On coverings of Banach spaces and their subsets by hyperplanes

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Terminology

We consider only real Banach spaces of dimension bigger than 1.

Definition

A **hyperplane** of X $=$ a one-codimensional closed subspace of X $=$ kernel of a non-zero bounded functional on X.

We denote the set of all hyperplanes of X by $\mathcal{H}(X)$.

We define the σ-ideal of subsets of X that can be covered by countably many hyperplanes:

$$\mathcal{H}_\sigma(X) = \{Y \subseteq X : \exists \mathcal{F} \subseteq (X) \ \ Y \subseteq \bigcup \mathcal{F},\ \mathcal{F} \text{ countable}\}.$$

Terminology

$X \notin \mathcal{H}_\sigma(X)$, hyperplanes are nowhere dense subsets of X so the Baire cat. thm applies.

- $\mathfrak{add}(X)$ = the minimal cardinality of a family of sets from $\mathcal{H}_\sigma(X)$ whose union is not in $\mathcal{H}_\sigma(X)$
- $\mathfrak{cov}(X)$ = the minimal cardinality of a family of sets from $\mathcal{H}_\sigma(X)$ whose union is equal to X = minimal number of hyperplanes that we need to cover X.
- $\mathfrak{non}(X)$ = the minimal cardinality of a subset of X that is not in $\mathcal{H}_\sigma(X)$
- $\mathfrak{cof}(X)$ = the minimal cardinality of a family of sets from $\mathcal{H}_\sigma(X)$ such that each member of $\mathcal{H}_\sigma(X)$ is contained in some element of that family.

Obs: $cov(X) \geq \omega_1$. $A_\alpha \in \mathcal{H}_\sigma(X)$

If we have $\{A_\alpha\}_{\alpha<\kappa}$, $\bigcup A_\alpha = X$, then we extend $A_\alpha \subseteq \bigcup_n H_{\alpha,n}$

$$\bigcup_{\substack{\alpha<\kappa \\ n\in\mathbb{N}}} H_{\alpha,n} = X$$

$\mathfrak{add}$ and $\mathfrak{cof}$

Theorem

For any Banach space X we have: $\mathfrak{add}(X) = \omega_1, \mathfrak{cof}(X) = |\mathcal{H}| = |X^*|$.

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Lemma

If a hyperplane $H \in \mathcal{H}(X)$ is included in a countable union of hyperplanes $\bigcup H_i$, $H_i \in \mathcal{H}(X)$, then $H = H_i$ for some i.

Proof: Assume that $H \neq H_i$ for $i \in \mathbb{N}$. Consider $H \cap H_i \neq H$.

hyperplanes in H

$H \cap H_i$ is a nowhere dense subset of H.

Baire category thm $\Rightarrow$ $\bigcup H \cap H_i \not\supseteq H$ – contradiction.

$\mathfrak{add}$ and $\mathfrak{cof}$

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Lemma

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Proof that add$(X) = \omega_1$: Take ω_1 hyperplanes $\{H_\alpha : \alpha < \omega_1\}$ and assume that

$\bigcup_{\alpha<\omega_1} H_\alpha \subseteq \bigcup_{n\in\mathbb{N}} G_n$. By Lemma we get that $H_\alpha = G_n$ for some n, so $\{H_\alpha : \alpha < \omega_1\}$

is in fact countable.

For cof(X) use the fact that $|X^*| = |X^*|^\omega$

Separable spaces

Proposition

If X is a separable Banach space, then $\mathfrak{cov}(X) = \mathfrak{c}$ and $\mathfrak{non}(X) = \omega_1$.

Both equalities follow from the following Klee's result:

Theorem

Let X be a separable Banach space. Then there is a set $Y \subseteq X$ of cardinality $\mathfrak{c}$ such that each infinite subset of Y is linearly dense in X. In particular $H \cap Y$ is finite for every $H \in \mathcal{H}(X)$

Linearly dense means that $\operatorname{Lin}\{y : y \in Y\}$ is dense. If $|H \cap Y| = \infty$ then $H \cap Y$ is linearly dense, but H is closed linear subspace, so $H = X$.

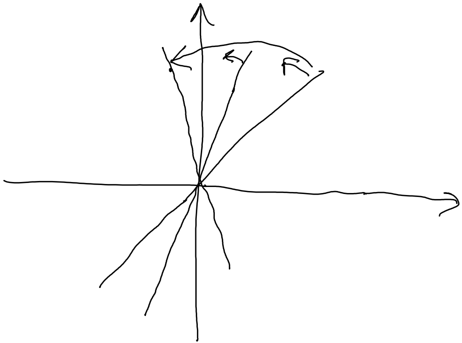
Proof that $\operatorname{non}(X) = \omega_1$: Take any $Z \subseteq Y$, $|Z| = \omega_1$. If $Z \subseteq \bigcup_{n \in \mathbb{N}} H_n$, then $|Z \cap H_n| < \infty$, so $\sum |Z \cap H_n| \leq \omega$. We get contradiction with $|Z| = \omega_1$.

For $\operatorname{cov}(X)$ assume that X can be covered by $\kappa < \mathfrak{c}$ many hyperplanes and take Z as above but of cardinality κ.

Nonseparable spaces: $\mathfrak{cov}$

Proposition

$\mathfrak{cov}(X) \leq \mathfrak{c}$ for every Banach space X. In particular CH implies that $\mathfrak{cov}(X) = \omega_1$ for each X.



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Proposition

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Theorem

Equality $\mathfrak{cov}(X) = \omega_1$ holds in ZFC for the following classes of nonseparable Banach spaces:

- spaces of density ω_1

Take any dense $\{x_\alpha : \alpha < \omega_1\} \subseteq X$. Consider spaces $X_\alpha = \overline{\mathrm{lin}\{x_\beta : \beta < \alpha\}}$

Each $x \in X$ is in some X_α ($X = \bigcup_{\alpha < \omega_1} X_\alpha$).

There is a countable $A \subseteq \omega_1$ such that $x = \mathrm{lin}_{\alpha \in A}\, x_\alpha$. $x \in X_\gamma$, $\gamma = \sup(A) + 1$.

Now we can we can extend X_α to hyperplanes.

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Theorem

Equality $\mathfrak{cov}(X) = \omega_1$ holds in ZFC for the following classes of nonseparable Banach spaces:

- spaces of density ω_1
- spaces with fundamental biorthogonal system

$(x_i, x_i^*)_{i \in I}$ $x_i \in X$, $x_i^* \in X^*$ is biorthogonal system if $x_i^*(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

We say that such system is fundamental if $X = \overline{\mathrm{lin}\{x_i : i \in I\}}$.

If X - a Hilbert space, $\{x_i : i \in I\}$ - orthonormal basis, $x_i^* = \langle x_i, - \rangle$

ℓ_p, c_0, $\ell_p(\kappa) = \{ (x_\alpha)_{\alpha < \kappa} : \sum |x_\alpha|^p < \infty \}$ with the norm $\|(x_\alpha)_{\alpha<\kappa}\| = \left(\sum |x_\alpha|^p\right)^{\frac{1}{p}}$.

Nonseparable spaces: $\mathfrak{cov}$

Proposition

$\mathfrak{cov}(X) \leq \mathfrak{c}$ for every Banach space X. In particular CH implies that $\mathfrak{cov}(X) = \omega_1$ for each X.

Theorem

Equality $\mathfrak{cov}(X) = \omega_1$ holds in ZFC for the following classes of nonseparable Banach spaces:

- spaces of density ω_1
- spaces with fundamental biorthogonal system
- spaces $C(K)$ where K is compact and does not have small diagonal
- spaces X such that B_{X^*} (with the weak* topology) does not have small diagonal

$C(K) = \{ f : K \to \mathbb{R} \text{ cont.} \}$ with $\|f\| = \sup_{x \in K} \{ |f(x)| \}$.

Small diagonals

Definition

We say that a topological space K has small diagonal if for every uncountable $A \subseteq K^2 \backslash \Delta(K)$ there is uncountable $B \subseteq A$ such that $\overline{B} \subseteq K^2 \backslash \Delta(K)$.

$\Delta(K) = \{(x,x) : x \in K\}$

Small diagonals

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Consider the following sentence:

$$\text{For every compact Hausdorff space } K, K \text{ is metrizable } \iff K \text{ has small diagonal.} \qquad (*)$$

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Consider the following sentence:

$$\text{For every compact Hausdorff space } K, K \text{ is metrizable} \iff K \text{ has small diagonal.} \qquad (*)$$

Theorem (Dow-Juhasz-Szentmiklossy)

- PFA $\implies (*)$
- $(*)$ is consistent with any possible size of $\mathfrak{c}$ – start with $V \models CH$ and add κ many Cohen reals.

Open question: Is $(*)$ provable in ZFC?

Nonseparable spaces: $\mathfrak{cov}$

Proposition

Assume that all compact spaces with small diagonals are metrizable. Then $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.

X - nonseparable $\Rightarrow$ B_{X^*} is compact and not metrizable $\Rightarrow$ B_{X^*} does not have small diagonal $\Rightarrow$ $\mathrm{cov}(X) = \omega_1$.

Nonseparable spaces: $\mathfrak{cov}$

Proposition

Assume that all spaces with small diagonals are metrizable. Then $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.

Corollary

- PFA implies that $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.
- It is consistent with any possible size of $\mathfrak{c}$ that $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.

Question

Is it provable in ZFC that $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces?

Nonseparable spaces: $\mathfrak{non}$

General inequality

Let X be a nonseparable Banach space of density κ. Then

density of X = the minimal cardinality of dense subset in X.

$$\kappa \leq \mathfrak{non}(X) \leq \mathrm{cf}([\kappa]^\omega).$$

If κ has countable cofinality, then $\mathfrak{non}(X) > \kappa$.

- $[\kappa]^\omega$ = the family of all countable subsets of κ
- $\mathrm{cf}([\kappa]^\omega)$ = the minimal cardinality of a family $\mathcal{F} \subseteq [\kappa]^\omega$ such that each member of $[\kappa]^\omega$ is included in some member of $\mathcal{F}$.

Proof of $\kappa \leq \mathrm{non}(X)$: Assume that $Y \subseteq X$, $|Y| < \kappa$. So $\overline{\mathrm{lin}\{y : y \in Y\}}$ is a proper closed subspace of X, so it is contained in some hyperplane.

If $\mathrm{cf}(\kappa) = \omega$ and we have $Y \subseteq X$, $|Y| = \kappa$, then $Y = \bigcup_{n \in \mathbb{N}} Y_n$, $|Y_n| < \kappa$. $Y_n \subseteq H_n$ so $Y \subseteq \bigcup_{n \in \mathbb{N}} H_n$.

$\mathrm{non}(X) \le \mathrm{cf}([\kappa]^\omega)$: $\qquad \mathrm{dens}(X)=\kappa$.

We start with a dense $\{x_\alpha : \alpha<\kappa\} \subseteq X$.

We fix $\mathcal{F}$ - a cofinal family in $[\kappa]^\omega$, $|\mathcal{F}| = \mathrm{cf}([\kappa]^\omega)$

For each $F \in \mathcal{F}$ consider the $X_F = \overline{\mathrm{lin}\{x_\alpha : \alpha \in F\}}$ - separable.

Pick $Y_F \subseteq X_F$ such that Y_F can't be covered by ω hyperplanes in X_F.

$|Y_F| = \omega_1$. $\quad Y = \bigcup_{F\in\mathcal{F}} Y_F$. $\quad |Y| = |Y_F| \cdot \mathrm{cf}([\kappa]^\omega) = \omega_1 \cdot \mathrm{cf}([\kappa]^\omega) = \mathrm{cf}([\kappa]^\omega)$.

We need to show that Y can't be covered by countably many hyperplanes.

Assume $Y \subseteq \bigcup_{n\in\mathbb{N}} H_n$. For each n pick α_n s.t. $x_{\alpha_n} \notin H_n$. There is a set $F \in \mathcal{F}$ such that $\{\alpha_n : n \in \mathbb{N}\} \subseteq F$. Look at X_F. For each n $X_F \cap H_n$ is a hyperplane in X_F.

So $Y_F \subseteq \bigcup_{n\in\mathbb{N}} (H_n \cap X_F)$ - contradiction.

Values of $\mathrm{cf}([\kappa]^\omega)$

Proposition

$\mathrm{cf}([\omega_n]^\omega) = \omega_n$ for $n = 1, 2, 3, \ldots$

$\mathrm{cf}([\omega_1]^\omega) \geqslant \omega_1$.

$\{\alpha : \alpha < \omega_1\}$ is a cofinal family in $[\omega_1]^\omega$.

$|A| = \omega \Rightarrow \quad A \subseteq \sup(A)+1 \in \{\alpha : \alpha < \omega_1\}$.

For ω_n use induction.

Values of $\mathrm{cf}([\kappa]^\omega)$

Proposition

$\mathrm{cf}([\omega_n]^\omega) = \omega_n$ for $n = 1, 2, 3, \ldots$

Corollary

If X is a Banach space of density ω_n, then $\mathfrak{non}(X) = \omega_n$.

Values of $\mathrm{cf}([\kappa]^\omega)$

Proposition

$\mathrm{cf}([\omega_n]^\omega) = \omega_n$ for $n = 1, 2, 3, \ldots$

Corollary

If X is a Banach space of density ω_n, then $\mathfrak{non}(X) = \omega_n$.

Theorem

Assume GCH or MM. Then for $\kappa > \omega$

- $\mathrm{cf}([\kappa]^\omega) = \kappa$, if $\mathrm{cf}(\kappa) > \omega$
- $\mathrm{cf}([\kappa]^\omega) = \kappa^+$, if $\mathrm{cf}(\kappa) = \omega$

In particular, if X has density κ, then $\mathfrak{non}(X) = \mathrm{cf}([\kappa]^\omega)$.

GCH $\Rightarrow$ $\kappa^\omega = \kappa$ if $\mathrm{cf}(\kappa) > \omega$.

If one of those eq. fails then there is an inner model with a measurable cardinal.

Theorem (Magidor)

Assume that there is a supercompact cardinal. Then for each $n \in \mathbb{N}$ it is consistent that $\mathrm{cf}([\omega_\omega]^\omega) = \omega_{\omega+n}$.

Question

Is it possible that $\mathfrak{non}(X) < \mathrm{cf}([\kappa]^\omega)$ for some Banach space of density $\kappa > \omega$?

Values of $\mathrm{cf}([\kappa]^\omega)$

Theorem (Magidor)

Assume that there is a supercompact cardinal. Then for each $n \in \mathbb{N}$ it is consistent that $\mathrm{cf}([\omega_\omega]^\omega) = \omega_{\omega+n}$.

Question

Is it possible that $\mathfrak{non}(X) < \mathrm{cf}([\kappa]^\omega)$ for some Banach space of density $\kappa > \omega$?

Proposition

If X has density $\kappa > \omega$ and admits a fundamental biorthogonal system, then $\mathfrak{non}(X) = \mathrm{cf}([\kappa]^\omega)$.

$\ell_\infty^c(\Gamma)$ – the subspace of $\ell_\infty(\Gamma)$ consisting of sequences with countable supports.

Definition

Let X be a Banach space of density κ. We say that a subset $Y \subseteq X$ is overcomplete, if $|Y| = \kappa$ and each subset of Y of cardinality κ is linearly dense in X.

Examples of spaces that admit overcomplete sets:

- separable Banach spaces – result by Klee
- $\ell_p(\omega_1)$ for $p \in (1, \infty)$ } special cases of the result by P Koszmider on WLD spaces.
- $c_0(\omega_1)$

$\shortparallel$

$\{(x_\alpha)_{\alpha<\omega_1} : \forall_{\varepsilon>0}$ we have $|x_\alpha| > \varepsilon$ only for finitely many $\alpha\}$.

$\|(x_\alpha)_{\alpha<\omega_1}\| = \sup_{\alpha<\omega_1} |x_\alpha|$

Overcomplete sets

Proposition

If $\mathrm{cf}(\mathrm{dens}(X)) > \mathfrak{cov}(X)$, then X does not admit an overcomplete set.

Corollary

Assume that all compact Hausdorff spaces with small diagonals are metrizable. Then no Banach space of density ω_2 admits an overcomplete set.