Fraïssé limits and their applications

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A better answer: The rational numbers $(\mathbb{Q}, \leqslant)$ also contain every finite linear order. Moreover, this order

- has the extension property: for any finite $A, B \subseteq \mathbb{Q}$, and embedding $f : A \longrightarrow B$, there is an embedding $g : B \longrightarrow \mathbb{Q}$ such that $g \circ f = Id$,
- contains every countable linear order.

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The class $\mathcal{K}$

- is **hereditary** if $B \in \mathcal{K}$, $A \leqslant B$ implies $A \in \mathcal{K}$,
- is **directed** if any $A_0, A_1 \in \mathcal{K}$ can be embedded in some $B \in \mathcal{K}$,
- has **amalgamation** if for any $A, B_0, B_1 \in \mathcal{K}$ and embeddings $f_0 : A \to B_0$, $f_1 : A \to B_1$ there are $C \in \mathcal{K}$ and embeddings $g_0 : B_0 \to C$, $g_1 : B_1 \to C$ such that $g_0 \circ f_0 = g_1 \circ f_1$.

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If $\mathcal{K}$ satisfies the above conditions, it is called a **Fraïssé class**.

Fraïssé limits

Theorem (Fraïssé): Let $\mathcal{K}$ be a hereditary, and countable up to isomorphism class of finite structures in a fixed signature. $\mathcal{K}$ is a Fraïssé class iff there is a unique up to isomorphism countable chain F from $\mathcal{K}$, i.e., $F = \bigcup_n F_n$, $F_n \subseteq F_{n+1}$, $F_n \in \mathcal{K}$, that contains a copy of every element of $\mathcal{K}$, and that has the extension property.

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Remark: If there is a Katětov functor for $\mathcal{K}$, then F also contains a copy of every countable chain C from $\mathcal{K}$, and $\mathrm{Aut}(F)$ contains a copy of $\mathrm{Aut}(C)$.

Examples of Fraïssé limits

- the rationals $(\mathbb{Q}, \leqslant)$ (finite linear orders),
- the Rado graph (finite graphs),
- the Hall group (finite groups),
- the countable atomless Boolean algebra (finite Boolean algebras).

Metric Fraïssé limits (Ben Yaacov; Kubiś)

- The Urysohn space (finite metric spaces),
- the separable Hilbert space (fin-dim Hilbert spaces),
- the Gurarii space (fin-dim Banach spaces).

Inverse Fraïssé limits (Irvine-Solecki)

Let A, B be structures in the same relational signature L. A surjection $f : B \to A$ is an **epimorphism** if

$$R(a_1, \ldots, a_n) \Leftrightarrow \exists b_1, \ldots, b_n \, (p(b_i) = a_i \,\&\, R(b_1, \ldots, b_n))$$

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A countable (up to isomorphism) class $\mathcal{K}$ of finite structures in the same signature is an **inverse Fraïssé class** if

- it is directed, i.e., for any $A_0, A_1 \in \mathcal{K}$ there is $B \in \mathcal{K}$ and epimorphisms $f_0 : B \to A_0$, $f_1 : B \to A_1$,
- has amalgamation, i.e., for any $A, B_0, B_1 \in \mathcal{K}$ epimorphisms $f_0 : B_0 \to A$, $f_1 : B_1 \to A$ there is $C \in \mathcal{K}$ and epimorphisms $g_0 : C \to B_0$, $g_1 : C \to B_1$ such that $f_0 \circ g_0 = f_1 \circ g_1$.

Inverse Fraïssé limits

The inverse Fraïssé limit F of $\mathcal{K}$ is the unique inverse sequence of elements from $\mathcal{K}$ that has the extension property, and that epimorphically projects on every element of $\mathcal{K}$. Topologically, F is the Cantor set $2^{\mathbb{N}}$.

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- The linearly ordered Cantor set (finite linear orders),
- a good measure μ (as defined by Akin) on $2^{\mathbb{N}}$ for a given set V_μ of values on clopens (finite probability measures with values in V_μ),
- The pseudo-arc is of the form $P/\sqcap$, where P is the inverse Fraïssé limit of finite linear graphs, and $\sqcap$ is the graph relation on P. In fact, every compact metric space X can be presented as a quotient of an inverse limit of graphs.

Spectra of ω-posets (Bartoš-Bice-Vignati)

Let $(P, \leqslant)$ be an ω-poset. A set $A \subseteq P$ is called a **cap** if it is $\leqslant$-refined by a finite maximal antichain in P. It is called a **selector** if it intersects every cap (equivalently: $P \setminus A$ is not a cap).

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Moreover, every second countable compact T_1 space is the spectrum of some ω-poset.

ω-posets as Fraïssé sequences of graphs

A surjective and co-surjective relation $\sqsupset \subseteq A \times B$ on reflexive graphs A, B, with edge relations $\sqcap$, is a graph morphism if

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In order to develop a Fraïssé theory for categories of graphs with graph morphism relations, the notion of amalgamation needs to be relaxed (lax-amalgamation):

$$\text{for all } \sqsupset_0 \subseteq A \times B_0, \sqsupset_1 \subseteq A \times B_1$$

$$\text{there are } \sqsupset'_0 \subseteq B_0 \times C, \sqsupset'_1 \subseteq B_1 \times C \text{ such that } \sqsupset_0 \circ \sqsupset'_0 \subseteq \sqsupset_1 \circ \sqsupset'_1 .$$

ω-posets as Fraïssé sequences of graphs

A sequence F of graphs from a category $\mathcal{K}$ is a **(lax-)Fraïssé sequence** if there is a morphism from F on every element of $\mathcal{K}$, and it has the (lax-)extension property. All the (lax-)Fraïssé sequences from $\mathcal{K}$ have the same spectrum (up to homeomorphism).

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- The arc is the spectrum of the Fraïssé sequence of the category of finite linear graphs with monotone graph morphisms,
- the pseudo-arc is the spectrum of the Fraïssé sequence of the category of finite linear graphs with all graph morphisms.

Automorphism groups of Fraïssé limits

Let F be the Fraïssé limit of a Fraïssé class $\mathcal{K}$. The automorphism group $G = \mathrm{Aut}(F)$ is a Polish topological group. Let $\mathcal{K}_1$ be the class of partial isomorphisms of elements of $\mathcal{K}$.

Theorem (Kechris, Rosendal):

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Theorem (Kechris, Rosendal):

- G has a dense conjugacy class (the Rokhlin property) iff $\mathcal{K}_1$ is directed.
- G has a residual conjugacy class (the strong Rokhlin property) iff $\mathcal{K}_1$ is directed and has weak amalgamation.

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Orbits under the action of a group G on G^n by diagonal conjugation are called n-diagonal conjugacy classes. Analogous characterizations hold for n-diagonal conjugacy classes.

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Remark: Diagonal conjugacy classes are isomorphism classes of representations of free groups in G. One can also consider representations of other countable groups in G.

Automatic continuity

A Polish group G has **ample generics** if it has a residual n-diagonal conjugacy class for every n; G has the **automatic continuity property** if every algebraic homomorphism of G into a separable group is continuous.

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Theorem (Kechris, Rosendal): If G has ample generics, then it has the automatic continuity property.

- $\mathrm{Aut}(R)$, where R is the Rado graph, has ample generics (Hrushovski),
- $\mathrm{Homeo}(2^{\mathbb{N}})$ has ample generics (Kwiatkowska) but no generic representation of $\mathbb{Z}^n$, $n > 1$ (Hochman),
- $\mathrm{Aut}(\mathbb{Q})$ i $\mathrm{Homeo}_+([0,1])$ do not have ample generics but they have a residual conjugacy class and automatic continuity (Rosendal, Solecki),
- $\mathrm{Iso}(\mathbb{U})$ does not have a residual conjugacy class but it has automatic continuity (Sabok).

My research

Past:

- Characterization of Polish ultrametric spaces X for which $\mathrm{Iso}(X)$ has ample generics,
- $\mathrm{Aut}(T)$, where T is the Fraïssé limit of finite tournaments has a residual conjugacy class but it has no generic representation of $\mathbb{Z}^n$, $n > 1$, (with M. Doucha, CAS).

My research

Present:

- A characterization of good measures, and $\mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ with a residual conjugacy class, where μ is a good measure with rational values on clopens (joint with M. Doucha CAS, D. Kwietniak, UJ and Piotr Niemiec UJ).
- Homeo(the pseudo-arc) has a dense conjugacy class (with z T. Bice, CAS),
- Homeo(the Lelek fan) has a residual conjugacy class, and the generic homeomorphism has zero topological entropy.

Good measures

A set $V \subseteq [0,1]$ is **group-like** if $V = [0,1] \cap G$, where G is a subgroup of $\mathbb{R}$.

Let μ be a full probability measure on $2^{\mathbb{N}}$, and let $V_\mu \subseteq [0,1]$ be the set of its values on clopens.

The measure μ is **ultrahomogeneous** if for any clopen partitions P, R of $2^{\mathbb{N}}$ such that there is a bijection $f\colon P \to R$ satisfying $\mu(x) = \mu(f(x))$ for $x \in P$, there exists a homeomorphism $\varphi \in \mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ such that $\varphi[x] = f(x)$ for $x \in P$.

It is **maximal** if for every tuple $v_1, \ldots, v_n$ of non-zero elements of $V(\mu)$ such that $\sum_{i=1}^n v_i = 1$ there exists a clopen partition $\{x_1, \ldots, x_n\}$ of $2^{\mathbb{N}}$ such that $\mu(x_i) = v_i$.

Theorem

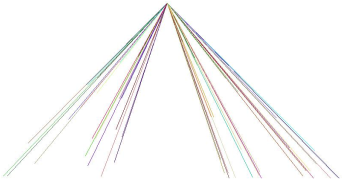
The measure μ is good iff V_μ is group-like, and μ is ultrahomogeneous and maximal.

Theorem

1. *(Akin) Suppose that V_μ is $\mathbb{Q}$-linear space-like. Then $\mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ has a comeager conjugacy class.*
2. *Suppose that $V_\mu \subseteq \mathbb{Q}$. Then $\mathrm{Homeo}(2^{\mathbb{N}}, \mu)$ has a comeager conjugacy class iff V_μ is ring-like.*

The Lelek fan

The **Lelek fan** is a subfan of the Cantor fan C such that the set of its end points is dense in C. Let $\mathcal{L}$ be the category of finite fan graphs with monotone-on-branches morphisms. Then $\mathcal{L}$ has amalgamation, and, for its Fraïssé sequence L, $\mathrm{Sp}(L)$ is the Lelek fan.



Theorem

$\mathrm{Homeo}(\mathrm{Sp}(L))$ *has a residual conjugacy class, and the generic homeomorphism has zero topological entropy.*

Thank You!