Classification of compact, smooth manifolds up to cobordism via power series

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2023-11-16

Definition

A smooth manifold is a space which is a patchwork of the real affine spaces $\mathbb{R}^n$ glued together with smooth threads. A map between manifolds is smooth if it is smooth on each patch.

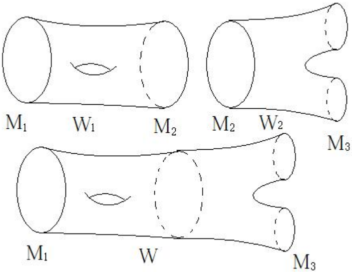
A cartesian product and a direct sum (coproduct) of manifolds is a manifold.

Classification of compact, connected manifolds in low dimensions up to diffeomorphism:

- $n = 0$ Point
- $n = 1$ Circle S^1
- $n = 2$ Sphere, connected sums of tori (orientable) and of projective planes (non-orientable)
- $n = 3$ Very complicated.... W. Thurston (Fields medal 1986), G. Perelman (Fields medal 2006, not accepted)
- $n = 4$ Impossible in general. (simply connected case: M. Freedman, S. Donaldson (Fields medals 1986)

Definition

Manifolds M i N are (co-)bordant if there exist a manifold with boundary W such that $\partial W = M \sqcup N$. (Co-)bordism is an equivalence relation.



Example for analysts: If $\mathbb{R}^m \supset U \xrightarrow{f} \mathbb{R}^n$ is a smooth map and $y_0, y_1 \in \mathbb{R}^n$ are regular values of f. Then if U is connected then the inverse images $f^{-1}(y_0), f^{-1}(y_1) \subset U$ are cobordant manifolds.

Theorem (V.A.Rokhlin, 1952)

Every 3-dimensional compact manifold bounds.

1. There are algebraic (homotopical) invariants detecting cobordism. (a corollary of the main theorem)
2. If two manifolds are cobordant then one can be constructed from the other via surgery.
3. Surgery is controlled via the Morse functions.

Classify compact manifolds up to cobordism.

To classify compact manifolds up to bordism R. Thom introduced the cobordism ring N^* whose elements are bordism classes of compact manifolds. Addition is defined defined by the disjoint sum $\sqcup$ and multiplication by the cartesian product $\times$.

Cobordism ring

The set of diffeomorphism classes of compact manifolds $\mathcal{M}^*$ is a graded, abelian semiring with

- Addition: disjoint sum (coproduct) $M_1 \sqcup M_2$
- Multiplication: cartesian product $M_1 \times M_2$
- Empty manifold as neutral element with respect to addition
- One-point manifold as neutral element with respect to multiplication
- Gradation is defined by the dimension of the manifold: $\mathcal{M}^* = \oplus \mathcal{M}^q$ where $\mathcal{M}^q$ - the semigroup of $-q$ dimensional manifolds.

Very important!

For an arbitrary manifold M its double $M \sqcup M$ bounds the manifold $M \times [0,1]$. Thus when we pass to cobordism classes we obtain the cobordism graded ring

$$N^* = \bigoplus_{q=-\infty}^{q=+\infty} N^q.$$

where N^q consists of $-q$-dimensional manifolds. It is even an algebra over the two-element field $\mathbb{F}_2$.

Find generators and relations in the ring N^*.

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Twierdzenie (R. Thom (1954),.... , D. Quillen (1969/1971)

Cobordism ring is isomorphic to a polynomial algebra over $\mathbb{F}_2$ of infinitely many variables in negative dimensions $\neq 1 - 2^j$

$$N^* \simeq \mathbb{F}_2[x_i \mid i \leqslant 0,\ i \neq 1 - 2^j]$$

Generators ... (Projective spaces in even dimensions, Milnor, Dold manifolds. . .)

Projective space of dim n: $\mathbb{P}^n$ is a space of all 1-dim subspaces in $\mathbb{R}^{n+1}$. It is diffeomorphic to $S^n/\{x \sim -x\}$ i.e. it is a compact manifold.

Idea of Quillen's proof

Twierdzenie (R. Thom,.... , D. Quillen)

There is a $\mathbb{F}_2$*-algebra isomorphism* $N^* \simeq \mathbb{F}_2[x_i \mid \deg x_i = -i, i \neq 1 - 2^j]$.

Study modules over the ring N^* *(in fact extend it to a functor into a category of the graded* N^**-modules). On the functor introduce additional algebraic structures defined by geometry to make it unique.*

Milestones.

1. Define a (bi-)functor defined on the category of manifolds and smooth maps $N^*(-): \mathcal{M} \to N^*$-modules such that $N^*(pt) = N^*$.
2. Calculate $N^*(X \times \mathbb{P}^n) \simeq N^*(X)[x_n]/(x_n^{n+1})$, where $x_n = e(H_n)$
3. The Segre map $\sigma_{n,m}: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ defines a power series (formal group) $\sum a_{ij} x^i y^j \in N^*[[x,y]] =: F_{N^*}(x,y)$, $\deg x = \deg y = 1$, $a_{ij} \in N^{1-i-j}$.
4. The Stiefel-Whitney classes and Steenrod squares in $N^*(-) \implies a_{ij} \in N^*$ are generators.
5. Input: Lazard theorem: structure of the universal formal group $(\mathcal{L}, F_{\mathcal{L}})$, $\mathcal{L} = \mathbb{Z}_2[x_i \mid \deg x_i = -i, i \neq 2^j - 1]$ and existence of its logarithm.
6. The Landweber-Novikov operations in $N^*(-) \implies$ the ring homomorphism $\mathcal{L} \to N^*$ classifying the formal group F_{N^*} is an isomorphism.

□

Definicja

A continuous map $f\colon X \to Y$ is proper iff for every compact subset $K \subset Y$ its inverse image $f^{-1}(K) \subset X$ is compact.

Definicja

Let $\varphi_i\colon V_i \to Y, \quad i = 0, 1$ be two smooth proper maps. They are cobordant if there exists a manifold with boundary W and a proper map $F : W \to Y$ such that $V_0 \sqcup V_1 = \partial W$ and $F|V_i = \varphi_i$. Cobordism of proper maps is an equivalence relation.

1. $N^k(X) := \{\varphi\colon V \to X \mid \varphi\text{-proper}, \dim X - \dim V = k\}/ \sim_{\textit{proper cobordism}}$
2. $[\varphi] + [\psi] := [\varphi \sqcup \psi]$, N^k is a $\mathbb{F}_2$-vector space.
3. $f\colon X \to Y$ a smooth map. We define $f^*\colon N^*(Y) \to N^*(X)$:

$$f^*([\varphi\colon V \to Y]) := [\overline{\varphi}\colon X \times_Y V \to X]$$

$$\begin{array}{ccc} X \times_Y V & \xrightarrow{\bar{f}} & V \\ \downarrow \bar{\varphi} & \varphi \pitchfork f & \downarrow \varphi \\ X & \xrightarrow{f} & Y \end{array}$$

Homotopic maps induce the same homomorphism.

4. The Mayer-Vietoris exact sequence for decomposition $X = U_1 \cup U_2$

$$\dots \to N^n(X) \to N^n(U_1) \oplus N^n(U_2) \to N^n(U_1 \cap U_2) \xrightarrow{\delta} N^{n+1}(X) \to \dots$$

Let $g\colon X \to Y$ be a smooth proper map $\dim g = \dim X - \dim Y =: m$. $\forall_k$ we define "Umkehr" (transfer) homomorphism:

$$g_\#\colon N^k(X) \to N^{k-m}(Y), \quad g_\#([\varphi : V \to X]) := [V \xrightarrow{\varphi} X \xrightarrow{g} Y]$$

Proposition

The assignment $g\colon X \to Y \mapsto g_\#\colon N^*(X) \to N^{*-m}(Y)$ is a covariant homotopy functor from the category of proper smooth maps to the category of $\mathbb{F}_2$-vector spaces. A transversal pull-back where g is a proper map:

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\bar{f}} & Z \\ \downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array} \quad \textit{defines c.d.} \quad \begin{array}{ccc} N^k(X \times_Y Z) & \xleftarrow{\bar{f}^*} & N^k(Z) \\ \downarrow{\scriptstyle \bar{g}_\#} & & \downarrow{\scriptstyle g_\#} \\ N^{k-m}(X) & \xleftarrow{f^*} & N^{k-m}(Y) \end{array} \qquad (1)$$

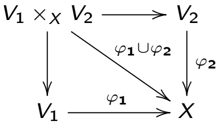
External product is defined by the cartesian product of maps:

$$\times\colon N^n(X) \otimes N^m(Y) \to N^{n+m}(X \times Y)$$

Definition (Ring structure on $N^*(X)$)

$$N^n(X) \otimes N^m(X) \xrightarrow{\times} N^{n+m}(X \times X) \xrightarrow{\Delta^*} N^{n+m}(X)$$

Cup-product of two transversal maps $\varphi\colon V_i \to X, \quad i = 1, 2,$ is represented by the diagonal in the pull-back diagram:



(2)

Proposition

1. *Induced homomorphisms $f^*\colon N^*(Y) \to N^*(X)$ are ring homomorphisms.*
2. *For a proper map $g\colon X \to Y$ the transfer $g_\#\colon N^*(X) \to N^{*-\dim g}(Y)$ is a $N^*(Y)$-module homomorphism i.e. $g_\#(g^*([\psi]) \cup [\varphi]) = [\psi]g_\#([\varphi])$.*

Proposition

For any $n \geqslant 0$.

1. *The cobordism ring of a sphere* $N^*(S^n)$ *is a free* N^**-module with two generators* $[S^n \xrightarrow{id} S^n] \in N^0(S^n)$ *and* $\iota_n := [pt \to S^n] \in N^n(S^n)$.
2. *For* $n > 0$ *there is an isomorphism of the graded rings* $N^*(S^n) = N^*[\iota_n]/(\iota_n^2)$.

Theorem

Let $\mathbb{P}^n = \mathbb{R}P^n$ *be a n-dim real projective space.*

- *The homomorphism of the graded rings where* $\deg(x_n) = 1$

$$N^*[x_n]/(x_n^{n+1}) \to N^*(\mathbb{P}^n),$$

s.t. $x_n \mapsto [\mathbb{P}^{n-1} \subset \mathbb{P}^n] \in N^1(\mathbb{P}^n)$ *is an isomorphism.*

- *Projections on factors define a ring isomorphism:*

$$N^*(\mathbb{P}^n \times \mathbb{P}^m) \simeq N^*(\mathbb{P}^n) \otimes_{N^*} N^*(\mathbb{P}^m) \simeq N^*[x_n, y_m]/(x_n^{n+1}, y_m^{m+1})$$

Definicja (Segre map / embedding)

Let $\mathbb{P}(\mathbb{R}^{n+1}) := \mathbb{P}^n$ *be a projective space i.e. space of 1-dim subspaces of* $\mathbb{R}^{n+1}$.

$$\sigma_{n,m}\colon \mathbb{P}(\mathbb{R}^{n+1}) \times \mathbb{P}(\mathbb{R}^{m+1}) \to \mathbb{P}(\mathbb{R}^{n+1} \otimes \mathbb{R}^{m+1}) = \mathbb{P}(\mathbb{R}^{(n+1)(m+1)}) =: \mathbb{P}^d$$

$$\sigma_{n,m}((L_1, L_2)) := L_1 \otimes L_2$$

$$\sigma^*_{n,m}\colon N^*(\mathbb{P}^d) = N^*[z]/(z^{d+1}) \to N^*[x,y]/(x^{n+1}, y^{m+1})$$

Passing to infinity $n, m \to +\infty$ we obtain a power series of two variables:

$$\sigma^*(z) = F_{N^*}(x,y) = x + y + \sum_{i,j\geqslant 1}^{+\infty} a_{ij} x^i y^j \in N^*[[x,y]]$$

where $\deg x = \deg y = 1$, $a_{ij} \in N^{1-i-j}$.

Proposition

$$F_{N^*}(x,y) = \frac{x + y + \sum_{m,n>0}^{\infty} [H(m,n)] x^m y^n}{\mathbb{P}(x)\mathbb{P}(y)}$$

where $\mathbb{P}(x) := \sum_i [\mathbb{P}^i] x^i$ *and* $H(m,n) := \{(x,y) \in \mathbb{P}^n \times \mathbb{P}^m \mid x_0 y_0 + \ldots + x_m z_m = 0\}$.

1. R^* - a graded $\mathbb{F}_2$-algebra such that $R^q = 0$ for $q > 0$.
2. Graded ring of formal power series $R^*[[x, y]]$ in two variables x, y in gradation 1.
3. Series of gradation n are of the form $\sum_{i,j\geqslant 0} a_{ij}x^iy^j$ where $\deg(a_{ij}) = n - (i + j)$.

Definition

A commutative formal group law (FGL) over a graded $\mathbb{F}_2$-algebra R^* is a power series $F \in R^*[[x, y]]$ satisfying the following conditions:

1. **(Neutral Element)** $F(x, 0) = x$, $F(0, y) = y$.
2. **(Associativity)** $F(F(x, y), z) = F(x, F(y, z))$,
3. **(Commutativity & Antisymmetry)** $F(x, y) = F(y, x)$, $F(x, x) = 0$

Usually one writes FGL's in the form $F(x, y) = x + y + \sum_{i,j>0} a_{ij}x^iy^j$.

Examples of FGL

- Additive $F_+(x, y) = x + y$ (FGL of singular cohomology)
- Multiplicative $F_m(x, y) = x + y + xy$ (FGL of complex K-theory)
- Formal group of cobordism $F_{N^*}(x, y) \in N^*[[x, y]]$

The set of all FGL over R^* is the set of objects of a category, denoted $FGL(R^*)$

Definition (Morphisms of formal groups over fixed ring)

For $F, G \in \text{ob } FGL(R^*)$

$$\text{Hom}_{R^*}(F,G) = \{\alpha(x) = \sum_{i>0} a_i x^i \mid \deg \alpha = 1,\ \alpha(F(x,y)) = G(\alpha(x), \alpha(y))\}$$

Composition is defined as the composition of series. Identity is the series $\iota(x) = x$.

Definition (Push forward of formal groups via homomomorphisms)

Any ring homomorphism $h\colon R^* \to S^*$ *defines push-forward functor* $h_*\colon FGL(R^*) \to FGL(S^*)$

$$h_* F(x,y) := x + y + \sum h(a_{ij}) x^i y^j$$

.

Proposition (Universal formal group)

There exists $(\mathcal{L}, F_{\mathcal{L}})$ *such that the map*

$$\text{Hom}_{Rings^*}(\mathcal{L}, S) \ni h \mapsto h_* F_{\mathcal{L}} \in FGL(R^*)$$

is bijective i.e. for every FGL (R^*, F) *there is the unique ring homomorphism* $h_F\colon \mathcal{L} \to R^*$ *such that* $h_F * F_{\mathcal{L}} = F$.

Logarithm of a formal group

Definition

Logarithm of $F \in FGL(R^*)$ *is a series* λ *which defines an isomorphism* $\lambda\colon F \to F_+$ *i.e.*

$$\lambda(F(\lambda^{-1}(x), \lambda^{-1}(y)) = x + y =: .F_+(x, y)$$

Theorem (M. Lazard)

Let R^* *be a graded ring of char. 2, and* $F(x, y)$ *an FGL over* R^**. Then* F *jest is isomorphic to the additive formal group. Moreover there exists a unique logarithm* $\lambda_F(x) = x + a_1x^2 + a_2x^3 + \ldots$ *such that* $a_j = 0$ *if* $j = 2^i - 1$ *for some* i*.*

Theorem (Universal formal group)

Let $\mathcal{L} := \mathbb{F}_2[a_2, a_4, a_5, ..]$ *be a graded polynomial ring such that* $\deg a_i = -i$ *and* $i \neq 2^j - 1$*. Let* $\lambda_{\mathcal{L}}(x) \in \mathcal{L}[[x]]$ *be defined as* $\lambda_{\mathcal{L}}(x) := x + \sum_{i>0} a_i x^{i+1}$ *and*

$$F_{\mathcal{L}}(x, y) := \lambda_{\mathcal{L}}^{-1}(\lambda_{\mathcal{L}}(x) + \lambda_{\mathcal{L}}(y)).$$

Then $(\mathcal{L}, F_{\mathcal{L}})$ *is a universal FGL for FGL's over graded* $\mathbb{F}_2$*-algebras.*

In particular it exists unique homomorphism

$$h\colon \mathcal{L} = \mathbb{F}_2[a_2, a_4, a_5, ..] \to N^* \quad \text{such that} \quad h_* F_{\mathcal{L}} = F_{N^*}$$

We'd like to prove that h is an isomorphism!

- Proof that $h\colon \mathcal{L} \to N^*$ is an epimorphism proceeds by induction w.r.t. dimension of manifold. Tool: Steenrod operations (squares) in cobordism.
- Proof that $h\colon \mathcal{L} \to N^*$ is an isomorphism uses cobordism functors twisted by formal groups. Tool: Landweber-Novikov operations.

Let $(\mathcal{L}, F_{\mathcal{L}})$ be the universal FGL and $h\colon \mathcal{L} \to N^*$ be the homomorphism corresponding to the formal group of cobordism. It defines a $\mathcal{L}$-module structure on N^*, thus on $N^*(X)$. For an arbitrary homomorphism $a\colon \mathcal{L} \to R$ (i.e. a FGL) we define a functor:

$$N_a^*(X) := R^* \otimes_{\mathcal{L},a} N^*(X).$$

Proposition

The functor $N_a^*(-)$ is a (bi-)functor. FGL of N_a^* is equal to $F_a \otimes 1$, where F_a is FGL defined by the homomorphism $a\colon \mathcal{L} \to R$.

Cobordism twisted by FGL - functoriality with respect to FGL

Theorem

Any isomorphism of FGL over R, $\theta\colon F_a \to F_b$ *(i.e.* $F_a(x,y) = \theta F_b(\theta^{-1}(x), \theta^{-1}(y))$*) defines a natural transformation of Mackey functors with values in R-modules*

$$\widehat{\theta}\colon N_a^*(X) \to N_b^*(X)$$

Let $F_a \xrightarrow{\varphi} F_b \xrightarrow{\psi} F_c$ *be morphisms of FGL's, and* $a, b, c\colon \mathcal{L} \to R$ *the corresponding homomorphisms. Then the following diagram commutes:*

$$\begin{array}{ccc} N_a^*(X) & \xrightarrow{\widehat{\varphi}} & N_b^*(X) \\ & \searrow^{\widehat{\psi\circ\varphi}} & \downarrow{\widehat{\psi}} \\ & & N_c^*(X) \end{array} \tag{3}$$

Hence any isomorphism θ *defines an isomorphism* $\widehat{\theta}\colon N_a(X) \to N_b(X)$.

Dowód.

Proof uses the Landweber-Novikov operations in cobordism. □

$(\mathcal{L}, F_{\mathcal{L}})$ - universal FGL, $\varepsilon\colon \mathcal{L} \to \mathbb{Z}_2$ - augumentation homomorphism.
The composition $\mathcal{L} \xrightarrow{\varepsilon} \mathbb{Z}_2 \xrightarrow{\iota} \mathcal{L}$ defines the additive FGL on $\mathcal{L}$.

Theorem

Let λ be the canonical logarithm of $F_{\mathcal{L}}$. Then

$$\widehat{\lambda}\colon \mathcal{L} \otimes (\mathbb{Z}_2 \otimes_{\mathcal{L}} N^*(X)) \xrightarrow{\simeq} N^*(X)$$

is a natural isomorphism of rings. In particular,

$$\mathcal{L} \simeq N^*$$

Moreover $\mathbb{Z}_2 \otimes_{N^*} N^*(X) \simeq H^*(X; \mathbb{Z}_2)$.

Dowód.

$\mathcal{L} \otimes (\mathbb{Z}_2 \otimes_{\varepsilon} N^*(X)) = \mathcal{L} \otimes_{\iota \circ \varepsilon} N^*(X) \xrightarrow{\widehat{\lambda}} \mathcal{L} \otimes_{id} N^*(X) = N^*(X).$ □

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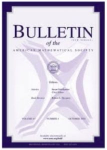

ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

ADVANCES IN MATHEMATICS **7**, 29–56 (1971)

Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations*

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Geometryczny kobordyzm i grupy formalne [1000-1M21GKG]
Semestr zimowy 2021/22
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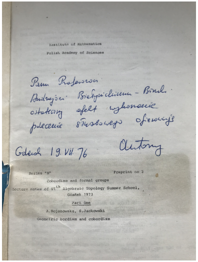

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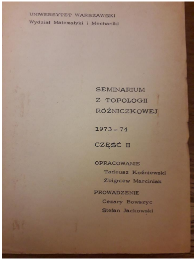

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50 years teaching at WMIM. Thank you!