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Ellipticity in geometric variational problems

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$$\Phi : \{\text{geometric objects}\} \to [0, \infty]$$

Goal: Study critical points (in particular minima) of Φ.

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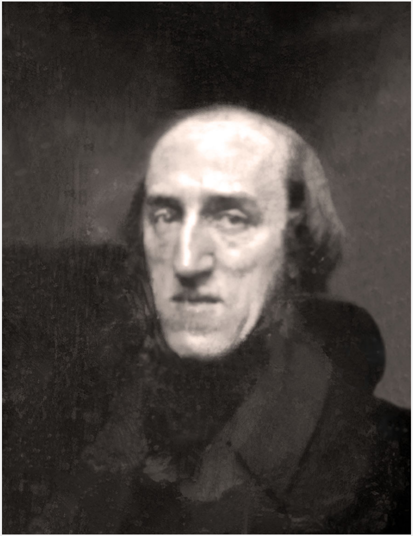
Examples:

1. The Plateau problem

$$\begin{aligned} \text{geometric objects} \quad &- \quad \text{surfaces with a fixed boundary} \\ \Phi \quad &- \quad \text{measure of the surface} \end{aligned}$$

2. The isoperimetric problem

$$\begin{aligned} \text{geometric objects} \quad &- \quad \text{open sets with fixed volume} \\ \Phi \quad &- \quad \text{measure of the boundary} \end{aligned}$$



Joseph Plateau 1801 – 1883
Image by Albert Callisto
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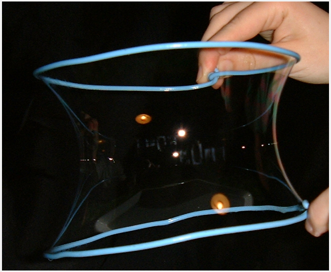

Catenoid

1930s parameterised surfaces

J. Douglas (Fields medal) and T. Radó

1960s rise of GMT: rectifiable currents and varifolds, Caccioppoli sets

H. Federer, W. Fleming, F. Almgren, E. R. Reifenberg, E. De Giorgi et al.

2010s new models and new solutions, more abstract setting

G. David, J. Harrison, C. De Lellis, F. Maggi, G. De Philippis et al.

$$\Phi : \{\text{geometric objects}\} \to [0, \infty]$$

$$S_i \in \operatorname{dmn} \Phi\,, \qquad \Phi(S_i) \xrightarrow{i \to \infty} \inf \operatorname{im} \Phi$$

- Is there a convergent sub-sequence of $\{S_1, S_2, \dots\}$?
- If so, is the limit in the domain of Φ?

Definition

Let $X \subseteq \mathbf{R}^N$. Define $\mathscr{K}(X)$ to be the space of those continuous functions $f : X \to \mathbf{R}$ which satisfy

$$\operatorname{Clos}\{x \in X : f(x) \neq 0\} \text{ is compact}.$$

$\mathscr{K}(X)$ is a locally convex topological vector space.

Definition

Radon measure over X is any continuous linear functional $\mu : \mathscr{K}(X) \to \mathbf{R}$ such that $\mu(f) \geq 0$ whenever $f \geq 0$.

Definition

Radon measures $\mu_1, \mu_2, \ldots$ over X *converge weakly* to μ if

$$\lim_{i \to \infty} \mu_i(f) = \mu(f) \quad \text{for all } f \in \mathscr{K}(X).$$

Examples

Let M be a $\mathscr{C}^1$-smooth submanifold of $\mathbf{R}^n$ of dimension k and such that $M \cap K$ has finite measure whenever $K \subseteq \mathbf{R}^n$ is compact.

1. One can associate a Radon measure μ_M to M by setting

$$\mu_M(f) = \int f \, \mathrm{dvol}_M \quad \text{for } f \in \mathscr{K}(\mathbf{R}^n) \, .$$

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$$\mu_M(f) = \int f \, \mathrm{dvol}_M \quad \text{for } f \in \mathscr{K}(\mathbf{R}^n) \, .$$

2. *Varifold* associated to M is the Radon measure $\mathbf{v}_k(M)$ over $\mathbf{R}^n \times \mathbf{G}(n,k)$

$$\mathbf{v}_k(M)(f) = \int f(x, \mathrm{Tan}(M,x)) \, \mathrm{dvol}_M(x)$$

for $f \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n,k))$, where

$$\begin{aligned} \mathbf{G}(n,k) &= \{ T : T \text{ a linear subspace of } \mathbf{R}^n, \dim T = k \} \\ &\simeq \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{ T : T \circ T = T, \, T^* = T, \, \mathrm{trace}\, T = k \} \, . \end{aligned}$$

Definition

A k-dimensional *varifold* in an open set $U \subseteq \mathbf{R}^n$ is a Radon measure over $U \times \mathbf{G}(n,k)$.

Definition

A varifold V is called *rectifiable* if for $i = 1, 2, \ldots$ there are positive numbers a_i and $\mathscr{C}^1$-manifolds M_i such that

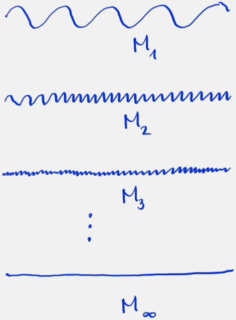
$$V = \textstyle\sum_{i=1}^{\infty} a_i \mathbf{v}_k(M_i)\,.$$

The *weight measure* of V is the Radon measure $\|V\|$ over U s.t.

$$\|V\|(A) = V(A \times \mathbf{G}(n,k)) \quad \text{for } A \subseteq U\,.$$

The *k-density* of V at x is defined (if the limit exists) by

$$\boldsymbol{\Theta}^k(\|V\|, x) = \lim_{r \downarrow 0} \frac{\|V\|\mathbf{B}(x,r)}{\boldsymbol{\alpha}(k) r^k}\,.$$



μ_{M_1}	$\mathbf{V}_k(M_1)$
μ_{M_2}	$\mathbf{V}_k(M_2)$
μ_{M_3}	$\mathbf{V}_k(M_3)$
$\vdots$	$\vdots$
$\lambda \mu_{M_\infty}$	$V(f) = \lambda \int \int f(x,T)\, d\nu(T)\, d\mathrm{vol}_{M_\infty}$

for some $\lambda \in \mathbb{R}$

Not a rectifiable varifold

$$V(f) = \lambda \int \int f(x,T)\, d\nu(T)\, d\mathrm{vol}_{M_\infty}$$

for some $\lambda \in \mathbb{R}$ and $\nu \in \mathrm{Prob}(G(n,k))$

Competitors: family $\mathcal{C}$ of varifolds associated to properly embedded $\mathscr{C}^1$-submanifolds of $U \subseteq \mathbf{R}^n$ with a given boundary.

Functional: $\Phi : \mathcal{C} \to \overline{\mathbf{R}}$ defined by $\Phi(V) = V(1)$, i.e., the total mass of $V \in \mathcal{C}$.

1. choose a minimising sequence $V_i \in \mathcal{C}$, i.e., $\lim_{i\to\infty} \Phi(V_i) = \inf \operatorname{im} \Phi$
2. use the Banach-Alaoglu theorem to find a weakly convergent sub-sequence
3. pass to the limit $V_i \xrightarrow{i\to\infty} V$

Questions. Is V rectifiable? Is it associated to a single manifold?

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Questions. Is V rectifiable? Is it associated to a single manifold?

Remark. One can consider $\Phi_F(V) = V(F)$, where $F : U \times \mathbf{G}(n,k) \to \mathbf{R}$ is continuous and s.t. $0 < \inf \operatorname{im} F < \sup \operatorname{im} F < \infty$.

Theorem (Almgren, Ann. of Math, 1968)

Assume $\mathcal{C}$ is a class of rectifiable varifolds "spanning" a given boundary, $F : U \times \mathbf{G}(n,k) \to \mathbf{R}$ is of class $\mathscr{C}^k$ and s.t. $0 < \inf \operatorname{im} F < \sup \operatorname{im} F < \infty$, $\Phi_F(V) = V(F)$ for $V \in \mathcal{C}$, V is a minimiser of Φ_F, and F is *uniformly elliptic*.
Then V is regular (of class $\mathscr{C}^{k-1}$) almost everywhere, i.e., there exists a manifold M of class $\mathscr{C}^{k-1}$, which is dense in $\operatorname{spt} \|V\|$ and such that $\|V\|(\operatorname{spt} \|V\| \sim M) = 0$.

Theorem (Almgren's big regularity paper, 2000)

If $F \equiv 1$, then the singular set has Hausdorff dimension $\leq k-2$.

Theorem (Federer, Bull. AMS, 1970)

If $F \equiv 1$ and $k = n-1$, then the singular set has Hausdorff dimension $\leq k-7$.

Assume

1. μ is a Borel regular measure over $\mathbf{R}^n$ (i.e. open sets are μ-measurable and each set is a subset of a Borel set with the same measure),
2. μ is translation invariant,
3. μ is finite on any bounded subset of any $T \in \mathbf{G}(n,k)$,
4. μ is continuous w.r.t. $\mathscr{C}^1$-deformations, i.e., given $f_k \xrightarrow{k\to\infty} 0$ in $\mathscr{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and a bounded set $A \subseteq \mathbf{R}^n$ with $\mu(A) < \infty$ there holds

$$\mu\big((\mathrm{id}_{\mathbf{R}^n} + f_k)[A]\big) \xrightarrow{k\to\infty} \mu(A)\,.$$

Then there exists $F : \mathbf{G}(n,k) \to [0,\infty)$ s.t.

$$\mu(M) = \int F(\mathrm{Tan}(M,x))\,\mathrm{dvol}_M(x) \quad \text{for any } \mathscr{C}^1\text{-manifold } M \subseteq \mathbf{R}^n\,.$$

Let $\phi : \mathbf{R}^n \to \mathbf{R}$ be a (non-Euclidean) norm. The k-dimensional *Hausdorff measure* is

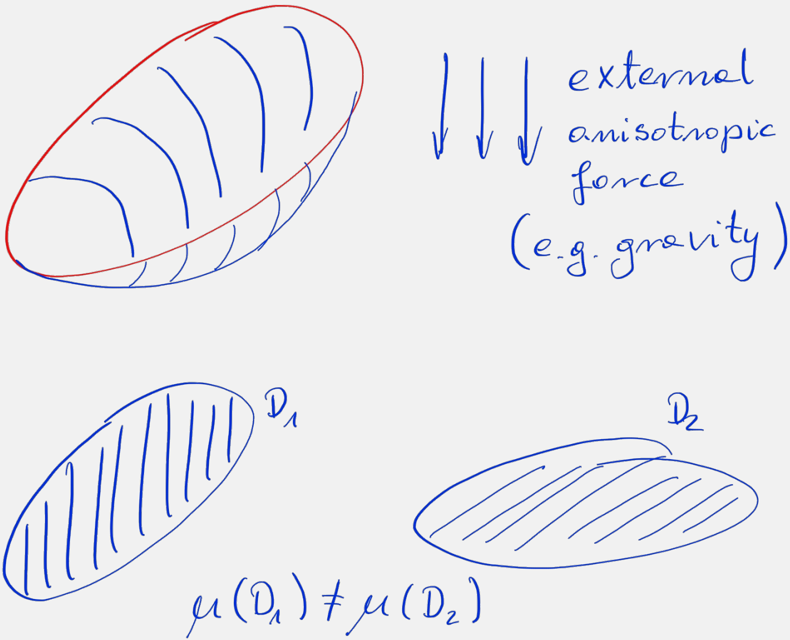
$$\mathscr{H}^k_\phi(A) = \liminf_{\delta \downarrow 0} \left\{ \frac{\boldsymbol{\alpha}(k)}{2^k} \sum_{E \in \mathcal{F}} \operatorname{diam}_\phi(E)^k : \begin{array}{c} \mathcal{F} \subseteq \mathbf{2}^{\mathbf{R}^n} \text{ countable}, \\ A \subseteq \bigcup \mathcal{F}, \\ \operatorname{diam}_\phi(E) < \delta \text{ for } E \in \mathcal{F} \end{array} \right\}$$

Lemma (Busemann, Ann. of Math, 1947)

For any $\mathscr{C}^1$-manifold $M \subseteq \mathbf{R}^n$ there holds $\mathscr{H}^k_\phi(M) = \int F^{\mathrm{BH}}(\operatorname{Tan}(M,x)) \, \mathrm{dvol}_M(x)$, where

$$F^{\mathrm{BH}}(T) = \frac{\boldsymbol{\alpha}(k)}{\mathscr{H}^k(\mathbf{B}^\phi(0,1) \cap T)} .$$

Remark. $\mathscr{H}^n = \mathscr{L}^n$ over $\mathbf{R}^n$.


external
anisotropic
force
(e.g. gravity)
D1
D2
μ(D1) ≠ μ(D2)

Assume V is a k-varifold in $U \subseteq \mathbf{R}^n$ s.t. for any choice of

$$g : U \to \mathbf{R}^n \quad \text{a } \mathscr{C}^1\text{-vectorfield with compact support},$$

defining

$$f_t(x) = x + tg(x) \quad \text{for } x \in U \text{ and } t \in (-\varepsilon, \varepsilon),$$

there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_F(f_{t\#}V) = 0\,.$$

Question. What can we say about the regularity of V?

For simplicity for the rest of my talk assume $F(x, T) = F(T)$.

$$g : U \to \mathbf{R}^n , \quad f_t(x) = x + tg(x) ,$$

$$\begin{aligned} \delta_F V(g) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_F(f_{t\#}V) \\ &= \int \operatorname{trace}\big(P_F(T) \circ \mathrm{D}g(x)\big) F(T) \,\mathrm{d}V(x, T) , \end{aligned}$$

where $P_F(T) \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is a (non-orthogonal) projection onto T.

Definition

We say that V is F-stationary if $\delta_F V = 0$.

Remark. In case $F \equiv 1$ and $M \subseteq \mathbf{R}^3$ is a *minimal surface*, then $\delta_F V = 0$.

Definition

(S, D) is a *test pair* if S is a k-rectifiable set, D is a flat k-disc, and S cannot be retracted onto ∂D.

Definition

$\boxed{F \in \mathrm{AE}}$ iff. for any test pair (S, D) with $\mathscr{H}^k(S) > \mathscr{H}^k(D)$

$$\Phi_F(S) > \Phi_F(D)\,.$$

$\boxed{F \in \mathrm{UAE}}$ iff. there is $c > 0$ s.t. for any test pair (S, D)

$$\Phi_F(S) - \Phi_F(D) \geq c(\mathscr{H}^k(S) - \mathscr{H}^k(D))\,.$$

$\boxed{F \in \mathrm{BC}}$ iff. for any $T \in \mathbf{G}(n,k)$ and μ a probability measure over $\mathbf{G}(n,k)$

$$\delta_F(\mathscr{H}^k \llcorner T \times \mu) = 0 \implies \mu = \mathrm{Dirac}(T)\,,$$

i.e.,

> any stationary varifold supported in some $T \in \mathbf{G}(n,k)$ which is translation invariant in T is rectifiable.

Let

$$\mathcal{G}_F = \{P_F(T) : T \in \mathbf{G}(n,k)\} \subseteq \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)\,.$$

$\boxed{F \in \mathrm{AC}}$ iff. $\mathcal{G}_F$ is the set of extreme points of $\operatorname{conv} \mathcal{G}_F$ and

$$\operatorname{conv} \mathcal{G}_F \cap \{A : \dim \operatorname{im} A \le k\} = \mathcal{G}_F\,.$$

Theorem (De Philippis, De Rosa, Ghiraldin, CPAM, 2018)

Assume $F \in \mathrm{AC}$, $c > 0$, V is a k-varifold such that $\boldsymbol{\Theta}^k(\|V\|, x) > c$ for $\|V\|$ almost all x and $\delta_F V$ is representable by integration. Then V is rectifiable.

Remark. In particular, any F-stationary varifold with k-density bounded away from zero is rectifiable.

Remark. $F \in \mathrm{AC}$ is also a necessary condition for the above implication.

Theorem (De Rosa, K., CPAM, 2020)

$\mathrm{AC} = \mathrm{BC} \subseteq \mathrm{AE}$

$\boxed{F \in \mathrm{mUSAC}}$ iff. there is $c > 0$ s.t. for each $T \in \mathbf{G}(n,k)$ there exists $N \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ s.t.

$$N \perp \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{A : T \subseteq \mathrm{im}\, A\}$$
$$\text{and} \quad N \bullet P_F(S) \geq c\|S - T\|^2 .$$

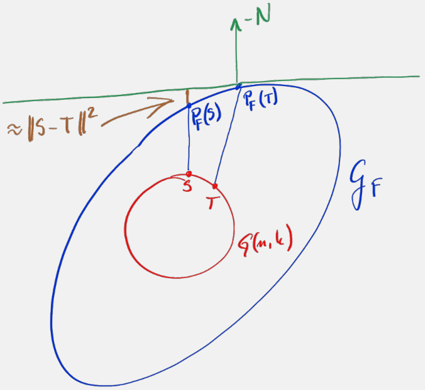
Theorem (De Rosa, Tione, Invent. math., 2022)

Assume $p > k$, $F \in \mathrm{mUSAC}$, $\Omega \subseteq \mathbf{R}^k$, V is associated to a graph of a Lipschitz function $f : \Omega \to \mathbf{R}^{n-k}$, and $\delta_F V$ representable by integration against some $\mathbf{L}_p(\|V\|, \mathbf{R}^n)$ function. Then there exists an open set $\Omega_0 \subseteq \Omega$ of full measure s.t. $f|\Omega_0$ is of class $\mathscr{C}^{1,\alpha}$, where $\alpha = 1 - k/p$.

$F \in \mathrm{mUSAC}$ iff. there is $c > 0$ s.t. for each $T \in \mathbf{G}(n,k)$ there exists $N \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ s.t.

$$N \perp \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{A : T \subseteq \operatorname{im} A\},$$

$$\text{and} \quad N \bullet P_F(S) \geq c\|S - T\|^2 \quad \text{for } S \in \mathbf{G}(n,k).$$



Theorem (Allard, Ann. of Math., 1972)

Assume $p > k$, $F \equiv 1$, V is a k-varifold with density bounded away from zero, $\delta_F V$ representable by integration against some $\mathbf{L}_p(\|V\|, \mathbf{R}^n)$ function. Then the set of regular points is open and dense in $\operatorname{spt} \|V\|$.

Theorem (Wickramasekera, Ann. of Math, 2014)

Assume $k = n - 1$, $F \equiv 1$, V is an integral k-varifold which is stationary and *stable*. Then the singular set consists of points which have a neighbourhood in which V is made of at least three $\mathscr{C}^{1,\alpha}$ hypersurfaces meeting along their common boundary and the rest which has Hausdorff dimension at most $k - 7$.

If $k = n - 1$, then $\mathbb{S}^k \xrightarrow{\pi} \mathbf{RP}(k) \simeq \mathbf{G}(n,k)$.

Fact. $F \in \mathrm{AC}$ iff. there exists a strictly convex norm G s.t. $F(\pi(v)) = G(v)$ for $v \in \mathbb{S}^k$.

Fact. $F \in \mathrm{UAE}$ iff. there exists a uniformly convex norm G s.t. $F(\pi(v)) = G(v)$ for $v \in \mathbb{S}^k$.

Remark. In higher co-dimension no particular examples of $F \in \mathrm{AC}$ are known!

- $F \in \mathrm{UAE}$ gives partial regularity of minimisers
- $F \in \mathrm{AC} = \mathrm{BC} \subseteq \mathrm{AE}$ gives rectifiability of critical points
- $F \in \mathrm{mUSAC}$ gives partial regularity for critical graphs
- $F \equiv 1$
  - singular set of a critical point is open and dense in the support
  - singular set of a minimiser is of dimension $\leq k - 2$
  - if $n = k + 1$, singular set of a minimiser is of dimension $\leq k - 7$
  - if $n = k + 1$, singularities of a stable critical point are classified into two categories; the dimension of one of them is $\leq k - 7$, the other has dimensions $k - 1$ is unavoidable and is well understood.

1. AE = AC?
2. mUSAC $\subseteq$ UAE?
3. $F^{BH} \in$ AC?
4. How to construct examples of $F \in$ AC?
5. Does any analogue of the Allard regularity theorem holds for $F \in$ mUSAC?
6. Is the singular set of measure zero? (not known even for $F \equiv 1$)
7. What is the precise dimension of the singular set?

Thank you for listening.