Between the Pytkeev and Fréchet-Urysohn properties of function spaces

Lyubomyr Zdomskyy

Institute of Mathematics, University of Vienna

Warsaw Topology and Set Theory seminar
Online, November 18, 2020

Based on a joint work in progress with S. Bardyla and J. Šupina.

Fréchet-Urysohn spaces

All spaces are Tychonoff.

Definition

- Y is a *Fréchet-Urysohn space* (or, equivalently, has the *Fréchet-Urysohn property*) if for every $A \subset Y$ and $y \in \bar{A} \setminus A$ there exists a sequence $\langle a_n : n \in \omega \rangle \in A^\omega$ converging to y.
- Y has *countable tightness* (equivalently, $t(Y) \leq \omega$) if for every $A \subset Y$ and $y \in \bar{A} \setminus A$ there exists $B \in [A]^\omega$ with $y \in \bar{B}$. □

$$Y \text{ is FU} \Longrightarrow t(Y) \leq \omega$$

Example. All metrizable spaces are FU. More generally, all first-countable spaces are FU. All spaces X with $t(X) \leq \omega$ and character $< \mathfrak{p}$ are FU. □

Theorem (Hrusak-Ramos Garcia 2014)

It is consistent that all separable FU topological groups are metrizable.

Definition

$C_p(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous }\}$. We consider $C_p(X)$ with the topology inherited from the Tychonoff product $\mathbb{R}^X$. □

Easy Fact. $C_p(X)$ is metrizable iff $C_p(X)$ is first-countable iff $|X| \leq \omega$. □

Theorem (Arkhangel'skii-Pytkeev 1980s)

$t(C_p(X)) \leq \omega$ iff X^n is Lindelöf for all $n \in \omega$. In particular, $t(C_p(X)) \leq \omega$ for every metrizable separable space.

Definition

A family $\mathcal{U} \subset \mathcal{P}(X)$ is called

- an ω-cover of X, if $X \notin \mathcal{U}$ and for every $K \in [X]^{<\omega}$ there exists $U \in \mathcal{U}$ such that $K \subset U$.
- a γ-cover of X, if for every $x \in X$, the set $\{U \in \mathcal{U} : x \notin U\}$ is at most finite.

A space X is called a *γ-space,* if every open ω-cover of X contains a γ-subcover. □

Example. $\{(n, n+2) : n \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ which is not an ω-cover.
$\{(-n, n) : n \in \omega\}$ is a γ-cover of $\mathbb{R}$.
$\{U \subset \mathbb{R} : U$ is open and $\mu(U) < 1\}$ is an ω-cover of $\mathbb{R}$ without any γ-subcover, so $\mathbb{R}$ is not a γ-space. □

Theorem (Gerlits-Nagy 1982)

$C_p(X)$ is FU iff X is a γ-space.

Intuition behind the proof of the Gerlits-Nagy theorem

For $U \subset X$ and $x \in X$ set $\chi_U(x) = 0$ if $x \in U$ and $\chi_U(x) = 1$ otherwise. Let $\mathcal{U}$ be a family of clopen subsets of X.

Easy to check:
$0 \in \overline{\{\chi_U : U \in \mathcal{U}\}} \setminus \{\chi_U : U \in \mathcal{U}\}$ iff $\mathcal{U}$ is an ω-cover of X.
Indeed, pick a basic open neighbourhood
$W := [1/2; x_0, \ldots, x_n] = \{f \in C_p(X) : \forall i \leq n(|f(x_i)| < 1/2)\}$ of 0 in $C_p(X)$ and note that $\chi_U \in W$ iff $\{x_0, \ldots, x_n\} \subset U$.

Equally easy to check:
For $\{U_n : n \in \omega\} \subset \mathcal{U}$, $\{\chi_{U_n} : n \in \omega\}$ converges to 0 iff $\{U_n : n \in \omega\}$ is a γ-cover of X.

This proves the Gerlits-Nagy theorem for $C_p(X, 2)$.

Examples of γ-spaces

Observation

If X is a metrizable separable space of size $< \mathfrak{p}$, then X is a γ-space. $\square$

Theorem (Gerlits-Nagy 1982)

If $X \subset \mathbb{R}$ is a γ-space, then X has the strong measure zero. In particular, there are no uncountable metrizable γ-spaces in the Laver model. $\square$

Theorem (Galvin-A. Miller 1984)

If $\mathfrak{p} = \mathfrak{c}$, then there exists a γ-space $X \subset 2^\omega$ of size $\mathfrak{p}$. $\square$

Theorem (Orenshtein-Tsaban 2011)

If $\mathfrak{p} = \mathfrak{b}$, then there exists a γ-space $X \subset 2^\omega$ of size $\mathfrak{p}$. $\square$

Previously was unknown even under $\mathfrak{d} = \omega_1$!

γ-spaces and forcing

Theorem (A. Miller 2005)

In the Hechler model there are no uncountable metrizable γ-spaces. In particular, the existence of uncountable strong measure zero sets of reals does not imply the existence of uncountable γ-spaces of reals. □

Theorem (A. Miller-Tsaban-Z. 2016)

Metrizable γ-spaces are preserved by Cohen forcing. □

Later, in a joint work with Repovš we have introduced a property of proper posets which is satisfied by Cohen, Miller, and Sacks forcings, and such that metrizable γ-spaces are preserved by countable support iterations of posets with this property.

Question (A. Miller)

Suppose that $X \subset \mathbb{R}$ is a γ-space and $Y \subset \mathbb{R}$, $|Y| < \mathfrak{p}$. Is $X \times Y$ a γ-space? What if $|Y| = \omega_1$ and MA plus non-CH hold?

Weakening the FU property

Definition

A space Y is

- sequential, if for every non-closed $A \subset Y$ there exists $y \in \bar{A} \setminus A$ and a sequence of elements of A convergent to y.
- subsequential, if it can be embedded into a sequential space.
- Pytkeev, if for every $A \subset Y$ and $y \in \bar{A} \setminus A$ there exists a countable family $\mathcal{B}$ of **infinite** subsets of A such that for every open $O \ni y$ there exists $B \in \mathcal{B}$ with $B \subset O$. □

Proposition (Pytkeev 1984)

Subsequential spaces have the Pytkeev property. □

Theorem (Pytkeev 1982)

$C_p(X)$ *is sequential iff it is FU.* □

Weakening the FU property, continued

FU $\implies$ sequential $\implies$ subsequential $\implies$ Pytkeev $\implies$ countable tightness.

Question (Arkhangel'skii 198?)

Is the subsequentiality equivalent to the FU property for C_p-spaces?
□

Theorem (Malykhin 1999)

$C_p([0,1])$ *is not Pytkeev, and hence it is not subsequential.* □

Theorem (A. Miller 2008)

Let X be a metrizable space. If $C_p(X)$ is Pytkeev, then X has the strong measure zero with respect to any totally bounded continuous metric on it. Therefore X is zero-dimensional. Moreover, in the Laver model $|X| = \omega$ iff $C_p(X)$ is Pytkeev. □

Question (Malykhin-Tironi 2000)

Does there exist a non-subsequential Pytkeev space in ZFC? □

M. Sakai 2006: Are the Pytkeev and FU properties equivalent for C_p-spaces?

Theorem (Bardyla-Šupina-Z. 2020)

(CH) There exists $X \subset 2^\omega$ such that $C_p(X)$ has the Pytkeev property but fails to be FU. □

Theorem (Simon-Tsaban 2008)

*The minimal cardinality of a set $X \subset \mathbb{R}$ such that $C_p(X)$ does **not** have the Pytkeev property is equal to $\mathfrak{p}$.* □

Thus no solution to Sakai's problem by playing around with cardinal characteristics.

Ingredient 1 of the proof: Simon-Tsaban characterization

Theorem (Simon-Tsaban 2008)

TFAE for a zero-dimensional space X:

- $C_p(X)$ *has the Pytkeev property;*
- *Each clopen ω-cover $\mathcal{U}$ of X contains infinite subsets $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \ldots$ such that $\{\cap \mathcal{U}_n : n \in \omega\}$ is an ω-cover of X.* □

The latter property of X will be denoted by (π).

Note that if there exists an infinite $\mathcal{U}_\infty \subset \mathcal{U}$ such that $\mathcal{U}_\infty \subset^* \mathcal{U}_n$ for all n, then $\mathcal{U}_\infty$ is a γ-cover of X. This will give us later a hint which witnesses for (π) we should have in a potential counterexample.

Remark. In the characterization above it is enough to demand that for all finite $K \subset X$ there exists n such that $K \subset U$ for almost all $U \in \mathcal{U}_n$.

Ingredient 2 of the proof: Galvin-Miller Lemma

Lemma (Galvin-Miller 1984)

Let $\mathcal{U}$ be an ω-cover of $[\omega]^{<\omega}$ consisting of clopen subsets of $\mathcal{P}(\omega)$. Then there exist an increasing number sequence $\langle k_n : n \in \omega \rangle$ and $\langle U_n : n \in \omega \rangle \in \mathcal{U}^\omega$ such that for every n and $x \subset \omega$, if $x \cap [k_n, k_{n+1}) = \emptyset$, then $x \in U_n$.

Proof. Enough to prove the following:

> *For every $k \in \omega$ there exists $k' > k$ and $U \in \mathcal{U}$ such that for every $x \subset \omega$, if $x \cap [k, k') = \emptyset$, then $x \in U$.*

Hint: take $U \in \mathcal{U}$ such that $U \supset \mathcal{P}(k)$ and using that it is open find suitable k'. $\square$.

Sufficient conditions for getting a counterexample

Lemma

Let $\mathcal{G}$ be an ultrafilter on ω. If the set $\{a_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{G}^*$ satisfies

$$\forall x \in [\omega]^\omega \exists \alpha \, (|x \cap a_\alpha| = \omega),$$

then $X := \{a_\alpha : \alpha < \mathfrak{c}\} \cup [\omega]^{<\omega}$ is not a γ-space.

Proof. Note that $\mathcal{O} = \{O_n : n \in \omega\}$, where $O_n = \{x \subset \omega : n \notin x\}$ is an ω-cover of X. The assumption on X states *literally* that $\mathcal{O}$ has no γ-subcover of X. □

Sufficient conditions for getting a counterexample II

Lemma

(CH). Let $\{k_\alpha : \alpha < \omega_1\}$ *be an enumeration of all increasing sequences from* ${}^\omega\omega$ *(each sequence repeated* ω_1 *many times) and* $\mathcal{G}$ *be an ultrafilter on* ω*. If for a partition* $\{I_n : n \in \omega\} \subset [\omega]^\omega$ *of* ω*, the set* $\{a_\alpha : \alpha < \omega_1\}$ *satisfies*

$$\big\{n \in \omega : (\forall^\infty j \in I_n)\ [k_\beta(j), k_\beta(j+1)) \cap a_\alpha = \emptyset\big\} \in \mathcal{G}$$

for all $\beta \le \alpha$*, then* $X := \{a_\alpha : \alpha < \omega_1\} \cup [\omega]^{<\omega}$ *has* (π)*.*

Proof. Given an ω-cover $\mathcal{U}$ of X, find β and $\langle U_j : j \in \omega\rangle \in \mathcal{U}^\omega$ such that $x \cap [k_\alpha(j), k_\alpha(j+1)) = \emptyset$ implies $x \in U_j$. . Set $\mathcal{U}_n = \{U_j : j \in I_n\}$ and pick a finite $s \subset \omega_1 \setminus \beta$. For every $\alpha \in s$ the set

$$G_\alpha := \big\{n \in \omega : (\forall^\infty j \in I_n)\ [k_\beta(j), k_\beta(j+1)) \cap a_\alpha = \emptyset\big\}$$

belongs to $\mathcal{G}$, and note that

$$G_\alpha \subset \big\{n \in \omega : (\forall^\infty j \in I_n)\, a_\alpha \in U_j\big\}.$$

Pick $n \in \bigcap_{\alpha\in s} G_\alpha$ and note that $\{a_\alpha : \alpha \in s\} \subset U_j$ for almost all $j \in I_n$, i.e., for almost all $U_j \in \mathcal{U}_n$. Thus,

$$\{\cap\mathcal{V} : \mathcal{V} \text{ is a cofinite subset of } \mathcal{U}_n \text{ for some } n\}$$

is an ω-cover of $\{a_\alpha : \alpha < \omega_1\} \cup [\omega]^{<\omega}$. This is almost (π) for X :-) □

Bringing together two halves

Let $\{k_\alpha : \alpha < \omega_1\}$ be an enumeration of all increasing sequences from ${}^\omega\omega$ (each sequence repeated ω_1 many times) and $\mathcal{G}$ be an ultrafilter on ω. Let $\{I_n : n \in \omega\} \subset [\omega]^\omega$ be a partition of ω.

If the set $\{a_\alpha : \alpha < \omega_1\} \subset \mathcal{G}^*$ satisfies

$$\left\{n \in \omega : (\forall^\infty j \in I_n)\ [k_\beta(j), k_\beta(j+1)) \cap a_\alpha = \emptyset\right\} \in \mathcal{G}$$

for all $\beta \leq \alpha$, and

$$\forall x \in [\omega]^\omega \exists \alpha\, (|x \cap a_\alpha| = \omega),$$

then $X := \{a_\alpha : \alpha < \mathfrak{c}\} \cup [\omega]^{<\omega}$ is not a γ-space but still satisfies (π).

Now just do a rather straightforward transfinite construction to get the two conditions above fulfilled...

Remark. The construction could be done under $\mathfrak{p} = \mathfrak{c}$, but there is a model of $\mathfrak{p} = \mathfrak{b}$ where all sets we can get in *such* constructions are γ-sets.

Is our result optimal/satisfactory?

No!!!

The property (π) is typical combinatorial covering property (selection principle), and being a γ-space is the strongest "standard" one. The weakest "standard" selection principle is the following Menger property:

> *For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X.*

Question

Does (π) imply Menger? What about the subsets of 2^ω? □

Would be very helpful (among other, sufficient for the answer of the question above) to know the answer to the following

Question

Is (π) for subsets of 2^ω preserved by the Cohen forcing? □

If yes, then in ZFC we would be able to conclude that (π) implies all finite powers being both Rothberger and Hurewicz, thus pushing it rather close to γ-spaces in the Scheepers diagram.

The last slide

Dziękuję za uwagę.

Thank you for your attention.