Dissipative dynamical systems and their attractors

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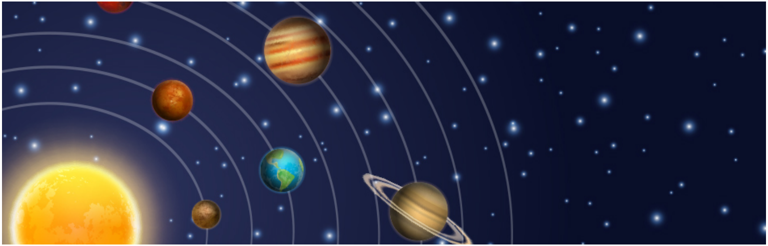
Plan of the talk

- Context: conservative and dissipative systems
- 3 Basic notions
- 10 Important problems of the theory.

Evolution of a conservative system

Example 1. Motivation: Is our solar system stable? (physical system)



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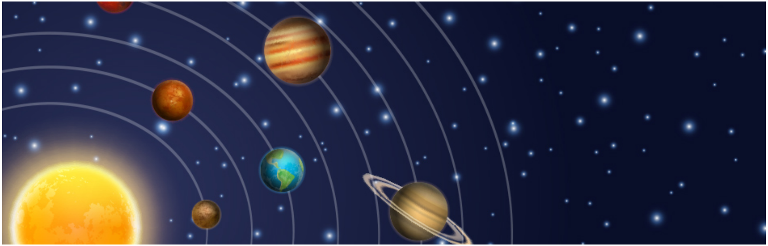
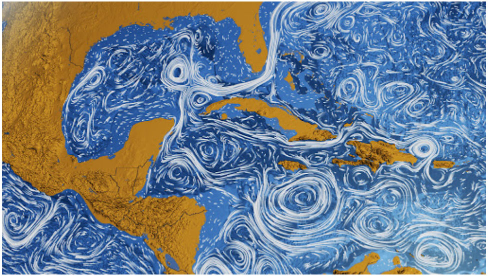


- Newtonian mechanics ∼ conservative system of ODEs in R^n
- system reversible in time ∼ groups ∼ deterministic chaos
- From P. S. de Laplace to H. Poincaré, and ...

Example 2. Motivation: How does turbulence in fluids develop?



Evolution of a dissipative system

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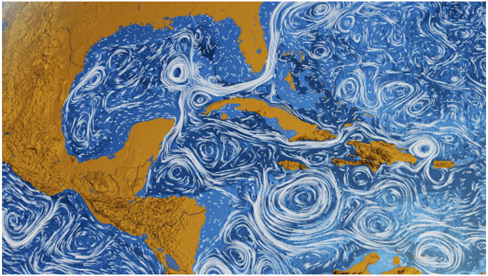

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Let us compare the above problems

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(Physics $\to$ Modelling $\to$ Qualitative theory of differential eqs. $\to$
$\to$Topological dynamics $\to$ Phenomenology $\to$ Physics)

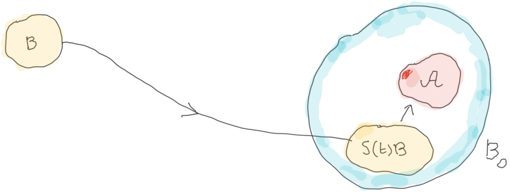
Dissipative system, time asymptotics, global attractor

DISSIPATIVENESS $\sim$ existence of a universal absorbing set B_0.

B_0 is open and bounded.

GLOBAL ATTRACTOR $\mathcal{A}$ (if it exists!) is a subset of the absorbing set B_0.

$\mathcal{A}$ is invariant, compact, and attracts all bounded sets of the phase space H.



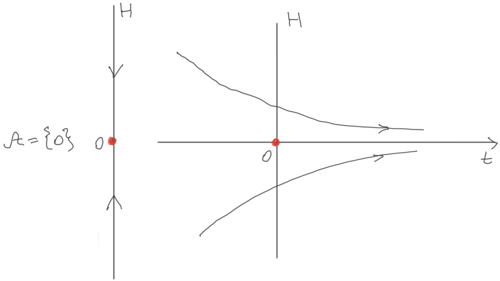
The simplest example (trivial dynamics)

Example 3. For $k > 0$,

$$\frac{du}{dt} = -ku, \quad u(t) = e^{-kt} u_0 = S(t)u_0, \quad \mathcal{A} = \{0\}, \quad B_0 = \mathcal{O}_\epsilon(\mathcal{A}).$$

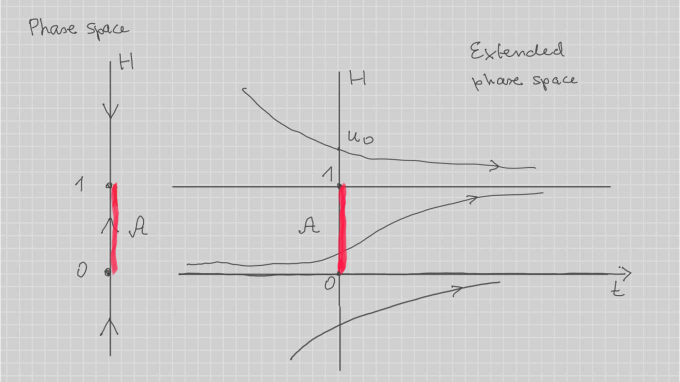
For $k \leq 0$ the system is not dissipative (for $k = 0$ is conservative).

Here $\{S(t) : t \in R\}$ is a group, $S(0) = Id, S(t+s) = S(t)S(s)$.



Example 4.

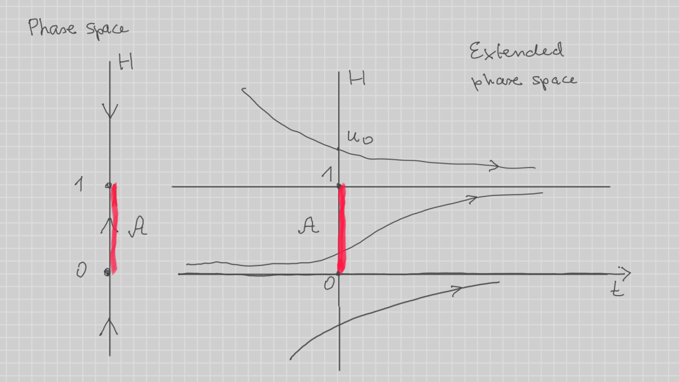
$$\frac{du}{dt} = |u|(1-u), \quad u(t) = S(t)u_0, \quad \mathcal{A} = [0,1], \quad B_0 = \mathcal{O}_\epsilon(\mathcal{A}).$$



$\mathcal{A}$ is maximal bounded invariant subset of H.

$\mathcal{A}$ is minimal closed attracting subset of H.

$\mathcal{A}$ is connected. If $\dim H = \infty$ then $\mathcal{A}$ has empty interior.



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- Structure of $\mathcal{A}$.

- Best estimates the fractal dimension of $\mathcal{A}$ (related to the complexity of the fluid flow).

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- How fast $\mathcal{A}$ attracts bounded sets?
  (related to numerical analysis of the fluid flow).
- How the dimension of $\mathcal{A}$ is related to boundary conditions ?
  (related to testing the model equations).

Some extensions of the theory

There are several theories. For example:

- We do not need the uniqueness of solutions. We can consider multivalued flows.
- We do not need the system to be autonomous. The system can be nonautonomous.

Reference: e.g. my book with Piotr Kalita:

G.Ł, P. Kalita: *Navier-Stokes Equations. An Introduction with Applications*, Springer 2016.

THANK YOU FOR YOUR ATTENTION !

谢谢