Uncountable homogeneous structures

Wiesław Kubiś

Institute of Mathematics, CAS, Prague, Czechia

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The setup

Definition

- We fix a Fraïssé class $\mathcal{F}$ of finitely generated structures in a countable language.
- We denote by $\sigma\mathcal{F}$ the class of all unions of countable chains in $\mathcal{F}$, in other words, the class of all countable structures whose age is in $\mathcal{F}$.
- We denote by $\overline{\mathcal{F}}$ the class of all structures whose age is in $\mathcal{F}$.
- We denote by $\mathbb{U}$ the Fraïssé limit of $\mathcal{F}$.

Definition

A structure M is homogeneous if every isomorphism between its finitely generated substructures extends to an automorphism of M.

Main questions

What can we say about uncountable homogeneous structures in $\overline{\mathcal{F}}$?
Do they always exist?

An example from basic group theory

Example

Let $\mathcal{F}$ be the class of all torsion-free cyclic groups. Then $\sigma\mathcal{F} = \overline{\mathcal{F}}$.

Note that, up to isomorphism, $\mathcal{F} = \{1, \langle \mathbb{Z}, + \rangle\}$.
Note that $\sigma\mathcal{F}$ consists of all locally cyclic torsion-free groups. All of them are isomorphic to subgroups of the Fraïssé limit $\mathbb{U} = \langle \mathbb{Q}, + \rangle$.

Fraïssé-Jónsson theory

Theorem

Assume CH. If $\sigma\mathcal{F}$ has the amalgamation property then there exists a unique structure $\mathbb{U}_{\omega_1} \in \overline{\mathcal{F}}$ of cardinality $\aleph_1$ that is $\sigma\mathcal{F}$-homogeneous and universal.
Furthermore, every structure in $\overline{\mathcal{F}}$ of cardinality $\leq 2^{\aleph_0}$ embeds into $\mathbb{U}_{\omega_1}$.

Remark

This may fail without CH.

Theorem (cf. Keisler 1964)

Assume the language of $\mathcal{F}$ is finite and relational and fix a non-principle ultrafilter p over ω. Then the ultrapower $\mathbb{U}^\omega/p$ is homogeneous and is a member of $\overline{\mathcal{F}}$.

Theorem (cf. Kubiś & Mašulović 2017)

Assume there exists a functor $K \colon \sigma\mathcal{F} \to \sigma\mathcal{F}$ together with a natural transformation η from the identity to K such that $K(\mathbb{U}) \approx \mathbb{U}$ and $\eta_{\mathbb{U}} \colon \mathbb{U} \to K(\mathbb{U})$ is a nontrivial embedding.
Then $K^{\omega_1}(\mathbb{U})$ is homogeneous.

The monoid of embeddings

Definition

Let $\mathcal{E}$ denote the monoid of all self-embeddings of $\mathbb{U}$. We say that $\mathcal{E}$ is nontrivial if $\mathcal{E} \neq \mathrm{Aut}(\mathbb{U})$.

Claim

$\overline{\mathcal{F}} \neq \sigma\mathcal{F} \iff \mathcal{E}$ is nontrivial.

Claim

If the language of $\mathcal{F}$ is finite and relational then $\mathcal{E}$ is nontrivial.

Example

Let $\mathcal{F}$ consist of all pairs of the form $\langle F, f\rangle$, where $f \colon F \to \omega$ is a one-to-one function. Then $\mathbb{U} \approx \langle \omega, \mathrm{id}_\omega\rangle$ and $\mathcal{E} = \{\mathrm{id}_{\mathbb{U}}\}$.

Main result

Theorem

Assume $\mathcal{M}$ is a sub-monoid of $\mathcal{E}$ containing $\mathrm{Aut}(\mathbb{U})$ as well as at least one nontrivial embedding. Furthermore, assume that $\mathcal{M}$ has the amalgamation property.
Then there exists an uncountable homogeneous structure $\mathbb{W}$ with age $\mathcal{F}$.

Remark

The assumptions above are fulfilled if the language of $\mathcal{F}$ is finite and relational.

Question

Does there exist a relational Fraïssé class $\mathcal{F}$ such that no uncountable structure with age $\mathcal{F}$ is homogeneous?

Claim

There exists a relational Fraïssé classes $\mathcal{F}$ such that no $X \in \sigma\mathcal{F}$ is an amalgamation base.

Proof.

One of the examples is the class of all finite anti-metric spaces with natural values. □

THE END