Countable spaces, realcompactness, and cardinal characteristics

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Definitions and basic facts

All spaces are Tychonoff.

A space X is *realcompact*, if X can be closely embedded into $\mathbb{R}^\kappa$ for some cardinal κ.

The minimal κ as above is denoted by $Exp(X)$.

All Lindelöf spaces are realcompact. Thus all metrizable separable and σ-compact spaces are realcompact.

$kc(X)$ is the minimal cardinality of a cover of X by compact subsspaces. $kc^*(X) := kc(\beta X \setminus X)$.

Motivation

Theorem (van Douwen 1984)

For every $\kappa \leq \mathfrak{c}$ there exists a metrizable separable space X with $Exp(X) = \kappa$.

Theorem (Hechler)

$Exp(\mathbb{Q}) = \mathfrak{d}$.

Question

Which cardinals can be realized as $Exp(X)$ for a countable crowded space X?

Main result

Theorem (AMZ 2023)

Let κ be an infinite cardinal. Then there exists a countable crowded space X with $Exp(X) = \kappa$ iff $\mathfrak{p} \leq \kappa \leq \mathfrak{c}$. $\square$

The proof consists of four parts:

- No $\kappa < \mathfrak{p}$ can serve as $Exp(X)$ for a countable crowded space;
- Producing an example for $\kappa = \mathfrak{p}$; *the core of the proof;*
- Modifying an example for $\mathfrak{p}$ in such a way that it gets Exp equal to any given $\kappa \in [\mathfrak{p}, \mathfrak{c}]$;
- No $\kappa > \mathfrak{c}$ can serve as $Exp(X)$ for a countable crowded space.

Proposition

Let X be a Lindelöf space. Then $Exp(X) = \max\{w(X), kc^*(X)\}$.

Proof. Set $\kappa = Exp(X)$ and $\kappa' = \max\{w(X), kc^*(X)\}$.

$\kappa \leq \kappa'$: Fix a compactification γX such that $w(\gamma X) = w(X)$. Fix a compact cover $\{K_\xi : \xi \in \kappa'\}$ of $\gamma X \setminus X$.
Using the lindelöfness, find (exercise) a continuous $f_\xi : \gamma X \longrightarrow [0,1]$ for $\xi \in \kappa'$ such that $f_\xi(z) = 0$ for every $z \in K_\xi$ and $f_\xi(z) > 0$ for every $z \in X$. Set $\mathcal{F} = \{f_\xi : \xi \in \kappa'\}$.

Fix a collection $\mathcal{G}$ of size at most $w(\gamma X) = w(X)$ consisting of continuous functions $g : \gamma X \to \mathbb{R}$ that separates points of γX.
Define $\phi : \gamma X \to \mathbb{R}^{\mathcal{F} \cup \mathcal{G}}$ by $\phi(z)(f) = f(z)$, where $z \in \gamma X$ and $f \in \mathcal{F} \cup \mathcal{G}$.

ϕ is an embedding "thanks" to $\mathcal{G}$. Thus

$$\phi[X] = \phi[\gamma X] \cap \left((0, \infty)^{\mathcal{F}} \times \mathbb{R}^{\mathcal{G}}\right),$$

and hence $\kappa = Exp(X) = Exp(\phi[X]) \leq \max\{|\mathcal{F}|, |\mathcal{G}|\} = \kappa'$.

Main tool, P. 2

$\kappa' \leq \kappa$: Assume that X is a closed subspace of $(0,1)^\kappa$, and let $Z = cl(X)$, where the closure is taken in $[0,1]^\kappa$. Z is a compactification of X.

Denote by $\pi_\xi : [0,1]^\kappa \longrightarrow [0,1]$ the natural projection on the ξ-th coordinate. For every $z \in Z \setminus X$ there exists $\xi \in \kappa$ such that $z(\xi) \in \{0,1\}$. Therefore

$$Z \setminus X = \bigcup_{\xi \in \kappa} (\pi_\xi^{-1}[\{0,1\}] \cap Z).$$

Each $\pi_\xi^{-1}[\{0,1\}] \cap Z$ is compact, hence $kc^*(X) \leq \kappa$. Since also $w(X) \leq w\big((0,1)^\kappa\big) = \kappa$, it follows that $\kappa' \leq \kappa$. $\square$

Van Douwen's and Hechler's results

- Let $X \subset [0,1]$ be a Bernstein set, $\kappa \leq \mathfrak{c}$ an infinite cardinal, and $X_\kappa \supset X$ such that $|[0,1] \setminus X_\kappa| = \kappa$. Then $Exp(X_\kappa) = \kappa$.

- Since $kc^*(\mathbb{Q}) = kc([0,1] \setminus \mathbb{Q}) = \mathfrak{d}$, $Exp(\mathbb{Q}) = \mathfrak{d}$.

Main tool 2

Proposition

Let X be a Lindelöf space. Assume that $n \in \omega$ and $X_0, \ldots, X_n$ are Lindelöf subspaces of X such that $X = X_0 \cup \cdots \cup X_n$. Then

$$Exp(X) \leq \max\{Exp(X_0), \ldots, Exp(X_n), w(X)\}.$$

Proof.

A straightforward verification that $kc^*(X)$ is also bounded by the maximum above, using a compactification γX of X of weight $w(\gamma X) = w(X)$. □

$< \mathfrak{p}$ is impossible, P. 1

Theorem

Let $\kappa < \mathfrak{p}$ be an infinite cardinal, and let X be a countable crowded subspace of ω^κ. Then X is not closed in ω^κ.

Proof. Define

$$\mathbb{P} = \{x \upharpoonright a : x \in X \text{ and } a \in [\kappa]^{<\omega}\}.$$

Given $s, t \in \mathbb{P}$, declare $s \leq t$ if $s \supseteq t$.

$\mathbb{P}$ is σ-centered: $\mathbb{P} = \bigcup_{x \in X} \{x \upharpoonright a : a \in [\kappa]^{<\omega}\}$. Given $x \in X$ and $a \in [\kappa]^{<\omega}$, define

- $D_x = \{s \in \mathbb{P} : s(\xi) \neq x(\xi) \text{ for some } \xi \in \mathrm{dom}(s)\}$,
- $D_a = \{s \in \mathbb{P} : s = x \upharpoonright b \text{ for some } x \in X \text{ and } b \in [\kappa]^{<\omega} \text{ such that } b \supseteq a\}$.

D_x's and D_a's are dense in $\mathbb{P}$. Bell's Theorem yields a filter G on $\mathbb{P}$ that meets all of these dense sets. Then $\bigcup G \in cl(X) \setminus X$, where cl denotes closure in ω^κ. □

$< \mathfrak{p}$ is impossible, P. 2

Suppose that X is a countable crowded closed subspace of $\mathbb{R}^\kappa$, where $\kappa < \mathfrak{p}$.

For every ξ pick a countable dense $Q_\xi \subset \mathbb{R} \setminus pr_\xi[X]$.

Then $\mathbb{R} \setminus Q_\xi \equiv \omega^\omega$, and thus X is a closed subspace of $\prod_{\xi\in\kappa}(\mathbb{R} \setminus Q_\xi) \equiv (\omega^\omega)^\kappa$, which is impossible.

There exists a crowded countable X with $Exp(X) = \mathfrak{p}$, P. 1

Let $\mathbb{P}$ and $\mathbb{P}'$ be posets. $i : \mathbb{P} \longrightarrow \mathbb{P}'$ is a *pleasant embedding*, if

(1) $i(1_{\mathbb{P}}) = 1_{\mathbb{P}'}$,
(2) $\forall p, q \in \mathbb{P} \left(p \leq q \to i(p) \leq i(q)\right)$,
(3) $\forall p, q \in \mathbb{P} \left(p \perp q \leftrightarrow i(p) \perp i(q)\right)$.

We will say that i is a *dense embedding* if it satisfies all of the above conditions plus the following:

(4) $i[\mathbb{P}]$ is dense in $\mathbb{P}'$.

Also recall that $\mathbb{P}$ is *separative*, if for all $p, q \in \mathbb{P}$ such that $p \not\leq q$ there exists $r \in \mathbb{P}$ such that $r \leq p$ and $r \perp q$.

$\mathbb{P}$ is *meet-friendly* if whenever $p, q \in \mathbb{P}$ are compatible, $\{p, q\}$ has a greatest lower bound, which we denote by $p \wedge q$.

Note that $p \wedge q \in \mathcal{F}$ whenever $\mathcal{F}$ is a filter on $\mathbb{P}$ and $p, q \in \mathcal{F}$.

Notice that $\mathbb{P}$ is meet-friendly iff every centered finite subset $\{p_0, \ldots, p_n\}$ of $\mathbb{P}$ has a greatest lower bound, which we will denote by $p_0 \wedge \cdots \wedge p_n$.

Example: $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$, where $\mathbb{B}$ is a boolean algebra.

If $\mathbb{P}$ is meet-friendly and $\mathcal{C}$ is a non-empty centered subset of $\mathbb{P}$,

$$\mathcal{F} = \{p \in \mathbb{P} : p_0 \wedge \cdots \wedge p_n \leq p \text{ for some } n \in \omega \text{ and } p_0, \ldots, p_n \in \mathcal{C}\}$$

is the (smallest) filter $\mathcal{F}$ generated by $\mathcal{C}$.

Let $\mathbb{P}$ and $\mathbb{P}'$ be meet-friendly. A pleasant embedding $i : \mathbb{P} \to \mathbb{P}'$ is *meet-preserving* if

(5) $\forall p, q \in \mathbb{P} \left(p \not\perp q \to i(p \wedge q) = i(p) \wedge i(q)\right)$.

Lemma

Let $\mathbb{P}$ be a meet-friendly partial order, and let $\mathcal{F}$ be a filter on $\mathbb{P}$. Then the following conditions are equivalent:

(A) *$\mathcal{F}$ is an ultrafilter,*

(B) $\forall p \in \mathbb{P} \setminus \mathcal{F}\, \exists q \in \mathcal{F}\, (p \perp q)$. □

Lemma

Let $\mathbb{P}$ and $\mathbb{P}'$ be meet-friendly partial orders, and let $i : \mathbb{P} \to \mathbb{P}'$ be a meet-preserving pleasant embedding. If $\mathcal{G}$ is a filter on $\mathbb{P}'$ then $i^{-1}[\mathcal{G}]$ is a filter on $\mathbb{P}$. $\square$

Lemma

Let $\mathbb{P}$ be a meet-friendly partial order, let $\mathbb{B}$ be a boolean algebra, and let $i : \mathbb{P} \to \mathbb{B} \setminus \{0\}$ be a pleasant embedding. Assume that $i[\mathbb{P}]$ generates $\mathbb{B}$ as a boolean algebra. If $\mathcal{U}$ is an ultrafilter on $\mathbb{P}$ then $i[\mathcal{U}]$ generates an ultrafilter on $\mathbb{B} \setminus \{0\}$. $\square$.

Given $a, b \in [\omega]^{<\omega}$, we will write $a \preccurlyeq b$ to mean $a \subseteq b$ and $b \setminus a \subseteq \omega \setminus \max(a)$. We will also write $a \prec b$ to mean $a \preccurlyeq b$ and $a \neq b$.

Given a subset $\mathcal{C}$ of $[\omega]^{\omega}$ with the SFIP, define

$$\mathbb{P}(\mathcal{C}) = \{(a, F) : a \in [\omega]^{<\omega} \text{ and } F \in [\mathcal{C}]^{<\omega}\}.$$

Order $\mathbb{P}(\mathcal{C})$ by declaring $(a, F) \leq (b, G)$ if the following conditions hold:

- $b \preccurlyeq a$, $G \subseteq F$,
- $a \setminus b \subseteq \bigcap G$.

This is the standard partial order that generically produces a pseudointersection of $\mathcal{C}$. $\mathbb{P}(\mathcal{C})$ is meet-friendly:
If $(a, F) \not\perp (b, G)$, then $(a \cup b, F \cup G)$ is the greatest lower bound of $\{(a, F), (b, G)\}$.
Recall that $\mathcal{A} \subset [\omega]^\omega$ is *independent*, if $\bigcap_{i \in n} A_i^{\delta_i}$ is infinite for any injective $\langle A_i : i \in n \rangle \in \mathcal{A}^n$ and $\langle \delta_i : i \in n \rangle \in \{0, 1\}^n$, where $A^0 = A$ and $A^1 = \omega \setminus A$.

Proposition (Nyikos)

There exists an independent family of size $\mathfrak{p}$ with no pseudointersection.

Proof. Fix an independent family $\mathcal{A}$ of size $\mathfrak{p}$, subset $\mathcal{C}$ of $[\omega]^\omega$ of size $\mathfrak{p}$ with the SFIP and no pseudointersection. Let $\mathcal{A} = \{A_\xi : \xi < \mathfrak{p}\}$ and $\mathcal{C} = \{C_\xi : \xi < \mathfrak{p}\}$ be injective enumerations. Set $\Delta^+ = \{(m, n) \in \omega \times \omega : m \leq n\}$. Then $\{(A_\xi \times C_\xi) \cap \Delta^+ : \xi < \mathfrak{p}\}$ is as required. $\square$

There exists a crowded countable X with $Exp(X) = \mathfrak{p}$, P. 5

Now we can pass to the actual construction

Fix an independent family $\mathcal{A}$ of size $\mathfrak{p}$ with no pseudointersection. Wlog, for every $n \in \omega$ there exists $A \in \mathcal{A}$ such that $n \notin A$. Set $\mathbb{P} = \mathbb{P}(\mathcal{A})$. For $a \in [\omega]^{<\omega}$, denote by $\mathcal{U}_a$ the filter on $\mathbb{P}$ generated by $\{(a, F) : F \in [\mathcal{A}]^{<\omega}\}$.

Claim 1. Each $\mathcal{U}_a$ is an ultrafilter on $\mathbb{P}$.
Proof. Enough to check that if $(b, G) \in \mathbb{P}$ is compatible with every element of $\mathcal{U}_a$, then $b \preccurlyeq a$ and $a \setminus b \subseteq \bigcap G$, hence $(a, G) \leq (b, G)$. □

Claim 2. $\mathbb{P}$ is separative.
Proof. Routine, using the independence of $\mathcal{A}$. □

Given $p \in \mathbb{P}$, we set $p\downarrow = \{q \in \mathbb{P} : q \leq p\}$. $U \subseteq \mathbb{P}$ is *open* if $p\downarrow \subseteq U$ for every $p \in U$.

$RO(\mathbb{P})$ is the *regular open algebra* of $\mathbb{P}$.
The map $i : \mathbb{P} \to RO(\mathbb{P}) \setminus \{0\}$ such that $i(p) = p\downarrow$ for $p \in \mathbb{P}$, is known to be well-defined, dense and meet-preserving embedding, and the following stronger form of condition (2) holds:

$(2')$ $\forall p, q \in \mathbb{P} \left(p \leq q \leftrightarrow i(p) \leq i(q)\right)$.

There exists a crowded countable X with $Exp(X) = \mathfrak{p}$, P. 6

Let $\mathbb{B}$ be the boolean subalgebra of $RO(\mathbb{P})$ generated by $i[\mathbb{P}]$, and let Z be the Stone space of $\mathbb{B}$. Given $b \in \mathbb{B}$, we will denote by $[b] = \{\mathcal{V} \in Z : b \in \mathcal{V}\}$ the corresponding basic clopen subset of Z. It follows that each $i[\mathcal{U}_a]$ generates an ultrafilter on $\mathbb{B}$, which we will denote by $\mathcal{V}_a$. Finally, set

$$X = \{\mathcal{V}_a : a \in [\omega]^{<\omega}\}.$$

Claim 3. Z is crowded.
Proof. This is equivalent to showing that $\mathbb{B}$ has no atoms, which follows from $\mathbb{P}$ having no atoms and $(2')$. ■

Claim 4. X is a countable dense subset of Z.
Proof. $\bigcup_{a\in[\omega]^{<\omega}} \mathcal{U}_a = \mathbb{P}$, and hence $\bigcup_{a\in[\omega]^{<\omega}} \mathcal{V}_a = \mathbb{B} \setminus \{0\}$. ■

It follows from Claims 3 and 4 that X is a countable crowded space, and that Z is a compactification of X. Furthermore, $w(X) \leq w(Z) = |\mathbb{B}| = \mathfrak{p}$. Since $Exp(X) \geq \mathfrak{p}$, it requires to show that $kc(Z \setminus X) = kc^*(X) \leq p$.

Fix an enumeration $\mathcal{A} = \{A_\xi : \xi \in \mathfrak{p}\}$. For every ξ set

$$U_\xi = \bigcup_{a \in [\omega]^{<\omega}} [(a, A_\xi) \downarrow],$$

an open subset of Z.

Claim 5. $X = \bigcap_{\xi \in \mathfrak{p}} U_\xi$.

Proof. $\subseteq$ is straightforward. In order to prove $\supseteq$, pick $\mathcal{V} \in \bigcap_{\xi \in \mathfrak{p}} U_\xi$. Thus, for every $\xi \in \mathfrak{p}$ we can fix $a_\xi \in [\omega]^{<\omega}$ such that $(a_\xi, A_\xi) \downarrow \in \mathcal{V}$. Set $\mathcal{U} = i^{-1}[\mathcal{V}]$, and observe that $\mathcal{U}$ is a filter on $\mathbb{P}$. Also, $(a_\xi, A_\xi) \in \mathcal{U}$ for all ξ.

Set $a = \bigcup_{\xi \in \mathfrak{p}} a_\xi$. a is finite, because $a \setminus \max(a_\xi) \subseteq A_\xi$ for all ξ.

Fix $\xi \in \mathfrak{p}$ such that $a = a_\xi$, We check that $\mathcal{U} = \mathcal{U}_a$, which would give that $\mathcal{V} = \mathcal{V}_a$, thus concluding the proof.
Since $\mathcal{U}_a$ is an ultrafilter, it will be enough to show that $\mathcal{U}_a \subseteq \mathcal{U}$.
So pick $(a, F) \in \mathcal{U}_a$, where $F = \{A_{\xi_0}, \ldots, A_{\xi_k}\}$. Note that

$$(a, F \cup \{A_\xi\}) = (a_{\xi_0}, A_{\xi_0}) \wedge \cdots \wedge (a_{\xi_k}, A_{\xi_k}) \wedge (a_\xi, A_\xi) \in \mathcal{U},$$

which clearly implies $(a, F) \in \mathcal{U}$, as desired.

Open questions

Using the CDH property of $\mathbb{R}^\kappa$ for $\kappa < \mathfrak{p}$, we get that any countable dense $X \subset \mathbb{R}^\kappa$ has a closed copy of any countable space of weight $\leq \kappa$, is a topological group, and $Exp(X) = \mathfrak{d}$.

Using our Main Tool one can show that $Exp(X) = \kappa$ for any countable dense $X \subset \mathbb{R}^\kappa$, provided that $\mathfrak{d} \leq \kappa \leq \mathfrak{c}$. In particular, this is true for dense countable subgroups of $\mathbb{R}^\kappa$.

This motivates the following

Question

For which cardinals κ such that $\mathfrak{p} \leq \kappa < \mathfrak{d}$ does there exist a countable crowded topological group (homogeneous space) X such that $Exp(X) = \kappa$? $\square$

The last slide

Thank you for your attention.