Closed groups generated by generic measure preserving transformations

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Introduction

Basics

Polish space = a separable completely metrizable space

Polish group = a topological group whose group topology is Polish

Standard Borel space = a set X equipped with the σ-algebra of sets Borel with respect to a Polish topology on X

Borel measure = a measure defined on the σ-algebra of a standard Borel space

All atomless Borel probability measures are isomorphic to each other, so we can think of such a measure as Lebesgue measure λ on $[0, 1]$.

The group of measure preserving transformations

(X, γ) = a standard Borel space with an atomless Borel probability measure

Aut = the Polish group of all measure preserving transformations of (X, γ)

Measure preserving transformations are identified if they coincide on a set of full measure.

Aut is taken with composition and is topologized so that

$$T_n \to T \ \text{ iff } \ \gamma\big(T_n(A) \triangle T(A)\big) \to 0, \ \text{for each Borel } A \subseteq X.$$

Genericity

X a Polish space, P a property

A **generic** $x \in X$ **has** P if $\{x \in X \mid x$ does not have $P\}$ is meager.

The subject matter of the talk

For a Polish group G and $g \in G$, let

$$\langle g \rangle_c = \mathrm{closure}\big(\{g^n \mid n \in \mathbb{Z}\}\big).$$

We study **closed subgroups of** Aut **generated by generic elements of** Aut, that is, groups of the form

$$\langle T \rangle_c,$$

for a generic measure preserving transformation T.

Boolean actions

G a Polish group

A **boolean action of G on** (X, γ) is a continuous homomorphism $\zeta\colon G \to \mathrm{Aut}$.

The word action is justified by viewing G as acting on the boolean algebra of measure classes of measurable subsets of (X, γ) by

$$gB = \zeta(g)\big(B\big).$$

For example, the Polish group $\langle T \rangle_c$, for $T \in \mathrm{Aut}$, has a natural boolean action being a subgroup of Aut.

Two more groups

The group of measurable functions

$\mathbb{T}$ = the group of all complex numbers of unit length taken with multiplication

λ = Lebesgue measure on $[0, 1]$

$\mathbf{L^0}(\lambda, \mathbb{T})$ = the Polish group of all measurable functions from $[0, 1]$ to $\mathbb{T}$

$L^0(\lambda, \mathbb{T})$ is taken with pointwise multiplication and the topology of convergence in measure.

Notation: We write L^0 for $L^0(\lambda, \mathbb{T})$.

$L^0(\lambda, \mathbb{R})$ = the Polish linear space of all measurable functions from $[0,1]$ to $\mathbb{R}$

There is a continuous surjective homomorphism

$$L^0(\lambda, \mathbb{R}) \to L^0,$$

namely,

$$f \to \exp(if).$$

The unitary group

H = the separable, infinite dimensional, complex Hilbert space

$\boldsymbol{\mathcal{U}}$ = the Polish group of unitary transformations of H

$\mathcal{U}$ is taken with composition and the strong operator topology.

The question and the theorem

Recall, for $T \in \mathrm{Aut}$,

$$\langle T \rangle_c = \mathrm{closure}\big(\{ T^n \mid n \in \mathbb{Z}\}\big).$$

Glasner–Weiss: Is it the case that for a generic $T \in \mathrm{Aut}$, $\langle T \rangle_c$ is isomorphic to L^0?

Motivation for the question

Qualifications

Glasner: L^0 is monothetic.

Analogy

Melleray–Tsankov: $\langle U \rangle_c$ is isomorphic to L^0 for a generic $U \in \mathcal{U}$.

Structure

Ageev: For a generic $T \in \mathrm{Aut}$, each finite abelian group embeds into $\langle T \rangle_c$.

S.: For a generic $T \in \mathrm{Aut}$, there is a Polish linear space L_T and a continuous surjective homomorphism $L_T \to \langle T \rangle_c$.

Dynamics

Glasner–Weiss: For a generic $T \in \mathrm{Aut}$, the natural boolean action of $\langle T \rangle_c$ on (X, γ) is whirly.

Theorem (S., 2021)

For a generic transformation $T \in \mathrm{Aut}$,
the group $\langle T \rangle_c$ is **not** *isomorphic to L^0.*

A rough outline of the proof

Prove the following two points.

1. If $L^0 \cong \langle T \rangle_c < \mathrm{Aut}$, for a generic $T \in \mathrm{Aut}$, then **some** ergodic boolean action of L^0 has **spectral properties** similar to spectral properties of a generic $T \in \mathrm{Aut}$.

2. No ergodic boolean actions of L^0 has **spectral properties** similar to spectral properties of a generic $T \in \mathrm{Aut}$.

Spectral behavior

Spectral behavior of a generic $T \in \mathrm{Aut}$

$\nu(T) =$ maximal spectral type of $T \in \mathrm{Aut}$.

Building on earlier work of Choksi–Nadkarni, Katok, and Stepin, del Junco–Lemańczyk proved the following theorem.

Theorem (del Junco–Lemańczyk, 1992)

For a generic $T \in \mathrm{Aut}$ and $\ell_1, \ldots, \ell_p, \ell'_1, \ldots, \ell'_{p'} \in \mathbb{N}$, if $(\ell_1, \ldots, \ell_p)$ and $(\ell'_1, \ldots, \ell'_{p'})$ are not rearrangements, then

$$\nu(T^{\ell_1}) * \cdots * \nu(T^{\ell_p}) \perp \nu(T^{\ell'_1}) * \cdots * \nu(T^{\ell'_{p'}}).$$

Call the condition above the **del Junco–Lemańczyk condition**.

Spectral behavior of L^0

A unitary representation of L^0 can be constructed as follows. Given $\phi \in L^0$, let

$$L^2(\lambda) \ni f \to \phi \cdot f \in L^2(\lambda).$$

This is a unitary representation in $\mathcal{U}\big(L^2(\lambda)\big)$.

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$$L^2(\mu) \ni f \to \phi^k \cdot f \in L^2(\mu),$$

for $\mu \preceq \lambda$ and $k \in \mathbb{Z}$. This is a unitary representation in $\mathcal{U}\big(L^2(\mu)\big)$.

$\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$

$\mathbb{N}[\mathbb{Z}^\times]$ consists of all finite functions x such that

$$\emptyset \neq \mathrm{dom}(x) \subseteq \mathbb{Z}^\times \text{ and } \mathrm{rng}(x) \subseteq \mathbb{N}$$

For $x \in \mathbb{N}[\mathbb{Z}^{\times}]$, let

$$D(x) = \{(k, i) \mid k \in \mathrm{dom}(x),\ i \leq x(k)\},$$

and

$$C_x = [0, 1]^{D(x)}.$$

For $(k, i) \in D(x)$, let

$$\pi_{k,i} \colon C_x \to [0, 1] = \text{projection onto coordinate } (k, i).$$

A permutation δ of $D(x)$ is **good** if, for each $(k, i) \in D(x)$, $\delta(k, i) = (k, j)$ for some j.

A finite Borel measure μ on C_x is **compatible with** $x \in \mathbb{N}[\mathbb{Z}^\times]$ if

— the marginals of μ given by $\pi_{k,i}$, for $(k, i) \in D(x)$, are absolutely continuous with respect to λ;

— μ is invariant under good permutations of $D(x)$;

— all diagonals of C_x have measure zero with respect to μ.

μ compatible with x

$\widetilde{L^2}(\mu)$ = the closed subspace of $L^2(\mu)$ consisting of all functions invariant under good permutations

For $x \in \mathbb{N}[\mathbb{Z}^\times]$ and $\phi \in L^0$,

$$R_x(\phi) = \prod_{(k,i) \in D(x)} (\phi \circ \pi_{k,i})^k.$$

If μ is compatible with x, then

$$f \in \widetilde{L^2}(\mu) \Rightarrow R_x(\phi) \cdot f \in \widetilde{L^2}(\mu).$$

Fix $\xi : L^0 \to \mathcal{U}$ a unitary representation without non-zero fixed points.

Theorem (S., 2014)

ξ is determined by finite Borel measures $(\mu_x)_{x \in N[\mathbb{Z}^\times]}$ with μ_x compatible with x.

ξ is isomorphic to the ℓ^2-sum over $x \in \mathbb{N}[\mathbb{Z}^\times]$ of the representations

$$L^0 \times \widetilde{L^2}(\mu_x) \ni (\phi, f) \to R_x(\phi) \cdot f \in \widetilde{L^2}(\mu_x).$$

The sequence $(\mu_x)_{x \in N[\mathbb{Z}^\times]}$ is unique up to mutual absolute continuity of its entries.

The above is true modulo multiplicity.

Theorem (S., 2014)

ξ is determined by a sequence of finite Borel measures $(\mu_x^j)_{x \in N[\mathbb{Z}^\times], j \in \mathbb{N}}$ such that, for each j,

$$\mu_x^j \text{ is compatible with } x, \text{ and } \mu_x^{j+1} \preceq \mu_x^j.$$

ξ is isomorphic to the ℓ^2-sum over $x \in \mathbb{N}[\mathbb{Z}^\times]$ and $j \in \mathbb{N}$ of the representations

$$L^0 \times \widetilde{L^2}(\mu_x^j) \ni (\phi, f) \to R_x(\phi) \cdot f \in \widetilde{L^2}(\mu_x^j).$$

The sequence $(\mu_x^j)_{x \in N[\mathbb{Z}^\times], j \in \mathbb{N}}$ is unique up to mutual absolute continuity of its entries.

The del Junco–Lemańczyk condition for L^0

$\mathbb{N}[\mathbb{Z}^\times]$ comes equipped with coordinatewise addition

$$x \oplus y.$$

Given: μ on C_x compatible with x, and ν on C_y compatible with y
Since $C_{x\oplus y} \sim C_x \times C_y$, we can define the "symmetried" product

$$\mu \otimes \nu$$

on $C_{x\oplus y}$ compatible with $x \oplus y$.

Recall: we have $\xi : L^0 \to \mathcal{U}$, a unitary representation without non-zero fixed points, with H separable.

Theorem (Etedadialiabadi, 2016/20)

Assume: *for a generic* $\phi \in L^0$ *and* $\ell_1, \ldots, \ell_p, \ell'_1, \ldots, \ell'_{p'} \in \mathbb{N}$ *such that* $(\ell_1, \ldots, \ell_p)$ *and* $(\ell'_1, \ldots, \ell'_{p'})$ *are not rearrangements, we have*

$$\nu(\phi^{\ell_1}) * \cdots * \nu(\phi^{\ell_p}) \perp \nu(\phi^{\ell'_1}) * \cdots * \nu(\phi^{\ell'_{p'}}),$$

where $\nu(\psi)$ = *maximal spectral type of* $\xi(\psi)$.

Then: *for* $x_1, \ldots, x_p \in \mathbb{N}[\mathbb{Z}^\times]$ *with* $p > 1$, *we have*

$$\mu_{x_1} \otimes \cdots \otimes \mu_{x_p} \perp \mu_{x_1 \oplus \cdots \oplus x_p}.$$

Theorem on Koopman representations of L^0

Given a boolean action $\zeta \colon G \to \mathrm{Aut}$, the **Koopman representation associated with** ζ is given by

$$G \ni g \to U_g \in \mathcal{U}\big(L^2(\gamma)\big),$$

where, for $f \in L^2(\gamma)$,

$$U_g(f) = f \circ \big(\zeta(g)\big)^{-1}.$$

The proposition below gives a connection of Etedadialiabadi's theorem with the Glasner–Weiss question.

Proposition (S., 2021)

Assume that there is a non-meager set of $T \in \mathrm{Aut}$ such that $\langle T \rangle_c$ is isomorphic to L^0.

There exists an ergodic boolean action of L^0 on (X, γ), whose Koopman representation is such that

$$\mu_x \otimes \mu_x \perp \mu_{x \oplus x}, \text{ for all } x \in \mathbb{N}[\mathbb{Z}^\times].$$

Theorem (S. 2021)

ξ = the Koopman representation associated with an ergodic boolean action of L^0.

Then, for $x_1, \dots, x_p \in \mathbb{N}[\mathbb{Z}^\times]$, we have

$$\mu_{x_1} \otimes \cdots \otimes \mu_{x_p} \preceq \mu_{x_1 \oplus \cdots \oplus x_p}.$$

In particular, for ergodic **Koopman** representations of L^0

$$\mu_x \otimes \mu_x \preceq \mu_{x \oplus x}, \text{ for all } x.$$

Contrast the above statement with

$$\mu_x \otimes \mu_x \perp \mu_{x \oplus x}, \text{ for all } x,$$

for the ergodic Koopman representation of L^0 found in the proposition.

Questions

Is there a Polish group G such that $\langle T \rangle_c$ is isomorphic to G, for a generic $T \in \mathrm{Aut}$?

Glasner–Weiss: Is the group $\langle T \rangle_c$ a Lévy group for a generic $T \in \mathrm{Aut}$?