# Approximation in the calculus of variations

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# Variational functionals

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$$u \mapsto \mathcal{F}[u] = \int_\Omega F(x, u, Du)\, dx\,, \quad \Omega \subset \mathbb{R}^n\,, \quad u : \Omega \to \mathbb{R}\,.$$

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is attained and what are the properties of the minimizer
(e.g. $C^{0,\alpha}$, $C^{1,\alpha}$ regularity, partial regularity, higher integrability).

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**Typical regularity method**

- consider nice enough $\mathcal{F}_\varepsilon$ and find a minimizer $u_\varepsilon$ (for every $\varepsilon$)
- show that $u_\varepsilon \to u$ and $\mathcal{F}_\varepsilon[u_\varepsilon] \to \mathcal{F}[u]$ well enough
- show that the limit function $u$ shares regularity with each $u_\varepsilon$

# Natural space to minimize a functional 1/2

We note that

$$\inf_{\textit{all}\ u} \mathcal{F}[u] \leq \inf_{\textit{regular}\ u} \mathcal{F}[u]\,,$$

where we think about all functions that make the functional finite.

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$(u_{\min}(x) = x^{\frac{1}{3}})$

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Denoting

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Previous example showed the functional with the **gap** between absolutely continuous functions $AC([0,1])$ and $C^1([0,1])$.

# No gap in the classical case

**Dirichlet principle.** The scalar Euler–Lagrange equation

$$-\Delta u = 0 \qquad \text{in} \;\; \Omega$$

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It does not matter if we minimize over

$$W(\Omega) := \left\{ \text{all functions with } Du \text{ such that } \int_\Omega |Du|^2 \, dx < \infty \right\}$$

$$H(\Omega) := \overline{C_0^\infty(\Omega)}^{W}$$

$$\text{because} \qquad \inf_{u \in u_0 + H} \mathcal{F}[u] = \inf_{u \in u_0 + W} \mathcal{F}[u] \,.$$

# No gap in the power case

Consider a functional

$$u \mapsto \mathcal{F}[u] = \int_\Omega F(x, u, Du)\, dx, \quad \Omega \subset \mathbb{R}^n$$

with the growth of $F$ governed by a power function for $1 < p < \infty$:

$$\nu|\xi|^p \leq F(x, s, \xi) \leq L\big(|\xi|^p + 1\big)\,.$$

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$$\inf_{u \in u_0 + H} \mathcal{F}[u] = \inf_{u \in u_0 + W} \mathcal{F}[u]\,.$$

This property **may fail** if $F$ is governed by **not regular enough** inhomogeneous function, e.g. $|\xi|^{p(x)}$ or $|\xi|^p + a(x)|\xi|^q$.

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**2. Functional analysis**

Can any function from an unconventional space be approximated?

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### 2. Functional analysis

Can any function from an unconventional space be approximated?

### 3. PDEs

What existence results one can provide thanks to the density properties?

# Calculus of variations motivation

**minimization of** $u \mapsto \int_\Omega F(x, u, Du)\, dx$ **for** $u : \Omega \to \mathbb{R}$

## Gap or no gap for minimizers

- Lavrentiev '26, Mania '34
- Gossez '82, Zhikov 80'-10'
- Buttazzo & Mizel '95, Belloni & Buttazzo '92
- Fonseca, Malý, Mingione '04, Esposito, Leonetti, Mingione '04, Balci, Diening, Surnachev '20
- Esposito, Leonetti, Petricca '19, Leonetti, De Filippis '22, Koch '22
- Bousquet '23
- 2018+ **via density**: Ahmida, Alberico, Borowski, Buliček, Chlebicka (Skrzypczak), Cianchi, Gwiazda, Skrzeczkowski, Świerczewska-Gwiazda, Wróblewska-Kamińska, Youssfi, Zatorska-Golstein

# Real-world motivation

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## cheese issue

Energy density: $\int_\Omega |Du|^p + a(x)|Du|^q \, dx$ with nasty weight $a$

**Figure:** Inhomogeneous medium

- **not** enough approximation properties of the function space
- **no** regularity of minimizers or solutions

## Examples of spaces with functions that cannot be approximated 1/2

### Variable exponent spaces

$$W := \{f \in W^{1,1}_{loc} : \ |Df|^{p(x)} \in L^1\}$$

when the exponent $p$ is not log-Hölder continuous [Zhikov1986]

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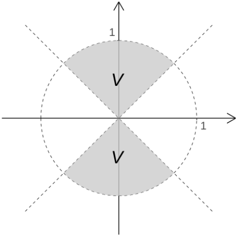
- Checkerboard on a 2d-plane
  $u_0$ – nice trace
- Nasty exponent: $p > 2$ in $V$
  and $p < 2$ outside $V$
- Bad $u_* \in W$ has $Du_* \equiv 0$ in $V$
  (but it jumps)




Then

$$\inf_{u \in u_0 + W} \mathcal{F}[u] \le \mathcal{F}[u_*] < \inf_{u \in u_0 + H} \mathcal{F}[u]$$

## Examples of spaces with functions that cannot be approximated 2/2

### Double phase spaces

$$W := \{f \in W^{1,1}_{loc} : \ |Df|^p + a(x)|Df|^q \in L^1\}$$

with $a : \Omega \to [0, \infty)$, $a \in C^{0,\alpha}$,

when powers do not satisfy $p < n < n + \alpha < q$

see [Zhikov1995], [Esposito, Leonetti, & Mingione, JDE2004]

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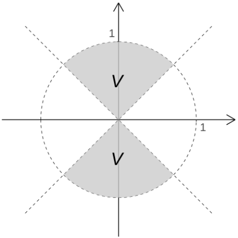
- Checkerboard on a 2d-plane, extended to n-d; $u_0$ – nice trace
- Nasty weight $a \in C^{0,\alpha}$ with $\operatorname{supp} a \subset V$.
- Bad $u_* \in W$ with $Du_* \equiv 0$ in $B_1 \setminus V$ (but it jumps).



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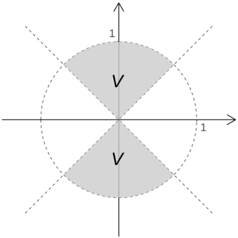


Use of fractals to get rid of the dimensional threshold

[Balci, Diening, & Surnachev, CalcVar2020]

# Consequences for minimizers

**Double phase spaces**

**[Fonseca, Malý, & Mingione, ARMA2004]**

For a functional $\int_\Omega |Du|^p + a(x)|Du|^q \, dx$ with $a \in C^{0,\alpha}$,

if $p < n < n + \alpha < q$

($=$ the closeness condition on the weights is not satisfied),

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**It is the fault of $a$.**

# Real-world motivation

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## thick soup case

Energy density: $\int_\Omega |Du|^p + a(x)|Du|^q \, dx$ with **nice** weight $a$

**Figure:** Inhomogeneous medium

- **good** approximation properties of the function space
- **possible** for study regularity of minimizers or solutions

# Regularity of minimizers

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if any of those holds true

- $\frac{q}{p} < 1 + \frac{\alpha}{n}$,
- a priori $u \in L^\infty$ and $q < p + \alpha$,
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series of works by Baroni, Colombo, Mingione 2015-18
continued by Harjulehto, Hästö, Byun, and their collaborators

# PDE motivation

## heating up a thick soup

### Goal

General theory for nonlinear diffusion equations in inhomogeneous media.

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Well-posedness of problems like

$$\partial_t u - \mathrm{div}\mathcal{A}(t, x, Du) = f(t, x)$$

with $\mathcal{A}(t, x, \xi)$ of growth given by $M : [0, T] \times \Omega \times \mathbb{R}^n \to [0, \infty)$.

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If $M$ is regular enough, then for any $u \in W$ there exists $\{u_k\} \subset C_0^\infty$:

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In the classical case of $W^{1,p}$ we have $M(t,x,\xi) \equiv |\xi|^p$.
Then the above density is in norm. It can be obtained by mollification.

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general growth (Orlicz) $\Delta_A$, anisotropic $\Delta_{\vec{p}}$ and more...

# Sufficient conditions on $M$ for $H = W$

## thick soup case

By the many efforts including [Ahmida, C, Gwiazda, Youssfi, JFA 2018], [Hästo, Harjulehto, Springer Lecture Notes 2019], [C, Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska, Springer Monographs in Mathematics 2021], [Borowski & C, JFA 2022], [Buliček, Gwiazda, Skrzeczkowski, ARMA 2022] we know that

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**Balance condition $(B)$**

For $M : \Omega \times \mathbb{R}^n \to [0, \infty)$ there exists a constant $C_M > 1$ such that

$$\text{there holds} \qquad \sup_{y \in B(x)} M(y, \xi) \leq M(x, C_M \xi)$$

for prescribed ranges of $x$ and $\xi$ $\qquad \leftarrow$ we fight for these ranges

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# Sufficient conditions on $M$ for $H = W$

## thick soup case

By the many efforts including [Ahmida, C, Gwiazda, Youssfi, JFA 2018], [Hästo, Harjulehto, Springer Lecture Notes 2019], [C, Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska, Springer Monographs in Mathematics 2021], [Borowski & C, JFA 2022], [Buliček, Gwiazda, Skrzeczkowski, ARMA 2022] we know that

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**Balance condition** $(B)$

For $M : \Omega \times \mathbb{R}^n \to [0, \infty)$ there exists a constant $C_M > 1$ such that

$$\text{there holds} \qquad \sup_{y \in B(x)} M(y, \xi) \leq M(x, C_M \xi)$$

for prescribed ranges of $x$ and $\xi$ $\quad \leftarrow$ we fight for these ranges

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implies needed **modular** density of $H$ in $W$.

# Basic ideas

**thick soup case**

1) Approximation is based on convolution
2) $\Omega$ is decomposed to star-shaped sub-domains $\Omega_i$
3) For $\Omega_i$ star-shaped with respect to $B_i(0,r)$, $\xi \in \mathcal{L}_M(\Omega_i)$ with $\operatorname{supp}\xi \subset \Omega$ and $\kappa_\delta < 1$, we define

$$S_\delta \xi(x) = \int_{\Omega_i} \rho_\delta(x-y)\xi(y/\kappa_\delta)\, dy.$$

4) **we need a kind of Jensen inequality to take convolution $S_\delta$ with respect to $x$ from inside of $M$ to get**

$$M\left(x, \frac{DS_\delta\varphi(x)}{\lambda}\right) \lesssim S_\delta M\left(\cdot, \frac{D\varphi(\cdot)}{\tilde{\lambda}}\right)(x) + 1$$

**but $M$ depends on $x$** (our space is defined via $\int_\Omega M(x,\xi)\,dx$)
this step essentially requires a balance condition $(B)$.

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**Answer YES is coming soon**

Borowski, C, Miasojedow, De Filippis, *Absence and presence of Lavrentiev's phenomenon in double phase functionals for every choice of exponents.*

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**Figure:** Nasty M = bad medium

**Figure:** **Nice** M = good medium

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- regularity and other properties of minimizers to

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- well-posedness of

$$\partial_t u - \mathrm{div}\mathcal{A}(t,x,Du) = f(t,x)\,,$$

  where $\mathcal{A}(t,x,\xi)$ has growth given by $M(t,x,\xi)$; then theory of PDEs (local and global qualitative properties of solutions like uniqueness, multiplicity, symmetry, local regularity, optimal transfer regularity from data to solutions, asymptotic behaviour...)

**Thank you for your attention!**