# Dynamical Systems and some problems in Complex Analysis and Geometry.

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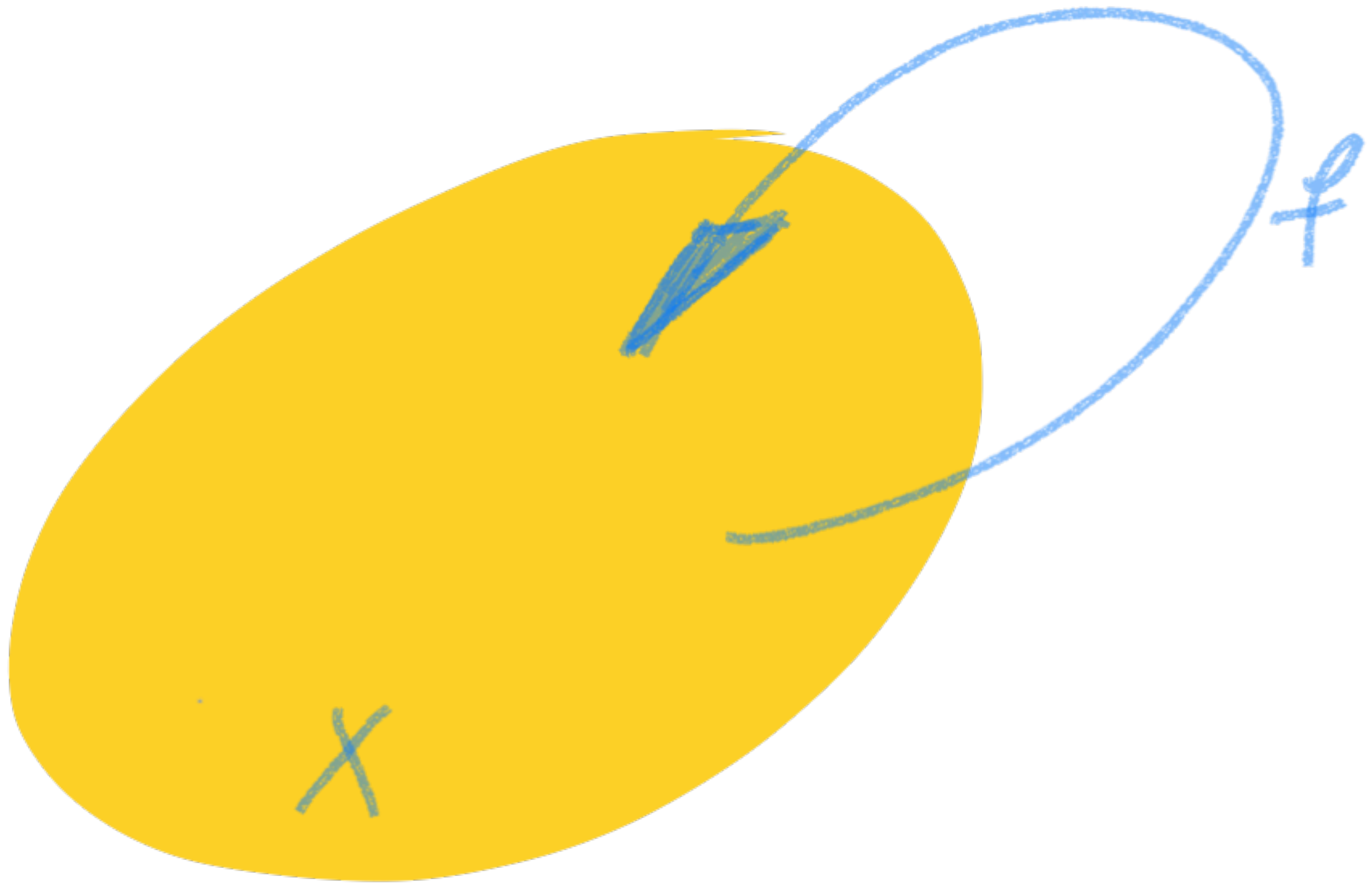
# Dynamical Systen

What is a dynamical system?

Most generally speaking, a dynamical system is a pair $(X, f^t)$, where $X$ is some space and $(f^t)_{t \in T}$ is a (semi)group of maps $f^t : X \to X$. Today, we shall talk about *discrete dynamical systems* generated by a single map $f\colon X \to X$. Then $T = \mathbb{N}$ and $f^n = f \circ f \circ \ldots \circ f$ ($f$ iterated $n$ times).



# Holomorphic maps

Reminder: we denote by $\mathbb{C}$ the set of complex numbers $z = a + bi$. We denote by $\hat{\mathbb{C}}$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$ endowed with the structure of a complex manifold.

Reminder: a function $f\colon U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{C}$ is called *holomorphic* if it is (complex) differentiable at every point $z \in U$.

# Motivation: Kleinian groups

Reminder: a Möbius map (homography) is a map given by the formula $h(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Möbius maps are (the only) holomorphic automorphisms of the Riemann sphere $\hat{\mathbb{C}}$.

Let $G$ be a subgroup of the group of Möbius transformations acting on the Riemann sphere. Then $G$ acts also as a group of isometries of the hyperbolic 3-ball $B$. The group is called discrete if identity is an isolated element of $G$. Then $G$ is called a Kleinian group.

The ordinary set $\Omega(G) \subset \hat{\mathbb{C}}$ is the set of points at which the group acts discontinously, i.e. the set of such points $z$ for which there exists a disc around $z$ which hits itself under the action only of a finite number of elements of the group. The limit set is defined as $\Lambda(G) = \hat{\mathbb{C}} \backslash \Omega(G)$. The limit set may be finite with 0, 1 or 2 elements. Otherwise it is infinite.

The set $\Omega(G)$ can be equivalently described as an equicontinuity region: the set of points for which there exists a neighborhood $U$ such that the family of maps $G_{|U}$ is a normal family (each sequence has a subsequence converging uniformly on compact sets).

Informally speaking, the action of the group $G$ divides the sphere into two invariant sets: open set $\Omega(G)$ on which the action of the group is „regular", and closed set $\Lambda(G)$ on which the action is „chaotic".

Study of the geometry of limit sets of Kleinian groups, and their relation with the geometry and topology of Riemann surfaces and hyperbolic 3-manifolds has been for decades one of the leading areas of research in mathematics.

In contrary, dynamics of a single Möbius map $h$ in $\hat{\mathbb{C}}$ is not interesting and easy to describe: the group $G = \{h^n\}_{n \in \mathbb{Z}}$ generated by $h$, called elementary, has finite set $\Lambda(G)$.

For example: if $h(z) = 2z$ then $\Lambda(G) = \{0, \infty\}$.

However, when, instead of of a single Möbius map, we consider iterations of a single holomorphic non-invertible function $f$: $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the situation resembles the previous one- the action of a Kleinian group.

# Dynamics of rational maps

Reminder: Holomorphic maps on the Riemann sphere are rational functions, i.e., $f(z) = \frac{p(z)}{q(z)}$ where $p, q$ are polynomials

Definition: For a rational function $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d > 1$ we define the Fatou set $F(f)$ (equicontinuity set) as the set of points for which there exists a neighborhood $U$ such that the family of maps $f^n_{|U}$ is a normal family.

Definition: The Julia set $J(f)$ is defined as the complement of the Fatou set:

$$J(f) = \hat{\mathbb{C}} \backslash F(f)$$

G.Julia, P.Fatou, 1920's

Vocabulary: $\Omega(G) \asymp F(f)$; $\Lambda(G) \asymp J(f)$

## Some properties of Fatou and Julia sets

- The Fatou set is open, the Julia set is closed.
- The Julia set is non-empty.
- The Julia set is a perfect set (no isolated points). If $\mathrm{int}(J(f)) \neq \emptyset$ then $J(f) = \hat{\mathbb{C}}$.
- Each connected component of the Fatou set $F(f)$ is mapped by $f$ onto some component of $F(f)$. There are no wandering components. This means that every connected component $U$ of $F(f)$ is eventually periodic: $f^{n+k}(U) = f^k(U)$ for some $k, n \in \mathbb{N}$. D.Sullivan, 1980's, using tools of PDEs
- Informally speaking, the Julia set $J(f)$ is the locus of „chaotic behaviour" of iterates of the map $f$.

Thus, the Fatou set (equicontinuity set) is a union of all periodic components (there is a finite number of them) and all their preimages. If $f^k(U) = U$ then $U$ is either:

classification: P.Fatou,M.Herman

- basin of attraction: for $z \in U$ $f^{nk}(z) \xrightarrow[n\to\infty]{} p$ where $p \in U$, $f^k(p) = p$ and $|(f^k)'(p)| < 1$, or
- parabolic basin: for $z \in U$ $f^{nk}(z) \xrightarrow[n\to\infty]{} p$ where $p \in \partial U$, $f^k(p) = p$ and $|(f^k)'(p)|$ is a root of unity, or
- Siegel discs: after a holomorphic change of coordinates $f^k_{|U}$ becomes a rotation on a disc by an angle $2\pi\theta$, $\theta \notin \mathbb{Q}$, or
- Herman ring: after a holomorphic change of coordinates $f^k_{|U}$ becomes a rotation on an annulus by an angle $2\pi\theta$, $\theta \notin \mathbb{Q}$

## Dynamics of polynomials

If $f$ is a polynomial of degree $d > 1$, then the point at infinity is an attracting fixed point: for all points $z$ with sufficiently large modulus $f^n(z) \xrightarrow[n\to\infty]{} \infty$

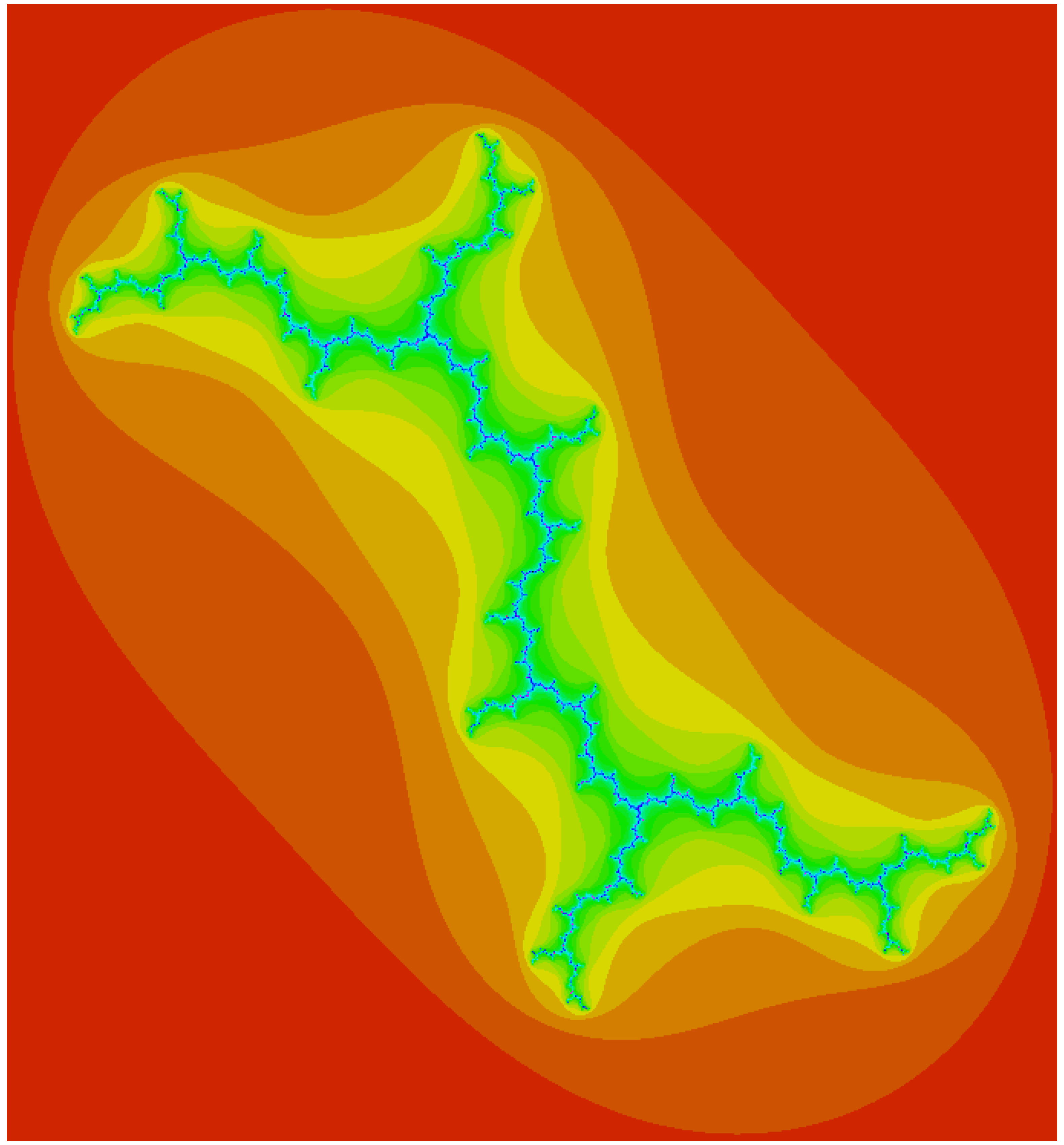
Denote by $\mathscr{A}_\infty(f)$ the basin of attraction of infinity, i.e. the set of points $z \in \hat{\mathbb{C}}$ such that $f^n(z) \xrightarrow[n\to\infty]{} \infty$. Denote by $K(f)$ the complement of $\mathscr{A}_\infty(f)$. The set $\mathscr{A}_\infty(f)$ is open and connected. For polynomials we have the following alternative (equivalent) definition of the Julia set:

$$J(f) = \partial\mathscr{A}_\infty(f) = \partial K(f)$$

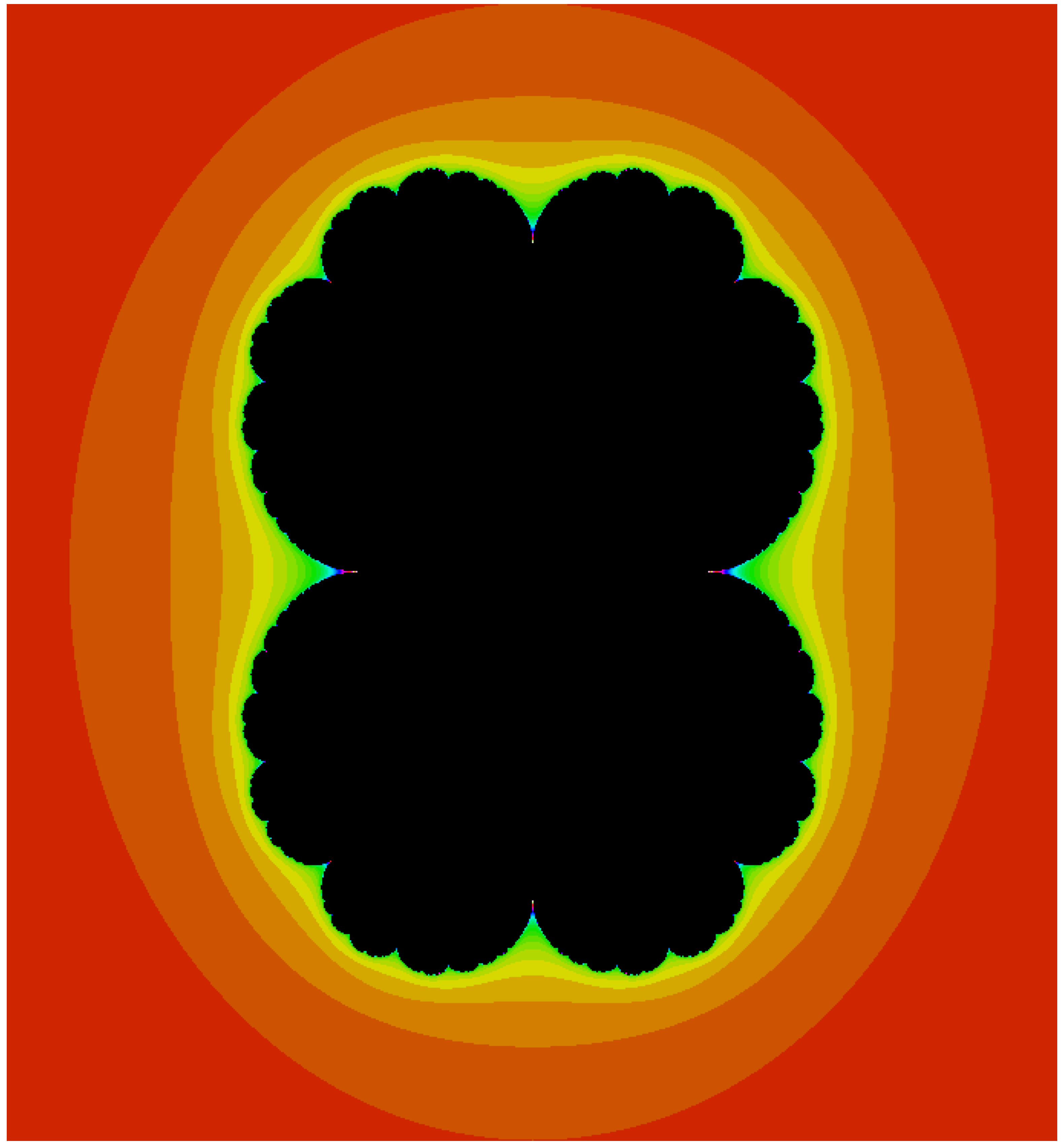
(which means that $J(f)$ is the common boundary of the „escaping" and „non escaping" sets)

# Examples of Julia sets: quadratic polynomials:

escaping set: in red/yellow/green ,
interior of $K(f)$ in black

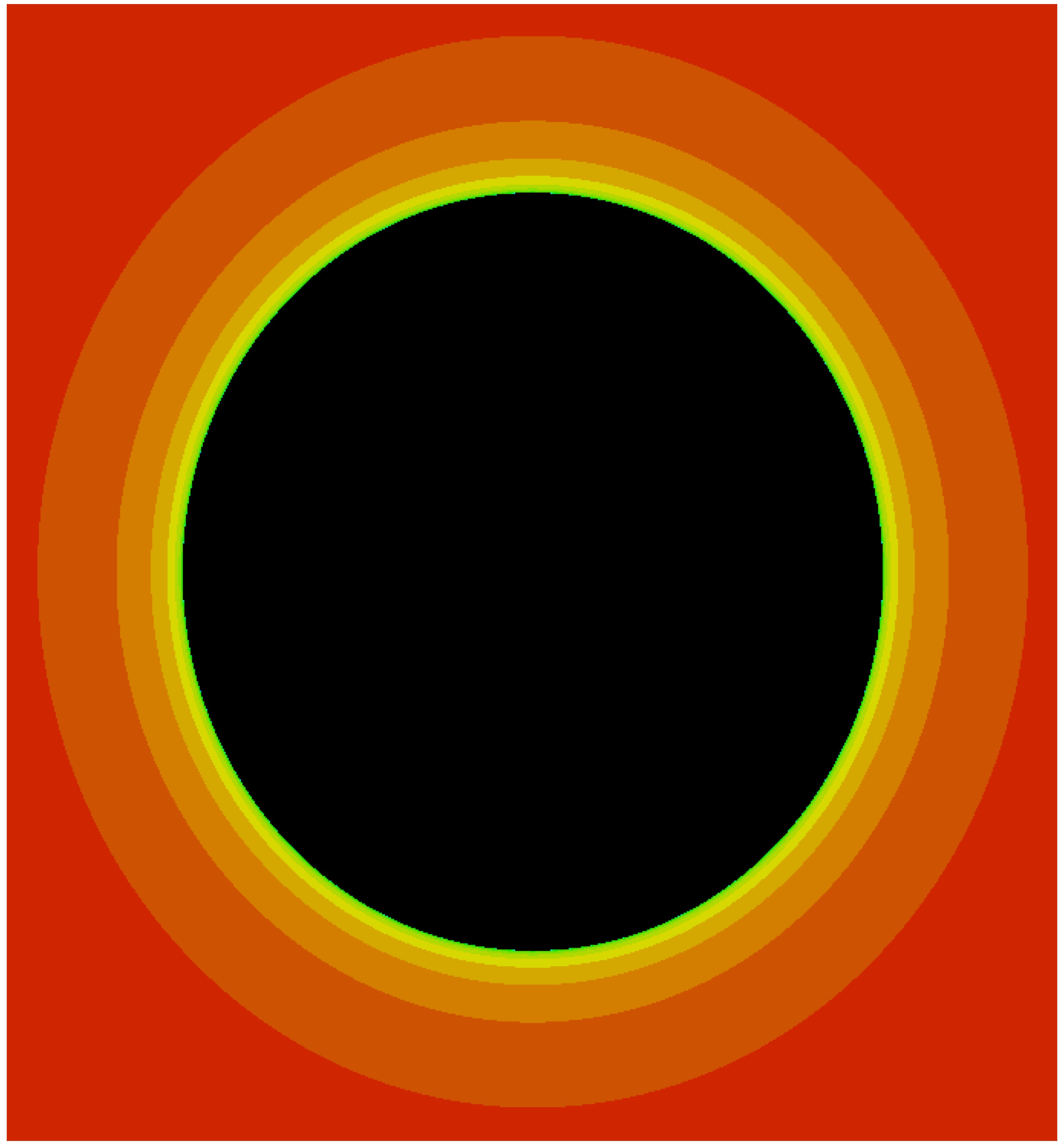


$$f(z) = z^2 + i;$$
$$0 \mapsto i \mapsto -1 + i \mapsto -i \mapsto -1 + i$$

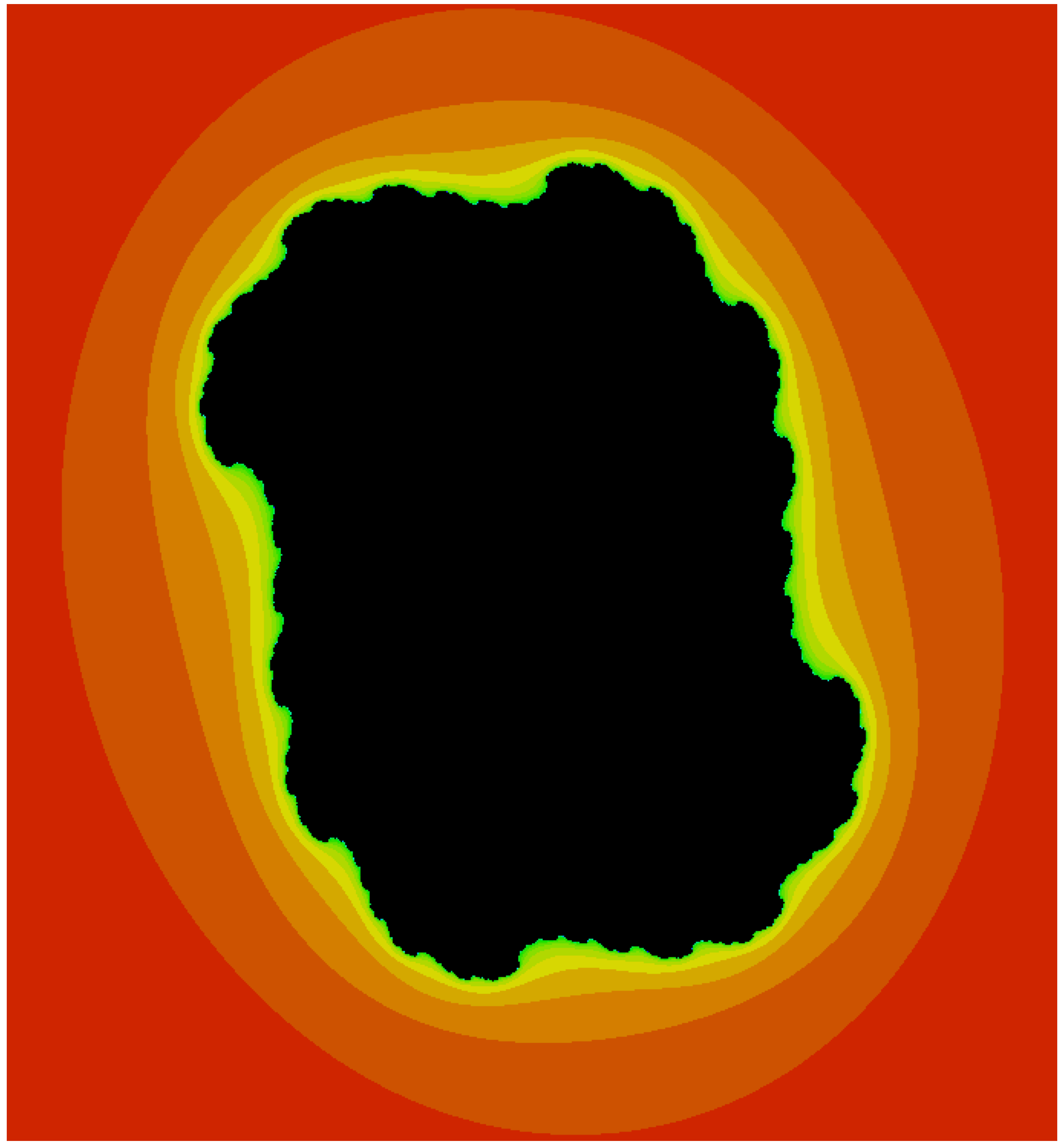


$$f(z) = z^2 + 1/4;$$
$$f(1/2) = 1/2,$$
$$f^n(0) \xrightarrow[n\to\infty]{} 1/2$$

# More examples of Julia sets (quadratic polynomials)

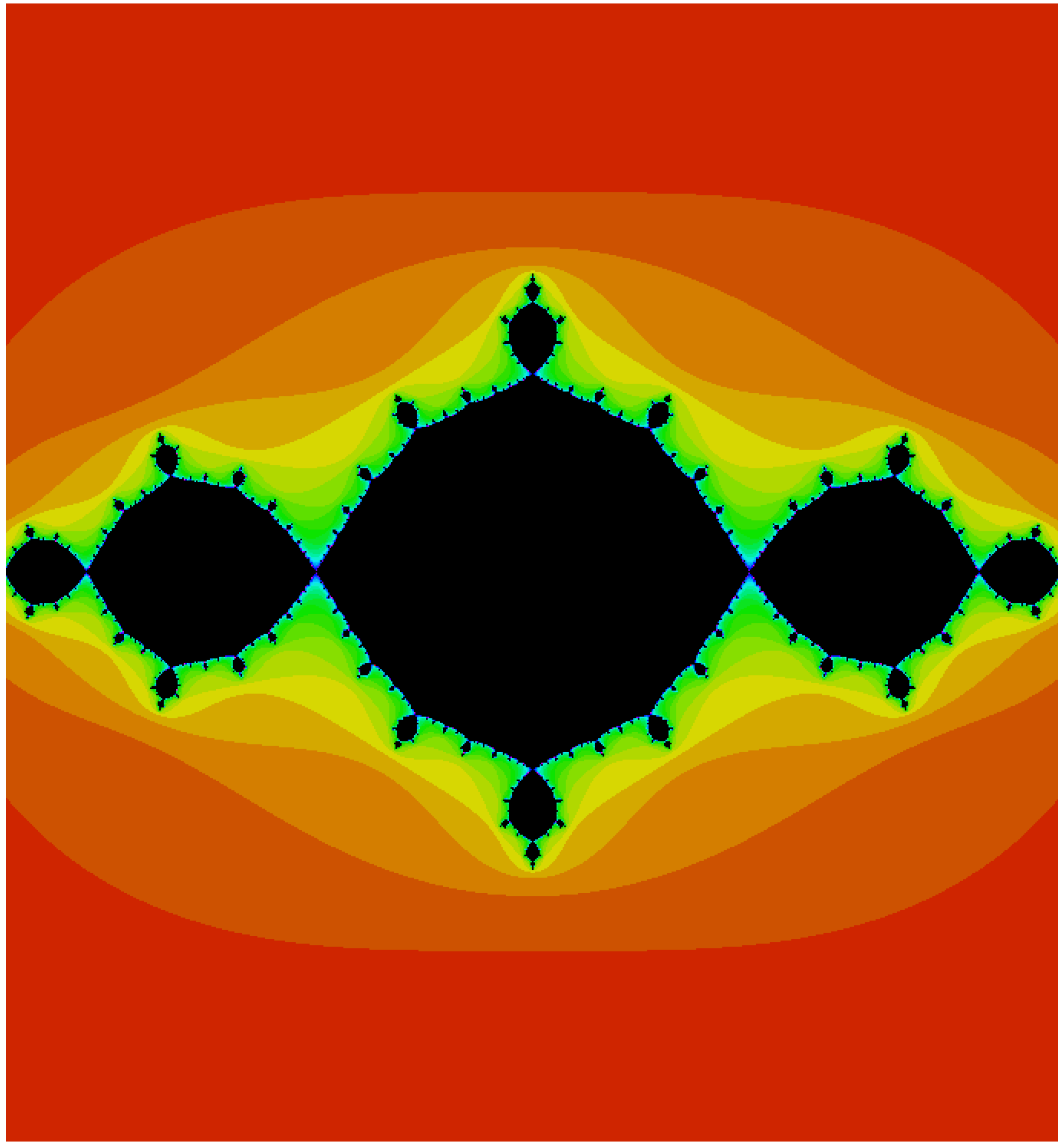


$f(z) = z^2$



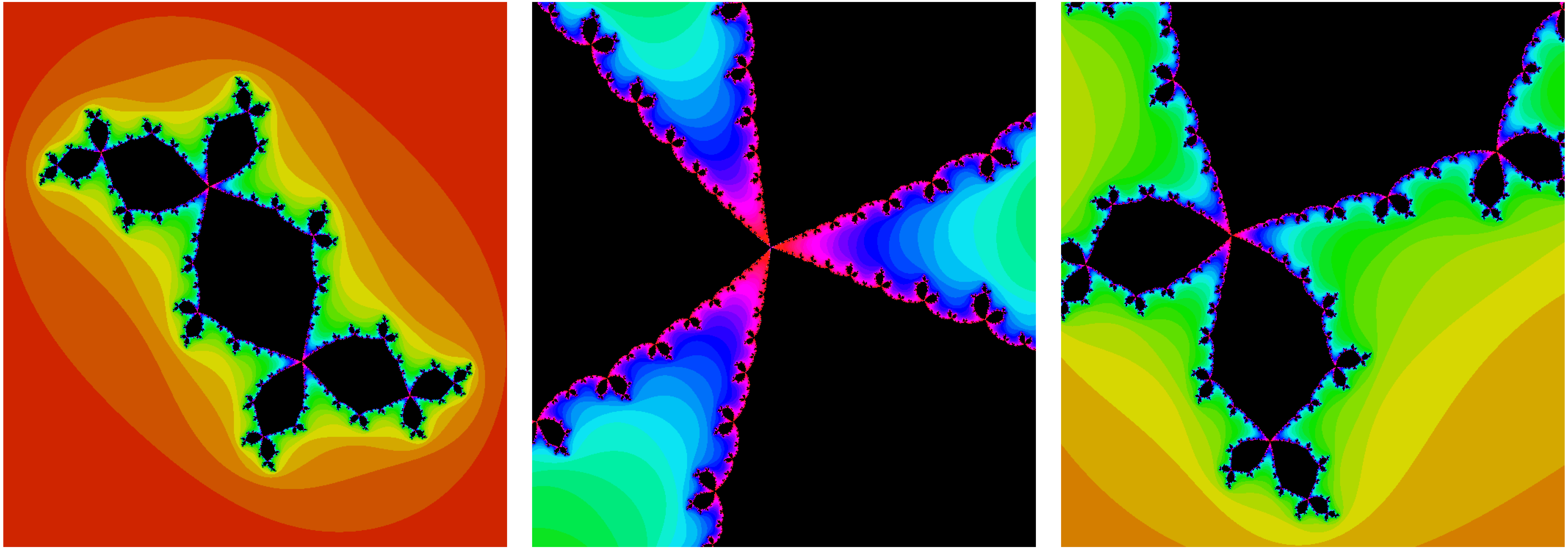
$f(z) = z^2 + 0.2 + 0.2i$

## One more example:

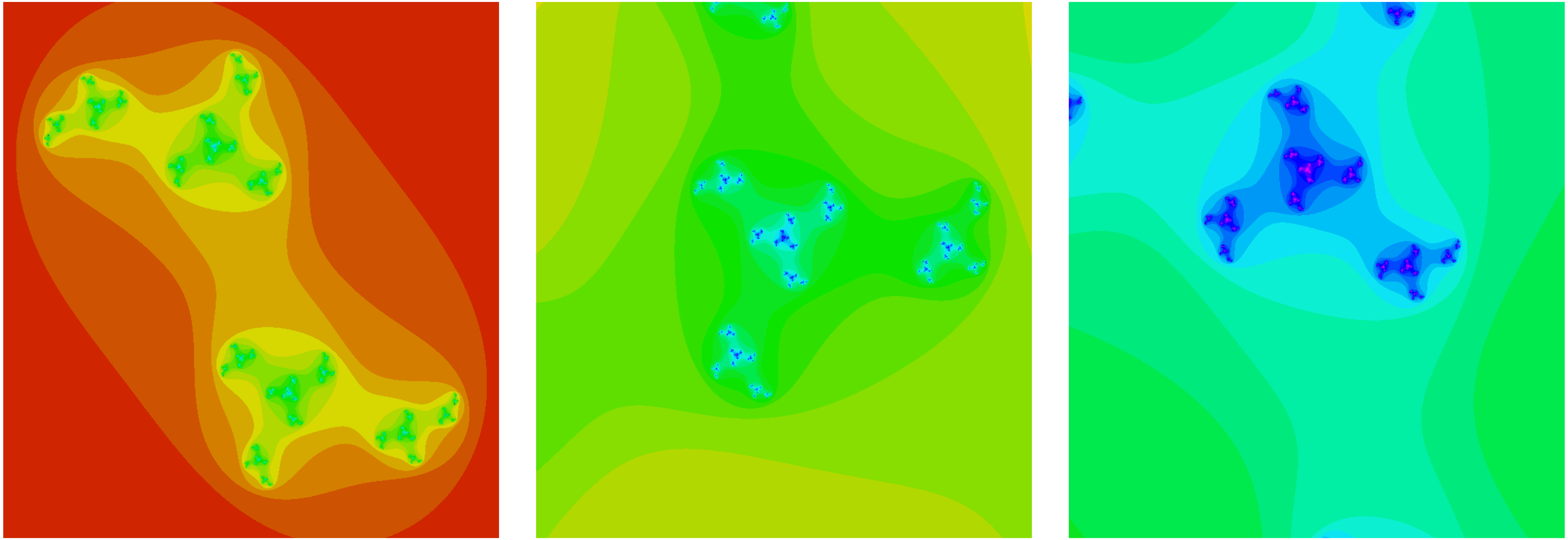


„Bassilica": $f(z) = z^2 - 1$; the critical point $0$ is mapped to $-1$and the back to $0$. So, this is a periodic attracting point of period 2. The central black region is the component of the  basin of attraction; it is mapped  by $f$ to the attached  component to the left of it, and then back onto itself. Other black regions are the preimages of these two components under iterates $f^k, k \in \mathbb{N}$.

# Structure of Julia sets. Example the „rabbit": $f(z) = -0.12z^2 + 0.75i$



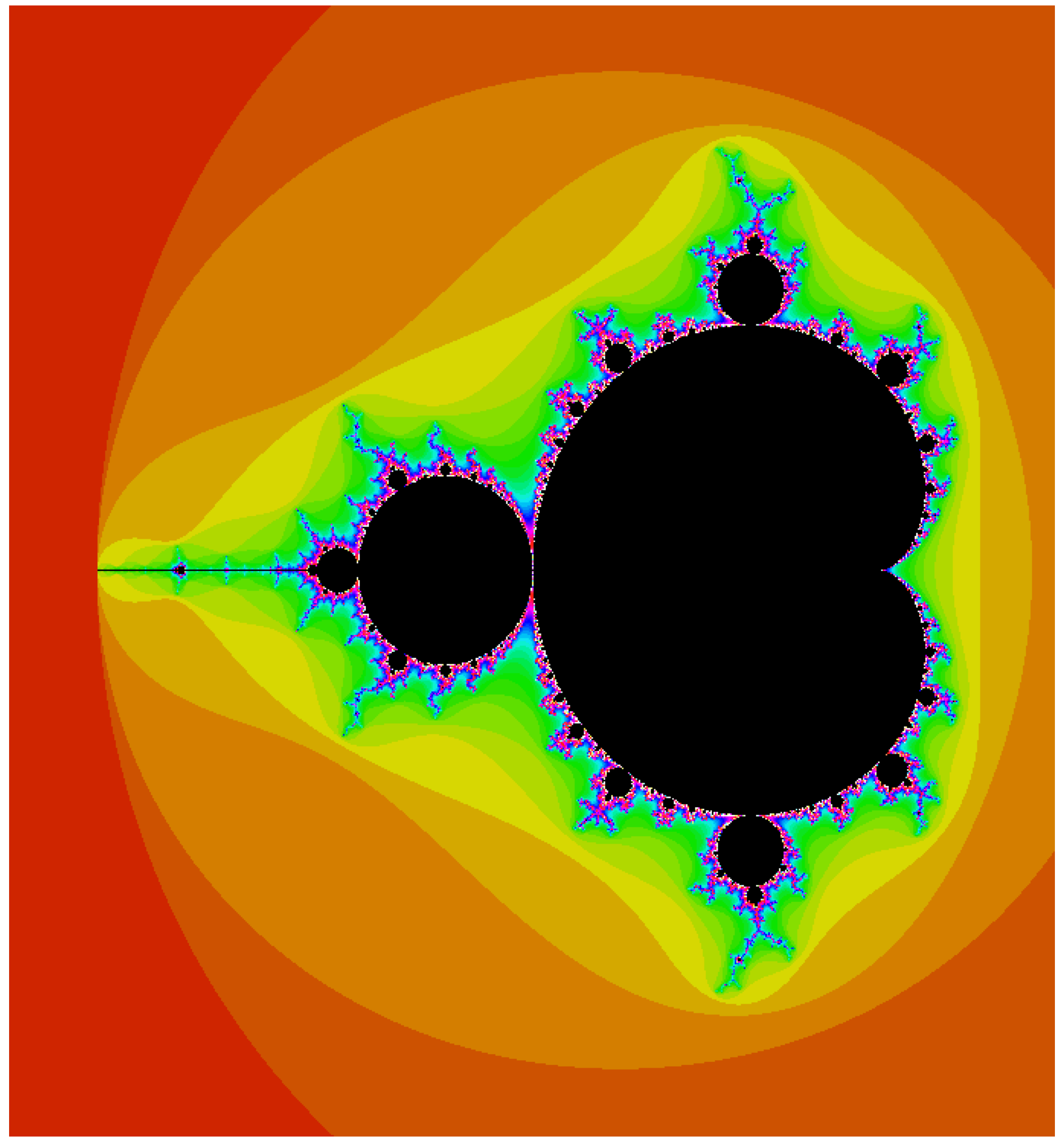
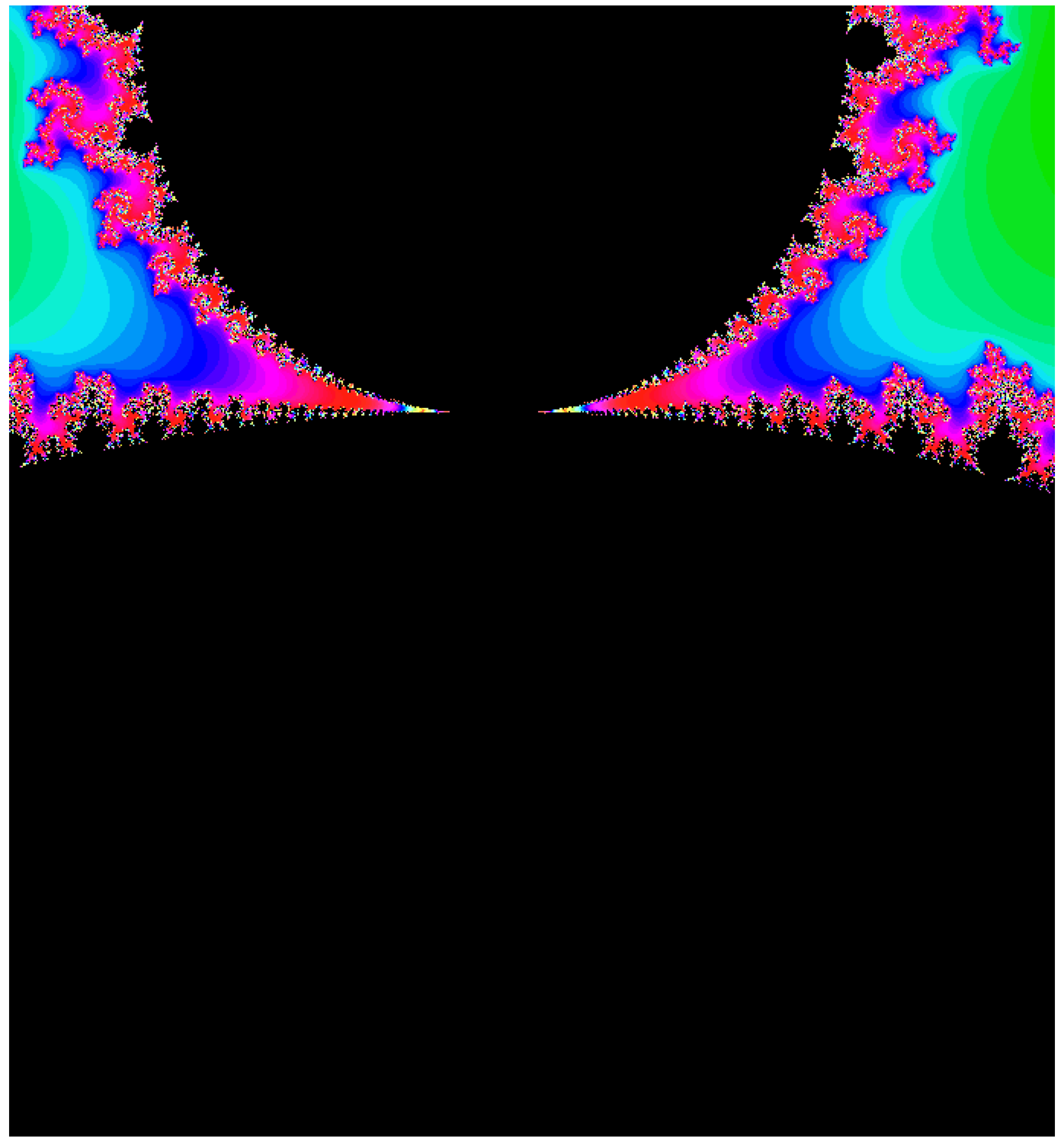
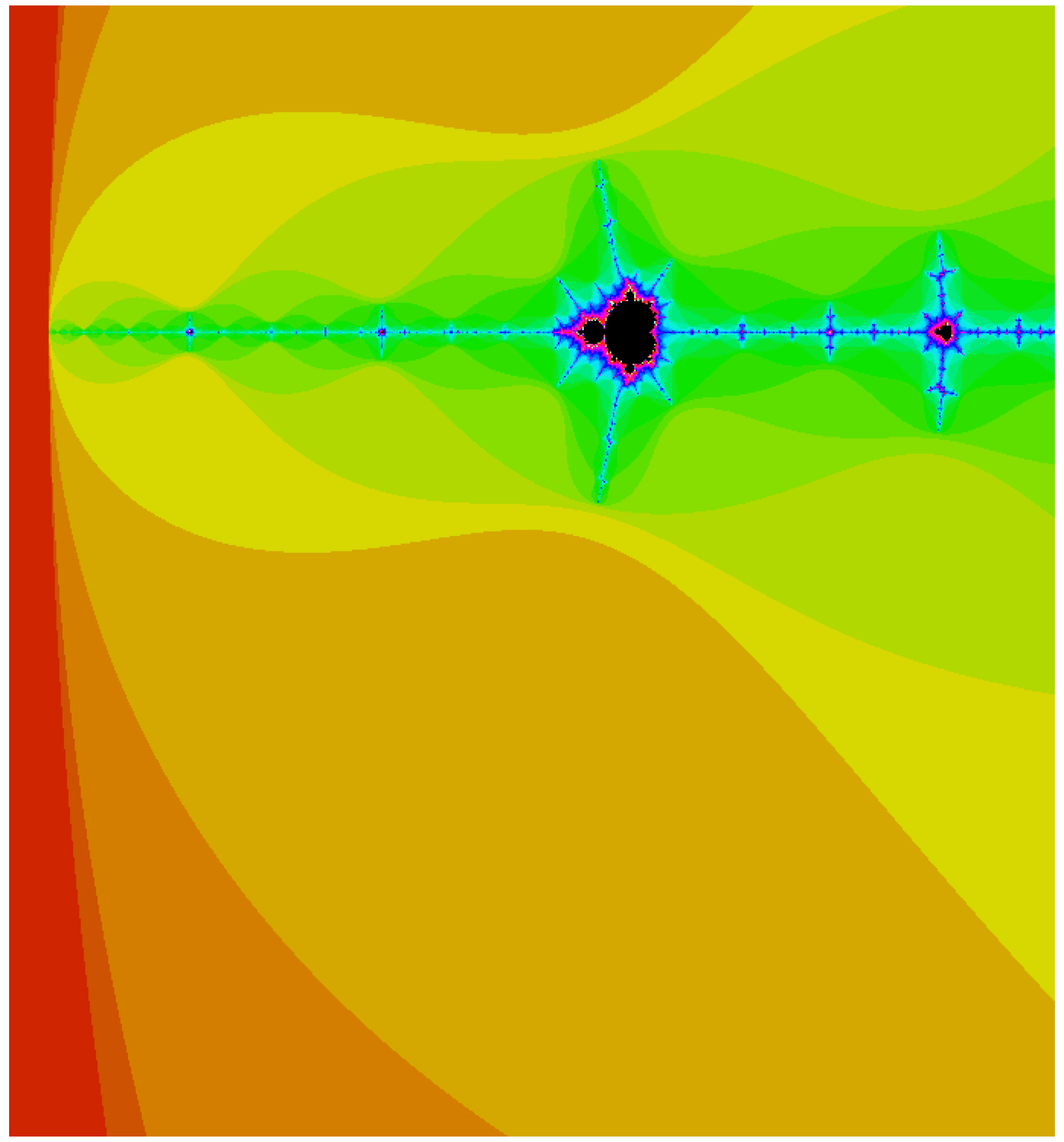
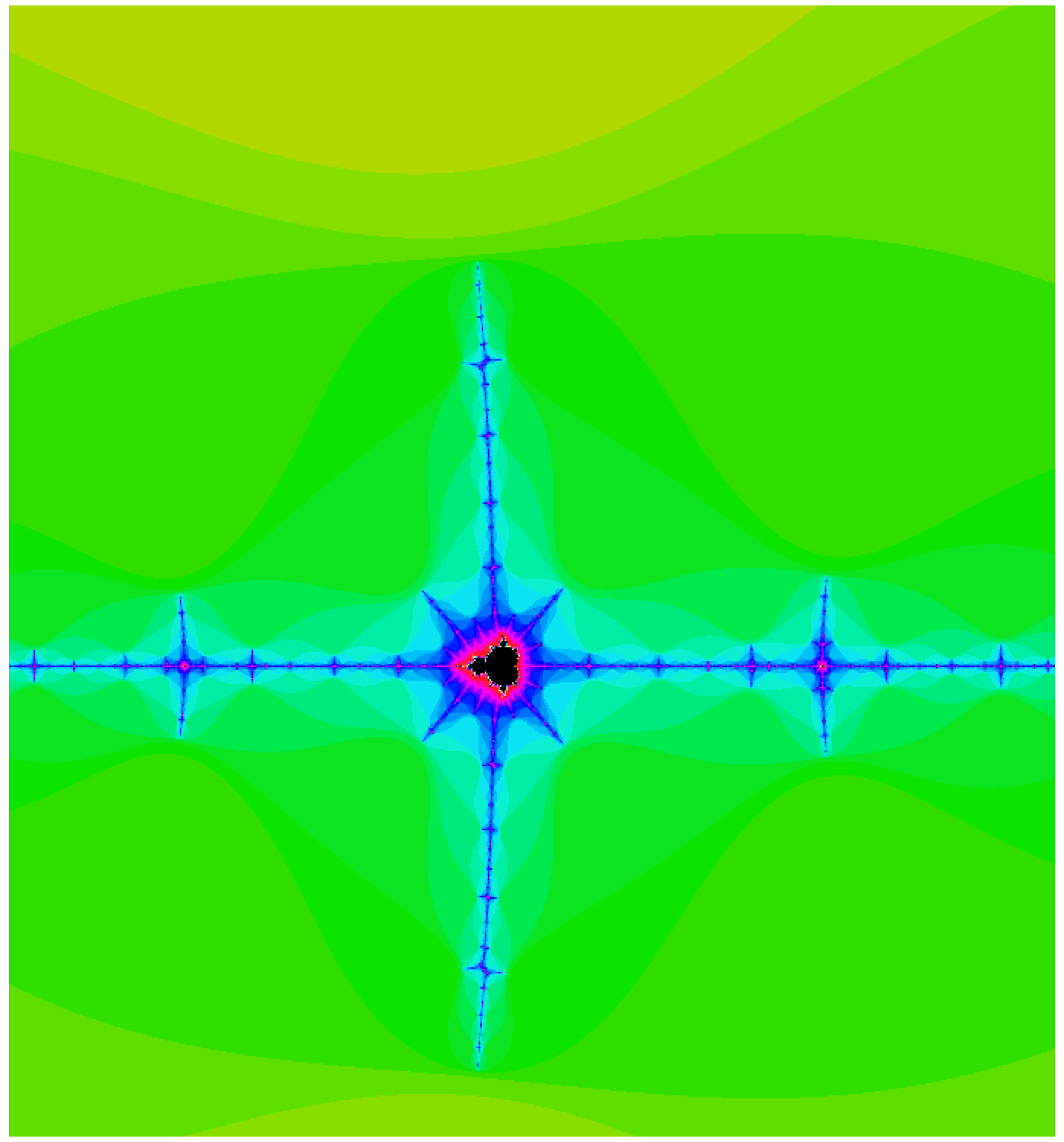
# Example: totally disconnected Julia set



$$f(z) = z^2 + 0.2 + i$$

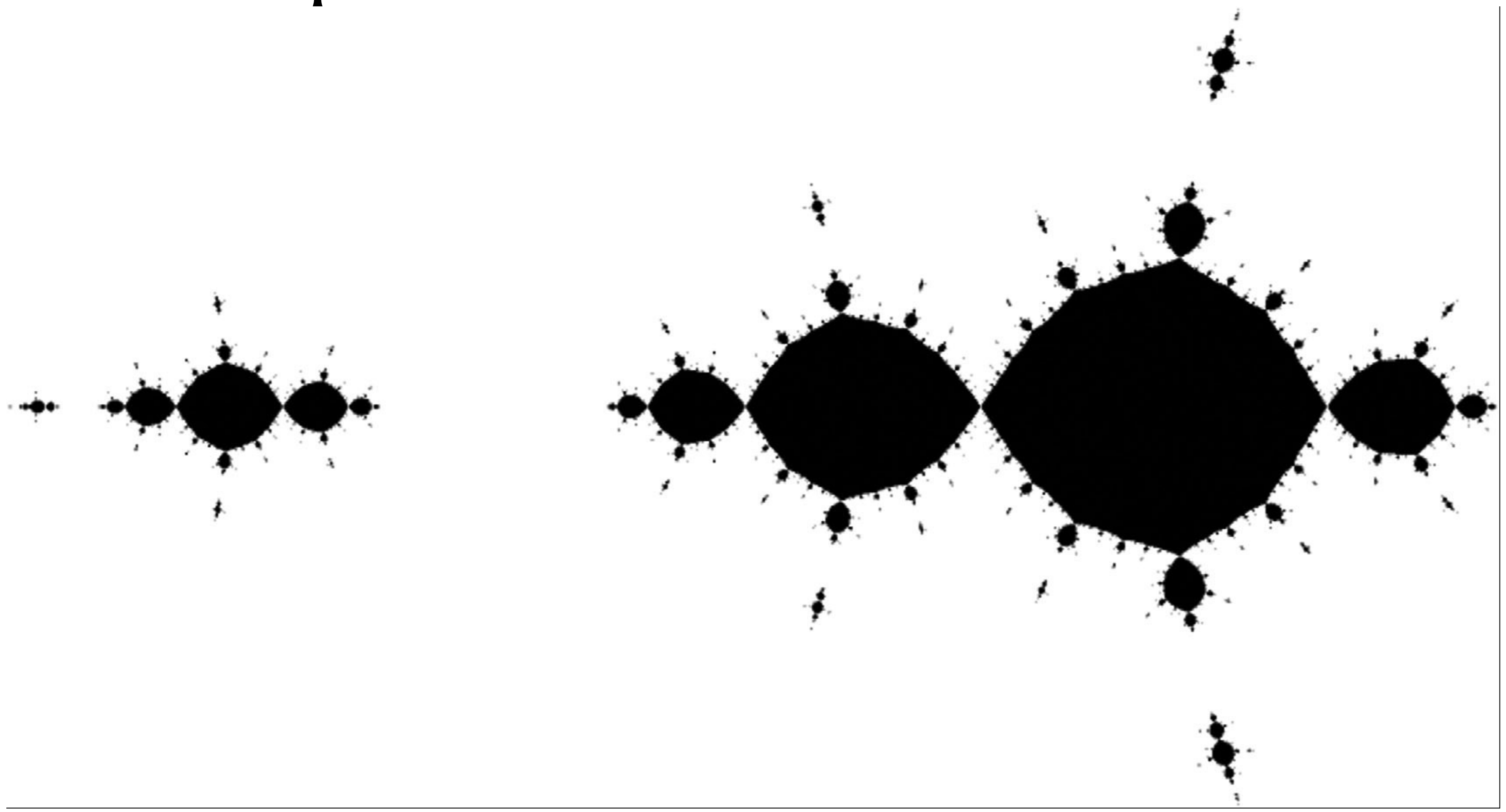
# The famous Mandelbrot set

$\mathscr{M}$ is connected: Douady, Hubbard; Sibony (independent)

Mandelbrot set $\mathscr{M}$ is in the space of parameters $c$: $c \in \mathscr{M}$ iff the Julia set of the polynomial $f_c(z) = z^2 + c$ is connected.

For quadratic polynomials the Julia set is either connected of totally disconnected.
For higher degree polynomials the situation and geometry of Julia sets is more involved. Here is the example:



$f(z) = az^3 + z^2 - b$, where $a = 0.215, b = 1.45$

Observe that the non- trivial components of $J(f)$ look similar to the „basilica" (there is in fact a quasiconformal homeomorphism mapping such a component to the basilica. There are also (uncountably many) components of $J(f)$ consisting of a single point! (not visible on this picture..)

## Tools to describe the geometry of these complicated „fractal" shapes?

Hausdorff dimension is a way of distinguishing the size of sets of (planar) measure zero. All Julia sets displayed on previous slides have planar measure zero (although, there are very specific, even quadratic polynomials for which the Julia set has positive Lebesgue measure).

Motivating example: The „standard" Cantor set with the interval $(1/3,2/3)$ and its copies removed has Hausdorff dimension $\dfrac{\log 2}{\log 3}$, while the Cantor set with the interval $(3/5,4/5)$ and its copies removed, has Hausdorff dimension $\dfrac{\log 4}{\log 5}$ which is larger.

## Formal definition of the (outer) Hausdorff measure in $\mathbb{R}^n$ is similar to that of Lebesgue measure:

Let $A \subset \mathbb{R}^n$. To define $h$ dimensional outer Hausdorff measure $\mathscr{H}^h(A)$ we

- choose $\delta > 0$
- consider all possible countable covers of $A$ with sets $C_i$ of diameter less than $\delta$
- calculate the sum $\sum_i (\mathrm{diam} A_i)^h$
- take infimum of these sums over all such covers, denote it by $\mathscr{H}^h_\delta(A)$
- let $\delta$ tend to zero, take the limit of $\mathscr{H}^h_\delta(A)$.

So, given a set $A \subset \mathbb{R}^n$, we can calculate $\mathscr{H}^h(A)$ for values $h \geq 0$. The value $h_0 = \dim_H(A)$ is uniquely determined by the condition: $\mathscr{H}^h(A) = \infty$ for $h < h_0$, $\mathscr{H}^h(A) = 0$ for $h > h_0$.

Informally speaking: $h_0 = \dim_H(A)$ is the right scale for measuring the „fractal" set $A$.

Example: every smooth, every rectifiable curve has Hausdorff dimension 1.

Hausdorff dimension of a measure: given a measure $\mu$ supported on a set $A$ we can (and we do) ask about Hausdorff dimension of the measure $\mu$:

$$dim_h(\mu) = \inf\{\dim_h(B) : B \subset A, \mu(B) > 0\}$$

Informally speaking: this value tells us what portion of the set $A$ is „seen" by the measure $\mu$.

## Hausdorff dimension of Julia set for a polynomial.

Theorem [rigidity:complicated or analytic]: Let $f$ be a polynomial with not totally disconnected Julia set. Then $dim_H(J(f)) > 1$, with only two exceptions:

$f(z) = z^d$ (then $J(f) = \{ |z| = 1 \}$ or

$f(z) = \pm$ Chebychev polynomial (then $J(f) = [-1,1]$)

(both: up to an affine change of variable).

Z. (connected) (1990)
Przytycki, Z. (not totally disconnected (2022)

Remark: it may even happen that every non- trivial connected component of $J(f)$ is an analytically embedded interval, but, still, the dimension of the whole Julia set is larger than one.

## Hausdorff dimension of basins of attraction

Z.(1991), Przytycki (2006)

Theorem [rigidity:complicated or a circle] Let $f$ be a rational function. If a simply connected domain $U$ is a (connected component of) the basin of attraction of an attracting periodic point then $dim_H(\partial U) > 1$ with only the (well described) few exceptions when the boundary of the basin is a circle or the interval.

Observation: actually, both above theorems say more: the hyperbolic dimension of $J(f)$ and of $\partial U$ is larger than 1. Hyperbolic dimension is the supremum of dimensions of *invariant measures* supported on $J(f)$ or $\partial U$, respectively.

Remark: A seemingly similar question: whether a connected Julia set of a rational function has dimension larger than 1 (except the above special cases) still remain open...

# Harmonic measure

Somewhat surprisingly, the above mentioned results are based on (very interesting themselves) questions about distribution of the harmonic measure.

Harmonic measure: If $U$ is a domain in $\hat{\mathbb{C}}$ (with non- polar boundary) and $h : \partial U \to \mathbb{R}$ - a continuous function then there exists a unique harmonic function $\hat{h} : U \to \mathbb{R}$ (harmonic extension) such that $\lim_{z\to\zeta} \hat{h}(z) = h(\zeta)$ for „nearly all" $\zeta \in \partial U$.

Harmonic measure „seen from a point $w \in U$":

$$\int_{\delta U} h(\zeta)\omega(z, d)(d\zeta) = \omega(z, U)(h) := \hat{h}(z)$$

Probabilistic approach: start a Brownian motion („random walk”) from a point $z \in U$. Then, with probability one, the trajectory exits the domain $U$. This defines a measure on $\partial U$: for (Borel) set $A \subset \partial U$ $\omega(z, U)(A)$ is the probability that the trajectory exits $U$ „through” $A$.

informally: open set „without holes”

If $U$ is simply connected then Riemann mapping theorem says that there exists a holomorphic bijection $R : \mathbb{D} \to U$. In general, this map does not extend to the boundary, but for (Lebesgue) almost all $\zeta \in \partial\mathbb{D}$ there is a radial (also: non-tangential) limit $\lim_{r\to 1} R(r\zeta)$. This defines (a.e) the extended map $\overline{R}\colon \partial\mathbb{D} \to \mathbb{C}$. The harmonic measure is just the push- forward of the Lebesgue measure under $\overline{R}$.

Hausdorff dimension of harmonic measure? What part of the boundary is „seen" by the Brownian motion? This question has been the subject of intense study for decades.

Theorem [Hausdorff dimension of harmonic measure] If U is a simply connected domain in the plane and $\omega = \omega(z, U)$ is the harmonic measure on $\partial U$, then $\dim_H(\omega) = 1$. For an arbitrary domain $U$ with non-polar boundary $\dim_H(\omega) \leq 1$ (Makarov, Rohde, Jones, Wolff)

Makarov's proof was based on probability tools: If $R : \mathbb{D} \to U$ is the Riemann map, consider the function $\log R'$. One can associate to this function a (complex valued) martingale. The growth of $R'$ was then controlled using a version of the Law of Iterated Logarithm for (real part of) this martingale.

Makarov's result says, in particular that there exists a universal constant $C > 0$ such that the harmonic measure $\omega(z, U)$ on $\partial U$ is absolutely continuous with respect to the Hausdorff measure $\mathscr{H}^\varphi$ associated to the function

$$\varphi(t) = t \exp\sqrt{C \log \frac{1}{t} \log\log\log \frac{1}{t}}.$$

Przytycki, Urbański, Z., Z.

For the „dynamical domains" (basins of attraction) discussed above we have, however, using also Law of Iterated Logarithm for some sequence of weakly dependent random variables, that for some $c \in (0,C)$ $\omega(z, U)$ is singular with respect to the Hausdorff measure $\mathscr{H}^\varphi$ associated to the function $\psi(t) = t \exp\sqrt{c \log \frac{1}{t} \log\log\log \frac{1}{t}}$.

Informally speaking, this means that these „dynamical domains" are the worst possible domains from the point of view of harmonic measure.

## Rigidity of harmonic measure

The theorem about rigidity of dimension can be reformulated and generalized in the following way:

Theorem. The Hausdorff dimension of the Julia set of a polynomial is larger than the Hausdorff dimension of the harmonic measure on it (apart from the mentioned analytic exceptions). The same applies for basins of attraction of a periodic attracting point. Informally speaking, most of the boundary is hidden and not visible for the Brownian motion starting from inside the domain. Z.

This general question whether (and when) dimension of harmonic measure is smaller than dimension of the set has been considered in various other settings, many of them not related to dynamics, with sometimes surprisingly difficult proofs (Carleson, Volberg, Batakis, Batakis-Z., Urbański-Z., Z., Tolsa, Azzam, David..) and with a lot of still open questions.

## One more open question related to the structure of harmonic measure

Brennan's conjecture: Let $G$ be a simply connected domain in $\hat{\mathbb{C}}$. The conjecture is about the growth of integral means for the Riemann map $R : \mathbb{D} \to G$

$$\beta_G(t) = \limsup_{r\to 1} \frac{\log \int |R'|^t(r\zeta)\,|d\zeta|}{\log \frac{1}{1-r}},$$ which in turn translates

into estimates of how many disjoint balls $B(z_j, \rho)$ with large harmonic measure (large: means: close to $\sqrt{\rho}$) the boundary may have („tops of spikes")

Equivalently, we ask about the integral $\int\int_{\mathbb{D}} |R'|^p\,dxdy$. The conjecture says

that this integral is finite for $-2 < p < 2/3$ (indeed, for $-1 < p < 2/3$ this is an easy consequence of classical Koebe distortion estimates).

Theorem: Brennan's conjecture is true for Julia set of a quadratic polynomial $f_c(z) = z^2 + c$ (provided it is connected). In other words: for every parameter in the Mandelbrot set.

Barański, Volberg, Z.

The proof is based on quite surprising geometric observations:

- to prove the conjecture it is enough to check that the filled-in Julia set $K(f)$ is contained in the area bounded by the ellipse with foci $\pm c$: $|z+c| + |z-c| = 4$. When $c$ is close to $-2$ then both the ellipse and $J(f_c)$ are close to the interval $[-2,2]$.
- to prove that the above condition holds we formulate another sufficient condition: the Mandelbrot set is contained in the closure of the area bounded by the curve $|c|^2(\mathrm{Re}c + 3) = 4$