a positive answer, cf. [7], [9]. The question is, however, open in its full generality.

Twisted sums of c_0 and $C(K)$ spaces are related to compactifications of the discrete space of natural numbers or, more generally, to discrete extensions of compacta. Let L be a compact space and $K \subseteq L$ its closed subspace; write $C(L|K) = \{f \in C(L) : f|K \equiv 0\}$. Note that if $L \setminus K$ is countable and discrete then $C(L|K)$ is isometric to c_0 so $C(L)$ is a twisted sum of c_0 and $C(K)$. Such a twisted sum is trivial if and only if there is an extension operator $E : C(K) \to C(L)$, that is a bounded operator such that $Eg|K = g$ for every $g \in C(K)$.

If, in the above setting, $L \setminus K$ is countable and discrete then we say that L is a countable discrete extension of K. Hence the natural way of constructing a nontrivial twisted sum of c_0 and $C(K)$ is to find a countable discrete extension L of K without a corresponding extension operator. We shall explore this approach and construct nontrivial twisted sums of c_0 and $C(K)$ for several classes of nonmetrizable compacta: dyadic spaces (section 7), linearly ordered compact spaces (section 8), scattered compacta of finite height (section 9). In this way we extend results from Castillo [7] and Correa and Tausk [9] or present their alternative proofs.

There are twisted sums of c_0 and $C(K)$ spaces that cannot be obtained in the above manner. For instance, there is a nontrivial twisted sum of c_0 and $C(2^{\mathfrak{c}})$ ([9]) but for every countable discrete extension L of $2^{\mathfrak{c}}$ there is an extension operator $E : C(2^{\mathfrak{c}}) \to C(L)$ simply because there is a retraction $L \to 2^{\mathfrak{c}}$. In section 3 we investigate the following more general construction: Let L denote the unit ball in $C(K)^*$ equipped with the *weak** topology. Then $C(K)$ embeds into $C(L)$ and if L' is a countable discrete extension of L then $C(L')$ contains, in a canonical way, a twisted sum of c_0 and $C(K)$. This enables us to formulate a sufficient condition under which every twisted sum of c_0 and $C(K)$ is trivial, see Theorem 3.4. Then in section 5 we prove that under Martin's axiom and the negation of the continuum hypothesis (CH) every twisted sum of c_0 and $C(2^{\omega_1})$ is trivial, hence giving the first consistent negative solution to Problem 1.1. We also show an analogous result for K being the scattered compactum defined by an almost disjoint family in ω of size ω_1. Our results are based on an auxiliary theorem on approximating nearly additive functions on Boolean algebras by finitely additive signed measures — this is Theorem 4.6. We prove it in section 4; here the use of Martin's axiom is essential and our argument makes use of several technical lemmas on extensions of finitely additive measures. Some of them build on a result from [4] and are discussed in Appendix.

Section 6 contains related results on the unit ball in $C(K)^*$; we show that such a ball of signed measures on K is never an absolute retract (and is even not a Dugundji space) whenever K is not metrizable.

The second author is very grateful to the anonymous referee of [12] who pointed out interesting connection of the results presented there with Problem 1.1.

2. Preliminaries

If K is a compact space then $C(K)$ is the familiar Banach space of continuous real-valued functions on K. We always identify $C(K)^*$ with the space $M(K)$ of signed Radon measures on K with finite variation. $M_1(K)$ stands for the unit ball of $M(K)$, equipped with the *weak** topology inherited from $C(K)^*$. The symbol $P(K)$ denotes the subspace of $M_1(K)$ consisting of all probability measures; given $x \in K$, $\delta_x \in P(K)$ is the Dirac measure, a point mass concentrated at the point x.

It will be convenient to use the following notion.

Definition 2.1. If L is a compact space then a compact superspace $L' \supseteq L$ will be called a *discrete countable extension* of L if $L' \setminus L$ is countable and discrete.

We shall write $L' \in \mathrm{CDE}(L)$ to say that L' is such an extension of L. Typically, when $L' \setminus L$ is dense in L', L' is a compactification of ω such that its remainder is homeomorphic to L. Unless stated otherwise, if $L' \in \mathrm{CDE}(L)$ and $L' \setminus L$ is infinite, then we usually identify $L' \setminus L$ with the set of natural numbers ω.

For the future reference we state the following simple observations on countable discrete extensions of compacta.

Lemma 2.2. *If $L' \in \mathrm{CDE}(L)$ and $h_1, h_2 \in C(L')$ agree on L then $\lim_n (h_1(n) - h_2(n)) = 0$.*

Proof. Otherwise, $|h_1(n) - h_2(n)| \geq \varepsilon$ for $\varepsilon > 0$ and n from some infinite set $N \subseteq \omega$. Taking an accumulation point x of $N \subseteq L'$ we get $x \in L$ and $h_1(x) \neq h_2(x)$, a contradiction. $\square$

Lemma 2.3. *Let $L' \in \mathrm{CDE}(L)$ and let $f_1, \ldots, f_k \in C(L')$ for some k. Then for every $\varepsilon > 0$ there is n_0 such that for every $n \geq n_0$ there is $x \in L$ such that $|f_i(x) - f_i(n)| < \varepsilon$ for every $i \leq k$.*

Proof. For every $x \in L$ take an open set $V_x \subseteq L'$ such that $|f_i(x) - f_i(y)| < \varepsilon$ for every $y \in V_x$ and $i \leq k$. Then $L \subseteq V = \bigcup_{j \leq m} V_{x_j}$ for some m and $x_1, \ldots, x_m \in L$. Then $L' \setminus V$ must be finite and we are done. $\square$

Given compact spaces L, L' with $L \subseteq L'$, an extension operator $E : C(L) \to C(L')$ is a bounded linear operator such that $Eg|L = g$ for every $g \in C(L)$. Recall the following standard facts (see e.g. [12], Corollary 2.3 and Lemma 2.4).

Lemma 2.4. *If $L' \in \mathrm{CDE}(L)$ and there is no extension operator $E : C(L) \to C(L')$ then $C(L')$ is a nontrivial twisted sum of c_0 and $C(L)$.*

Proof. Note that $L' \setminus L$ must be infinite; we can identify it with ω. Define an embedding $i : c_0 \to C(L')$, sending the unit vector $e_n \in c_0$ to $\chi_n \in C(L')$. Then $Z = i[c_0]$ is the subspace of $C(L')$ of all functions vanishing on L and hence $C(L')/Z$ is isomorphic to $C(L)$. The subspace Z of $C(L')$ is not complemented. Indeed, supposing that P is a projection from $C(L')$ onto Z one can easily define an extension operator $E : C(L) \to C(L')$ by $Eg = \widehat{g} - P\widehat{g}$, where $\widehat{g}$ is *any* extension of $g \in C(L)$ to a continuous function on L' with the same norm. The point is that if $\widetilde{g} \in C(L')$ also extends $g \in C(L)$ then $\widehat{g} - \widetilde{g}$ vanishes on L so $P(\widehat{g} - \widetilde{g}) = \widehat{g} - \widetilde{g}$, and therefore E is well-defined. $\square$

Remark 2.5. Let $L' \in \mathrm{CDE}(L)$ be as in Lemma 2.4 and K be a compact space containing L and such that $K \cap (L' \setminus L) = \emptyset$. Then we can treat $K' = K \cup (L' \setminus L)$ as a countable discrete extension of K. If we additionally assume that there exists a extension operator $E : C(L) \to C(K)$ then it can be easily observed that there is no extension operator $E : C(K) \to C(K')$, hence $C(K')$ is a nontrivial twisted sum of c_0 and $C(K)$.

Following [12] we say that a compactification $\gamma\omega$ of the discrete space ω is *tame* if the natural copy of c_0 in $C(\gamma\omega)$, consisting of all functions from vanishing on the remainder $K = \gamma\omega\backslash\omega$, is complemented in $C(\gamma\omega)$. This is equivalent to saying that $\gamma\omega \in \mathrm{CDE}(K)$ and there is a corresponding extension operator. We include an easy observation related to Lemma 2.4.

Proposition 2.6. *Let L be a separable compact space and $L' \in \mathrm{CDE}(L)$ be such that there is no extension operator $E : C(L) \to C(L')$. Then there exists a non-tame compactification $\gamma\omega$ with the remainder $\gamma\omega \setminus \omega$ homeomorphic to L.*

Proof. Let $\{d_n : n \in \omega\}$ be a countable dense subset of L. Consider the following subset of the product $L' \times [0, 1]$:

$$K = L' \times \{0\} \cup \left\{ \left(d_n, \frac{1}{n+k+1}\right) : k, n \in \omega \right\}.$$

Obviously, $C = (L' \setminus L) \times \{0\} \cup \{(d_n, 1/(n+k+1)) : k, n \in \omega\}$ is a countable discrete space and K is a compactification of C with the required properties. $\square$

The following standard fact reduces the problem of defining an extension operator for $L' \in \mathrm{CDE}(L)$ to a problem of finding a suitable sequence of measures.

Lemma 2.7. *Let $L' = L \cup \omega$ be a countable discrete extension of a compact space L. The following are equivalent*

(i) there is a extension operator $E : C(L) \to C(L')$ with $\|E\| \le r$;
(ii) there is a continuous map $\varphi : L' \to rM_1(L)$ such that $\varphi(x) = \delta_x$ for any $x \in L$;
(iii) there is a sequence $(\nu_n)_n$ in $M(L)$ such that $\|\nu_n\| \le r$ for every n and $\nu_n - \delta_n \to 0$ in the weak* topology of $M(L')$.

Proof. $(i) \to (ii)$. Let $\varphi(x) = E^*\delta_x$, for $x \in L'$. Clearly, the map φ is continuous and takes values in $rM_1(L)$. If $x \in L$ then, for any $f \in C(L)$, $E^*\delta_x(f) = Ef(x) = f(x)$, hence $\varphi(x) = \delta_x$.

$(ii) \to (iii)$. Define $\nu_n = \varphi(n)$. Take any $f \in C(L')$ and $g = f|L \in C(L)$. Then

$$\nu_n(f) - \delta_n(f) = \nu_n(g) - f(n) = \varphi(n)(g) - f(n) \to 0.$$

Indeed, if the set $N = \{n \in \omega : |\varphi(n)(g) - f(n)| \geq \varepsilon\}$ were infinite then it would have an accumulation point $t \in L$ and $|\varphi(t)(g) - f(t)| = |g(t) - f(t)| \geq \varepsilon$, a contradiction.

(iii) $\to$ *(i)*. We can extend a function $g \in C(L)$ to $Eg \in C(L')$ setting $Eg(n) = \nu_n(g)$ for $n \in \omega$. By (ii) Eg is indeed continuous, and E is a bounded operator since ν_n are bounded. □

The following result is essentially due to Kubiś, see [20].

Theorem 2.8. *(a) If $\gamma\omega$ is a tame compactification of ω then its remainder $\gamma\omega \setminus \omega$ supports a strictly positive measure.*

(b) Let K be a compact space if weight ω_1 which does not support a measure. Then there is a nontrivial twisted sum of c_0 and $C(K)$.

Proof. The first assertion follows from Lemma 2.7, see [20, Theorem 17.3] or [12, Theorem 5.1].

By the Parovicenko theorem, K satisfying the assumptions from (b) is a continuous image of $\beta\omega\setminus\omega$ so there is a compactification $\gamma\omega$ which remainder is homeomorphic to K. It follows from (a) that the compactification $\gamma\omega$ is not tame so by Lemma 2.4 $C(\gamma\omega)$ is a nontrivial twisted sum of c_0 and $C(K)$. □

Recall that a compact space K is an *absolute retract* if K is a retract of any compact space L containing K (equivalently, of any completely regular space X containing K). Clearly, if $L' \in \mathrm{CDE}(L)$ and L is an absolute retract that, taking a retraction $r : L' \to L$, we get a norm-one extension operator $E : C(L) \to C(L')$, where $Eg = g \circ r$.

We shall often discuss Boolean algebras and their Stone spaces, using the classical Stone duality. Given an algebra $\mathfrak{A}$; its Stone space (of all ultrafilters on $\mathfrak{A}$) is denoted by $\mathrm{ult}(\mathfrak{A})$. If K is compact and zerodimensional then $\mathrm{Clop}(K)$ is the algebra of clopen subsets of K.

We write $M(\mathfrak{A})$ for the space of all signed *finitely* additive functions on an algebra $\mathfrak{A}$; likewise, for any $r \geq 0$, $M_r(\mathfrak{A})$ denotes the family of $\mu \in M(\mathfrak{A})$ for which $\|\mu\| \leq r$. Here, as usual, $\|\mu\| = |\mu|(X)$ and $|\mu|$ is the variation of μ. Recall that $M(\mathfrak{A})$ may be identified with $M(\mathrm{ult}(\mathfrak{A}))$ because every $\mu \in M(\mathfrak{A})$ defines via the Stone isomorphism an additive function on $\mathrm{Clop}(\mathfrak{A})$ and such a function extends uniquely to a Radon measure.

Given any subfamily $\mathcal{F}$ of an algebra $\mathfrak{A}$, we denote by $\langle \mathcal{F} \rangle$ the subalgebra of $\mathfrak{A}$ generated by $\mathcal{F}$.

3. On twisted sums

We show here that Lemma 2.4 can be, to some extend, reversed.

Definition 3.1. We shall say that a compact space K has property (#) if for every $L' \in \mathrm{CDE}(M_1(K))$ there is a bounded operator $E : C(K) \to C(L')$ such that $Eg(\nu) = \nu(g)$ for every $g \in C(K)$ and $\nu \in M_1(K)$.

Let us note that $C(K)$ may be seen as a subspace of $C(M(K))$ by the usual identification of an element of a Banach space with the corresponding element of its second dual. The operator E as in Definition 3.1 will be called an *extension operator, the one that extends $g \in C(K)$ treated as an element of $C(M(K))$ to $C(L')$.

Lemma 3.2. *Given K and $L' \in \mathrm{CDE}(M_1(K))$, the following are equivalent*

(i) there is an *extension operator $E : C(K) \to C(L')$;

(ii) there is a bounded sequence $(\nu_n)_n$ in $M(K)$ such that for every $g \in C(K)$, if $\widehat{g} \in C(L')$ is any function extending g, treated as a function on $M_1(K)$, then

$$\lim_n (\nu_n(g) - \widehat{g}(n)) = 0.$$

Proof. $(i) \to (ii)$. Consider an *extension operator $E : C(K) \to C(L')$ and the conjugate operator $E^* : M(L') \to M(K)$. We put $\nu_n = E^*\delta_n$ for $n \in \omega \subseteq L'$; then $(\nu_n)_n$ is a bounded sequence in $M(K)$.

Take any $g \in C(K)$ and its extension $\widehat{g} \in C(L')$. Then $\nu_n(g) = Eg(n)$ so

$$\nu_n(g) - \widehat{g}(n) = Eg(n) - \widehat{g}(n) \to 0,$$

by Lemma 2.2, since Eg and $\widehat{g}$ are two continuous extensions of the same function, of g acting on $M_1(K)$, and $L' \in \mathrm{CDE}(M_1(K))$.

For $(ii) \to (i)$ take any $g \in C(K)$, put $Eg(\nu) = \nu(g)$, for $\nu \in M_1(K)$, and define $Eg(n) = \nu_n(g)$, for $n \in \omega$. Then the function Eg is continuous on L'. $\square$

We shall also need the following general fact.

Lemma 3.3. *Let $T : X \to Y$ be a bounded linear surjection between Banach spaces X and Y. Then*

$$T^*[Y^*] = \ker(T)^\perp = \{x^* \in X^* : x^*|\ker(T) \equiv 0\}.$$

Theorem 3.4. *If a compact space K has property (#) then every twisted sum of c_0 and $C(K)$ is trivial.*

Proof. Take an isomorphic embedding $i : c_0 \to X$; let $Z = i[c_0]$ and suppose that $T : X \to C(K)$ is a bounded linear surjection such that $Z = \ker(T)$.

Write e_n for the n-th unit vector in c_0 and $e_n^* \in \ell_1 = (c_0)^*$ be the corresponding dual functional. Let $x_n = i(e_n)$ for every n. Then there is a norm bounded sequence $(x_n^*)_n$ in X^* such that $i^*x_n^* = e_n^*$. Suppose that $\|x_n^*\| \le r_0$ for every n.

Note that the set $\{x_n^* : n \in \omega\}$ is *weak** is discrete. Let

$$L = T^*[r \cdot M_1(K)] \subseteq X^*,$$

where $r > 0$ is taken big enough so that L contains $\{x^* \in Z^\perp : \|x^*\| \le r_0\}$.

We now consider $L' = L \cup \{x_n^* : n \in \omega\}$ and equip L' with the *weak** topology.

Claim. L' is a countable discrete extension of L.

Indeed, it is enough to notice that if x^* is an accumulation point of $\{x_n^* : n \in \omega\}$ then $x^* \in Z^\perp$ but this follows from the fact that for $n > k$

$$x_n^*(i(e_k)) = i^*x_n^*(e_k) = e_n^*(e_k) = 0.$$

Consider a mapping

$$h : L'' = M_1(K) \cup \omega \to L' = T^*[M_r(K)] \cup \{x_n^* : n \in \omega\},$$

defined by $h(\nu) = T^*(r\nu)$ for $\nu \in M_1(K)$ and $h(n) = x_n^*$ for $n \in \omega$. Then h is a bijection since T^* is injective and $x_n^* \neq x_k^*$ for $n \neq k$. We topologize L'' so that h becomes a homeomorphism; clearly $M_1(K)$ gets its usual *weak** topology when treated as a subspace of L''.

Since K has property (#), by Lemma 3.2 there is a bounded sequence $(\nu_n)_n$ in $M(K)$ satisfying 3.2(ii). Let $z_n^* = T^*(r\nu_n)$ for $n \in \omega$. Then $(z_n^*)_n$ is a bounded sequence in X^* and the following holds.

Claim. $z_n^* - x_n^* \to 0$ in the *weak** topology of X^*.

Take any $x \in X$; then (thinking that $x \in X^{**}$), $x \circ h \in C(L'')$ and

$$x \circ h(\nu) = T^*(r\nu(x)) = \nu(rTx),$$

for $\nu \in M_1(K)$. This means that $x \circ h$ is an extension of $rTx \in C(K)$ treated as a function on $M_1(K)$. Therefore

$$z_n^*(x) - x_n^*(x) = \nu_n(rTx) - x \circ h(n) \to 0,$$

as required.

Define now

$$P : X \to X, \quad Px = \sum_n (x_n^*(x) - z_n^*(x)) \cdot x_n.$$

Note that $Px_k = x_k$ since $x_n^*(x_k) = 1$ if $n = k$ and is 0 otherwise; moreover, $z_n^*(x_k) = 0$ for every n and k. Using Claim above, we conclude that P is indeed a projection onto Z, and the proof is complete. □

Remark 3.5. Using the construction from the above proof we can show that if X is a twisted sum of c_0 and $C(K)$ then X is isomorphic to a subspace of $C(L')$, where L' is a countable discrete extension of $M_1(K)$.

4. Asymptotic measures on Boolean algebras

We consider here an algebra $\mathfrak{A}$ of subsets of some set X. In the sequel, $\mathfrak{B}$ (with possible indices) always denotes a finite subalgebra of $\mathfrak{A}$. We introduced in section 2 the symbol $M(\mathfrak{A})$ denoting the space of all signed finitely additive functions $\mathfrak{A} \to \mathbb{R}$ of bounded variation. We shall moreover write $M^{\mathbb{Q}}(\mathfrak{B})$ (or $M_r^{\mathbb{Q}}(\mathfrak{B})$) for the set of signed measures having rational values (and having the norm $\le r$, respectively).

Given *any* real-valued partial functions φ, ψ on $\mathfrak{A}$ and an algebra $\mathfrak{B}$ contained in their domains we write

$$\mathrm{dist}_{\mathfrak{B}}(\varphi, \psi) = \sup_{B \in \mathfrak{B}} |\varphi(B) - \psi(B)|.$$

Notation 4.1. For the rest of this section we fix a sequence $(\varphi_n)_n$ of *any* set functions $\varphi_n : \mathfrak{A} \to [-1, 1]$. For any $\mathfrak{B}$ and n we define

$$o_n(\mathfrak{B}) = \inf\{\mathrm{dist}_{\mathfrak{B}}(\nu, \varphi_n) : \nu \in M_1^{\mathbb{Q}}(\mathfrak{B})\}.$$

Of course, in the formula defining $o(\mathfrak{B})$ we might as well replace $M_1^{\mathbb{Q}}(\mathfrak{B})$ by $M_1(\mathfrak{B})$ but it will be convenient to consider in the sequel measures on finite algebras having only rational values.

We shall show that, under some assumptions on $\mathfrak{A}$, if $\lim_n o_n(\mathfrak{B}) = 0$ for every finite $\mathfrak{B} \subseteq \mathfrak{A}$ then the Martin's axiom implies that there is a bounded sequence $(\mu_n)_n$ in $M(\mathfrak{A})$ such that $\mu_n(A) - \varphi_n(A) \to 0$ for every $A \in \mathfrak{A}$. In the prove below we use the following parameters.

Definition 4.2. Fix $r > 1$. We define $O_n(\mathfrak{B})$ (positive real or $+\infty$) for $n \in \omega$ and finite $\mathfrak{B} \subseteq \mathfrak{A}$ by induction on $|\mathfrak{B}|$. Set $O_n(\mathfrak{C}) = 1/(n + 1)$ for every n in the case of the trivial algebra $\mathfrak{C}$. Suppose that $O_n(\mathfrak{C})$ has been defined for every proper subalgebra $\mathfrak{C}$ of $\mathfrak{B}$. Then we put

$$O_n(\mathfrak{B}) = C_0 + o_n(\mathfrak{B}) + 1/(n + 1),$$

where C_0 is the infimum of $C > 0$ such that

(i) whenever $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{B}$ are proper subalgebras, the measures $\nu_i \in M_r^{\mathbb{Q}}(\mathfrak{A}_{\mathfrak{B}_i})$ agree on $\mathfrak{B}_1 \cap \mathfrak{B}_2$ and satisfy $\mathrm{dist}_{\mathfrak{B}_i}(\nu_i, \varphi_n) < O_n(\mathfrak{B}_i)$ for $i = 1, 2$, then there is $\mu \in M_r^{\mathbb{Q}}(\mathfrak{B})$ such that μ is a common extension of ν_1 and ν_2 and $\mathrm{dist}_{\mathfrak{B}}(\mu, \varphi_n) \le C$;

(ii) for any proper subalgebra $\mathfrak{C} \subseteq \mathfrak{B}$ and a measure $\nu \in M_r^{\mathbb{Q}}(\mathfrak{C})$ with $\mathrm{dist}_{\mathfrak{C}}(\nu, \varphi_n) < O_n(\mathfrak{C})$ there is an extension of ν to $\mu \in M^{\mathbb{Q}}(\mathfrak{B})$ such that $\|\mu\| \le \max(\|\nu\|, 1)$ and $\mathrm{dist}_{\mathfrak{B}}(\mu, \varphi_n) \le C$.

Remark 4.3. The definition of O_n depends on the chosen parameter r; we write O_n rather than O_n^r for simplicity.

Note that in case of (ii) above the set of such μ is always nonempty, see Lemma A.2 from Appendix at the end of the paper. However, there may be no common extension of ν_1, ν_2 considered in (i) which would satisfy $\|\mu\| \le r$; in such a case we understand that $O_n(\mathfrak{B}) = +\infty$.

Lemma 4.4. *If* $\lim_n o_n(\mathfrak{B}) = 0$ *for every finite algebra* $\mathfrak{B} \subseteq \mathfrak{A}$ *then* $\lim_n O_n(\mathfrak{B}) = 0$ *for every such* $\mathfrak{B}$.

Proof. We argue by induction on $|\mathfrak{B}|$. If $\mathfrak{B}$ is trivial then $O_n(\mathfrak{B}) = 1/(n + 1)$. Suppose that $\mathfrak{B}$ is nontrivial and $\lim_n O_n(\mathfrak{C}) = 0$ for any proper subalgebra $\mathfrak{C}$ of $\mathfrak{B}$.

Let $N \ge 2$ be the number of atoms of $\mathfrak{B}$. Fix $\varepsilon > 0$ and take $\delta > 0$ such that

$$4N\delta < r - 1 \text{ and } (4N + 1)\delta < \varepsilon.$$

By the inductive assumption and the fact that $\lim_n o_n(\mathfrak{B}) = 0$ there is n_0 such that for $n \ge n_0$ we have $o_n(\mathfrak{B}) < \delta$ and $O_n(\mathfrak{C}) < \delta$ for all proper subalgebras $\mathfrak{C}$ of $\mathfrak{B}$. We shall check that then $O_n(\mathfrak{B}) \le \varepsilon$ whenever $n \ge n_0$. Given such n, we will verify that $C = \varepsilon$ satisfies conditions (i-ii) of Definition 4.2.

Consider a pair $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{B}$ of proper subalgebras and a pair $\nu_i \in M_r^{\mathbb{Q}}(\mathfrak{B}_i)$ of consistent measures as in Definition 4.2(i). Take $\lambda \in M_1^{\mathbb{Q}}(\mathfrak{B})$ witnessing $o_n(\mathfrak{B}) < \delta$. Then $\mathrm{dist}_{\mathfrak{B}_i}(\nu_i, \lambda) < 2\delta$ so by Lemma A.1 there is a common extension of ν_1, ν_2 to a measure $\lambda' \in M^{\mathbb{Q}}(\mathfrak{B})$ such that $\|\lambda - \lambda'\| < 4N\delta$. This implies

$$\|\lambda'\| \le \|\lambda\| + 4N\delta \le 1 + r - 1 = r;$$

$$\mathrm{dist}_{\mathfrak{B}}(\lambda', \varphi_n) \le \mathrm{dist}_{\mathfrak{B}}(\lambda', \lambda) + \mathrm{dist}_{\mathfrak{B}}(\lambda, \varphi_n) < 4N\delta + \delta < \varepsilon,$$

as required.

Consider now $\mathfrak{C}$ and $\nu \in M_r^{\mathbb{Q}}(\mathfrak{C})$ as in Definition 4.2(ii).

Let, again, $\lambda \in M_1^{\mathbb{Q}}(\mathfrak{B})$ witnesses that $o_n(\mathfrak{B}) < \delta$. We have $\mathrm{dist}_{\mathfrak{C}}(\nu, \lambda) < 2\delta$ so Lemma A.2 gives us a measure $\lambda' \in M^{\mathbb{Q}}(\mathfrak{B})$ extending ν with $\|\lambda'\| \le \max(\|\nu\|, 1)$ and such that $\mathrm{dist}_{\mathfrak{B}}(\lambda', \lambda) < 6\delta$. It follows that $\mathrm{dist}_{\mathfrak{B}}(\lambda', \varphi_n) < 7\delta < (4N+1)\delta < \varepsilon$, as required. $\square$

Definition 4.5. Let us say that a Boolean algebra $\mathfrak{A}$ has LEP(r) (local extension property) for some $r > 1$ if there is a family $\mathbb{B}$ of finite subalgebras of $\mathfrak{A}$ such that

(i) for every finite algebra $\mathfrak{C} \subseteq \mathfrak{A}$ there is $\mathfrak{B} \in \mathbb{B}$ with $\mathfrak{B} \supseteq \mathfrak{C}$;
(ii) whenever $\mathbb{B}' \subseteq \mathbb{B}$ is uncountable then there are distinct $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathbb{B}'$ such that any pair of consistent measures $\nu_i \in M_1^{\mathbb{Q}}(\mathfrak{B}_i)$ admits a common extension to a measure $\nu \in M_r^{\mathbb{Q}}(\langle \mathfrak{B}_1 \cup \mathfrak{B}_2 \rangle)$.

We are now ready for the main result of this section. As usual, for a given cardinal number κ, $\mathsf{MA}(\kappa)$ denotes the Martin's axiom for κ many dense sets in a partially ordered set with *ccc*, see e.g. Fremlin [15].

Theorem 4.6. *Suppose that $\mathfrak{A}$ is an algebra with $|\mathfrak{A}| = \kappa$. Suppose further that $\mathfrak{A}$ has* LEP(r) *for some $r > 1$ and that $\mathrm{ult}(\mathfrak{A})$ is separable. Let (as in 4.1) $\varphi_n : \mathfrak{A} \to [-1, 1]$ be a sequence of functions such that $\lim_n o_n(\mathfrak{B}) = 0$ for every finite subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.*

Assuming $\mathsf{MA}(\kappa)$, there is a sequence $(\mu_n)_n$ in $M_r(\mathfrak{A})$ such that

$$\lim_{n\to\infty} \big(\varphi_n(A) - \mu_n(A)\big) = 0,$$

for every $A \in \mathfrak{A}$.

Proof. Let $\mathbb{B}$ be a family of subalgebras granted by LEP(r).

We consider a partially ordered set $\mathbb{P}$ of conditions

$$p = (\mathfrak{B}, n, (\nu_i)_{i \le n}, k), \text{ where}$$

(i) $\mathfrak{B} \in \mathbb{B}$ and n, k are positive integers;
(ii) for every $i \le n$, the measure ν_i is in $M_r^{\mathbb{Q}}(\mathfrak{B})$;
(iii) $\mathrm{dist}_{\mathfrak{B}}(\nu_i, \varphi_i) < O_i(\mathfrak{B})$ for any $i \le n$;
(iv) $O_m(\mathfrak{B}) < 1/k$ for every $m \ge n$.

Consider two conditions

$$p = (\mathfrak{B}, n, (\nu_i)_{i \le n}, k), \quad p' = (\mathfrak{B}', n', (\nu_i')_{i \le n'}, k') \in \mathbb{P}.$$

We shall say that p' is a simple extension of p if $k' \ge k$ and

— either $\mathfrak{B}' = \mathfrak{B}$ and $n' \geq n$, $\nu_i' = \nu_i$ for $i \leq n$,
— or $n' = n$, $\mathfrak{B} \subseteq \mathfrak{B}'$ and ν_i' extends ν_i for every $i \leq n$.

Then we define a partial order on $\mathbb{P}$ declaring $p \leq p'$ if there are $s \in \omega$ and a sequence p_j, $j = 0, \ldots, s$, in $\mathbb{P}$ such that $p_0 = p$, $p_s = p'$ and p_{j+1} is a simple extension of p_j for every $j < s$. Note that $\leq$ is indeed a partial order on $\mathbb{P}$.

Claim A. Let $A \in \mathfrak{B}$ and let $p, p' \in \mathbb{P}$ be specified as above. If $p \leq p'$. then

$$|\nu_i'(A) - \varphi_i(A)| < 1/k \text{ whenever } n \leq i \leq n'.$$

Note that if p' is a simple extension of p with $\mathfrak{B}' = \mathfrak{B}$ then for $i \geq n$ we have

$$|\nu_i'(A) - \varphi_i(A)| < O_i(\mathfrak{B}') = O_i(\mathfrak{B}) < 1/k,$$

by (iv). If p' is a simple extension of p with $n' = n$ then the inequality holds as $\nu_i'(A) = \nu_i(A)$ for $i \leq n$. Hence the assertion follows by induction on the number of simple extensions leading from p to p'.

Note that $p \leq p'$ means that the condition p' is stronger; accordingly, we consider *ccc* and other properties to be defined upwards.

Claim B. $\mathbb{P}$ is *ccc*.

Consider an uncountable family $P \subseteq \mathbb{P}$ of conditions

$$p = (\mathfrak{B}^p, n^p, (\nu_i^p)_{i \leq n^p}, k^p).$$

Shrinking P if necessary, we can assume that $n^p = n$ and $k^p = k$ are constant for $p \in P$.

Let S be a countable dense subset of $\mathrm{ult}(\mathfrak{A})$. Every $x \in S$ defines a 0-1 probability measure $\delta_x \in M(\mathfrak{A})$, where $\delta_x(B) = 1$ iff $B \in x$. Let M^S be the countable family of all measures on $\mathfrak{A}$ that are rational linear combinations of δ_x's with $x \in S$. Note that any measure $\nu \in M^{\mathbb{Q}}(\mathfrak{B})$ on a finite algebra $\mathfrak{B}$ can be represented as a restriction of some $\widetilde{\nu} \in M^S$ to $\mathfrak{B}$, where $\|\widetilde{\nu}\| = \|\nu\|$.

Using the above remark, thinning P out again, we can assume that for every $i \leq n$ there is $\widetilde{\nu}_i \in M^S$ such that $\nu_i^p = \widetilde{\nu}_i|\mathfrak{B}^p$ and $\|\nu_i^p\| = \|\widetilde{\nu}_i|\mathfrak{B}^p\|$ for every $p \in P$.

Finally, we apply LEP(r) to choose distinct $p, q \in P$ so that $\mathfrak{B}^p$ and $\mathfrak{B}^q$ have the property granted by Definition 4.5(ii). We put $\mathfrak{B}_0 = \mathfrak{B}_1 \cap \mathfrak{B}_2$ and, using 4.5(i) choose $\mathfrak{B} \in \mathbb{B}$ containing $\mathfrak{B}_1 \cup \mathfrak{B}_2$. By Lemma 4.4 there is $n_1 \geq n$ such that $O_m(\mathfrak{B}) < 1/k$ for every $m \geq n_1$. We shall check that p and q have a common extension in $\mathbb{P}$.

For every i such that $n < i \leq n_1$ we choose a measure $\pi_i \in M_1^{\mathbb{Q}}(\mathfrak{B}_0)$ such that

$$\mathrm{dist}_{\mathfrak{B}_0}(\pi_i, \varphi_i) < o_i(\mathfrak{B}_0) + 1/(i+1) \leq O_i(\mathfrak{B}_0),$$

and then by part (ii) of Definition 4.2 extend π_i to measures

$$\nu_i^p \in M_1^{\mathbb{Q}}(\mathfrak{B}^p) \text{ and } \nu_i^q \in M_1^{\mathbb{Q}}(\mathfrak{B}^q) \text{ such that}$$

$$\mathrm{dist}_{\mathfrak{B}^p}(\nu_i^p, \varphi_i) < O_i(\mathfrak{B}^p), \mathrm{dist}_{\mathfrak{B}^q}(\nu_i^q, \varphi_i) < O_i(\mathfrak{B}^q).$$

Then there is ν_i in $M_r^{\mathbb{Q}}(\mathfrak{B})$ which is a common extension of ν_i^p and ν_i^q and such that $\mathrm{dist}_{\mathfrak{B}}(\nu_i, \varphi_i) < O_i(\mathfrak{B})$; indeed, if $O_i(\mathfrak{B}) < +\infty$, then this follows from Definition 4.2(i).

In case $O_i(\mathfrak{B}) = +\infty$ we may take *any* extension granted by 4.5(ii) and the way we have chosen $\mathfrak{B}^p$ and $\mathfrak{B}^q$, and extend it to $\mathfrak{B}$ preserving its norm.

For $i \le n$ we choose $\nu_i \in M_r^{\mathbb{Q}}(\mathfrak{B})$ applying Definition 4.2 to the pair ν_i^p, ν_i^q. Note that if $O_i(\mathfrak{B}) = +\infty$ then we may use the fact that both ν_i^p and ν_i^q are represented by the same measure $\widetilde{\nu}_i \in M^S$ so $\widetilde{\nu}_i|\mathfrak{B}$ is their common extension to $\mathfrak{B}$ with norm $\le r$.

In this way we get simple extensions

$$p_1 = (\mathfrak{B}^p, n_1, (\nu_i^p)_{i\le n_1}, k), \quad q_1 = (\mathfrak{B}^q, n_1, (\nu_i^q)_{i\le n_1}, k),$$

of p and q, respectively. In turn,

$$s = (\mathfrak{B}, n_1, (\nu_i)_{i\le n_1}, k) \in \mathbb{P}$$

satisfies $p_1, q_1 \le s$, and this finishes the proof of Claim B.

Claim C. For every $k_0, n_0 \in \omega$ and finite $\mathfrak{B}_0 \subseteq \mathfrak{A}$, the set

$$\mathbb{D}(\mathfrak{B}_0, n_0, k_0) = \{p = (\mathfrak{B}^p, n^p, (\nu_i^p)_{i\le n^p}, k^p) \in \mathbb{P} : \mathfrak{B}^p \supseteq \mathfrak{B}_0, n^p \ge n_0, k^p \ge k_0\},$$

is upwards dense in $\mathbb{P}$.

Take any $p = (\mathfrak{B}^p, n^p, (\nu_i^p)_{i\le n^p}), k^p) \in \mathbb{P}$ and consider a triple $\mathfrak{B}_0$, n_0, k_0; we can assume that $k_0 \ge k^p$ and $n_0 \ge n^p$.

Find $\mathfrak{B} \in \mathbb{B}$ containing $\mathfrak{B}^p \cup \mathfrak{B}_0$ and $n_1 \ge n_0$ such that $O_m(\mathfrak{B}) < 1/k_0$ for $m \ge n_1$. Then, arguing as in the proof of Claim B we define appropriate ν_i and ν_i' so that

$$p \le (\mathfrak{B}^p, n_1, (\nu_i)_{i\le n_1}, k_0) \le (\mathfrak{B}, n_1, (\nu_i')_{i\le n_1}, k_0).$$

Indeed, for $n^p < i \le n_1$ we pick a measure $\nu_i \in M_1^{\mathbb{Q}}(\mathfrak{B}^p)$ such that

$$\mathrm{dist}_{\mathfrak{B}^p}(\nu_i, \varphi_i) < o_i(\mathfrak{B}^p) + 1/(i+1) \le O_i(\mathfrak{B}^p),$$

and then by part (ii) of Definition 4.2 extend ν_i to a measure $\nu_i' \in M_1^{\mathbb{Q}}(\mathfrak{B})$ such that $\mathrm{dist}_{\mathfrak{B}}(\nu_i', \varphi_i) < O_i(\mathfrak{B})$. Accordingly, for $i \le n$ we suitably extend every ν_i to $\nu_i' \in M_r^{\mathbb{Q}}(\mathfrak{B})$.

With Claim B and C at hand, we apply Martin's axiom $\mathsf{MA}(\kappa)$ to get a directed set $\mathbb{G} \subseteq \mathbb{P}$ such that $\mathbb{G} \cap \mathbb{D}(\mathfrak{B}_0, n_0, k_0) \ne \emptyset$ for every finite $\mathfrak{B}_0 \subseteq \mathfrak{A}$ and positive integers n_0, k_0, This means that, for every i, we get a consistent family of measures ν_i of variation $\le r$. Their domains cover all of $\mathfrak{A}$ so they extend uniquely to a measure $\mu_i \in M_r(\mathfrak{A})$.

For any $A \in \mathfrak{A}$ there is $\mathfrak{B}_0 \in \mathbb{B}$ such that $A \in \mathfrak{B}_0$. Given $\varepsilon > 0$, take k_0 such that $1/k_0 < \varepsilon$ and

$$p = (\mathfrak{B}, n, (\nu_i)_{i\le n}, k) \in \mathbb{G} \cap \mathbb{D}(\mathfrak{B}_0, 1, k_0).$$

Then $\mu_n(A) = \nu_n(A)$ so

$$|\mu_n(A) - \varphi_n(A)| < O_n(\mathfrak{B}) < 1/k \le 1/k_0 < \varepsilon.$$

For every $m > n$ there is $p' = (\mathfrak{B}', n', (\nu_i')_{i\le n'}, k') \in \mathbb{G}$ such that $p \le p'$ and $m \le n'$. Then $\mu_m(A) = \nu_m'(A)$ and $|\nu_m(A) - \varphi_m(A)| < 1/k_0$ by Claim A. This shows that

$$\mu_n(A) - \varphi_n(A) \to 0,$$

and the proof is complete. $\square$

Proposition 4.7. *For any cardinal number κ, the algebra $\mathfrak{A} = \mathrm{Clop}(2^\kappa)$ has* LEP(2).

Proof. For any finite set $F \subseteq \kappa$ we let $\mathfrak{B}_F$ be the finite subalgebra of $\mathfrak{A}$ of all sets that are determined by in coordinates in F; thus $\mathfrak{B}_F$ is generated by its atoms A of the form

$$A = \{t \in 2^\kappa : t|F = \tau|F\},$$

for some function $\tau : F \to 2$. Clearly the family $\mathbb{B}$ of all such $\mathfrak{B}_F$ is cofinal in $\mathfrak{A}$ so it is enough to check that any $\mathfrak{B}_{F_1}, \mathfrak{B}_{F_2}$ satisfy (ii) of Definition 4.5.

Consider $\nu_i \in M_1^{\mathbb{Q}}(\mathfrak{A}_{F_i})$, $i = 1, 2$ and suppose that ν_1 and ν_2 agree on $\mathfrak{B}_{F_1} \cap \mathfrak{B}_{F_2}$ which is $\mathfrak{B}_H$, where $H = F_1 \cap F_2$.

Let A_i, $i \leq 2^{F_1 \setminus H}$, be the list of all atoms of $\mathfrak{B}_{F_1 \setminus H}$ and, accordingly, B_j be the list of all atoms of $\mathfrak{B}_{F_2 \setminus H}$.

For a fixed atom C of $\mathfrak{A}_H$ we apply Lemma A.3 to $a_i = \nu_1(A_i \cap C)$ and $b_j = \nu_2(B_j \cap C)$. Note that

$$\sum_i a_i = \nu_1(C) = \nu_2(C) = \sum_j b_j.$$

This enables us to define $\overline{\nu}$ for all $A \in \mathfrak{A}_F$ contained in C, and

$$|\overline{\nu}|(C) \leq \max\big(|\nu_1|(C), |\nu_2|(C)\big) \leq |\nu_1|(C) + |\nu_2|(C),$$

so we get $\overline{\nu}$ with $\|\overline{\nu}\| \leq 2$. $\square$

Remark 4.8. In the proof of 4.7 one can alternatively check that ν_1 and ν_2 admit a common extension of norm ≤ 2 applying a result Basile, Rao and Shortt [4] described in Appendix. One can check that $SC(\nu_1, \nu_2) \leq 2$ basing on the following remark: If $B_i \in \mathfrak{B}_{F_i}$ and $B_1 \subseteq B_2$ then there is $C \in \mathfrak{B}_{F_1 \cap F_2}$ such that $B_1 \subseteq C \subseteq B_2$.

Proposition 4.9. *Let $\mathfrak{A}$ be an algebra of subsets of ω that is generated by an almost disjoint family $\mathcal{A}$ and all finite subsets of ω. Then the algebra $\mathfrak{A}$ has* LEP(3).

Proof. We consider the family $\mathbb{B}$ of finite subalgebras of $\mathfrak{A}$, where every $\mathfrak{B} = \langle n, A_1, \ldots, A_k \rangle \in \mathbb{B}$ is an algebra spanned by all subsets of $n = \{0, 1, \ldots, n-1\}$ and $A_i \in \mathcal{A}$ having the property that $A_i \cap A_j \subseteq n$ for $i \neq j$. Clearly $\mathbb{B}$ is cofinal in $\mathfrak{A}$; we shall check that (ii) of Definition 4.5 holds for $r = 3$.

If $\mathbb{B}' \subseteq \mathbb{B}$ is uncountable then there are two algebras in $\mathbb{B}'$ of the form

$$\mathfrak{B}_1 = \langle n, A_1, \ldots, A_m, B_1, \ldots, B_k \rangle, \quad \mathfrak{B}_2 = \langle n, A_1, \ldots, A_m, C_1, \ldots, C_k \rangle,$$

where $B_i, C_j \in \mathcal{A}$ are all distinct. Set

$$X = n \cup \bigcup_{i \leq m} A_i \cup \bigcup_{i,j \leq k} B_i \cap C_j;$$

note that $B_i \setminus X$ and $C_i \setminus X$ are infinite for $i \leq k$.

Take $\nu_1 \in M_1^{\mathbb{Q}}(\mathfrak{B}_1)$ and $\nu_2 \in M_1^{\mathbb{Q}}(\mathfrak{B}_2)$ which agree on $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \langle n, A_1, \ldots, A_m \rangle$. We can represent them as the restrictions of

$$\nu_1 = \nu_0 + \sum_{i \leq k} b_i \delta_{x_i} + b\delta_x, \quad \nu_2 = \nu_0 + \sum_{i \leq k} c_i \delta_{y_i} + c\delta_x, \text{ where}$$

$$x_i \in B_i \setminus X, y_i \in C_i \setminus X, x \in \omega \setminus \big(X \cup \bigcup_{i\le k}(B_i \cup C_i)\big).$$

Here ν_0 is defined as $\nu_0(A) = \nu_1(A \cap X) = \nu_2(A \cap X)$. Write $\overline{b} = \sum_{i\le k} b_i$ and $\overline{c} = \sum_{i\le k} c_i$, and consider the measure

$$\nu = \nu_0 + \sum_{i\le k} b_i\delta_{x_i} + \sum_{i\le k} c_i\delta_{y_i} + (b - \overline{c})\delta_x.$$

Then we have

$$\nu(\omega) = \nu_0(X) + \overline{b} + \overline{c} + b - \overline{c} = \nu_1(\omega) = \nu_2(\omega).$$

Moreover, $\nu(B_i) = \nu_0(B_i \cap n) + b_i = \nu_1(B_i)$, a similar argument holds for ν_2 and C_i. It follows that ν is a common extension of ν_1, ν_2. Clearly, $\|\nu\| \le 3$, so this finishes the proof. □

5. Trivial twisted sums of c_0 and $C(K)$

We conclude our considerations from previous section and show here that under Martin's axiom and the negation of the continuum hypothesis every twisted sum of c_0 and $C(2^{\omega_1})$ is trivial. Another example if this kind is an Aleksandroff-Urysohn space defined from an almost disjoint family of small size.

Theorem 5.1. *Let K be a zerodimensional separable compact space of weight $\kappa < \mathfrak{c}$, and such that $\mathfrak{A} = \mathrm{Clop}(K)$ has $\mathrm{LEP(r)}$ for some $r > 1$.*

Subject to $\mathsf{MA}(\kappa)$, K has property $(\#)$.

Proof. Set $L = M_1(K)$; fix $L' \in \mathrm{CDE}(L)$ and identify $L' \setminus L$ with ω.

For every $A \in \mathfrak{A}$, the function $M_1(K) \ni \nu \to \nu(A)$ is continuous on $M_1(K)$. Denote by θ_A some its extension to a continuous function $L' \to [-1, 1]$. Define set functions φ_n on $\mathfrak{A}$ as $\varphi_n(A) = \theta_A(n)$ for every n and $A \in \mathfrak{A}$.

Recall that $o_n(\mathfrak{B})$ is defined in 4.1. We have $\lim_n o_n(\mathfrak{B}) = 0$, for every finite subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ by Lemma 2.3 applied to the finite family $\{\theta_B : B \in \mathfrak{B}\}$. Now Theorem 4.6 says that there is a sequence $\mu_n \in M_r(\mathfrak{A})$ such that

$$\lim_n(\theta_A(n) - \mu_n(A)) = \lim_n \big(\varphi_n(A) - \mu_n(A)\big) = 0,$$

for every $A \in \mathfrak{A}$. Every measure μ_n extends uniquely to a Radon measure on K; we denote its extension by the same symbol.

Consider the family $\mathcal{G}$ of those $g \in C(K)$ such that whenever $\widehat{g} \in C(L')$ extends g as a function on $M_1(K)$ then $\mu_n(g) - \widehat{g}(n) \to 0$. As we have seen, for every $A \in \mathfrak{A}$, $\chi_A \in \mathcal{G}$, see Lemma 2.2. Using the same lemma we can easily verify that $\mathcal{G}$ is closed under finite linear combinations, hence every simple continuous function is in $\mathcal{G}$.

Note that if $g, h \in C(K)$ and $\|g - h\| < \varepsilon$ for some $\varepsilon > 0$ then, taking any extensions $\widehat{g}, \widehat{h}$ of g and h, respectively, we have $|\widehat{g}(n) - \widehat{h}(n)| < \varepsilon$ for almost all $n \in \omega$. This remark implies that the family $\mathcal{G}$ is closed under uniform limits and we hence $\mathcal{G} = C(K)$. Consequently, by Lemma 3.2, K has property $(\#)$, and this finishes the proof. □

Now Propositions 4.7 and 4.9, together with Theorem 5.1 yield the following.

Corollary 5.2. *Assume that $\kappa < \mathfrak{c}$ is such a cardinal number that $\mathsf{MA}(\kappa)$ holds. Then 2^κ has property (#) and hence, by Theorem 3.4, every twisted sum of c_0 and $C(2^\kappa)$ is trivial.*

Corollary 5.3. *Assume that $\kappa < \mathfrak{c}$ and $\mathsf{MA}(\kappa)$ holds. Let $\mathcal{A}$ be an almost disjoint family of subsets of ω with $|\mathcal{A}| = \kappa$. Let $K = \mathrm{ult}(\mathfrak{A})$ where $\mathfrak{A}$ is an algebra of subsets of ω generated by $\mathcal{A}$ and all finite sets.*

Then K has property (#) and hence, by Theorem 3.4, every twisted sum of c_0 and $C(K)$ is trivial.

The above results seem to give the first (consistent) examples of a nonmetrizable compact space K for which every twisted sum of c_0 and $C(K)$ is trivial. Correa and Tausk [9] proved, in particular, that $C(2^\mathfrak{c})$ admits a nontrivial twisted sum with c_0. Hence the question about nontrivial twisted sums of c_0 and $C(2^{\omega_1})$ cannot be decided within the usual set theory. This is also the case for compact spaces K as in Corollary 5.3, since Castillo proved that assuming CH, for such spaces K, there exists a nontrivial twisted sum of c_0 and $C(K)$, see Theorem 9.3.

The problem arises, if we can apply the above argument to any separable compactum of weight $< \mathfrak{c}$. In other words, we do not know if every small Boolean algebra having a separable Stone space has LEP(r) for some $r > 1$.

Problem 5.4. *Is there a ZFC example of a separable compact space K of weight ω_1 such that c_0 and $C(K)$ have a nontrivial twisted sum?*

Let us note that if we could, while examining property (#) of a compactum K, exchange $M_1(K)$ for $P(K)$, the space of probability measures on K, then the way to Corollary 5.3 would be much shorter, at least for $\kappa = \omega_1$. Indeed, $P(2^{\omega_1})$ is homeomorphic to $[0,1]^{\omega_1}$ and therefore $P(2^{\omega_1})$ is an absolute retract, In particular, for every $L' \in \mathrm{CDE}(P(2^{\omega_1}))$ there is a retraction $L' \to P(2^{\omega_1})$ so there is a norm-one extension operator $C(P(2^{\omega_1})) \to C(L')$. However, we prove in the next section that $M_1(2^{\omega_1})$ is not an absolute retract. Note that $M_1(2^{\omega_1})$ is clearly a dyadic space but this fact itself does not help as the examples given in Section 7 indicate.

6. On properties of $M_1(K)$

Recall that a compact space K is a *Dugundji space* if for every compact space L containing K there exists a *regular* extension operator $E : C(K) \to C(L)$, i.e. an extension operator of norm 1 preserving constant functions. It is well-known that a convex compact space K is a Dugundji space if and only if it is an absolute retract, cf. [16, Sec. 2]. For a nonmetrizable compact space K, the space $P(K)$ can be an absolute retract, namely Ditor and Haydon [10] proved that $P(K)$ has this property if and only if K is a Dugundji space of weight at most ω_1. We will show that this can never happen for the space $M_1(K)$.

Theorem 6.1. *If K is a nonmetrizable compact space, then the space $M_1(K)$ is not a Dugundji space, in particular, it is not an absolute retract.*

We will prove this theorem using spectral theorem of Shchepin, the key ingredient will be Proposition 6.2 below.

For a surjection $\varphi : L \to K$ between compact spaces K, L, $\varphi^* : M_1(L) \to M_1(K)$ denotes the canonical surjection associated with φ, i.e., the surjection given by the operator conjugate to the isometrical embedding of $C(K)$ into $C(L)$ induced by φ. In other words, for $\mu \in M(L)$, $\varphi^*(\mu) \in M(K)$ is defined by $\varphi^*(\mu)(B) = \mu(f^{-1}[B])$ for Borel sets $B \subseteq K$.

Proposition 6.2. *Let $\varphi : L \to K$ be a surjection of a compact space L onto an infinite space K. If φ is not injective, then the map $\varphi^* : M_1(L) \to M_1(K)$ is not open.*

Proof. We will consider two cases:

Case 1. There exist distinct points $x, y \in K$ such that $|\varphi^{-1}(x)| > 1$ and y is an accumulation point of K. Pick disjoint neighborhoods U_x of x and U_y of y. Take two distinct points $z_0, z_1 \in \varphi^{-1}(x)$ and a continuous function $f : L \to [0,1]$ such that $f(z_i) = i$ and $f^{-1}((0,1]) \subseteq \varphi^{-1}(U_x)$. Consider the open set $V = \{\mu \in M_1(L) : \mu(f) > 1/4\}$. We will show that its image $\varphi^*(V)$ is not open in $M_1(K)$. Clearly, we have

$$\mu = (1/2)\delta_{z_1} - (1/2)\delta_{z_0} \in V \text{ and } \varphi^*(\mu) = 0.$$

Let W be any open neighborhood of 0 in $M_1(K)$. Since y is an accumulation point, we can find $y' \in U_y \setminus \{y\}$ such that $\nu = (1/2)\delta_y - (1/2)\delta_{y'} \in W$. One can easily check that $\nu \notin \varphi^*(V)$.

Case 2. It is clear that if the Case 1 does not hold, then there exists an accumulation point x in K such that $\varphi^{-1}(x)$ is the only nontrivial fiber of φ. Take two distinct points $z_0, z_1 \in \varphi^{-1}(x)$ and disjoint neighborhoods U_0, U_1 of z_0, z_1, respectively, in L. Then we have $\varphi(U_0) \cap \varphi(U_1) = \{x\}$, hence there is an $i \in \{0,1\}$ such that $\varphi(U_i)$ is not a neighborhood of x. Find a a continuous function $f : L \to [0,1]$ such that $f(z_i) = 1$ and $f^{-1}((0,1]) \subseteq U_i$. We define the open set $V = \{\mu \in M_1(L) : \mu(f) > 1/4\}$ as in Case 1. Again, we have

$$\mu = (1/2)\delta_{z_i} - (1/2)\delta_{z_{1-i}} \in V \text{ and } \varphi^*(\mu) = 0.$$

Take any open neighborhood W of 0 in $M_1(K)$. Since x is an accumulation point, we can find distinct points $y, y' \in K \setminus \varphi(U_i)$ such that $\nu = (1/2)\delta_y - (1/2)\delta_{y'} \in W$. One can easily verify that $\nu \notin \varphi^*(V)$, hence $\varphi^*(V)$ is not open in $M_1(K)$. □

Proposition 6.3. *Let K be a compact space of weight ω_1. Then the space $M_1(K)$ is not a Dugundji space.*

Proof. Assume towards a contradiction that $M_1(K)$ is a Dugundji space. Then by a result of Haydon, cf. [16], [28], $M_1(K)$ is an inverse limit of a continuous inverse sequence $\langle L_\alpha, p_\alpha^\beta, \omega_1\rangle$, where all spaces L_α are metrizable and all bonding maps p_α^β are open. Let $\langle K_\alpha, q_\alpha^\beta, \omega_1\rangle$ be any continuous inverse sequence with all spaces K_α infinite metrizable, all bonding maps q_α^β non-injective, and the limit homeomorphic to K. Then one can easily verify that the inverse system $\langle M_1(K_\alpha), (q_\alpha^\beta)^*, \omega_1\rangle$ is continuous and its limit is homeomorphic to $M_1(K)$. Then, by Shchepin's spectral theorem [28] the sequences $\langle L_\alpha, p_\alpha^\beta, \omega_1\rangle$ and

$\langle M_1(K_\alpha), (q_\alpha^\beta)^*, \omega_1\rangle$ would contain isomorphic subsequences, which is impossible, since the maps p_α^β are open and the maps $(q_\alpha^\beta)^*$ are not open by Proposition 6.2. $\square$

We need to recall some notions and results from [22]. Let $X = \Pi_{\alpha<\kappa} X_\alpha$ be the product of metrizable compact spaces X_α, and let $r : X \to Y$ be a retraction. A subset $S \subseteq \kappa$ is *r-admissible* if $x|S = x'|S$ implies $r(x)|S = r(x')|S$ for all $x, x' \in X$. Obviously the union of any family of r-admissible subsets is r-admissible. For $y \in Y$ let $p_S : Y \to \Pi_{\alpha\in S} X_\alpha$ be defined by $p_S(y) = y|S$ and let $Y_S = p_S(Y)$. Kubiś proved in [22] that each countable subset of κ is contained in a countable r-admissible subset, and if $S \subseteq \kappa$ is r-admissible then the map $p_S : Y \to Y_S$ is right-invertible, i.e., there exists a continuous map $j : Y_s \to Y$ such that $p_S \circ j = \mathrm{id}_{Y_S}$, hence Y_S is homeomorphic to a retract $j(Y_S)$ of Y.

Proof of Theorem 6.1. Suppose that K is a nonmetrizable compact space such that the space $M_1(K)$ is a Dugundji space, hence an absolute retract. We will show that there exists a continuous image L of K of weight ω_1 such that $M_1(L)$ is homeomorphic to a retract of $M_1(K)$. This will give a contradiction with Proposition 6.3. Let $\mathcal{F} = \{f_t : K \to K_t : t \in T\}$ be the family of all continuous surjections of K onto a subspace of $[0,1]^\omega$. Then the diagonal map

$$\varphi = \triangle_{t\in T} f_t^* : M_1(K) \to \Pi_{t\in T} M_1(K_t),$$

is an embedding of $M_1(K)$ into the product of metrizable compacta. Let $Y = \varphi(M_1(K))$ and let $r : \Pi_{t\in T} M_1(K_t) \to Y$ be a retraction. Fix a subset $\{t_\alpha : \alpha < \omega_1\} \subseteq T$ such that the image of the diagonal map

$$\triangle_{\alpha<\omega_1} f_{t_\alpha} : K \to \Pi_{\alpha<\omega_1} K_{t_\alpha},$$

is of weight ω_1. For any countable subset $S \subseteq T$ fix a countable r-admissible subset $\eta(S) \subseteq T$ containing S. By induction we will define the family of r-admissible countable sets $S_\alpha \subseteq T$ for $\alpha < \omega_1$. We start with $S_0 = \eta(\{t_0\})$. Suppose that we have defined the sets S_β for $\beta < \alpha$. Put $P_\alpha = \bigcup\{S_\beta : \beta < \alpha\} \cup \{t_\alpha\}$ and let $s_\alpha \in T$ be such that $f_{s_\alpha} = \triangle_{t\in P_\alpha} f_t : K \to \Pi_{t\in P_\alpha} K_t$. We define $S_\alpha = \eta(P_\alpha \cup \{s_\alpha\})$. Finally we put $S = \bigcup\{S_\alpha : \alpha < \omega_1\}$. The set S is r-admissible, hence the map $p_S : Y \to Y_S$ is right-invertible, so Y_s is homeomorphic to a retract of Y. Let L be the image of K under the diagonal map $\triangle_{t\in S} f_t : K \to \Pi_{t\in S} K_t$. The use of indexes t_α in our construction guaranties that L has weight ω_1. A routine verification shows that Y_s is homeomorphic to $M_1(L)$. $\square$

7. Countable discrete extensions of dyadic compacta

If $\mathfrak{A}$ is a subalgebra of the algebra of all subsets of ω containing all finite sets then its Stone space $\mathrm{ult}(\mathfrak{A})$ can be seen as a compactification of ω because one can identify every $n \in \omega$ with the corresponding principal ultrafilter. Note that $\mathrm{ult}(\mathfrak{A}/\mathit{fin})$ is homeomorphic to the remainder of such a compactification. In this setting, the equivalence of conditions (i) and (iii) from Lemma 2.7 can be stated as follows (see Lemma 3.1 in [12]).

Lemma 7.1. *Let $\mathfrak{A}$ be an algebra such that $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$. Then the compactification $\mathrm{ult}(\mathfrak{A})$ of ω is tame if and only if there exists a bounded sequence $(\nu_n)_n$ in $M(\mathfrak{A})$ such that*

(i) $\nu_n|fin \equiv 0$ *for every* n*, and*
(ii) $\nu_n - \delta_n \to 0$ *on* $\mathfrak{A}$*, that is* $(\nu_n - \delta_n)(A) \to 0$ *for every* $A \in \mathfrak{A}$*.*

A Boolean algebra $\mathfrak{B}$ is called *dyadic* if it can be embedded into a free algebra $\mathrm{Clop}(2^\kappa)$ for some cardinal number κ, that is if $\mathrm{ult}(\mathfrak{A})$ is a dyadic compactum, i.e., a continuous image of some Cantor cube 2^κ ([14]). Recall that for $L = 2^\kappa$ and $L' \in \mathrm{CDE}(L)$ there is a retraction from L' onto L so, in particular, there is an extension operator $C(L) \to C(L')$. We give below examples showing that this is no longer true if we replace here 2^κ by its continuous image.

Lemma 7.2. *Let* $\mathfrak{B}$ *be a Boolean algebra generated by a family* $\mathcal{G}$ *of size* κ *such that* $\mathcal{G} = \bigcup_n \mathcal{G}_n$*, where every* $\mathcal{G}_n$ *is an independent family and for every* $k \neq n$*, if* $a \in \mathcal{G}_k$ *and* $b \in \mathcal{G}_n$ *then* $a \cap b = 0$*.*

Then $\mathfrak{B}$ *embeds into* $\mathrm{Clop}(2^\kappa)$*.*

Proof. Take a pairwise disjoint sequence $D(n)$ in $\mathrm{Clop}(2^\kappa)$ and for every n choose independent family $\{D_\xi(n) : \xi < \kappa\}$, where every $D_\xi(n)$ is a clopen subset of $D(n)$.

Write $\mathcal{G}_n = \{g_\xi(n) : \xi < \kappa_n\}$, where $\kappa_n \leq \kappa$. Define φ setting $\varphi(g_\xi(n)) = D_\xi(n)$ for every n and $\xi < \kappa_n$. Then φ extends to a Boolean embedding $\mathfrak{A} \to \mathrm{Clop}(2^\kappa)$ in a obvious way. □

Example 7.3. There is a dyadic compactum L of weight ω_1 and $L' \in \mathrm{CDE}(L)$ such that there is no extension operator $E : C(L) \to C(L')$ with $\|E\| < 2$.

Proof. Divide ω into three infinite sets P, Q_1, Q_2. Recall that, for $A, B \subseteq \omega$, $A \subseteq^* B$ means that the set $A \setminus B$ is finite.

On P we consider a Hausdorff gap, see e.g. [18], 29.7: take $A_\alpha, B_\alpha \subseteq P$, $\alpha < \omega_1$ such that

(a) $A_\alpha \subseteq^* A_\beta, B_\alpha \subseteq^* B_\beta$ for $\alpha < \beta < \omega_1$;
(b) $A_\alpha \cap B_\beta$ is finite for every $\alpha, \beta < \omega_1$;
(c) there is no $X \subseteq N$ satisfying $A_\alpha \subseteq^* X \subseteq^* N \setminus B_\beta$ for $\alpha, \beta < \omega_1$.

For $i = 1, 2$, we choose a family $\{C_\alpha(i) : \alpha < \omega_1\}$ of independent subsets of Q_i and define a subalgebra $\mathfrak{A}$ of $P(\omega)$ generated by *fin* and all the sets

$$G_\alpha(1) = C_\alpha(1) \cup A_\alpha, G_\alpha(2) = C_\alpha(2) \cup B_\alpha, \alpha < \omega_1.$$

By Lemma 7.2 the algebra $\mathfrak{A}/fin$ is dyadic.

Now $L' = \mathrm{ult}(\mathfrak{A})$ is a countable discrete extension of $L = \mathrm{ult}(\mathfrak{A}/fin)$. Suppose that there is an extension operator $E : C(L) \to C(L')$ such that $r = \|E\| < 2$. Take a sequence $(\nu_n)_n$ in $M(\mathfrak{A})$ as in Lemma 2.7(iii). Then $\|\nu_n\| \leq r < 2$ for every n. Take $\delta > 0$ such that $r < 2 - 2\delta$. For every $\alpha < \omega$ put

$$\widehat{A}_\alpha = \{n \in A_\alpha : \nu_n(G_\alpha(1)) > 1 - \delta\}.$$

Then $A_\alpha \subseteq^* \widehat{A}_\alpha$ since $\lim_{n \in A_\alpha} \nu_n(G_\alpha(1)) = 1$. Hence the set $X = \bigcup_{\alpha < \omega_1} \widehat{A}_\alpha$ almost contains every A_α. On the other hand, for every $\beta < \omega_1$, $B_\beta \cap X$ must be finite: otherwise, there is $n \in B_\beta \cap X$ such that $\nu_n(G_\beta(2)) > 1 - \delta$. Since $n \in X$, $n \in \widehat{A}_\alpha$ for some α so $\nu_n(G_\alpha(1)) > 1 - \delta$. But $G_\alpha(1) \cap G_\beta(2)$ is finite so $\nu_n(G_\alpha(1) \cap G_\beta(2)) = 0$. It follows that

$$\|\nu_n\| \geq \nu_n(G_\alpha(1)) + \nu_n(G_\beta(2)) > 2 - 2\delta > r,$$

contrary to our assumption.

In this way we have checked that X separates the gap, which is impossible. $\square$

It is a well-known fact from the theory of absolute retracts that a metrizable compactum M is an absolute retract, provided it is a union of two compact absolute retracts M_1, M_2 whose intersection $M_1 \cap M_2$ is also an absolute retract. It is also known that this is not the case without the metrizability assumption. Our Example 7.3 can be applied to demonstrate this.

Corollary 7.4. *Let $K = 2 \times [0,1]^{\omega_1}$, and x be a fixed point of $[0,1]^{\omega_1}$. The quotient space M obtained from K by identification of the points $(0,x)$ and $(1,x)$ is the union of two copies of $[0,1]^{\omega_1}$ intersecting at the single point, yet it is not an absolute retract.*

Proof. We adopt the notation from the proof of Example 7.3.

Observe that, since the cube $[0,1]^{\omega_1}$ is homogeneous, the space M does not depend on the choice of a point x. We can assume that $x \in \{0,1\}^{\omega_1}$. Let S be the subspace of M which is a quotient image of $2 \times \{0,1\}^{\omega_1} \subseteq K$. Using the fact that $\{0,1\}^{\omega_1}$ is a Dugundji space, we can easily obtain an extension operator $E' : C(S) \to C(M)$ of norm 1.

One can easily verify that S is homeomorphic to the space L from Example 7.3. Indeed, for $i = 1, 2$, let $\mathfrak{A}_i$ be the subalgebra of $P(\omega)$ generated by *fin* and the family of sets $G_\alpha(i)$, $\alpha < \omega_1$. These families are independent, hence $L_i = \mathrm{ult}(\mathfrak{A}_i/\mathit{fin})$ are homeomorphic to $\{0,1\}^{\omega_1}$. Since all intersections $G_\alpha(1) \cap G_\beta(2)$ are finite, we conclude that L is homeomorphic to S. Therefore, we can take $S' \in \mathrm{CDE}(S)$ such that there is no extension operator $E : C(S) \to C(S')$ with $\|E\| < 2$. We can assume that $S' \setminus S$ is disjoint from M. Let $M' = M \cup S'$. If there was a retraction $r : M' \to M$, then the assignment $f \mapsto E'(f) \circ r$ would define an extension operator from $C(S)$ to $C(S')$ of norm 1, a contradiction. $\square$

Proposition 7.5. *Let K be a compact space, such that, for some point $p \in K$, $K = \bigcup_{i=1}^n K_i$, where K_i is a Dugundji space, and $K_i \cap K_j = \{p\}$, for all $i, j \leq n, i \neq j$. Then, for any compact space L containing K, there exists an extension operator $E : C(K) \to C(L)$ with $\|E\| \leq 2n - 1$.*

Proof. For any $i \leq n$, let L_i be the quotient space obtained from L by identifying all points from $\bigcup_{j \neq i} K_j$ with the point p, and let $q_i : L \to L_i$ be the quotient map. Clearly, q_i maps K_i homeomorphically onto $q_i(K_i)$. Let $r_i : q_i(K_i) \to K_i$ be the inverse homeomorphism. By our assumption on K_i, we can find an extension operator $E_i : C(q_i(K_i)) \to C(L_i)$ of norm 1. Now, we can define the extension operator $E : C(K) \to C(L)$ by

$$E(f)(x) = \sum_{i=1}^n E_i(f|K_i \circ r_i)(q_i(x)) - (n-1)f(p),$$

for $f \in C(K)$ and $x \in L$. It is clear that, for each $f \in C(K)$, $E(f)$ is continuous on L. If $x \in K$, then $x \in K_i$, for some i, hence $q_j(x) = p$ and $E_j(f|K_j \circ r_j)(q_j(x)) = f(p)$ for $j \neq i$. Therefore $E(f)(x) = f(x)$. Obviously, we have $\|E\| \leq 2n - 1$, so E is as desired. □

From the above Proposition and the proof of Corollary 7.4 we immediately obtain the following

Corollary 7.6. *For the spaces L and L' from Example 7.3 there exists an extension operator $E : C(L) \to C(L')$ of norm 3.*

Our next example uses the concept of multigaps introduced by Avilés and Todorčević and partially builds on Theorem 29 from [2]. In what follows, we consider ideals $\mathcal{I}$ on ω containing all finite sets. Two ideals $\mathcal{I}_1, \mathcal{I}_2$ are orthogonal if $A_1 \cap A_2$ is finite for any $A_i \in \mathcal{I}_i$. Given k and a family $\mathcal{I}_1, \ldots, \mathcal{I}_k$ of mutually orthogonal ideals, they are said to constitute a k-gap if for any $X_1, \ldots, X_k \subseteq \omega$, if for every $i \leq k$ and $A \in \mathcal{I}_i$, $A \subseteq^* X_i$ then $\bigcap_{i \leq k} X_i \neq \emptyset$.

Note that a Hausdorff gap is, in particular, a 2-gap defined by ideals generated by ω_1 sets. Aviles and Todorcevic [2] proved that for every k there are k-gaps of $\mathfrak{c}$-generated ideals; on the other hand, under $\mathsf{MA}(\omega_1)$ there are no 3-gaps defined by ω_1-generated ideals.

Example 7.7. There is a dyadic compactum L of weight $\mathfrak{c}$ and $L' \in \mathrm{CDE}(L)$ such that there is no extension operator $E : C(L) \to C(L')$.

Proof. Take a partition $\omega = \bigcup_{k \geq 2} N_k$ into infinite sets. For every $k \geq 2$ divide N_k into infinite sets P_k, $Q_{k,j}$, $j \leq k$. Let $\mathcal{I}(k, j)$, $j \leq k$ be a family of mutually orthogonal ideals of subsets of P_k that constitutes a k-gap.

Fix k and $j \leq k$. Choose an independent family $\{C_\xi(k, j) : \xi < \mathfrak{c}\}$ of subsets of $Q_{k,j}$ and fix some enumeration $\{I_\xi(k, j) : \xi < \mathfrak{c}\}$ of $\mathcal{I}_{k,j}$.

We define $\mathfrak{A}$ to be an algebra of subsets of ω generated by finite sets and

$$G_\xi(k, j) = I_\xi(k, j) \cup C_\xi(k, j), \xi < \mathfrak{c}, k \geq 2, j \leq k.$$

By Lemma 7.2 $\mathfrak{A}/\mathit{fin}$ is a dyadic algebra (can be embedded into $\mathrm{Clop}(2^{\mathfrak{c}})$), so $L = \mathrm{ult}(\mathfrak{A}/fin)$ is a dyadic compactum of weight $\leq \mathfrak{c}$. We let $L' = L \cup \omega$ which is identified with $\mathrm{ult}(\mathfrak{A})$.

Suppose that there is an extension operator $E : C(L) \to C(L')$; such that $\|E\| < \infty$. Take a sequence $(\nu_n)_n$ in $M(\mathfrak{A})$ as in Lemma 7.1. Then $\|\nu_n\| \leq \|E\|$ for every n. Take $k > 2 \cdot \|E\|$.

Note that for every $j \leq k$ and $A \in \mathcal{I}(k, j)$ there is G_A such that $A \cup G_A \in \mathfrak{A}$; moreover, if $A \in \mathcal{I}(k, j)$ and $A' \in \mathcal{I}(k, j')$ with $j \neq j'$ then $G_A \cap G_{A'}$ is finite. For $A \in \mathcal{I}(k, j)$ put

$$\widehat{A} = \{n \in A : \nu_n(A \cup G_A) > 1/2\}.$$

Then $A \subseteq^* \widehat{A}$ since $\lim_{n \in A} \nu_n(A \cup G_A) = 1$. Hence the set

$$X_j = \bigcup_{A \in \mathcal{I}(k,j)} \widehat{A},$$

almost contains every $A \in \mathcal{I}(k, j)$. Since the family $\{\mathcal{I}(k, j) : j \leq k\}$ constitutes a k-gap, there is $n \in \bigcap_{j \leq k} X_j$. Then there are $A_j \in \mathcal{I}(k, j)$, $j \leq k$, such that $n \in \widehat{A}_j$ so

$\nu_n(A_j \cup G_{A_j}) > 1/2$ and $A_j \cup G_{A_j}$ are almost pairwise disjoint for different j's. Since ν_n vanishes on *fin*, this gives $\|\nu_n\| > k/2 > \|E\|$, a contradiction. □

Problem 7.8. *Can we, in ZFC, define L as in Example 7.7, but of weight ω_1?*

Correa and Tausk [9] proved that if a compact space K contains a copy of $2^{\mathfrak{c}}$, then $C(K)$ admits a nontrivial twisted sum with c_0. Gerlits and Efimov showed that every dyadic compactum K contains a copy of the Cantor cube 2^κ, for every regular cardinal number $\kappa \leq w(X)$, see [13, 3.12.12]. From these results easily follows

Theorem 7.9. *Assuming* CH, *for each nonmetrizable dyadic space K, c_0 and $C(K)$ have a nontrivial twisted sum.*

8. Linearly ordered compact spaces

The following theorem is a consequence of several known results.

Theorem 8.1. *Assuming* CH, *if L is a nonseparable linearly ordered compact space, then there is a nontrivial twisted sum of c_0 and $C(L)$.*

Proof. Recall that every measure on a linearly ordered compactum has a separable support, see [27] or [24]. Hence the space L does not support a strictly positive measure.

If L has *ccc* then it is first-countable ([14, 3.12.4]), and it follows that $|L| \leq \mathfrak{c}$ ([14, 3.12.11(d)]) so L is of weight $\omega_1 = \mathfrak{c}$. Therefore we obtain the desired conclusion by Theorem 2.8.

If L does not satisfy *ccc* then this follows from a theorem of Correa and Tausk stated in [9]. Namely, it is well-known that in such a case $C(L)$ contains an isometric copy of $c_0(\omega_1)$, and by [8, Corollary 2.7] this copy is complemented in $C(L)$. It remains to recall that there exists a nontrivial twisted sum of c_0 and $c_0(\omega_1)$, cf. [7]. □

The next result is an improvement of Theorem 7.1 from [12].

Theorem 8.2. *Let L be a separable linearly ordered compact space of weight κ such that $2^\kappa > \mathfrak{c}$. Then there is a non-tame compactification $\gamma\omega$ with remainder homeomorphic to L. Hence there is a nontrivial twisted sum of c_0 and $C(L)$.*

Corollary 8.3. *If L is a separable linearly ordered compact space of weight $\mathfrak{c}$, then there is a nontrivial twisted sum of c_0 and $C(L)$.*

Corollary 8.4. *Under* CH, *if K is a nonmetrizable linearly ordered compact space, then there is a nontrivial twisted sum of c_0 and $C(K)$.*

Note that Corollary 8.4 follows directly from Corollary 8.3 and Theorem 8.1. The rest of this section is devoted to proving Theorem 8.2.

We shall use the following well-known description of the class of separable linearly ordered compact spaces. Let A be an arbitrary subset of a closed subset K of the unit interval $I = [0,1]$. Put

$$K_A = (K \times \{0\}) \cup (A \times \{1\}),$$

and equip this set with the order topology given by the lexicographical order (i.e., $(s,i) \prec (t,j)$ if either $s < t$, or $s = t$ and $i < j$).

For $K = I$ and $A = (0,1)$ the space $\mathbb{K} = K_A$ is a well known double arrow space (some authors use this name for the space I_I).

It is known that the class of all spaces K_A coincides with the class of separable linearly ordered compact spaces. Namely, the following is a reformulation of the characterization due to Ostaszewski [26]:

Theorem 8.5 (Ostaszewski)**.** *The space L is a separable compact linearly ordered space if and only if L is homeomorphic to K_A for some closed set $K \subseteq I$ and a subset $A \subseteq K$.*

The next lemma seems to belong to the mathematical folklore, we include a short justification for the readers convenience.

Lemma 8.6. *Let L be a separable linearly ordered compact space of uncountable weight κ. Then L contains a topological copy of the space I_B, where B is a dense subset of $(0,1)$ of the cardinality κ.*

Proof. By Theorem 8.5 we can assume that $L = K_A$ for some closed $K \subseteq I$ and some subset A of K. From our assumption on the weight of K it easily follows that $|A| = \kappa$. Take a dense-in-itself subset C of A of cardinality κ. Let M be the closure of C in K and let $a = \inf M, b = \sup M$. Put $D = M \cap A \cap (a,b)$. Obviously, D is a dense subset of M of the cardinality κ and the space M_D is a subspace of K_A. Let $\{(a_n, b_n) : n < m\}$ be an enumeration of the family of all components of $[a,b] \setminus M$ for some $m \leq \omega$. Put

$$P = M_D \setminus (\{(a_n, 1) : a_n \in D\} \cup \{(b_n, 0) : b_n \in D\}) .$$

Then P is a closed, dense-in-itself subspace of M_D. Let $\sim$ be the equivalence relation on M defined by $a_n \sim b_n$ for $n < m$, and let $q : M \to M_{/\sim}$ be the quotient map. The space $S = M_{/\sim}$ is compact, linearly ordered, connected, and metrizable, hence there is a homeomorphism $h : M_{/\sim} \to I$ with $h(q(a)) = 0$. One can easily verify that P can be identified with I_B where $B = h(q(D \cup \{a_n : n < m\}))$. $\square$

Theorem 8.7. *Let B be a dense subset of $(0,1)$ of the cardinality κ such that $2^\kappa > \mathfrak{c}$. Then there is a non-tame compactification $\gamma\omega$ which remainder is homeomorphic to I_B.*

Proof. Let Q be a countable dense subset of $(0,1)$. For each $x \in B$ put $P_x = \{q \in Q : q \leq x\}$ and pick a strictly increasing sequence $(q_x^n)_{n\in\omega}$ in Q such that $\lim_n q_x^n = x$. Let $S_x = \{q_x^n : n \in \omega\}$. For any $f : B \to 2$ define

$$R_x^f = \begin{cases} P_x & \text{if } f(x) = 0, \\ P_x \setminus S_x & \text{if } f(x) = 1. \end{cases}$$

Let $\mathcal{A}^f$ be a subalgebra of $P(Q)$ generated by $\{R_x^f : x \in B\} \cup$ *fin*, where *fin* denotes the family of all finite subsets of Q. We shall check that, for any f, the Stone space $\mathrm{ult}(\mathcal{A}^f)$ is a compactification of a countable discrete space with remainder homeomorphic to I_B.

For any $q \in Q$ let u^f_q denote the ultrafilter in $\mathrm{ult}(\mathcal{A}^f)$ containing $\{q\}$. Let $r^f : \mathcal{A}^f \to \mathcal{A}^f/\mathit{fin}$ be the quotient map. It is well-known that $\mathrm{ult}(\mathcal{A}^f)$ is a compactification of its countable discrete subspace $\{u^f_q : q \in Q\}$ and its remainder can be identified with the space $\mathrm{ult}(\mathcal{A}^f/\mathit{fin})$. Observe that $\mathcal{A}^f/\mathit{fin}$ is generated by the family $\{r^f(R^f_x) : x \in B\}$. For $x, y \in B$, $x < y$ we have $P_x \subseteq P_y$, the difference $P_y \setminus P_x$ is infinite, and the intersection $P_x \cap S_y$ is finite. Therefore

$$R^f_x \subsetneq^* R^f_y \Leftrightarrow x < y \text{ for } x, y \in B \text{ and } f \in 2^B.$$

Let U be an ultrafilter in $\mathcal{A}^f/\mathit{fin}$. The set $T_U = \{x \in B : r(R^f_x) \in U\}$ is a final segment in $(B, <)$, hence, either

$$\begin{aligned} \exists z \in I \setminus B \quad T_U &= (z,1) \cap B \quad \text{or} \\ \exists y \in B \quad T_U &= [y,1) \cap B \quad \text{or} \\ \exists y \in B \quad T_U &= (y,1) \cap B\,. \end{aligned}$$

The ultrafilter U is uniquely determined by the set T_U. For $z \in I \setminus B$ let U^f_z be the ultrafilter in $\mathcal{A}^f/\mathit{fin}$ such that $T_{U^f_z} = (z,1) \cap B$. For $y \in B$ let $U^f_{y,0}$, $U^f_{y,1}$ be the ultrafilters such that $T_{U^f_{y,0}} = [z,1) \cap B$, $T_{U^f_{y,1}} = (z,1) \cap B$.

A routine verification shows that the map $\varphi^f : \mathrm{ult}(\mathcal{A}^f/\mathit{fin}) \to I_B$ given by

$$\varphi^f(U) = \begin{cases} (z,0) & \text{if} \quad U = U^f_z, \quad z \in I \setminus B, \\ (y,i) & \text{if} \quad U = U^f_{y,i}, \quad y \in B, i = 0,1 \end{cases}$$

is a homeomorphism. The map $\psi^f : \mathrm{ult}(\mathcal{A}^f/\mathit{fin}) \to \mathrm{ult}(\mathcal{A}^f)$, given by $\psi^f(U) = (r^f)^{-1}(U)$, for $U \in \mathrm{ult}(\mathcal{A}^f/\mathit{fin})$ is a homeomorphic embedding. Let

$$u^f_z = \psi^f(U^f_z), u^f_{y,i} = \psi^f(U_{y,i}) \text{ for } z \in I \setminus B, y \in B, i = 0,1.$$

Observe that if $f(x) = 0$, then $S_x \subseteq R^f_x$, otherwise $S_x \subseteq Q \setminus R^f_x$, hence S_x is contained in the element of the ultrafilter $u^f_{y,f(x)}$. It follows that the sequence $(u^f_{q^n_x})_{n\in\omega}$ converges to $u^f_{y,f(x)}$ in $\mathrm{ult}(\mathcal{A}^f)$. The space $M(I_B)$ has the cardinality $\mathfrak{c}$, which follows for instance from the fact that every probability measure on I_B has a uniformly distributed sequence, see Mercourakis [24]. Hence the family $\mathcal{E}$ of all maps $e : Q \to M(I_B)$ has the same cardinality.

Suppose that for all $f \in 2^B$ there is an extension operator

$$T^f : C(\psi^f(\mathrm{ult}(\mathcal{A}^f/\mathit{fin}))) \to C(\mathrm{ult}(\mathcal{A}^f)).$$

Consequently, by Lemma 2.7 there exists a continuous map $g^f : \mathrm{ult}(\mathcal{A}^f) \to M(\psi^f(\mathrm{ult}(\mathcal{A}^f/\mathit{fin})))$ such that, for any $U \in \mathrm{ult}(\mathcal{A}^f/\mathit{fin})$, $g^f(\psi^f(U)) = \delta_{\psi^f(U)}$. By continuity of g^f we have

$$\lim_n g^f(u^f_{q^n_x}) = \delta u^f_{y,f(x)},$$

for all $x \in B$. Let

$$\phi^f : M(\psi^f(\mathrm{ult}(\mathcal{A}^f/\mathit{fin}))) \to M(I_B),$$

be the isometry induced by the embedding ψ^f and the homeomorphism φ^f. Let $e^f : Q \to M(I_B)$ be defined by $e^f(q) = \phi^f(g^f(u^f_q))$. Then, for any $x \in B$, the sequence $(e^f(q^n_x))_{n\in\omega}$

converges to $\delta_{(x,f(x))}$. It follows that the assignment $f \mapsto e^f$ is an injection of 2^B into $\mathcal{E}$, a contradiction. □

Theorem 8.7 immediately implies the following.

Corollary 8.8. *Let $\mathbb{K}$ be the double arrow space. There is a non-tame compactification $\gamma\omega$ which remainder is homeomorphic to $\mathbb{K}$. Hence there is a nontrivial twisted sum of c_0 and $C(\mathbb{K})$.*

Let us remark that if K is a closed subset of a linearly ordered compact space L, then there is a regular extension operator $E : C(K) \to C(L)$, cf. [17]. Using this fact one can easily deduce Theorem 8.2 from Theorem 8.7, Lemma 8.6, Remark 2.5 and Proposition 2.6.

9. On scattered compact spaces

We start this section by presenting one more construction of non-tame compactifications of ω, based on an idea similar to that used in the proof of Theorem 8.7. For a compact space K we denote by $\mathrm{Auth}(K)$ the group of autohomeomorphisms of K.

Theorem 9.1. *Let L be a compact space such that*

(i) $|M(L)| = \mathfrak{c}$,
(ii) L contains a continuous image K of $\omega^* = \beta\omega \setminus \omega$ such that $|\mathrm{Auth}(K)| > \mathfrak{c}$.

Then there exists a countable discrete extension L' of L such that there is no extension operator $E : C(L) \to C(L')$; in particular, there is a nontrivial twisted sum of $C(L)$ and c_0.

Moreover, if $L = K$ then we may additionally require that L' is a non-tame compactification of ω which remainder is homeomorphic to L.

Proof. The assumption that K is a continuous image of ω^* is equivalent to the existence of a compactification $\gamma\omega$ of ω with the remainder $\gamma\omega \setminus \omega$ homeomorphic to K. We can assume that $\gamma\omega$ and L are disjoint. Let $S = L \cup \gamma\omega$ be the disjoint union of L and $\gamma\omega$. Consider the family $\mathcal{H}$ of all homeomorphisms $\varphi : \gamma\omega \setminus \omega \to K$. By our assumption $|\mathcal{H}| > \mathfrak{c}$. For any $\varphi \in \mathcal{H}$, let $\sim_\varphi$ be the equivalence relation on S given by $x \sim_\varphi \varphi(x)$ for all $x \in \gamma\omega \setminus \omega$ and let $q_\varphi : S \to S/_{\sim_\varphi}$ be a corresponding quotient map. Clearly $q_\varphi(L)$ is homeomorphic to L and $S/_{\sim_\varphi}$ is a countable discrete extension of $q_\varphi(L)$.

Suppose that for all $\varphi \in \mathcal{H}$ there is an extension operator

$$T_\varphi : C(q_\varphi(L)) \to C(S/_{\sim_\varphi}).$$

Consequently, by Lemma 2.7 for every such φ there exists a continuous map

$$h_\varphi : S/_{\sim_\varphi} \to M(q_\varphi(L)),$$

satisfying $h_\varphi(y) = \delta_y$ for every $y \in q_\varphi(L)$. Let $g_\varphi : M(q_\varphi(L)) \to M(L)$ be the isometry induced by the homeomorphism $q_\varphi|L : L \to q_\varphi(L)$. Define $e_\varphi : \omega \to M(L)$ by $e_\varphi(n) = g_\varphi(h_\varphi(q_\varphi(n)))$ for $n \in \omega$. Observe that the family of all maps from ω to $M(L)$ has the cardinality $\mathfrak{c}$, since $|M(L)| = \mathfrak{c}$. We will get the desired contradiction by showing that the assignment $\varphi \mapsto e_\varphi$ is one-to-one. Fix distinct $\varphi_0, \varphi_1 \in \mathcal{H}$, and take $x \in \gamma\omega \setminus \omega$ such that $\varphi_0(x) \neq \varphi_1(x)$. Pick $f \in C(L)$ with $f(\varphi_i(x)) = i$, $i = 0, 1$. Since $q_{\varphi_i}(x) = q_{\varphi_i}(\varphi_i(x)))$, we have $g_{\varphi_i}(h_{\varphi_i}(q_{\varphi_i}(x)))(f) = i$. By continuity of h_{φ_i} we can find a neighborhood U_i of x in $\gamma\omega$ such that

$$g_{\varphi_0}(h_{\varphi_0}(q_{\varphi_0}(z)))(f) < 1/2 \text{ for } z \in U_0;$$

$$g_{\varphi_1}(h_{\varphi_1}(q_{\varphi_1}(z')))(f) > 1/2 \text{ for } z' \in U_1.$$

Now, for any $n \in \omega$ such that $n \in U_0 \cap U_1$, we have $e_{\varphi_0}(n)(f) < 1/2 < e_{\varphi_1}(n)(f)$. □

We shall now consider the well-known class of compact spaces associated with uncountable almost disjoint families of subsets of ω — separable compact spaces K whose set of accumulation points is the one-point compactification of an uncountable discrete space. Such compact spaces were considered first by Aleksandrov and Urysohn [1] and for that reason we call them AU-compacta, cf. [23]. It is worth recalling that the space $C(K)$ for an AU-compactum K may have interesting structural properties, see Koszmider [21].

In section 5 we considered an AU-compactum described as the Stone space of the algebra of subsets of ω generated by finite sets and a given almost disjoint family. Below we use the following description of AU-compacta.

Let D be a countable set and let $\mathcal{A}$ be an uncountable almost disjoint family of infinite subsets of D, i.e. the intersection of any two distinct members of $\mathcal{A}$ is finite. Let $A \mapsto p_A$ be a one-to-one correspondence between members of $\mathcal{A}$ and points in some fixed set disjoint from D, an let ∞ be a point distinct from points in D and any point p_A. In the set

$$K_{\mathcal{A}} = D \cup \{p_A \colon A \in \mathcal{A}\} \cup \{\infty\}$$

we introduce a topology declaring that points of D are isolated, basic neighborhoods p_A are of the form $\{p_A\} \cup (A \setminus F)$, where $F \subset A$ is finite, and ∞ is the point at infinity of the locally compact space $D \cup \{p_A \colon A \in \mathcal{A}\}$.

From Theorem 9.1 one can easily derive the following

Corollary 9.2. *Let $\mathcal{A}$ be an almost disjoint family of subsets of ω of cardinality κ, where $2^\kappa > \mathfrak{c}$. Then there exists a non-tame compactification $\gamma\omega$ which remainder is homeomorphic to $K_{\mathcal{A}}$. Hence there is a nontrivial twisted sum of c_0 and $C(K_{\mathcal{A}})$.*

Proof. First, recall that every measure from $M(K_{\mathcal{A}})$ is purely atomic, hence $|M(K_{\mathcal{A}})| = \mathfrak{c}$. Second, observe that the subspace $K = \{p_A \colon A \in \mathcal{A}\} \cup \{\infty\}$ of $K_{\mathcal{A}}$ is homeomorphic to a one point compactification of a discrete space of cardinality κ, therefore $|\mathrm{Auth}(K)| = 2^\kappa > \mathfrak{c}$. Next, notice that K is a continuous image of ω^*, since $K_{\mathcal{A}}$ is a compactification of ω with remainder K. Finally, we can obtain the desired non-tame compactification using Theorem 9.1 and Proposition 2.6. □

Recall that a space X is *scattered* if no nonempty subset $A \subseteq X$ is dense-in-itself. For an ordinal α, $X^{(\alpha)}$ is the αth Cantor-Bendixson derivative of the space X. For a scattered space X, the scattered height $ht(X) = \min\{\alpha : X^{(\alpha)} = \emptyset\}$.

Using Theorem 9.1 we can also provide an alternative, more topological proof of the following result of Castillo.

Theorem 9.3 (Castillo [7]). *Under* CH*, if K is a nonmetrizable scattered compact space of finite height, then there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Our argument is based on the following two auxiliary facts.

Proposition 9.4. *Each nonmetrizble scattered compact space K contains a nonmetrizble retract of cardinality at most $\mathfrak{c}$.*

Proof. Consider the family of all uncountable (equivalently, nonmetrizable) clopen subspaces of K, and pick such a subspace L of minimal height α. By compactness of L, α is a successor ordinal, i.e., $\alpha = \beta + 1$. The set $L^{(\beta)}$ is finite, therefore we can partition L into finitely many clopen sets containing exactly one point from $L^{(\beta)}$. One of these sets must be uncountable, hence, without loss of generality we can assume that $L^{(\beta)} = \{p\}$. For every $x \in L \setminus \{p\}$ fix a clopen neighborhood U_x of x in L such that $p \notin U_x$. Clearly, the height of U_x is less than α, so, by our choice of L, U_x must be countable, hence metrizable. Since every point of $L \setminus \{p\}$ has a metrizable neighborhood it follows that

$$(\forall A \subset L)\,(\forall x \in \overline{A} \setminus \{p\})\,(\exists (x_n)) \quad x_n \in A \text{ and } x_n \to x\,. \tag{9.1}$$

For any subset $A \subseteq L \setminus \{p\}$ define

$$\varphi(A) = \overline{\bigcup \{U_x : x \in A\}} \setminus \{p\}\,.$$

Observe that, by $|U_x| \le \omega$ and (9.1), we have

$$|\varphi(A)| \le \mathfrak{c}, \text{ provided } |A| \le \mathfrak{c}\,. \tag{9.2}$$

Fix any subset $A \subseteq L \setminus \{p\}$ of cardinality ω_1. We define inductively, for any $\alpha < \omega_1$, sets $A_\alpha \subseteq L \setminus \{p\}$. We start with $A_0 = A$, and at successor stages we put $A_{\alpha+1} = \varphi(A_\alpha)$. If α is a limit ordinal we define $A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$. Finally we take $B = \bigcup\{A_\alpha : \alpha < \omega_1\}$. From (9.2) we conclude that $|B| \le \mathfrak{c}$. First, observe that B is open in L, since, for any $x \in B$, x belongs to some A_α, and then $U_x \subseteq A_{\alpha+1} \subseteq B$. Second, the union $M = B \cup \{p\}$ is closed in L. Indeed, if $x \in \overline{B} \setminus \{p\}$, then by (9.1), there is a sequence (x_n) in B converging to x. We have $\{x_n : n \in \omega\} \subseteq A_\alpha$, for some $\alpha < \omega_1$, therefore $x \in A_{\alpha+1} \subseteq B$. Now, we can define a retraction $r : L \to M$ by

$$r(x) = \begin{cases} x & \text{for } x \in M, \\ p & \text{for } x \in L \setminus M\,. \end{cases}$$

Then r is continuous since it is continuous on closed sets M and $L \setminus B$. It remains to observe that M is also a retract of K, since L is a retract of K. □

Lemma 9.5. *Every nonmetrizable scattered compact space K of finite height contains a copy of a one point compactification of an uncountable discrete space.*

Proof. Let $n+1$ be the height of K. Using the same argument as at beginning of the proof of Theorem 9.4, without loss of generality, we can assume that $K^{(n)} = \{p\}$ and every $x \in K \setminus \{p\}$ has a countable clopen neighborhood U_x in K. Let $k = \max\{i : |K^{(i)}| > \omega\}$. Consider

$$A = K^{(k)} \setminus \big(\bigcup\{U_x : x \in K^{(k+1)} \setminus \{p\}\} \cup \{p\}\big)\,.$$

Observe that by our choice of k, the set A is uncountable. One can easily verify that the set A is discrete and p is the unique accumulation point of A. Therefore $L = A \cup \{p\}$ is a one point compactification of an uncountable discrete space. □

Proof of Theorem 9.3. Let K be a nonmetrizable scattered compact space of finite height, and let L be a a nonmetrizable retract of K of cardinality at most $\mathfrak{c}$, given by Proposition 9.4. Obviously, in the presence of continuum hypothesis, we have $|L| = \mathfrak{c}$. Since L is a retract of K it is enough to justify the existence of a nontrivial twisted sum of c_0 and $C(L)$. Take a copy S in L of a one point compactification of an uncountable discrete space, given by Lemma 9.5. Obviously, we have $|S| = \mathfrak{c}$. Since L is scattered, every measure in $M(L)$ is purely atomic, hence $|M(L)| = \mathfrak{c}$. We also have $|\mathrm{Auth}(S)| = 2^{\mathfrak{c}}$, so we can apply Theorem 9.1 as in the proof of Corollary 9.2. □

Clearly, a compact scattered space supports measure if and only if it is separable. Therefore we have the following easy consequence of Lemma 2.8.

Corollary 9.6. *If K is a nonseparable scattered compact space of weight ω_1, then there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Theorem 9.3, Corollary 9.2, and Corollary 9.6 should be compared with the following direct consequence of Corollary 5.3.

Theorem 9.7. *Assume $\mathsf{MA}(\kappa)$ and let K be a separable scattered compact space of height 3 and weight κ. Then every twisted sum of c_0 and $C(K)$ is trivial.*

Proof. It is well-known that each infinite scattered compact space K contains a nontrivial convergent sequence, and hence $C(K)$ contains a complemented copy of c_0. Consequently, for any $n \in \omega$, the space $C(K)$ is isomorphic with $C(K) \oplus \mathbb{R}^n$.

If K is a separable scattered compact space of height 3, then the quotient space L obtained from K by gluing together all points in $K^{(2)}$ is an AU-compactum. Let $|K^{(2)}| = n$. A standard factorization argument shows that $C(K)$ is isomorphic to $C(L) \oplus \mathbb{R}^{n-1}$, hence it is isomorphic to $C(L)$, and we can apply Corollary 5.3. □

Recall that two families $\mathcal{A}$ and $\mathcal{B}$ of infinite subsets of ω are *separated* if there exists $S \subseteq \omega$ such that $A \subseteq^* S$, for each $A \in \mathcal{A}$, and $B \subseteq^* \omega \setminus S$ for each $B \in \mathcal{B}$. N.N. Luzin constructed (in ZFC) an almost disjoint family $\mathcal{L}$ of subsets of ω of cardinality ω_1 such that no two disjoint uncountable subfamilies of $\mathcal{L}$ are separated, see [11, Theorem 4.1] and [29] and references therein.

Proposition 9.8. *Let $\mathcal{A}$ be an almost disjoint family of subsets of ω which contains two separated disjoint uncountable subfamilies. Then there exists an $L \in \mathrm{CDE}(K_{\mathcal{A}})$ such that there is no extension operator $E : C(K_{\mathcal{A}}) \to C(L)$ with $\|E\| < 2$.*

Proof. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be disjoint uncountable subfamilies of $\mathcal{A}$ separated by a set $S \subseteq \omega$. Without loss of generality we may assume that $\mathcal{A}_i$ have the cardinality ω_1, so we can enumerate $\mathcal{A}_0 \cup \mathcal{A}_1$ as $\{A_\alpha : \alpha < \omega_1\}$. Let $\mathcal{L} = \{L_\alpha : \alpha < \omega_1\}$ be the Luzin almost disjoint family of subsets of a countable set ω'. We assume that the AU-compacta

$$K_{\mathcal{A}} = \omega \cup \{p_A \colon A \in \mathcal{A}\} \cup \{\infty\} \text{ and } K_{\mathcal{L}} = \omega' \cup \{r_L \colon L \in \mathcal{L}\} \cup \{\infty'\}$$

are disjoint. To simplify the notation we denote p_{A_α} by p_α and r_{L_α} by r_α for $\alpha < \omega_1$. Let L' be the disjoint union of $K_{\mathcal{A}}$ and $K_{\mathcal{L}}$ and L be the quotient space obtained from L' by the identification of p_α with r_α, for all $\alpha < \omega_1$, and ∞ with ∞'. Let $q : L' \to L$ be the quotient map. Clearly $q(K_{\mathcal{A}})$ is a topological copy of $K_{\mathcal{A}}$.

Suppose that there exists an extension operator

$$E : C(q(K_{\mathcal{A}})) \to C(L) \text{ with } \|E\| = a < 2.$$

Then by Lemma 2.7 there is a sequence $(\nu_p)_p$ in $M(q(K_{\mathcal{A}}))$ such that $\|\nu_p\| \le a$ for every $p \in \omega'$ and $\nu_p - \delta_{q(p)} \to 0$ in the *weak** topology of $M(L)$. Let

$$\Gamma = \{\alpha < \omega_1 : \nu_p(\{q(p_\alpha)\}) \ne 0 \text{ for some } p \in \omega'\}.$$

Obviously, the set Γ is countable. We put

$$T = \{p \in \omega' : |\nu_p|(S) > a/2\}, \quad T' = \{p \in \omega' : |\nu_p|(\omega \setminus S) > a/2\};$$

$$\mathcal{L}_i = \{L_\alpha : A_\alpha \in \mathcal{A}_i,\ \alpha \in \omega_1 \setminus \Gamma\} \text{ for } i = 0, 1.$$

We will obtain the desired contradiction by showing that the set T separates uncountable subfamilies $\mathcal{L}_i$ of $\mathcal{L}$. First, fix some $L_\alpha \in \mathcal{L}_0$. Take a finite set $F \subseteq \omega$ such that $A_\alpha \setminus F \subseteq S$. Note that the set $C = (A_\alpha \setminus F) \cup \{p_\alpha\}$ is clopen in $K_{\mathcal{A}}$ and the set $D = L_\alpha \cup \{r_\alpha\}$ is clopen in $K_{\mathcal{L}}$. Therefore the characteristic function f of $q(C \cup D)$ is continuous on L. For all $p \in L_\alpha$, we have $\delta_{q(p)}(f) = 1$, so $(\nu_p)_p(f) \to 1$. Since $\nu_p(\{q(p_\alpha)\}) = 0$, $\nu_p(A_\alpha \setminus F) > a/2$ for almost all $p \in L_\alpha$. It follows that $L_\alpha \subseteq^* T$. In the same way one can show that, for all $L_\alpha \in \mathcal{L}_1$, $L_\alpha \subseteq^* T'$. It remains to observe that the assumption that $\|\nu_p\| \le a$ implies that T' and T are disjoint. ☐

10. Remarks and open problems

Let us recall that a compact space is *Eberlein compact* if K is homeomorphic to a weakly compact subset of a Banach space. There are well-studied much wider classes of Corson and Valdivia compacta.

Given a cardinal number κ, the Σ-product $\Sigma(\mathbb{R}^\kappa)$ of real lines is the subspace of $\mathbb{R}^\kappa$ consisting of functions with countable supports. A compactum K is *Corson compact* if it can be embedded into some $\Sigma(\mathbb{R}^\kappa)$; K is *Valdivia compact* if for some κ there is an embedding $g : K \to \mathbb{R}^\kappa$ such that $g(K) \cap \Sigma(\mathbb{R}^\kappa)$ is dense in the image, see Negrepontis [25] and Kalenda [19].

It is known that if K is a nonmetrizable Eberlein compact space then c_0 admits a nontrivial twisted sum with $C(K)$, see [7]. The following generalization can be found in [7] and [9].

Theorem 10.1. *If K is a Valdivia compact space which does not satisfy the countable chain condition then c_0 admits a nontrivial twisted sum with $C(K)$.*

Let us note that 10.1 can be demonstrated as follows. If K is Valdivia compact without *ccc* then there is a retraction of K onto its subspace which has the weight ω_1 and still does not satisfy *ccc*. Then one can apply Theorem 2.8. This suggests the following question.

Problem 10.2. *Let K be Valdivia compact that does not support a measure. Does there exist a nontrivial twisted sum of c_0 and $C(K)$?*

The main obstacle here is that we do not know if every Valdivia compact space not supporting a measure has a retract of weight ω_1 which does not support a measure either.

We also recall a related class of compact spaces: a compactum K is *Gul'ko compact* if $C(K)$ equipped with the weak topology is countably determined, i.e., is the continuous image of a closed subset of a product of some subset S of the irrationals P and a compact space (cf. [25]). We have the following relations between the classes of compacta mentioned above

$$\text{metrizable} \Rightarrow \text{Eberlein} \Rightarrow \text{Gul'ko} \Rightarrow \text{Corson} \Rightarrow \text{Valdivia}$$

and none of the above implications can be reversed, cf. [25]. Since each Gul'ko compact space satisfying *ccc* is metrizable (cf. [25, 6.40]), Theorem 10.1 yields

Proposition 10.3. *For every nonmetrizable Gul'ko compact space K, there exists a nontrivial twisted sum of c_0 and $C(K)$.*

Correa and Tausk [9] proved that, assuming CH, the above result can be extended to the class of Corson compact space. It is well-known that under MA and the negation of CH, every Corson compact satisfying *ccc* is metrizable. Hence, using again 10.1, we can state the theorem of Correa and Tausk in a slightly stronger way:

Theorem 10.4 (Correa and Tausk)**.** *Assuming MA, for every nonmetrizable Corson compact space K, there exists a nontrivial twisted sum of c_0 and $C(K)$.*

It is natural to ask whether we can prove the above theorem in ZFC.

Let us, finally, summarize the open problems mentioned in the previous sections. On one hand, we were not able to prove in section 5 that under $\mathsf{MA}(\omega_1)$ no separable compactum K of weight ω_1 admits a nontrivial twisted sum of c_0 and $C(K)$, see Problem 5.4. On the other hand, our attempts at giving a ZFC construction of a separable compact space K of weight ω_1 and its countable discrete extension L admitting no extension operator failed for some combinatorial reasons, see Problem 7.8 and the assumption in Theorem 8.7. In all the cases we have considered one can construct such a pair K and L that there is no extension operator $E : C(K) \to C(L)$ of small norm. However, at each instance we needed some additional set-theoretic assumption to kill all the possible extension operators, see e.g. Proposition 9.8 and Theorem 9.1. Therefore the following question is worth considering.

Problem 10.5. *Does there exist a model of set theory in which every twisted sum of c_0 and $C(K)$ is trivial whenever K is a separable compactum of weight ω_1?*

Appendix A. Bounded common extensions

We discuss here some consequences of a result due to Basile, Rao and Shortt [4] on common extensions of finitely additive signed measures. Let $\mathfrak{B}$ be a Boolean algebras of subsets of X and $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{B}$ its two subalgebras. We consider $\nu_i \in M(\mathfrak{B}_i)$, $i = 1, 2$, where the measures ν_1, ν_2 are consistent, that is $\nu_1(B) = \nu_2(B)$ for every $B \in \mathfrak{B}_1 \cap \mathfrak{B}_2$.

Let η be a function defined on $\mathfrak{B}_1 \cup \mathfrak{B}_2$ by $\eta(B) = \nu_1(B)$ for $B \in \mathfrak{B}_1$ and $\eta(B) = \nu_2(B)$ for $B \in \mathfrak{B}_2$. We define

$$SC(\nu_1, \nu_2) = \sup \left\{ \sum_{i=1}^{n} |\eta(B_{i+1}) - \eta(B_i)| \right\},$$

where the supremum is taken over all $n \geq 0$ and all increaing chains $\emptyset = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_{n+1} = X$ from $\mathfrak{B}_1 \cup \mathfrak{B}_2$.

Theorem 1.5 from [4] asserts that there is a common extension of ν_1, ν_2 to a measure $\lambda \in M(\langle \mathfrak{B}_1 \cup \mathfrak{B}_2 \rangle)$ such that $\|\lambda\| = SC(\nu_1, \nu_2)$. Clearly, we can extend such λ to $\mathfrak{B}$ preserving its norm.

Lemma A.1. *Let $\mathfrak{B}$ be a finite algebra having N atoms. Suppose that $\mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathfrak{B}$ are subalgebras, $\nu_i \in M(\mathfrak{B}_i)$ for $i = 1, 2$ are two consistent measures, and $\delta > 0$.*

(a) If $|\nu_i(B)| < \delta$ for $B \in \mathfrak{B}_i$, $i = 1, 2$, then there is a common extension of ν_1, ν_2 to $\lambda \in M(\mathfrak{B})$ such that $\|\lambda\| \leq 2N\delta$.

(b) If $\lambda \in M(\mathfrak{B})$ is such a measure that $|\lambda(B) - \nu_i(B)| < \delta$ for $B \in \mathfrak{B}_i$, $i = 1, 2$, then there is a common extension of ν_1, ν_2 to $\lambda' \in M(\mathfrak{B})$ such that $\|\lambda - \lambda'\| \leq 2N\delta$.

(c) In the setting of (b), if moreover ν_1, ν_2 and λ have rational values then there is such λ' that also assumes only rational values.

Proof. To check (a) it is enough to notice that if $B_0, \ldots, B_{n+1}$ is a strictly increasing chain in $\mathfrak{B}_1 \cup \mathfrak{B}_2$ then $n + 1 \leq N$ so clearly $SC(\nu_1, \nu_2) \leq 2N\delta$.

To get (b) we can apply (a) to the measures $\nu_1' = \nu_1 - \lambda$ and $\nu_2' = \nu_2 - \lambda$ considered on $\mathfrak{B}_1$ and $\mathfrak{B}_2$, respectively.

For (c) we may also assume that $\delta \in \mathbb{Q}$. By (b) the set

$$E = \{\mu \in M(\mathfrak{B}) : \mu \text{ extends } \nu_1, \nu_2 \text{ and } \|\mu - \lambda\| \leq 2N\delta\},$$

is nonempty. The set E may be identified with a symplex in $\mathbb{R}^N$ defined by equations and inequalities with rational coefficients. Hence any extreme point of E has rational coefficients and defines the required measure λ'. $\square$

Lemma A.2. *Let $\mathfrak{B}$ be a finite algebra. For any subalgebra $\mathfrak{C} \subseteq \mathfrak{B}$ and a measure $\nu \in M(\mathfrak{C})$, if, for some $\delta > 0$, there is $\lambda \in M_1(\mathfrak{B})$ such that $|\lambda(C) - \nu(C)| < \delta$ for $C \in \mathfrak{C}$ then* *there is an extension of* ν *to* $\mu \in M(\mathfrak{B})$ *such that* $\|\mu\| \leq \max(1, \|\nu\|)$ *and* $|\mu(B) - \lambda(B)| \leq 3\delta$ *for every* $B \in \mathfrak{B}$*.*

If, moreover, ν *and* λ *have rational values then there is such* μ *with rational values.*

Proof. Note first that for any $\nu_1 \in M(\mathfrak{C})$, if $|\nu_1(C)| < \delta$ for every $C \in \mathfrak{C}$ then there is an extension $\widetilde{\nu_1} \in M(\mathfrak{B})$ of ν_1 such that $|\widetilde{\nu_1}(B)| < \delta$ for every $B \in \mathfrak{B}$. Indeed, we can define such $\widetilde{\nu_1}$ by the following procedure: If C is an atom of $\mathfrak{C}$ then choose any atom B of $\mathfrak{B}$ contained in C and set $\widetilde{\nu_1}(B) = \nu_1(C)$ and $\widetilde{\nu_1}(B_1) = 0$ for every $B_1 \in \mathfrak{B}$ contained in $C \setminus B$. Note also that then $\widetilde{\nu_1}$ satisfies $\|\widetilde{\nu_1}\| < 2\delta$.

We can now apply the preceding remark to $\nu_1 = \nu - \lambda$ considered on $\mathfrak{C}$ to get $\widehat{\nu_1}$ as above. Then the measure $\lambda' = \widehat{\nu_1} + \lambda$ extends ν and satisfies $\|\lambda'\| < \|\lambda\| + 2\delta \leq 1 + 2\delta$.

Now it is enough to check that we can appropriately lower the size of $\|\lambda'\|$. Consider first some atom C of $\mathfrak{C}$ and let B^+ be the union of all atoms B of $\mathfrak{B}$ contained in C for which $\lambda'(B) > 0$; set $B^- = C \setminus B^+$. Note that if $t \leq \min(|\lambda'(B^+), |\lambda'(B^-)|$ then we can modify λ' on C, assigning the value $\lambda'(B^+) - t$ to B^+ and $\lambda'(B^-) + t$ to B^-; this defines an extension of ν of norm $\|\lambda'\| - 2t$.

Let now $C_1, \ldots, C_m$ be the list of all atoms; we divide every C_j into B_j^+ and B_j^- as described above. Let

$$p = \sum_{j \leq m} \min\big(\lambda'(B_j^+), |\lambda'(B_j^-)|\big).$$

If $p \geq \delta$, by the procedure described above we shall get a measure μ extending ν with $\|\mu\| \leq 1$. Namely, we then choose numbers nonnegative $t_j \leq \min\big(|\lambda'(B_j^+), |\lambda'(B_j^-)|\big)$ such that $\sum_{j \leq m} t_j = \delta$, and apply the modification by t_j to C_j. If $p < \delta$ then the same procedure will give μ such that μ is either nonnegative or nonpositive on each C_i. In such a case

$$\|\mu\| = \sum_{j \leq m} |\mu(C_j)| = \sum_{j \leq m} |\nu(C_j)| = \|\nu\|.$$

In both cases we shall have $|\mu(B) - \lambda'(B)| \leq 2\delta$ for any $B \in \mathfrak{B}$ so μ will be the required measure.

For the final statement just note that we can assume that $\delta \in \mathbb{Q}$; the above argument shows that in such a case λ' and μ have values in $\mathbb{Q}$. $\square$

In the last auxiliary result we consider the the following set in a Euclidean space:

$$T(a,b) = \left\{ x \in \mathbb{R}^m \times \mathbb{R}^n : \sum_{j \leq n} x_{ij} = a_i \text{ for } i \leq m, \sum_{i \leq m} x_{ij} = b_j \text{ for } j \leq n \right\}.$$

Lemma A.3. *For and* $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$ *such that* $\sum_{i \leq m} a_i = \sum_{j \leq n} b_j$ *there is* $x \in T(a,b)$ *satisfying*

$$\sum_{i,j} |x_{ij}| \leq \max\big(\sum_{i \leq m} |a_i|, \sum_{j \leq n} |b_j|\big).$$

Proof. The assertion is clearly true of either $m = 1$ or $n = 1$. We argue by induction on $m + n$.

Since $\sum_{i\le m} a_i = \sum_{j\le n} b_j$ there are i and j such that a_i and b_j have the same sign. Suppose e.g. that this is the case for $i=j=1$. Moreover, let us assume that $0\le a_1\le b_1$; the other case may be treated by symmetric argument. Set

(i) $x_{11}=a_1$ and $x_{1,j}=0$ for $j>1$;
(ii) $b'_1=b_1-a_1$, $b'_j=b_j$ for $j>1$;
(iii) $a'=(a_2,\ldots,a_m)\in\mathbb{R}^{m-1}$.

Then for

$$r'=\max\Big(\sum_{2\le i\le m}|a_i|,\sum_{j\le n}|b'_j|\Big),$$

by the inductive assumption there is

$$x'=(x_{ij})_{2\le i\le m, 1\le j\le n},$$

such that $x'\in T(a',b')$ and $\|x'\|\le r'$. Note that

$$\sum_{2\le i\le m}|a_i|\le r-a_1,$$

$$\sum_{j\le n}|b'_j|=b_1-a_1+\sum_{j\le 2\le n}|b_j|\le b_1-a_1+r-b_1=r-a_1,$$

so $r'\le r-a_1$. Hence we can extend x' by the first row defined above and get the required vector x. □

References

[1] P.S. Aleksandrov and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akad. Wetensch., Amsterdam 14 (1929).

[2] A. Avilés and S. Todorcevic, *Multiple gaps*, Fund. Math. 213 (2011), 15–42.

[3] A. Avilés, F. Cabello Sánchez, J.M.F. Castillo, M. González, Y. Moreno, *Separably injective Banach spaces*, Lecture Notes in Mathematics 2132, Springer (2016).

[4] A. Basile, K.P.S. Bhaskara Rao, R.M. Shortt, *Bounded common extensions of charges*, Proc. Amer. Math. Soc. 121 (1994), 137–143.

[5] F. Cabello Sánchez, J.M.F. Castillo, N.J. Kalton, D.T. Yost, *Twisted sums with $C(K)$ spaces*, Trans. Amer. Math. Soc. 355 (2003), 4523–4541.

[6] F. Cabello Sánchez, J.M.F. Castillo, D.T. Yost, *Sobczyk's Theorem from A to B*, Extracta Math. 15(2) (2000), 391–420.

[7] J.M.F. Castillo, *Nonseparable $C(K)$-spaces can be twisted when K is a finite height compact*, Topology Appl. 198 (2016), 107–116.

[8] C. Correa and D.V. Tausk, *Extension property and complementation of isometric copies of continuous functions spaces*, Results Math. 67 (2015), 445–455.

[9] C. Correa and D.V. Tausk, *Nontrivial twisted sums of c_0 and $C(K)$*, J. Funct. Anal. 270 (2016), 842–853.

[10] S. Ditor and R. Haydon, *On absolute retracts, $P(S)$, and complemented subspaces of $C(D^{\omega_1})$*, Studia Math. 56 (1976), 243–251.

[11] E. van Douwen, *The integers and topology*, in: *Handbook of set-theoretic topology*, K. Kunen, J. Vaughan (eds.), North-Holland, Amsterdam (1984), 111–167.

[12] P. Drygier and G. Plebanek, *Compactifications of ω and the Banach space c_0*, Fund. Math. 237 (2017), 165–186.

[13] B.A. Efimov, *Dyadic bicompacta* (Russian), Trudy Moskov. Mat. Obšč. 14 (1965), 211–247.

[14] R. Engelking, *General Topology*, Heldermann Verlag, Berlin (1989).

[15] D.H. Fremlin, *Consequences of Martin's axiom*, Cambridge Tracts in Mathematics 84, Cambridge University Press, Cambridge (1984).

[16] R. Haydon, *On a problem of Pełczyński: Milutin spaces, Dugundji spaces and AE(0-dim)*, Studia Math. 52 (1974), 23–31.

[17] R.W. Heath and D.J. Lutzer, *Dugundji extension theorems for linearly ordered spaces*, Pacific J. Math. 55 (1974), 419–425.

[18] T. Jech, *Set theory. The third millennium edition, revised and expanded*, Springer-Verlag, Berlin (2003).

[19] O. Kalenda, *Valdivia compact spaces in topology and Banach space theory*, Extracta Math. 15 (2000), 1-85.

[20] J. Kąkol, W. Kubiś, M. López-Pellicer, *Descriptive topology in selected topics of functional analysis*, Developments in Mathematics 24, Springer, New York (2011).

[21] P. Koszmider, *On decompositions of Banach spaces of continuous functions on Mrwka's spaces*, Proc. Amer. Math. Soc. 133 (2005), 2137-2146.

[22] W. Kubiś, *A representation of retracts of cubes*, arXiv:math/0407196.

[23] W. Marciszewski and R. Pol, *On Banach spaces whose norm-open sets are F_σ-sets in the weak topology*, J. Math. Anal. Appl. 350 (2009), 708–722.

[24] S. Mercourakis, *Some remarks on countably determined measures and uniform distribution of sequences*, Monatsh. Math. 121 (1996), 79–111.

[25] S. Negrepontis, *Banach spaces and topology*, in: *Handbook of set-theoretic topology*, K. Kunen, J. Vaughan (eds.), North-Holland, Amsterdam (1984), 1045–1142.

[26] A.J. Ostaszewski, *A characterization of compact, separable, ordered spaces*, J. London Math. Soc. 7 (1974), 758–760.

[27] A. Sapounakis, *Measures on totally ordered spaces*, Mathematika 27 (1980), 225–235.

[28] E.V. Shchepin, *Topology of limit spaces with uncountable inverse spectra*, Uspekhi Mat. Nauk, 31 (1976), no. 5 (191), 191–226 (Russian Mathematical Surveys, 1976, 31:5, 155–191).

[29] S. Todorčević, *Analytic gaps*, Fund. Math. 150 (1996), 55–66.

Institute of Mathematics, University of Warsaw, Banacha 2
02–097 Warszawa, Poland
E-mail address: wmarcisz@mimuw.edu.pl

Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4,
50–384 Wrocław, Poland
E-mail address: grzes@math.uni.wroc.pl