Automatic continuity of homomorphisms on topological groups

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Warsaw, 26 January 2022

Motivation: the Cauchy equation

A function $f : \mathbb{R} \to \mathbb{R}$ is

- *additive* if $\forall x, y \in \mathbb{R} \quad f(x+y) = f(x) + f(y)$;
- *linear* if $\quad \exists a \in \mathbb{R} \quad \forall x \in \mathbb{R} \quad f(x) = ax$.

Theorem (Cauchy, 1821)

A function $f : \mathbb{R} \to \mathbb{R}$ is linear iff it is additive and continuous.

Problem (Cauchy)

Are there any nonlinear (and hence discontinuous) solutions of the Cachy function equation $f(x+y) = f(x) + f(y)$?

Example (Haar, 1905)

The Axiom of Choice implies the existence of a discontinuous additive function $f : \mathbb{R} \to \mathbb{R}$.

BP-Measurable homomorphisms are continuous

Theorem (Banach, Schwartz)

Any BP-measurable additive function between Banach spaces is continuous.

Def: A function $f : X \to Y$ between topological spaces is *BP-measurable* if for every open set $U \subseteq Y$ the preimage $f^{-1}[U]$ has the *Baire Property*, i.e., belongs to the smallest σ-algebra containing all open sets and all meager subsets of X.

It is clear that every continuous function is BP-measurable.

Theorem (Pettis, 1950)

Every BP-measurable homomorphism between Polish groups is continuous.

The reason of automatic continuity

Theorem (Piccard, 1939; Pettis, 1951)

If a subset A of a topological group X is not meager and has the Baire property, then AA^{-1} is a neighborhood of the identity in X.

Def: A topological group G is *ω-narrow* if for every nonempty open set $U \subseteq G$ there exists a countable set $C \subseteq G$ such that $UC = G$.

Corollary (Pettis)

Eacy BP-measurable homomorphism $h\colon X \to Y$ from a nonmeager topological group to an ω-narrow topological group is continuous.

Proof: Given any nbhd U of e_Y in Y, choose a nbhd $V \subseteq Y$ of e_Y such that $VV^{-1} \subseteq U$. By ω-narrowness of Y, find a countable set $C \subseteq X$ such that $Y = V \cdot h[C]$. Then $X = h^{-1}[V] \cdot C$ and hence $A = h^{-1}[V]$ is nonmeager in X. Since h is BP-measurable, A has the Baire property in X and then AA^{-1} is a neighborhood of e_X by the Piccard-Pettis Theorem. Then $h[AA^{-1}] = h[A]h[A]^{-1} \subseteq VV^{-1} \subseteq U$ and hence $h^{-1}[U] \supseteq AA^{-1}$ is a nbhd of e_X, so h is continuous.

Haar measurable homomorphisms are continuous

Theorem (Weil, 1965)

Every Haar-measurable homomorphism from a locally compact topological group to any ω-narrow topological group is continuous.

Reason?

Theorem (Steinhaus, 1920; Weil, 1965)

For every Haar-measurable set A of positive Haar measure in a locally compact topological group the set AA^{-1} is a neighborhood of the identity.

Therefore we have two results on automatic continuity of homomorphisms from Baire or locally compact groups with values in ω-narrow topological groups.

Are those results still true without ω-narrowness of the range group?

Homomorphisms between locally compact groups

Theorem (Kleppner, 1989)

Any Haar-measurable homomorphism between locally compact groups is continuous.

Theorem (Kuznetsova, 2012)

Under Martin's Axiom, every Haar-measurable homomorphism from a locally compact group to any topological group is continuous.

Problem (Kuznetsova, 2012)

Is the preceding theorem true in ZFC?

Main Results

Theorem (B., 2022)

Every Haar-measurable homomorphism from any locally compact topological group to any topological group is continuous.

A Tychonof space X is *Čech-complete* if it is a G_δ-set in its compactification.

Theorem (B., 2022)

Every BP-measurable homomorphism from an ω-narrow Čech-complete top group to any top group is continuous.

Corollary (B., 2022)

Every Borel homomorphism from a Čech-complete topological group to an arbitrary topological group is continuous.

Four Poles Theorem and its corollary

Theorem (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski, 1997)

Let $\mathcal{I}$ be a σ-ideal with a Borel base on a Polish space X. Any point-finite subfamily $\mathcal{J} \subseteq \mathcal{I}$ with $\bigcup \mathcal{J} \notin \mathcal{I}$ contains a subfamily $\mathcal{J}' \subseteq \mathcal{J}$ whose union $\bigcup \mathcal{J}'$ does not belong to the smallest σ-algebra $\mathcal{BI}$ containing all Borel sets and all sets in the ideal $\mathcal{I}$.

Corollary

Let $\mathcal{I}$ be a left-invariant σ-ideal with a Borel base on a Polish group X. For any $\mathcal{BI}$-measurable homomorphism $h : X \to Y$ to an arbitrary topological group Y and any neighborhood $U \subseteq Y$ of the identity we have $h^{-1}[U] \notin \mathcal{I}$.
If $\mathcal{I}$ is Steinhaus or ccc, then h is continuous.

What are nonmetrizable generalizations of those results?

Let $\mathcal{I}$ be a σ-ideal on a Polish space X. A subset $M \subseteq X$ is called *analytically $\mathcal{I}$-measurable* if for any analytic set $A \subseteq X$ with $A \cap M \notin \mathcal{I}$ there exists an analytic set $B \subseteq A \cap M$ such that $B \notin \mathcal{I}$.

Theorem (B., Ratowski, Żeberski, 2021)

Let $\mathcal{I}$ be a σ-ideal on a Polish space X. For any point-finite subfamily $\mathcal{J} \subseteq \mathcal{I}$ with $\bigcup \mathcal{J} \notin \mathcal{I}$ there exists a subfamily $\mathcal{J}' \subseteq \mathcal{I}$ whose union $\bigcup \mathcal{J}'$ is not analytically $\mathcal{I}$-measurable.

K-analytic spaces and groups

A Tychonoff space is *K-analytic* if it is a continuous image of a Lindelöf Čech-complete space.

It is known that a metrizable space is K-analytic iff it is analytic.

Theorem (B., 2022)

For a topological group X the following conditions are equivalent:

1. *X is K-analytic and Baire;*
2. *X is Lindelöf and Čech-complete;*
3. *X is ω-narrow and Čech-complete;*
4. *X is countably cellular and Čech-complete.*

Functionally Borel and (co)analytic sets

Let $\mathcal{P}$ is some property of subsets in Polish spaces, for example, Borel, analytic, coanalytic, etc.

A subset A of a topological space X is called *functionally $\mathcal{P}$* if there exists a continuous function $f : X \to Y$ to a Polish space Y and a set $B \subseteq Y$ with property $\mathcal{P}$ such that $A = f^{-1}[B]$.

Let $\mathcal{I}$ be a σ-ideal on a Tychonoff space X. A subset $M \subseteq X$ is called *K-analytically $\mathcal{I}$-measurable* if for any K-analytic subspace $A \subseteq X$ with $A \cap M \notin \mathcal{I}$ there exists a K-analytic subspace $B \subseteq A \cap M$ such that $B \notin \mathcal{I}$.

Theorem (B., 2022)

Let $\mathcal{I}$ be a left-invariant σ-ideal with functionally coanalytic base on a K-analytic topological group. Assume that $h : X \to Y$ is a homomorphism to a topological group Y such that for every open set $U \subseteq Y$ the preimage $h^{-1}[U]$ is K-analytically $\mathcal{I}$-measurable in X. Then for every neighborhood $U \subseteq Y$ of the identity we have $h^{-1}[U] \notin \mathcal{I}$. If $\mathcal{I}$ has is Steinhaus or ccc, then h is continuous.

Steinhaus ideals

A left-invariant σ-ideal $\mathcal{I}$ on a topological group X is defined to be

- *n-Steinhaus* if for any K-analytic subspace $A \notin \mathcal{I}$ in X the set $(AA^{-1})^n$ is a neighborhood of the identity in X;
- *Steinhaus* if it is n-Steinhaus for some $n \in \mathbb{N}$;
- *ccc* if X contains no uncountable family of $\mathcal{I}$-positive K-analytic subsets.

Example

1. By Piccard–Pettis Theorem, the σ-ideal $\mathcal{M}$ of meager sets any Baire topological group is 1-Steinhaus;
2. By Steinhaus–Weil Theorem, the ideal $\mathcal{N}$ of Haar-null sets in a locally compact group is 1-Steinhaus.
3. Every left-invariant ccc-ideal on a Baire topological group is 2-Steinhaus.

Main result

Theorem

Let X be an ω-narrow Čech-complete group and $\mathcal{I}$ be a Steinhaus left-invariant σ-ideal with a functionally coanalytic base on X. Every $\mathcal{BI}$-measurable homomorphism $h : X \to Y$ to any topological group Y is continuous.

Corollary

1. *Every Haar-measurable homomorphism from a locally compact group to any topological group is continuous.*
2. *Every BP-measurable homomorphism from any ω-narrow Čech-complete group to any topological group is continuous.*
3. *Every Borel homomorphism from any Čech complete group to any topological group is continuous.*

Thank you!