Approximate Fraïssé theory and MU-categories

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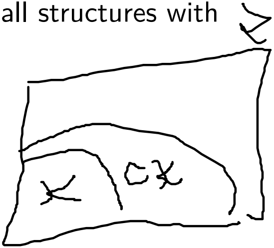
Topology and Set Theory Seminar
University of Warsaw (via Zoom), 20 January 2021

This is joint work in progress with Wiesław Kubiś,
part of the EXPRO project 20-31529X:
Abstract Convergence Schemes And Their Complexities

1. Fraïssé theory

Fraïssé theory – rough idea

Fraïssé theory links properties of families of small structures with properties of limit structures.



$\mathcal{K}$ – our family of small objects,

$\mathcal{L}$ – ambient family of large objects, $\mathcal{K} \subseteq \mathcal{L}$,

$\sigma\mathcal{K}$ – limit objects, approximated by $\mathcal{K}$, "countable unions of chains in $\mathcal{K}$".

Structures are organized through extensions:

- A simpler structure is embedded in a more complex structure: "$A \subseteq B$".
- We may use the language of category theory: "$A \to B$".

Fraïssé theory – rough idea

Examples of $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$

($\mathcal{L}$ is not that important):

- finite linear orders, countable linear orders;
- finite graphs, countable graphs;
- finite groups, *locally finite* countable groups;
- finite T_1 topological spaces, countable discrete spaces;
- finite topological spaces, countable Alexandrov-discrete spaces.

We may change the morphisms:

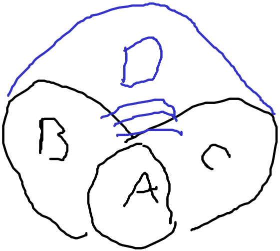
- finite T_1 topological spaces with continuous surjections, metrizable zero-dimensional compact spaces with continuous surjections.

Fraïssé theory – properties

Definition

$\mathcal{K}$ has the *amalgamation property (AP)* if for every $\mathcal{K}$-maps $f\colon A \to B$ and $g\colon A \to C$ there are $\mathcal{K}$-maps $f'\colon B \to D$ and $g'\colon C \to D$ such that $f' \circ f = g' \circ g$.

"Every two extensions $A \subseteq B$ and $A \subseteq C$ can be encompassed in a common extension $A \subseteq D$."

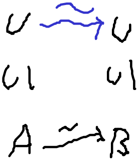


Fraïssé theory – properties

Definition

An $\mathcal{L}$-object U is *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-maps $f, g \colon A \to U$ from a $\mathcal{K}$-object there is an automorphism $h \colon U \to U$ such that $h \circ f = g$.

"Every isomorphism of small substructures can be extended to an automorphism of the whole structure."

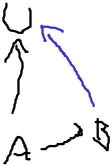


Fraïssé theory – properties

Definition

An $\mathcal{L}$-object U is *injective* or *extensive* or has the *extension property* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f: A \to U$ and every $\mathcal{K}$-map $g: A \to B$ there is an $\mathcal{L}$-map $h: B \to U$ such that $h \circ g = f$.

"Every small extension of a small substructure can be accommodated in the whole structure."

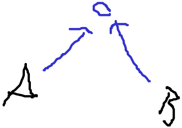


Fraïssé theory – three pillars

Theorem (existence of the Fraïssé sequence)

$\mathcal{K}$ has a *Fraïssé sequence* if and only if the following are true:

1. $\mathcal{K}$ has a countable *dominating* subcategory
   ($\sim$ "countably many isomorphism-types"),
2. $\mathcal{K}$ is *directed*
   ("joint embedding property"),
3. $\mathcal{K}$ has the amalgamation property.



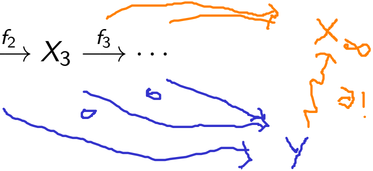
We call $\mathcal{K}$ a *Fraïssé category*.

Fraïssé theory – sequences

Recall

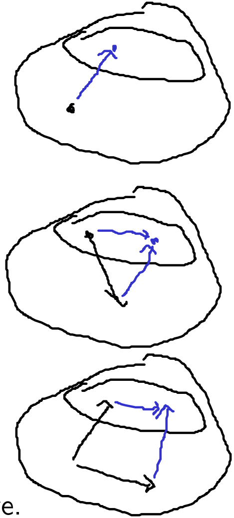
- A *(direct) sequence* $\langle X_*, f_* \rangle$ in a category is a diagram

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots$$



- f_n^m denotes the composition $X_n \to X_m$ for $n \leq m$.
- A *(co)cone* for $\langle X_*, f_* \rangle$ is a pair $\langle Y, g_* \rangle$ consisting of maps $g_n \colon X_n \to Y$ such that $g_m \circ f_n^m = g_n$ for $n \leq m$.
- A *(co)limit* or *direct limit* of $\langle X_*, f_* \rangle$ is a cone $\langle X_\infty, f_*^\infty \rangle$ such that for every cone $\langle Y, g_* \rangle$ for $\langle X_*, f_* \rangle$ there is a unique map $g_\infty \colon X_\infty \to Y$ such that $g_\infty \circ f_n^\infty = g_n$ for every n.

Fraïssé theory – properties



Definition

A subcategory or a sequence $\mathcal{S}$ in $\mathcal{K}$ is

universal/cofinal if for every $\mathcal{K}$-object K there is a $\mathcal{K}$-map $f: K \to S$ to an $\mathcal{S}$-object;

absorbing if for every $\mathcal{S}$-object S and a $\mathcal{K}$-map $f: S \to K$ there is a $\mathcal{K}$-map $g: K \to S'$ such that $g \circ f$ is an $\mathcal{S}$-map;

injective/extensive if for every $\mathcal{K}$-map $f: K \to S$ to an $\mathcal{S}$-object and every $\mathcal{K}$-map $g: K \to K'$ there is an $\mathcal{S}$-map $f': S \to S'$ and $\mathcal{K}$-map $g': K' \to S'$ such that $f' \circ f = g' \circ g$;

dominating if it is cofinal and absorbing;

extensively dominating if it is cofinal and extensive.

Fraïssé sequence is a dominating sequence in a category with (AP) or equivalently an extensively dominating sequence.

Fraïssé theory – three pillars

Theorem (uniqueness of the Fraïssé sequence)

The Fraïssé sequence in $\mathcal{K}$ is unique up to an isomorphisms of sequences via a back and forth construction.

$$\begin{array}{ccccccccccc} X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \cdots \\ Y_0 & \xrightarrow[g_0]{} & Y_1 & \xrightarrow[g_1]{} & Y_2 & \xrightarrow[g_2]{} & Y_3 & \xrightarrow[g_3]{} & Y_4 & \xrightarrow[g_4]{} & \cdots \end{array}$$

Fraïssé theory – three pillars

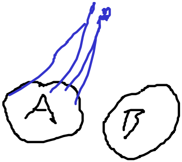
Theorem (characterization of the Fraïssé limit)

Let $\sigma\mathcal{K}$ be a *free completion* of $\mathcal{K}$. For an $\sigma\mathcal{K}$-object U = X the following conditions are equivalent:

1. X is the limit in $\sigma\mathcal{K}$ of a Fraïssé sequence in $\mathcal{K}$.
2. X is cofinal and homogeneous in $\langle\mathcal{K}, \sigma\mathcal{K}\rangle$.
3. X is cofinal and extensive in $\langle\mathcal{K}, \sigma\mathcal{K}\rangle$.

Such object U is unique, cofinal in $\sigma\mathcal{K}$, and every sequence in $\mathcal{K}$ with limit U is Fraïssé. U is called the *Fraïssé limit* of $\mathcal{K}$.

Fraïssé theory



Examples of $\mathcal{K}$, $\sigma\mathcal{K}$, and U

Fraïssé category $\mathcal{K}$, free completion $\sigma\mathcal{K}$, Fraïssé limit U, translation of the extension property:

- finite linear orders, countable linear orders, the rationals $\langle \mathbb{Q}, \leq \rangle$, dense linear order: $\forall x < y \; \exists z_0 < x < z_1 < y < z_2$;
- finite graphs, countable graphs, the Rado graph, $\forall A, B \subseteq U$ disjoint finite $\exists x \in U \setminus (A \cup B)$ with an edge to every point of A and with no edge to a point of B;
- finite discrete spaces with surjections, metrizable zero-dimensional compact spaces with continuous surjections, the Cantor space 2^ω, no isolated points.

Fraïssé theory – flavors

Classical/model-theoretic $\times$ abstract/category-theoretic

Classical:

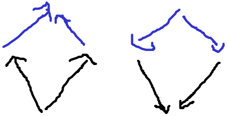
- finite or finitely generated first-order structures with embeddings;
- age, hereditary property, joint embedding property, countably many isomorphism-types.

Abstract:

- abstract objects and morphisms,
- language of category theory,
- a toy example – $\langle \mathbb{N}^+, \cdot \rangle$ (monoid as a category with one object), Fraïssé sequence: a sequence of numbers such that each prime divides infinitely many of them (note that the numbers in the sequence are morphisms, not objects).

Fraïssé theory – flavors

Injective × projective



Conceptually, there are two directions of arrows:

- domain → codomain (the morphism direction),
- simpler → more complex (the extension direction).

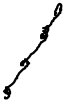
Fraïsse-theoretic notions should follow the extension direction.

Injective:

- The directions are the same, as with embeddings.
- Includes the classical Fraïssé theory.

Projective:

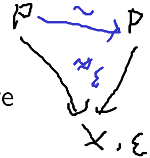
- The directions are opposite, as with quotients.
- Includes the projective Fraïssé theory introduced by Irvin and Solecki (2006) – deals with topological first-order structures and continuous epimorphisms.



- Finite connected linear graphs yield a pre-space of the *pseudo-arc* as Fraïssé limit.

Fraïssé theory – flavors

Discrete/strict × approximate/continuous



- So far everything was discrete – the diagrams were commuting strictly.
- Irwin and Solecki characterized the pseudo-arc as the unique arc-like continuum P such that for every arc-like continuum X, every $\varepsilon > 0$, and every two continuous surjections $f, g \colon P \to X$ there is a homeomorphism $h \colon P \to P$ such that $f \approx_\varepsilon g \circ h$.
- The characterization looks like approximate version of homogeneity. This can be formalized, and the whole Fraïssé theory can be done in approximate setting. Done by Kubiś (2012) in *metric-enriched* setting; now we generalize to *MU-categories*.
- Other examples include the *Urysohn space* (over finite metric spaces and isometric embeddings) and the *Gurarij space* (over finite-dimensional Banach spaces and isometric embeddings).

Fraïssé theory – flavors

Strong/ordinary × weak

There are weaker forms of the amalgamation property, extension property and homogeneity.

- $\mathcal{K}$ has the *cofinal amalgamation property (CAP)* if for every $\mathcal{K}$-object A there is a $\mathcal{K}$-map $A \to A'$ to an amalgamable object.
- $\mathcal{K}$ has the *weak amalgamation property (WAP)* if for every $\mathcal{K}$-object A there is an amalgamable $\mathcal{K}$-map $A \to A'$.

Allows to consider *weak Fraïssé limits* to accommodate more examples; is connected to the abstract Banach–Mazur game. There are also projective and/or approximate variants.

Fraïssé theory – flavors

Countable × uncountable

- So far, everything was countable – sequences, dominating subcategories.
- It is possible to consider uncountable sequences or directed diagrams.
- For example, under (CH), $\mathcal{P}(\omega)/\text{fin}$ is the Fraïssé limit of countable boolean algebras and embeddings.
- In the approximate setting we have used $\varepsilon \in (0, \infty)$; it might be appropriate to generalize to uniformities in the uncountable case.

Fraïssé theory – flavors

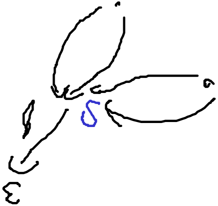
Our setting today:

classical	$\times$	abstract
injective	$\times$	projective
strict	$\times$	approximate
ordinary	$\times$	weak
countable	$\times$	uncountable

- In the abstract setting "injective $\times$ projective" is just a matter of convention. We choose projective because it fits our application.
- Since we want to build approximate Fraïssé theory in the abstract setting, we need a framework to deal with "$f \approx_\varepsilon g$" abstractly. This leads to *MU-categories*.

2. MU-categories

MU-categories



Definition

An *MU-category* is a category $\mathcal{K}$ such that

1. every hom-set $\mathcal{K}(X, Y)$ is an ∞-metric space,
2. for every $\mathcal{K}$-map f we have $d(g \circ f, h \circ f) \leq d(g, h)$ for every compatible $\mathcal{K}$-maps,
3. for every $\mathcal{K}$-map f and every $\varepsilon > 0$ there is $\delta > 0$ such that f is $\langle \varepsilon, \delta \rangle$-*continuous*: $d(g, h) < \delta$ implies $d(f \circ g, f \circ h) < \varepsilon$.

The letters "M" and "U" refer to "metric" and "uniformity".

Example

$\mathbf{Met_u}$, the category of all metric spaces and all uniformly continuous maps endowed with the supremum ∞-metric: $d(f, g) := \sup\{d(f(x), g(x)) : x \in X\}$, is an MU-category.

MU-categories

Example

Every category $\mathcal{K}$ may be endowed with the 0–1 discrete metric, turning it into a *discrete* MU-category.

Example

Every metric space X can be turned into an MU-category $\mathcal{K}_X$ with one nontrivial hom-set corresponding to X.

In this sense, MU-categories generalize both categories and metric spaces.

Example

MCpt, the category of all metrizable compact spaces and all continuous maps, and its subcategories can be viewed as MU-categories – formally, a compatible metric has to be fixed on every metrizable compact space, so **MCpt** becomes an MU-subcategory of $\mathbf{Met_u}$, but different choices lead to canonically MU-isomorphic MU-categories.

MU-categories

We consider the following types of maps in a MU-category $\mathcal{K}$.

- Every map is uniformly continuous.
- f is *non-expansive* if we have $d(f \circ g, f \circ h) \leq d(g, h)$ for every compatible maps g, h. $\mathcal{K}$ is a *metric-enriched* if every $\mathcal{K}$-map is non-expansive. Corresponds to enrichment over the symmetric monoidal category ∞-**Met**. Was considered by Kubiś.
- f is a *metric epimorphism* if we have $d(g \circ f, h \circ f) = d(g, h)$ for every compatible maps g, h. In $\mathbf{Met_u}$ corresponds to epimprphisms, i.e. maps with dense image. In discrete MU-categories also corresponds to epimorphisms.

MU-categories

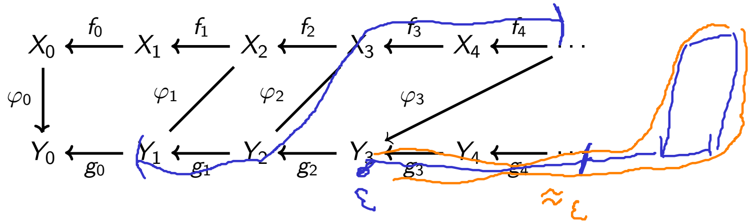
We consider the following types of functors $F\colon \mathcal{K} \to \mathcal{L}$ between MU-categories.

- F is *MU-continuous* or just an *MU-functor* if for every $\mathcal{K}$-object X and every $\varepsilon > 0$ there is $\delta > 0$ such that $d(f, g) < \delta$ implies $d(F(f), F(g)) < \varepsilon$ for every $\mathcal{K}$-maps to X.
- F is an *MU-isomorphism* if it is an MU-functor and there is an MU-functor $G\colon \mathcal{L} \to \mathcal{K}$ such that $G \circ F = \mathrm{id}_{\mathcal{K}}$ and $F \circ G = \mathrm{id}_{\mathcal{L}}$.
- F is *non-expansive* if $d(F(f), F(g)) \leq d(f, g)$ for every compatible $\mathcal{K}$-maps.
- F is a *local isometric embedding* if $d(F(f), F(g)) = d(f, g)$ for every compatible $\mathcal{K}$-maps.
- F is an *isometric embedding* if it is a local isometric embedding and it is one-to-one on objects.
- F is *locally dense* if $F[\mathcal{K}(X, Y)]$ is dense in $\mathcal{L}(F(X), F(Y))$ for every $\mathcal{K}$-objects X, Y.

MU-categories – transformations

Definition

Let $\langle X_*, f_* \rangle$ and $\langle Y_*, g_* \rangle$ be sequences in an MU-category $\mathcal{K}$.
A *pre-transformation* $\varphi_* \colon f_* \to g_*$ in $\mathcal{K}$ is any family of $\mathcal{K}$-maps $\varphi_n \colon Y_n \leftarrow X_{\varphi(n)}$, $n \in \omega$.



We put ${\varphi_{n,n'}}^m := g_{n,n'} \circ \varphi_n \circ f_{\varphi(n),m}$ for $n \le n'$ and $\varphi(n') \le m$.

φ_* is a *transformation* if $\varphi \colon \omega \to \omega$ is increasing and cofinal, and

$$(\forall n)(\forall \varepsilon > 0)(\exists n_0 \ge n)(\forall n'' \ge n' \ge n) \quad {\varphi_{n,n'}}^{\varphi(n'')} \approx_\varepsilon {\varphi_{n,n''}}^{\varphi(n'')}$$

Transformations are stable under composition.

MU-categories – transformations

Definition ($\sigma_0\mathcal{K}$)

We consider the category of all sequences in $\mathcal{K}$ and all transformations, and we endow the category with the following distance:

$$d(\varphi_*, \psi_*) = \bigvee_n d_n(\varphi_*, \psi_*) \wedge 1/n,$$

$$d_n(\varphi_*, \psi_*) = \bigwedge_{n_0 \geq n} \bigvee_{n', n'' \geq n} \bigwedge_{m \geq \varphi(n'), \psi(n'')} d(\varphi_{n,n'}{}^m, \psi_{n,n''}{}^m).$$

With the distance we have the axioms of MU-category with the catch that hom-sets are only ∞-pseudometric. The quotient by the corresponding equivalence is an MU-category denoted by $\sigma_0\mathcal{K}$. The construction σ_0 is functorial.

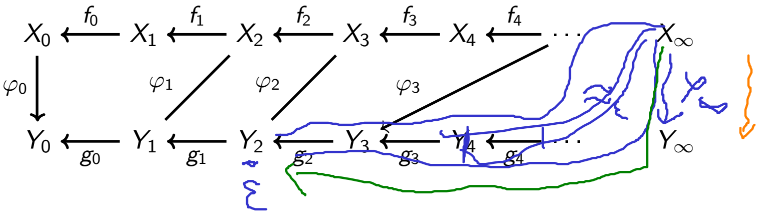
Let $J\colon \mathcal{K} \to \sigma_0\mathcal{K}$ be the functor assigning to every $\mathcal{K}$-object and $\mathcal{K}$-map the corresponding constant sequence and transformation, respectively. J is a locally dense isometric embedding.

$\sigma_0\mathcal{K}$ has twofold purpose – it is a place for sequences to live, and it serves as a completion construction.

MU-categories – completeness

Definition

Let $\langle X_*, f_* \rangle$ and $\langle Y_*, g_* \rangle$ be sequences in $\mathcal{K}$ having limits.
Let us fix their limits $\langle X_\infty, f_{*,\infty} \rangle$ and $\langle Y_\infty, g_{*,\infty} \rangle$.



A $\mathcal{K}$-map $\varphi_\infty \colon X_\infty \to Y_\infty$ is the *limit* of φ_* if

$$(\forall n)(\forall \varepsilon > 0)(\exists n_0 \geq n)(\forall n' \geq n_0) \quad {\varphi_{n,n'}}^{\infty} \approx_\varepsilon {\varphi_{n,\infty}}^{\infty}.$$

Note that the maps ${\varphi_{n,\infty}}^{\infty}$ are limits of the sequence of maps $\langle {\varphi_{n,n'}}^{\infty} \rangle_{n' \geq n}$, $\langle X_\infty, {\varphi_{*,\infty}}^{\infty} \rangle$ is a cone for g_*, and φ_∞ is the factorizing map.

MU-categories – completeness

Definition

We say that a pair $\mathcal{K} \subseteq \mathcal{L}$ of MU-categories is

- *complete* if every transformation in $\mathcal{K}$ has a limit in $\mathcal{L}$;
- *sequentially complete* if every sequence in $\mathcal{K}$ has a limit in $\mathcal{L}$;
- *locally complete* if every Cauchy sequence in $\mathcal{K}(X, Y)$ has a limit in $\mathcal{L}(X, Y)$.

$\mathcal{L}$ is (sequentially/locally) complete if $\langle \mathcal{L}, \mathcal{L} \rangle$ is.

Proposition

$\langle \mathcal{K}, \mathcal{L} \rangle$ is complete if and only if it is both sequentially complete and locally complete.

Example

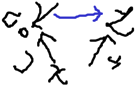
The MU-categories $\mathbf{CMet_u}$, $\mathbf{MCpt}$, and $\mathbf{MCpt_s}$ are complete; $\mathbf{CMet_{us}}$ is neither sequentially complete nor locally complete.

Theorem

$\sigma_0\mathcal{K}$ is a complete MU-category for every MU-category $\mathcal{K}$.

MU-categories – completeness

Given a complete MU-pair $\langle \mathcal{K}, \mathcal{L} \rangle$ and fixing an $\mathcal{L}$-limit $\langle X_\infty, f_{*,\infty} \rangle$ for every $\mathcal{K}$-sequence f_* (taking the canonical limit for $J(X)$) there is a unique functor $L\colon \sigma_0\mathcal{K} \to \mathcal{L}$ assigning to every transformation its limit – called *limit functor*. Hence, $\langle \mathcal{K}, \sigma_0\mathcal{K} \rangle$ can be viewed as a *free completion*.



Definition

An MU-functor $F\colon \mathcal{K} \to \mathcal{L}$ is an *MU-equivalence* if there exists an MU-functor $G\colon \mathcal{L} \to \mathcal{K}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{K}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{L}}$. Equivalently, F is essentially surjective, full, and *MU-faithful*:

$$(\forall X)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f, g\colon \to X)\ F(f) \approx_\delta F(g) \implies f \approx_\varepsilon g.$$

Definition

$\langle \mathcal{K}, \mathcal{L} \rangle$ is a *free completion* if the limit functor $L\colon \sigma_0\mathcal{K} \to \mathcal{L}$ is an MU-equivalence.

MU-categories – completeness

Characterization of the free completion $\langle \mathcal{K}, \mathcal{L} \rangle$

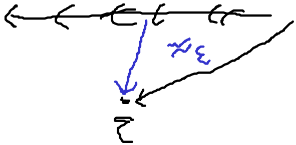
L exists	$\iff$	$\langle \mathcal{K}, \mathcal{L} \rangle$ is complete
L is essentially surjective	$\iff$	(O) every $\mathcal{L}$-object is a limit of a $\mathcal{K}$-sequence
L is full	$\implies$	(F1) *factorization existence condition*
		but modulo (F2) equivalent
L is MU-faithful	$\iff$	(F2) *factorization uniqueness condition*
L is MU-continuous	$\impliedby$	(S) *separation condition*
		but modulo other conditions equivalent

Definition ($\sigma\mathcal{K}$)

Given a complete MU-pair $\langle \mathcal{K}, \mathcal{L} \rangle$ we consider the MU-category $\sigma\mathcal{K} \subseteq \mathcal{L}$ consisting of all limits of $\mathcal{K}$-transformations and all limit-witnessing maps. This assures (O) for $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ while increasing the chance for the other conditions.

MU-categories – completeness

The conditions



(F1) $\forall \langle X_*, f_* \rangle \; \forall Y \; \forall \varepsilon > 0 \; \forall h \colon X_\infty \to Y$
$\exists g \colon X_n \to Y \quad g \circ f_{n,\infty} \approx_\varepsilon h.$
Holds for $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ if $\mathcal{K} \subseteq \mathbf{MCpt_s}$ is a full subcategory of polyhedra.

(F2) $\forall Y \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall \langle X_*, f_* \rangle$
$\forall g, g' \colon X_n \to Y \quad g \circ f_{n,\infty} \approx_\delta g' \circ f_{n,\infty}$
$\implies \; \exists n' \geq n \quad g \circ f_{n,n'} \approx_\varepsilon g' \circ f_{n,n'}.$
Holds if $\mathcal{L}$ consists of metric epimorphisms.

(S) $\forall \langle X_*, f_* \rangle \; \forall \varepsilon > 0 \; \exists n \; \exists \delta > 0 \; \forall h, h' \colon Y \to X_\infty$
$f_{n,\infty} \circ h \approx_\delta f_{n,\infty} \circ h' \implies h \approx_\varepsilon h'.$
Holds for $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ if $\mathcal{K} \subseteq \mathbf{MCpt}$.

MU-categories – Fraïssé theory

The three pillars of Fraïssé theory hold for MU-categories when the approximate definitions of amalgamation property, domination, homogeneity, and the extension property are used.

Theorem (characterization of the Fraïssé limit)

Let $\sigma\mathcal{K}$ be a free completion of $\mathcal{K}$. For an $\sigma\mathcal{K}$-object U the following conditions are equivalent:

1. X is the limit in $\sigma\mathcal{K}$ of a Fraïssé sequence in $\mathcal{K}$.
2. X is cofinal and homogeneous in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ or *equivalently in* $\sigma\mathcal{K}$.
3. X is cofinal and extensive in $\langle \mathcal{K}, \sigma\mathcal{K} \rangle$ or *equivalently in* $\sigma\mathcal{K}$.

Such object U is unique, and every sequence in $\mathcal{K}$ with limit U is Fraïssé. U is called the *Fraïssé limit* of $\mathcal{K}$.

Example

Let $\mathcal{K} \subseteq \mathbf{MCpt_s}$ be the full subcategory of finite spaces. Then $\sigma\mathcal{K} \subseteq \mathbf{MCpt_s}$ is the full subcategory of zero-dimensional spaces, $\mathcal{K}$ is Fraisé, $\sigma\mathcal{K}$ is its free completion, and the Cantor space is the Fraïssé limit.

3. Application:

Hereditarily indecomposable continua

Application – hereditarily indecomposable continua

Recall

- A *continuum* is a metrizable compact connected space.
- A continuum X is *hereditarily indecomposable* if for every subcontinua $C, D \subseteq X$ we have $C \subseteq D$ or $C \supseteq D$ or $C \cap D = \emptyset$.
- $\mathbb{I} := [0, 1]$ denotes the unit interval.
- A continuum X is *arc-like* if it is the limit of the sequence copies of $\mathbb{I}$ and continuous surjective bonding maps.

Theorem (Bing, 1951)

There is a unique hereditarily indecomposable arc-like continuum – the *pseudo-arc*.

Application – hereditarily indecomposable continua

Crookedness

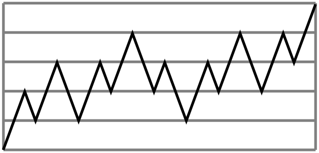
A map $f\colon \mathbb{I} \to \mathbb{I}$ is ε-*crooked* if for every $x \leq y \in \mathbb{I}$ there are $x \leq y' \leq x' \leq y$ such that $f(x) \approx_\varepsilon f(x')$ and $f(y) \approx_\varepsilon f(y')$.

Let $\mathcal{I}$ denote the MU-category of all continuous surjections on $\mathbb{I}$.

A sequence f_* is $\mathcal{I}$ is *crooked* if for every n and $\varepsilon > 0$ there is $n' \geq n$ such that $f_{n,n'}$ is ε-crooked.

Fact

For every $\varepsilon > 0$ there exists and ε-crooked $\mathcal{I}$-map.



Theorem

Let f_* be a sequence in $\mathcal{I}$ and let $\langle X, f_{*,\infty}\rangle$ be its limit. The arc-like continuum X is hereditarily indecomposable if and only if f_* is a crooked sequence.

(Holds much more generally – Brown, Krasinkiewicz, Minc, Maćkowiak.)

Application – hereditarily indecomposable continua

- We consider the MU-category $\mathcal{I} \subseteq \mathbf{MCpt_s}$.
- Then $\sigma\mathcal{I}$ is the full subcategory of $\mathbf{MCpt_s}$ of all arc-like continua, and it is a free completion of $\mathcal{I}$.
- Hence, we have the characterization of the Fraïssé limit (if it exists).
- Since $\mathcal{I}$ is clearly directed and has a countable dominating subcategory, the Fraïssé limit exists if and only if $\mathcal{I}$ has the amalgamation property.
- $\mathcal{I}$ has even strict (AP) for piecewise-linear maps (mountain-climbing theorem).
- Since arbitrarily crooked maps exist, the Fraïssé sequence is crooked (as it is absorbs everything).
- Hence, the Fraïssé limit is a hereditarily indecomposable arc-like continuum and so the pseudo-arc by the Bing's characterization.

Application – hereditarily indecomposable continua

Besides the Fraïssé theory and the facts about crookedness (which holds much more generally than presented) we used only two specific facts: the amalgamation property of $\mathcal{I}$ and the Bing's characterization of the pseudo-arc. Both facts follows from the following result, which can be proved directly.

Crookedness factorization theorem

For every $\mathcal{I}$-map g and every $\varepsilon > 0$ there is $\delta > 0$ such that for every δ-crooked $\mathcal{I}$-map f there is an $\mathcal{I}$-map h such that $f \approx_\varepsilon g \circ h$.

Application – hereditarily indecomposable continua

Pseudo-solenoids

- Let $\mathbb{S}$ denote the unit circle, and let $\mathcal{S} \subseteq \mathbf{MCpt_s}$ denote the MU-category of all continuous surjections on $\mathbb{S}$.
- Recall that a continuum is *circle-like* if it is a limit of an $\mathcal{S}$-sequence.
- So $\sigma\mathcal{S}$ is the full subcategory of $\mathbf{MCpt_s}$ of all circle-like continua.
- As with the unit interval, $\sigma\mathcal{S}$ is a free completion of $\mathcal{S}$.
- However, $\mathcal{S}$ does not have the amaglamation property.
- Every $\mathcal{S}$-map has a *degree* $k \in \mathbb{Z}$ (the winding number).
- For every P set of primes let $\mathcal{S}_P$ denote the subcategory of $\mathcal{S}$ consisting of maps of degrees with prime divisors in P.
- By the result of Rogers, $\mathcal{S}_P$ has the amalgamation property.
- Moreover, $\sigma\mathcal{S}_P \subseteq \sigma\mathcal{S}$ is a free completion of $\mathcal{S}_P$, and hence every $\mathcal{S}_P$ has a Fraïssé limit called the *P-adic pseudo-solenoid*.

Dziękuje!

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