Dual weak-type estimates for singular integral operators

Motivation

- Hardy-Littlewood max. operator: $f \in L^1_{loc}(\mathbb{R}^d)$

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f| \, dx \quad : \quad Q \text{ cube}, \quad x \in Q$$

- Weight: nonnegative locally integrable function ω

Associated measure: $\omega(A) = \int_A \omega \, dx$

Muckenhoupt's A_1 condition: $[\omega]_{A_1} = \operatorname{ess\,sup} \frac{M\omega}{\omega} < \infty$.

$\boxed{M\omega \leq [\omega]_{A_1} \omega}$ a.e.

Thm (Fefferman-Stein 1971) For any weight ω and any f,

$$\underbrace{\omega(\{x \in \mathbb{R}^d : \boxed{M} f(x) \geq 1\})}_{\|Mf\|_{L^{1,\infty}(\omega)}} \leq C_d \underbrace{\int_{\mathbb{R}^d} |f| \, M\omega \, dx}_{\|f\|_{L^1(M\omega)}}$$

Calderón – Zygmund singular integral operators

$K : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{ (x,x) : x \in \mathbb{R}^d \} \longrightarrow \mathbb{R}$

satisfying some boundedness and regularity conditions.

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^d} f(y) K(x,y)\, dy = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} f(y) K(x,y)\, dy$$

$f \in C_0^\infty(\mathbb{R}^d)$

If T admits an extension to $L^2(\mathbb{R}^d)$, then T is a C-Z operator.

<u>Prototypical example</u>

Hilbert transform on $\mathbb{R}$: $\quad Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$

$$K(x,y) = k(x-y) = \frac{1}{\pi} \cdot \frac{1}{x-y}.$$

Muckenhoupt - Wheeden conjectures

Strong: For any C-Z op. T there is a constant $C_{T,d} < \infty$ s.t. for all f, w

$$w(\{x \in \mathbb{R}^d : Tf(x) \geq 1\}) \leq C_{T,d} \int_{\mathbb{R}^d} |f| \, Mw \, dx$$

Weak: - - - - - - - - - - - - - - - - for all $f \in L^1_{loc}$, $w \in A_1$

$$w(\{x \in \mathbb{R}^d : Tf(x) \geq 1\}) \leq C_{T,d} [w]_{A_1} \int_{\mathbb{R}^d} |f| \, w \, dx$$

Both fail.

Lerner, Nazarov, Perez 2008

Substitute:

$$w(\{x \in \mathbb{R}^d : Tf(x) \geq 1\}) \leq C_{T,d} \underbrace{[w]_{A_1} \log(e + [w]_{A_1})}_{\text{LlogL optimal 2016}} \int_{\mathbb{R}^d} |f| w \, dx$$

Dual versions

Strong $\quad w(\{x\in\mathbb{R}^d : Tf(x) \geq \underline{M\overset{w}{\cancel{f}}(x)}\}) \leq C_{T,d}\int_{\mathbb{R}^d}|f|\,dx$ FAILS

Weak $\quad w(\{x\in\mathbb{R}^d : Tf(x) \geq w(x)\}) \leq C_{T,d}\,[w]_{A_1}\int_{\mathbb{R}^d}|f|\,dx$ HOLDS

Why dual?

If the strong conjecture were true, then interpolation would give

$$\int_{\mathbb{R}^d}|Tf|^p w\,dx \leq C_{T,d,p}\int_{\mathbb{R}^d}|f|^p\left(\frac{Mw}{w}\right)^p w\,dx \qquad 1<p<\infty$$

Duality:

$$\int_{\mathbb{R}^d}\left(\frac{|T^*f|}{Mw}\right)^{p'} w\,dx \leq C_{T,d,p}\int_{\mathbb{R}^d}\left(\frac{|f|}{\underline{w}}\right)^{p'}\underline{w}\,dx \qquad \underline{1}<p'<\infty$$

$p'\to 1$

Probabilistic setting

$(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_n)$ - atomic filtration ($\forall_n$, $\mathcal{F}_n$ is finite)
$\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathcal{A}$ - collection of all atoms of $(\mathcal{F}_n)_{n \geq 0}$. ($B \in \mathcal{A}$, if B is an atom of some $\mathcal{F}_n$)

Example $([0,1]^d, \mathcal{B}([0,1]^d), |\cdot|)$ with the dyadic filtration.

$$\mathcal{M}f(\omega) = \sup\left\{ \frac{1}{\mathbb{P}(A)} \int_A |f| \, d\mathbb{P} : A \in \mathcal{A}, \ \omega \in A \right\}$$

Carleson sequences

A sequence $\alpha = (\alpha_Q)_{Q \in \mathcal{A}} \subseteq [0,\infty)$ has the Carleson property, if

$$\forall_{R \in \mathcal{A}} \quad \sum_{\substack{Q \subseteq R \\ Q \in \mathcal{A}}} \alpha_Q \, \mathbb{P}(Q) \leq \mathbb{P}(R).$$

$\equiv$ there exists a collection $(E(Q))_{Q \in \mathcal{A}}$ of pairwise disjoint events s.t. $\forall_{Q \in \mathcal{A}}$, $E(Q) \subseteq Q$ and $\mathbb{P}(E(Q)) = \alpha_Q \mathbb{P}(Q)$.

The associated shift operator:

$$S^\alpha f = \sum_{Q \in \mathcal{A}} \alpha_Q \, \mathbb{E}(f|Q) \, \mathbb{1}_Q.$$

(Lerner ~ 2005)

Example: If we fix n, $\alpha_Q = 1$ if $Q \in \mathcal{A} \cap \mathcal{F}_n$, $\alpha_Q = 0$ otherwise.

$$\Rightarrow S^\alpha f = f_n.$$

Main theorem

$Mw \leq [w]_{A_1} w$ a.s.

If α is a Carleson sequence, then for any $f \in L^1$, $w \in A_1$,

$$w(S^\alpha f \geq w) \leq 8 [w]_{A_1} \int_\Omega |f| \, d\mathbb{P}.$$

$\hookrightarrow$ C-Z operators

Thm (Lerner - Nazarov)

If T is a C-Z operator and $f \in L^1(\mathbb{R}^d)$, then there exist 3^d dyadic lattices, and Carleson sequences s.t.

$$|Tf| \leq C_{T,d} \sum_{j=1}^{3^d} S^{\alpha^j} |f|.$$

$w \in \mathbb{R}^d$

- $Mw \leq [w]_{A_1} w$

$M_d w \leq [w]_{A_1} w.$

Thm

Suppose that $w \in A_1$ and α is a Carleson sequence. Then

$$\| S^{\alpha} f \|_{L^2(w^{-1})} \leq 4 [w]_{\underset{A_2}{\underline{\underline{A_1}}}} \| f \|_{L^2(w^{-1})} .$$

Proof: postponed.

$$w \in A_1 \Rightarrow \boxed{w \in A_2} \Longleftrightarrow \underline{w^{-1} \in A_2}$$

$$[w]_{A_2} \leq [w]_{A_1}$$

Bellman function method for shift operators

$V : [0,\infty)^3 \to \mathbb{R}$, $c \geq 1$ constant. Suppose we want

$$\mathbb{E}\, V(f, S^\alpha f, w) \leq 0 \qquad (*)$$

for all $f \in L^1$, Carleson sequences α, $w \in A_1$ with $[w]_{A_1} \leq c$.

($V(x,y,u) = u\,\mathbb{1}_{\{y \geq u\}} - 8c|x|$ $\longrightarrow$ desired estimate)

Let $D = \{ (x,y,u,v,t) \in [0,\infty)^4 \times [0,1] : c^{-1} v \leq u \leq v \}$

Suppose $B : D \to \mathbb{R}$ satisfies the following:

$$\mathbb{E}V(f, S^\alpha f, w) \overset{2^\circ}{\leq} \mathbb{E}B(\overbrace{f, S^\alpha f, w, *, *}) \overset{3^\circ+4^\circ}{\leq} \mathbb{E}B(\ldots\ldots) \overset{3^\circ+4^\circ}{\leq} \mathbb{E}B(\ldots\ldots) \leq \ldots\ldots \overset{3^\circ+4^\circ}{\leq} \mathbb{E}B(\ldots\ldots) \underset{1^\circ}{\leq} 0.$$

1° $B(x, 0, u, v, t) \le 0$

2° $B(x, y, u, v, t) \ge V(x, y, u)$

3° For all (x, y, u, v, t) and $\delta \in [0, t]$,

$$B(x, y, u, v, t) \geqslant B(x, y+\delta x, u, v, t-\delta)$$

4° For all (x, y, u, v, t), $(x_\pm, y, u_\pm, v_\pm, t_\pm) \in D$

and all $\lambda_\pm \in [0, 1]$ s.t. $\lambda_- + \lambda_+ = 1$, $x = \lambda_- x_- + \lambda_+ x_+$

$u = \lambda_- u_- + \lambda_+ u_+$, $t = \lambda_- t_- + \lambda_+ t_+$, $v_\pm = \max\{u_\pm, v\}$,

$$B(x, y, u, v, t) \geqslant \lambda_- B(x_-, y, u_-, v_-, t_-) + \lambda_+ B(x_+, y, u_+, v_+, t_+).$$

<u>Thm</u> (X) holds $\Longleftrightarrow$ <u>there exists B satisfying $1° - 4°$.</u>

How to find B ????

Hint. Burkholder's estimate for martingale transforms.

$f = (f_n)$, $g = (g_n)$, g is the transf. of f: $dg_n = \underbrace{v_n}_{\text{predictable}, \in [-1,1]} df_n$

Burkholder's method for martingale transforms:

If one wants to show $\boxed{\mathbb{E} V(f_n, g_n) \le 0,}$

it is enough to find B s.t.

1° $B(x,y) \le 0$ $\quad |y| \le |x|$

2° $B \ge V$

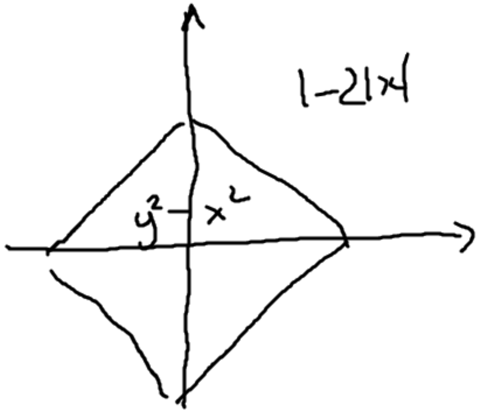
3° B is concave along lines of slope ± 1.

For example: $\quad \underline{\mathbb{E}|g_n|^2} \le \underline{\mathbb{E}|f_n|^2}$: $\quad V(x,y) = y^2 - x^2 = B(x,y)$

Another example: $\quad \mathbb{P}(|g_n| \geq 1) \le 2\,\mathbb{E}|f_n|$

$$V(x,y) = \mathbb{1}_{\{|y| \geq 1\}} - 2|x|$$

$$\underline{B(x,y)} = \begin{cases} y^2 - x^2 & |x| + |y| \le 1, \\ 1 - 2|x| & |x| + |y| > 1. \end{cases}$$



$$\boxed{= \begin{cases} \min\{y^2 - x^2,\ 1 - 2|x|\} & |x| \le 1, \\ 1 - 2|x| & |x| \geq 1 \end{cases}}$$

We know, that an L^2 bound for S^α holds.

$\Rightarrow$ there is an appropriate Bellman function for this estimate: $\overline{\mathbb{B}}$

For the weak-type estimate, one takes

$$\mathbb{B}(x,y,u,v,t) = \begin{cases} \min\{\overline{\mathbb{B}}(x,y,u,v,t),\ |u| - 8c|x|\} & |x| \le \frac{u}{4c} \\ u - 8c|x| & |x| > \frac{u}{4c} \end{cases}$$

$$V(x,y,u) = u\,\mathbb{1}_{\{|y| \geq u\}} - 8c|x|.$$