Let $\mathbb{M}_{\omega_2}$ be an iterated Miller forcing of length ω_2 with countable support and G_{ω_2} be an $\mathbb{M}_{\omega_2}$-generic over V satisfying CH.

Lemma 0.1. *For each set $X \subseteq 2^\omega \cap V[G_{\omega_2}]$ such that $X \in V[G_{\omega_2}]$, there is an ordinal number $\beta < \omega_2$ such that $X \cap V[G_\beta] \in V[G_\beta]$.*

Proof. Let $X = \{ x_\alpha : \alpha < \omega_2 \}$. In V we have a sequence $\langle \dot{x}_\alpha : \alpha < \omega_2 \rangle$, where each $\dot{x}_\alpha$ is a name for a real and $X = \{ \dot{x}_\alpha[G_{\omega_2}] : \alpha < \omega_2 \}$. Every antichain in $\mathbb{M}_{\omega_2}$ has size at most ω_1. Fix $\alpha < \omega_2$. Without loss of generality we may assume that $\dot{x}_\alpha$ is the set of all pairs $\langle \langle i, \check{k}^\alpha_{i,\xi} \rangle, p^\alpha_{i,\xi} \rangle$, where $i < \omega$ and $\xi < \omega_1$ such that:

- $k^\alpha_{i,\xi} \in \{0,1\}$,
- for each i, the set $A^\alpha_i := \{ p^\alpha_{i,\xi} : \xi < \omega_1 \}$ is a maximal antichain in $\mathbb{M}_{\omega_2}$,
- for each ξ, we have $p^\alpha_{i,\xi} \Vdash \dot{x}_\alpha(i) = k^\alpha_{i,\xi}$.

Since the set $\bigcup \{ \operatorname{supp}(p) : p \in A^\alpha_i, i < \omega \}$ has size at most ω_1, it is contained in some ordinal number $g(\alpha) > \alpha$. Let C_0 be the club of all fixed points of the map g, i.e., for all $\beta \in C_0$ and $\alpha < \beta$, we have $g(\alpha) < \beta$.

For each $\beta \in C_0$, we have $\{ x_\alpha : \alpha < \beta \} \in V[G_\beta]$: The sequence $\langle \dot{x}_\alpha : \alpha < \beta \rangle$ is in V. Since all antichains A^α_i for $\alpha < \beta$ and $i < \omega$ are subsets of $\mathbb{M}_\beta$, we have $G_{\omega_2} \cap A^\alpha_i = G_\beta \cap A^\alpha_i$ and thus,

$$\dot{x}_\alpha[G_{\omega_2}] = \{ \langle i, k^\alpha_{i,\xi} \rangle : i < \omega, \{p^\alpha_{i,\xi}\} = G_{\omega_2} \cap A^\alpha_i = G_\beta \cap A^\alpha_i \} = \dot{x}_\alpha[G_\beta]$$

for all $\alpha < \beta$.

For each ordinal number $\alpha < \omega_2$, there is an ordinal number $h(\alpha) > \alpha$ such that $X \cap V[G_\alpha] \subseteq \{ x_\beta : \beta < h(\alpha) \}$. Let C_1 be a club in ω_2 such that for all $\beta \in C_1$ and $\alpha < \beta$, we have $h(\alpha) < \beta$.

For each $\beta \in C_1$ with $\operatorname{cf}(C_1 \cap \beta) = \omega_1$, we have $X \cap V[G_\beta] \subseteq \{ x_\alpha : \alpha < \beta \}$: Fix a real $x \in X \cap V[G_\beta]$. Then there is an ordinal number $\gamma < \beta$ such that $x \in X \cap V[G_\gamma]$, and thus $x \in \{ x_\alpha : \alpha < h(\gamma) \} \subseteq \{ x_\alpha : \alpha < \beta \}$.

Let D be the set of all $\beta \in C_0 \cap C_1$ such that there is an increasing sequence in $C_0 \cap C_1$ of length ω_1 whose supremum is β. Then the set D is an ω_1-club. Pick $\beta \in D$. Since $\beta \in C_0$, we have $\{ x_\alpha : \alpha < \beta \} \in V[G_\beta]$, and thus $\{ x_\alpha : \alpha < \beta \} \subseteq X \cap V[G_\beta]$. On the other hand, since $\beta \in C_1$ and $\operatorname{cf}(C_1 \cap \beta) = \omega_1$, we have $X \cap V[G_\beta] \subseteq \{ x_\alpha : \alpha < \beta \}$. Finally, we have that $X \cap V[G_\beta] = \{ x_\alpha : \alpha < \beta \} \in V[G_\beta]$. □