An Arithmetic Inverse Result for Matrix Groups

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arXiv:2410.03444

Warsaw Algebra Seminar,
November 27, 2025

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(Fake) Motivation

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Burnside problem (1902)

Is every finitely generated torsion group finite?

No! (Golod–Shafarevitch 1964)

Let K be a field.

Theorem (Burnside–Schur)

Every finitely generated torsion subgroup $G \leq \mathrm{GL}_d(K)$ is finite.

Also true for f.g. periodic subsemigroups of $K^{d\times d}$.

(Fake) Motivation

Theorem (Burnside–Schur)

Every finitely generated torsion subgroup $G \leq \mathrm{GL}_d(K)$ is finite.

Let $K = \overline{K}$, e.g. $K = \mathbb{C}$ or $K = \overline{\mathbb{Q}}$.

Let the spectrum $\sigma(G)$ be the set of all eigenvalues of all matrices in G.

- G torsion implies that $\sigma(G)$ consists of roots of unity, so $|\sigma(G)| < \infty$ (using f.g.).
- Converse? Suppose G is irreducible (no proper G-invariant subspace of K^d).
  Then there exists a K-basis $A_1, \ldots, A_{d^2} \in G$ of $K^{d\times d}$, and

  $$\varphi\colon K^{d\times d} \to K^{d^2}, \quad X \mapsto \big(\mathrm{Tr}(XA_1), \ldots, \mathrm{Tr}(XA_{d^2})\big)$$

  is a vector space isomorphism.

  Since $\varphi(G) \subseteq (\underbrace{\sigma(G) + \cdots + \sigma(G)}_{d \text{ times}})^{d^2}$, the group G is finite.

(Fake) Motivation

Proposition

Let $K = \overline{K}$ and $G \leq \mathrm{GL}_d(K)$ be finitely generated.

1. If G is irreducible, then $|\sigma(G)| < \infty \Leftrightarrow |G| < \infty$.
2. $|\sigma(G)| < \infty$ if and only if there exists $T \in \mathrm{GL}_d(K)$ such that there is a block-structure

$$TGT^{-1} = \begin{bmatrix} G_1 & * & \cdots & * \\ 0 & G_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & G_r \end{bmatrix}$$

   with finite groups G_i. ("G is tame").

Weakening the restriction on $\sigma(G)$

Definition

Let $G \leq \mathrm{GL}_d(K)$. The spectrum $\sigma(G)$ is finitely generated (equivalently, satisfies the Pólya property) if there exists a finitely generated subgroup $\Gamma \leq \overline{K}^\times$ such that

$$\sigma(G) \subseteq \Gamma.$$

Lemma

If $K = \overline{K}$ and G is irreducible with f.g. spectrum, then there exists $M \geq 0$, f.g. $\Gamma \leq K^\times$ such that

$$G \subseteq (M \cdot \Gamma_0)^{d \times d} \quad \text{with} \quad \Gamma_0 \coloneqq \Gamma \cup \{0\}, \quad M \cdot \Gamma_0 = \underbrace{\Gamma_0 + \cdots + \Gamma_0}_{M \text{ times}}.$$

If $G \subseteq (M \cdot \Gamma_0)^{d \times d}$ (for some Γ, M), then G has the Bézivin property.

Examples

Bézivin groups ($G \subseteq (M \cdot \Gamma_0)^{d \times d}$):

- Finite groups.
- Finitely generated groups of monomial matrices.
- The Bézivin property is closed under conjugation (M, Γ may change).
- If G is Bézivin and $V \subseteq K^d$ is G-invariant, the induced $G|_V \leq \mathrm{GL}(V)$ and $\overline{G} \leq \mathrm{GL}(K^d/V)$ are Bézivin,

$$\begin{bmatrix} G_V & * \\ 0 & \overline{G} \end{bmatrix}.$$

- G is Bézivin if the representation $j\colon G \hookrightarrow \mathrm{GL}_d(K)$ is the epimorphic image of a monomial representation.

$$\begin{array}{ccc} & & \mathrm{GL}(V) \\ & \overset{\varphi}{\nearrow} & \downarrow \pi \\ G & \xrightarrow{\;j\;} & \mathrm{GL}_d(K) \end{array}$$

($\pi\colon V \twoheadrightarrow K^d$ such that $\pi(Av) = A\pi(v)$ for $A \in G, v \in V$)

Examples

Finitely generated spectrum:

- block-triangular groups with diagonal blocks from the previous list, e.g.,
- block-triangular groups with monomial diagonal blocks.
- closed under conjugation, epimorphic images.

Aside: Submultiplicative Spectrum

Definition

A semigroup $S \subseteq K^{d\times d}$ has submultiplicative spectrum if $\sigma(AB) \subseteq \sigma(A)\sigma(B)$ for A, $B \in S$ (Lambrou–Longstaff–Radjavi '92).

- A finitely generated S with submultiplicative spectrum has finitely generated spectrum.
- Submultiplicative spectrum is much more restrictive.

Theorem (Radjabalipour–Radjavi '99, Radjavi '00)

If $S \subseteq \mathbb{C}^{d\times d}$ is irreducible and has submultiplicative spectrum, then there is a finite nilpotent group $G \leq \mathrm{GL}_d(\mathbb{C})$ such that, up to conjugation,

$$\mathbb{C}S = \mathbb{C}G.$$

Kramar '04, '05, '06; Grunenfelder–Košir–Omladič–Radjavi '12

Problem and Main Result

The problem

Problem

(I) Which matrix groups are Bézivin?
($G \subseteq (M \cdot \Gamma_0)^{d \times d}$ with $M \geq 0$, $\Gamma \leq K^\times$ finitely generated)

(II) Which matrix groups have finitely generated spectrum?
($\sigma(G) \subseteq \Gamma$ with $\Gamma \leq \overline{K}^\times$ finitely generated)

Main Results

Reminder: G being Bézivin means $G \subseteq (M \cdot \Gamma_0^{d\times d})$.

Theorem (Puch-S. '24)

Let $K = \overline{K}$ and $G \leq \mathrm{GL}_d(K)$ finitely generated. The following are equivalent.

(a) G is Bézivin.

(b) $G \hookrightarrow \mathrm{GL}_d(K)$ is an epimorphic image of a monomial representation of G.

(c) G is virtually simultaneously diagonalizable.

Similar result with $\mathrm{char}\, K = 0$ characterizing linear groups (of diagonalizable matrices) with bounded generation (BG) by Corvaja–Demeio–Rapinchuk–Ren–Zannier '23.

G has the BG property if and only if $G = \langle A_1 \rangle \cdots \langle A_n \rangle$.

Main Results

Theorem (Puch-S. '24)

Let $K = \overline{K}$ and $G \leq \mathrm{GL}_d(K)$ finitely generated. The following are equivalent.

(a) $\sigma(G)$ is finitely generated (Pólya property).

(b) $G \hookrightarrow \mathrm{GL}_d(K)$ is the epimorphic image of a block-triangular representation with monomial diagonal blocks.

(c) G is virtually solvable.

- For irreducible G: G Bézivin $\Leftrightarrow \sigma(G)$ finitely generated.
- (a) $\Leftrightarrow$ (c) was observed before by Bernik '05 in characteristic 0.
- Tits' alternative: f.g. $G \leq \mathrm{GL}_d(K)$ is either virtually solvable or contains a non-cyclic free subgroup.

Main Results (More General)

Theorem (Puch-S. '24)

Let K be a field, and $S \subseteq \mathrm{GL}_d(K)$ a semigroup that is finitely generated or $\mathrm{char}\, K = 0$.

(I) The following are equivalent.
- (a) S is locally Bézivin and K is uniformly power-splitting for S.
- (b) $S \hookrightarrow \mathrm{GL}_d(K)$ is an epimorphic image of a monomial representation of S (over K).
- (c) $\langle S \rangle$ is virtually simultaneously diagonalizable (over K).

(II) The following are equivalent.
- (a) S has locally finitely generated spectrum and K is uniformly power-splitting for S.
- (b) $S \hookrightarrow \mathrm{GL}_d(K)$ is the epimorphic image of a block-triangular representation with monomial diagonal blocks (over K).

K is uniformly power-splitting for S if there exists $N \geq 1$ such that for all eigenvalues $\lambda \in \overline{K}$ of all $A \in S$, we have $\lambda^N \in K$.

On the Proof

Key Tool: Unit Equations

Let $\operatorname{char} K = 0$, and $\Gamma \le K^\times$ finitely generated.

Solve

$$a_1 X_1 + \cdots + a_n X_n = 0$$

in $\Gamma_0 = \Gamma \cup \{0\}$.

Theorem (Evertse '84, van der Poorten–Schlickewei '82, '91)

Unit equations have only finitely many non-degenerate solutions (as projective points).

- A solution $(x_1, \ldots, x_n)$ is non-degenerate if $\sum_{i \in I} a_i x_i \neq 0$ for all $\varnothing \neq I \subsetneq \{1, \ldots, n\}$.
- Each solution can be partitioned into non-degenerate solutions of subequations.
- Characteristic $p > 0$ is different: Derksen–Masser '12, Adamczewski–Bell '12.

A Special Case of Our Theorem

Proposition (Special Case)

Let $K = \overline{K}$, $\operatorname{char} K = 0$, and $G \leq \mathrm{GL}_d(K)$ Bézivin, say,

$$G \subseteq (M \cdot \Gamma_0)^{d \times d}.$$

Assume there exists $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_d) \in G$ **with** $(\lambda_i/\lambda_j)^n \neq 1$ **for all** $i \neq j$, $n \neq 0$**.**
Then every $B = (b_{ij}) \in G$ is monomial.

$$(B^k)_{i_0 i_k} = \sum_{1 \leq i_1, \ldots, i_{k-1} \leq d} \underbrace{b_{i_0 i_1} b_{i_1 i_2} \cdots b_{i_{k-1} i_k}}_{=: \beta(\mathbf{i}) \text{ with } \mathbf{i} = (i_0, i_1, \ldots, i_k)} = \sum_{1 \leq i_1, \ldots, i_{k-1} \leq d} \beta(\mathbf{i}).$$

Key Claim: For all $k \geq 0$: $\left| \{ \mathbf{i} : \beta(\mathbf{i}) \neq 0 \} \right| \leq M$.

$$(BA^{n_1} BA^{n_2} \cdots BA^{n_{k-1}} B)_{i_0 i_k} = \sum_{1 \leq i_1, \ldots, i_{k-1} \leq d} \beta(\mathbf{i}) \lambda_{i_1}^{n_1} \ldots \lambda_{i_{k-1}}^{n_{k-1}} \qquad \text{where} \qquad n_i \in \mathbb{Z}.$$

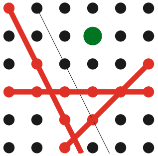
By the Bézivin property,

$$(BA^{n_1}BA^{n_2}\cdots A^{n_{k-1}}B)_{i_0 i_k} = \sum_{1\le i_1,\dots,i_{k-1}\le d} \beta(\mathbf{i})\lambda_{i_1}^{n_1}\cdots\lambda_{i_{k-1}}^{n_{k-1}} = \gamma_1(\mathbf{n}) + \cdots + \gamma_M(\mathbf{n}).$$

with $\gamma_i(\mathbf{n}) \in \Gamma_0$. **Unit equation!** Consider partitions into subequations:

1) Bad partitions: two terms $\mathbf{i} \neq \mathbf{j}$ on LHS in same non-degenerate subequation.
This is rare:

$$\varphi_{\mathbf{i},\mathbf{j}}\colon (\mathbb{Z}^{k-1}, +) \mapsto (\Gamma, \cdot), \quad \mathbf{n} \mapsto \left(\frac{\lambda_{i_1}}{\lambda_{j_1}}\right)^{n_1} \cdots \left(\frac{\lambda_{i_{k-1}}}{\lambda_{j_{k-1}}}\right)^{n_{k-1}}$$



has $\operatorname{rank} \operatorname{im} \varphi_{\mathbf{i},\mathbf{j}} \ge 1$, so $\operatorname{rank} \ker(\varphi_{\mathbf{i},\mathbf{j}}) \le k-2$.

Only possible for $\mathbf{n}$ in finitely many cosets (by unit equations).

2) Look at one $\mathbf{n}$ with a good partition:

- either $\mathbf{i}$ isolated (then $\beta(\mathbf{i}) = 0$), or
- $\mathbf{i}$ uses up at least one $\gamma_j(\mathbf{n})$ from RHS, so at most M nonzero $\beta(\mathbf{i})$.

We proved the **Key Claim:** For all $k \ge 0$: $|\{\mathbf{i} : \beta(\mathbf{i}) \neq 0\}| \le M$.

(We have: $\beta(\mathbf{i}) = b_{i_0 i_1} b_{i_1 b_2} \cdots b_{i_{k-1} i_k}$ with $B = (b_{ij})$ invertible.)

Know the **Key Claim:** For all $k \geq 0$: $|\{\mathbf{i} : \beta(\mathbf{i}) \neq 0\}| \leq M$.

Show: B is monomial.

Observe:

1. Each $\mathbf{i} = (i_0, \ldots, i_k)$ with $\beta(\mathbf{i}) \neq 0$ extends to some $\mathbf{i}' = (i_0, \ldots, i_k, i_{k+1})$ with $\beta(\mathbf{i}') \neq 0$. (Since row i_k of B is nonzero.)
2. For large enough k, these extension are unique. (By the claim!).
3. For each $1 \leq j \leq d$ and $k \geq 0$, there exist $\mathbf{i} = (i_0, \ldots, i_k)$ with $\beta(\mathbf{i}) \neq 0$ and $i_k = j$. (Since column j of B^k is nonzero)

So: for each i there exists exactly one j with $b_{ij} \neq 0$, i.e., B is monomial!

Beyond the Special Case

If such a nice A (all eigenvalues essentially distinct) does not exist:

- Decompose K^d into eigenspaces of any A^n, n sufficiently large.
- For every B, there exists suitable m such that B^m leaves the eigenspaces of A invariant (using a block variant of the key claim).
- Taking $D = \langle A^{n(A)} : A \in G \rangle$, the quotient G/D is torsion and linear.
- G/D is finite by Burnside–Schur.
- D is simultaneously diagonalizable by construction.

So G is virtually simultaneously diagonalizable.

More Generality

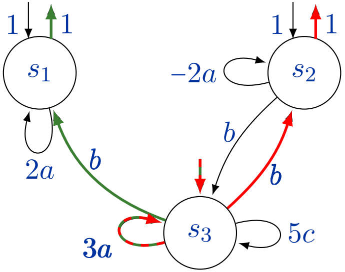
If K is not algebraically closed: descent from $\overline{K}$ (using uniform power-splitting).

If $\operatorname{char} K = p > 0$: the key claim still holds! By Derksen–Masser '12 unit equations have *few* solutions. The bad points are in a sufficiently sparse set of cosets of smaller rank.

The Actual Motivation/Application: Weighted Automata

Weighted Finite Automata

Let X be an alphabet, K a field.



$$f(a^2b) = 3 \cdot 3 \cdot 1 + 3 \cdot 3 \cdot 1 = 18,$$

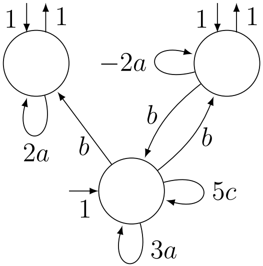
$$\sum_{w \in X^*} f(w)w = 2 + 2b + 8a^2 + 6ab + 2b^2 + \cdots$$
$$+ 10cb + 18a^2b + \cdots - 60b^3ababcb + \cdots$$
$$\in \mathbb{Q}\langle\langle a, b, c\rangle\rangle$$

Computational model

WFA computes a rational $f \colon X^* \to K$:

- Given $w \in X^*$, find all successful runs for w.
- On each run, take the product of all the weights, then sum over all runs.

Weighted Finite Automata



$$u = \begin{pmatrix} 1,1,1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mu(b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\mu(c) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Using matrices:

- two vectors $u \in K^{1\times d}$, $v \in K^{d\times 1}$,
- for each letter x a transition matrix $\mu(x) \in K^{d\times d}$
- $f(x_{i_1} \cdots x_{i_l}) = u\mu(x_{i_1}) \cdots \mu(x_{i_l})v$.

- $|X| = 1$ are precisely linear recurrence sequences (LRS) ($f(n) = uA^n v$).

Ambiguity

WFA can be

$\{$deterministic$\} \subsetneq \{$unambiguous$\} \subsetneq \{$finitely ambiguous$\} \subsetneq \{$polynomially ambiguous$\}$

or exponentially ambiguous.

Problem

Given a WFA $\mathcal{A}$ recognizing f, is there WFA of certain lower ambiguity class recognizing the same f?

E.g. is $\mathcal{A}$ determinizable?

Reutenauer's Conjecture

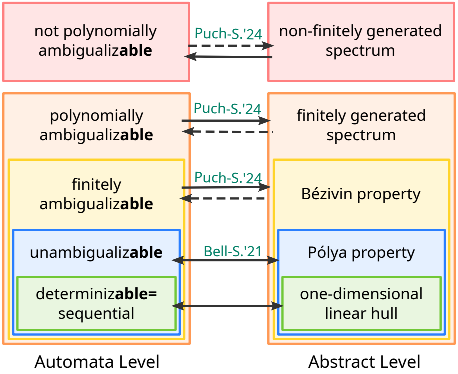
Reutenauer conjectured (1979): $\mathcal{A}$ unambigualizable $\Leftrightarrow$ $f(X^*)$ is Pólya ($\subseteq \Gamma_0$).

- Reutenauer proved it for $f(X^*)$ finite.
- For $|X| = 1$, i.e., LRS, known by Pólya 1920, Benzaghou 1970, Bézivin 1986.

Theorem (Bell-S. '21)

A WFA over a field is unambigualizable if and only if it its output is Pólya.

Ambiguity Hierarchy of Weighted Automata



Theorem

For WFA over (computable) fields,

1. (Bell-S. '21, '23) Determinizability and unambigualizability are decidable.
2. (Puch-S. '24) If the transition matrices are invertible, the full ambiguity hierarchy is decidable.