Algebra and Geometry in Liners

Taras Banakh

Lviv-Warsaw – 30 October 2024

Plan of the Talk

1. Liners
2. Desarguesian liners
3. Vectors and Scalars
4. non-Desarguesian planes
5. Selected Open Problems

Part I: Liners

Basic geometric structure: liner

A *liner* is a mathematical structure $(X, \mathcal{L})$ that consists of a set X whose elements are called points and a family $\mathcal{L}$ of subsets of X whose elements are called lines, such that the following axioms are satisfied:

- any two distinct points belongs to a unique line;
- every line contains at least two points.

For two distinct points x, y of a liner $(X, \mathcal{L})$ let $\overline{x\,y}$ denote the unique line $L \in \mathcal{L}$ containing these two points.
If $x = y$, then put $\overline{x\,y} := \{x\} = \{y\}$.
For subsets $A, B \subseteq X$ let

$$\overline{A\,B} := \bigcup_{a \in A} \bigcup_{b \in B} \overline{a\,b}$$

be the union of all lines connecting the points of the sets A, B.

Basic geometric structure: liner

A *liner* is a mathematical structure $(X, \mathcal{L})$ that consists of a set X whose elements are called points and a family $\mathcal{L}$ of subsets of X whose elements are called lines, such that the following axioms are satisfied:

- any two distinct points belongs to a unique line;
- every line contains at least two points.

For two distinct points x, y of a liner $(X, \mathcal{L})$ let $\overline{x\,y}$ denote the unique line $L \in \mathcal{L}$ containing these two points.
If $x = y$, then put $\overline{x\,y} := \{x\} = \{y\}$.
For subsets $A, B \subseteq X$ let

$$\overline{A\,B} := \bigcup_{a \in A} \bigcup_{b \in B} \overline{a\,b}$$

be the union of all lines connecting the points of the sets A, B.

Flats and hulls

Definition

A subset A of a liner X is called a flat if $\forall x, y \in A \;\; (\overline{x\,y} \subseteq A)$.

Definition

The flat hull $\overline{A}$ of a subset $A \subseteq X$ of a liner $(X, \mathcal{L})$ is the smallest flat that contains the set A.
This flat is equal to the intersection of all flats that contain the set A.
It is also equal to the union of the increasing sequence of sets $(A_n)_{n\in\omega}$, defined by the recursive formula:

$$A_0 = A \;\text{ and }\; A_{n+1} = \overline{A_n\,A_n} \;\text{ for }\; n \geq 0.$$

Flats and hulls

Definition

A subset A of a liner X is called a flat if $\forall x, y \in A\ \ (\overline{x\,y} \subseteq A)$.

Definition

The flat hull $\overline{A}$ of a subset $A \subseteq X$ of a liner $(X, \mathcal{L})$ is the smallest flat that contains the set A.

This flat is equal to the intersection of all flats that contain the set A.

It is also equal to the union of the increasing sequence of sets $(A_n)_{n\in\omega}$, defined by the recursive formula:

$$A_0 = A \ \text{ and } \ A_{n+1} = \overline{A_n\, A_n} \ \text{ for } \ n \geq 0.$$

Flats and hulls

Definition

A subset A of a liner X is called a flat if $\forall x, y \in A \;\; (\overline{x\,y} \subseteq A)$.

Definition

The flat hull $\overline{A}$ of a subset $A \subseteq X$ of a liner $(X, \mathcal{L})$ is the smallest flat that contains the set A.
This flat is equal to the intersection of all flats that contain the set A.
It is also equal to the union of the increasing sequence of sets $(A_n)_{n\in\omega}$, defined by the recursive formula:

$$A_0 = A \;\text{ and }\; A_{n+1} = \overline{A_n\,A_n} \;\text{ for }\; n \geq 0.$$

Rank and Dimension

Definition

For a subset $A \subseteq X$ of a liner X, the cardinal

$$\|A\| := \min\{|B| : B \subseteq X,\ A \subseteq \overline{B}\}$$

is called the rank of the set A, and the cardinal

$$\dim(A) := \|A\| - 1$$

is called the dimension of the set A in the liner X.

Example

Lines are flats of rank 2 and dimension 1.

Definition

Flats of dimension 2 are called planes.

Rank and Dimension

Definition

For a subset $A \subseteq X$ of a liner X, the cardinal

$$\|A\| := \min\{|B| : B \subseteq X,\ A \subseteq \overline{B}\}$$

is called the rank of the set A, and the cardinal

$$\dim(A) := \|A\| - 1$$

is called the dimension of the set A in the liner X.

Example

Lines are flats of rank 2 and dimension 1.

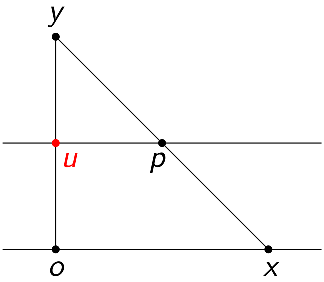
Definition

Flats of dimension 2 are called planes.

Parallel Postulates: affine liners

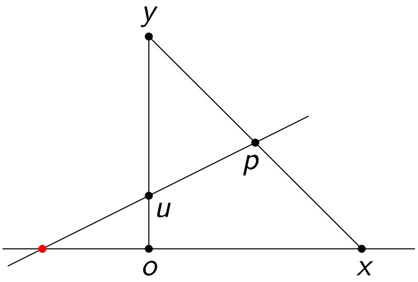
Definition

A liner $(X, \mathcal{L})$ is called affine if
$\forall o, x, y \in X \;\; \forall p \in \overline{x\,y} \setminus \overline{o\,x} \;\; \exists! u \in \overline{o\,y} \quad (\overline{u\,p} \cap \overline{o\,x} = \varnothing)$.



Projective and proaffine liners

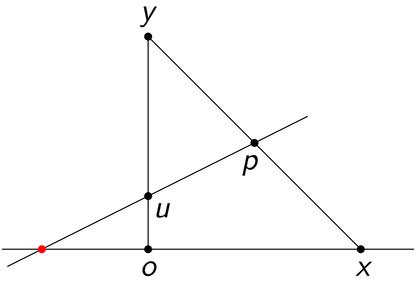
Definition: A liner $(X, \mathcal{L})$ is called projective if
$\forall o, x, y \in X \ \forall p \in \overline{x\,y} \ \forall u \in \overline{o\,y} \setminus \{p\} \quad (\overline{u\,p} \cap \overline{o\,x} \neq \varnothing)$.



Definition: A liner $(X, \mathcal{L})$ is called proaffine if
$\forall o,x,y \in X \ \forall p \in \overline{x\,y} \ \exists v \in \overline{o\,y} \ \forall u \in \overline{o\,y} \setminus \{v\} \ (\overline{u\,p} \cap \overline{o\,x} \neq \varnothing)$.

Projective and proaffine liners

Definition: A liner $(X, \mathcal{L})$ is called projective if
$\forall o, x, y \in X \;\; \forall p \in \overline{x\,y} \;\; \forall u \in \overline{o\,y} \setminus \{p\} \quad (\overline{u\,p} \cap \overline{o\,x} \neq \varnothing).$



Definition: A liner $(X, \mathcal{L})$ is called proaffine if
$\forall o,x,y \in X \; \forall p \in \overline{x\,y} \; \exists v \in \overline{o\,y} \; \forall u \in \overline{o\,y} \setminus \{v\} \; (\overline{u\,p} \cap \overline{o\,x} \neq \varnothing).$

Order of affine and projective liners

Observation

Any two lines in an affine liner contain the same number of points. This number is called the order of an affine liner.

Definition

A liner $(X, \mathcal{L})$ is n-long if every line $L \in \mathcal{L}$ has cardinality $|L| \geq n$.

Theorem (simple)

Any two lines in a 3-long projective liner contain the same number of points.

Definition: The order of a 3-long projective liner is the number of points in a line minus 1.

Order of affine and projective liners

Observation

Any two lines in an affine liner contain the same number of points. This number is called the order of an affine liner.

Definition

A liner $(X, \mathcal{L})$ is n-long if every line $L \in \mathcal{L}$ has cardinality $|L| \geq n$.

Theorem (simple)

Any two lines in a 3-long projective liner contain the same number of points.

Definition: The order of a 3-long projective liner is the number of points in a line minus 1.

Order of affine and projective liners

Observation

Any two lines in an affine liner contain the same number of points. This number is called the order of an affine liner.

Definition

A liner $(X, \mathcal{L})$ is n-long if every line $L \in \mathcal{L}$ has cardinality $|L| \geq n$.

Theorem (simple)

Any two lines in a 3-long projective liner contain the same number of points.

Definition: The order of a 3-long projective liner is the number of points in a line minus 1.

Order of affine and projective liners

Observation

Any two lines in an affine liner contain the same number of points. This number is called the order of an affine liner.

Definition

A liner $(X, \mathcal{L})$ is n-long if every line $L \in \mathcal{L}$ has cardinality $|L| \geq n$.

Theorem (simple)

Any two lines in a 3-long projective liner contain the same number of points.

Definition: The order of a 3-long projective liner is the number of points in a line minus 1.

Prime and prime-power liners

Definition

An affine or projective liner is called

- prime if its order is a prime number;
- prime-power if its order is a power of a prime number.

Empirical Fact:

All known finite affine or projective liners are prime-power. Moreover, every finite 3-long projective liner of rank $\|X\| \geq 4$ is prime-power.

Problem

Is every finite 3-long projective plane prime-power?

Prime and prime-power liners

Definition

An affine or projective liner is called

- prime if its order is a prime number;
- prime-power if its order is a power of a prime number.

Empirical Fact:

All known finite affine or projective liners are prime-power. Moreover, every finite 3-long projective liner of rank $\|X\| \geq 4$ is prime-power.

Problem

Is every finite 3-long projective plane prime-power?

Prime and prime-power liners

Definition

An affine or projective liner is called

- prime if its order is a prime number;
- prime-power if its order is a power of a prime number.

Empirical Fact:

All known finite affine or projective liners are prime-power. Moreover, every finite 3-long projective liner of rank $\|X\| \geq 4$ is prime-power.

Problem

Is every finite 3-long projective plane prime-power?

(Strongly) regular liners

Definition

A liner $(X, \mathcal{L})$ is strongly regular if $\overline{A \cup \{x\}} = \overline{A\,x}$ for every flat $A \subseteq X$ and point $x \in X \setminus A$, that have a common point.

Theorem

A liner is strongly regular if and only if it is projective.

Definition

A liner $(X, \mathcal{L})$ is regular if $\overline{A \cup L} = \overline{A\,L}$ for ev ery flat $A \subseteq X$ and line $L \subseteq X$with $L \cap A \neq \varnothing$.

Theorem (non-trivial)

Every 4-long affine liner is regular.

(Strongly) regular liners

Definition

A liner $(X, \mathcal{L})$ is strongly regular if $\overline{A \cup \{x\}} = \overline{A\,x}$ for every flat $A \subseteq X$ and point $x \in X \setminus A$, that have a common point.

Theorem

A liner is strongly regular if and only if it is projective.

Definition

A liner $(X, \mathcal{L})$ is regular if $\overline{A \cup L} = \overline{A\,L}$ for ev ery flat $A \subseteq X$ and line $L \subseteq X$with $L \cap A \neq \varnothing$.

Theorem (non-trivial)

Every 4-long affine liner is regular.

(Strongly) regular liners

Definition

A liner $(X, \mathcal{L})$ is strongly regular if $\overline{A \cup \{x\}} = \overline{A\,x}$ for every flat $A \subseteq X$ and point $x \in X \setminus A$, that have a common point.

Theorem

A liner is strongly regular if and only if it is projective.

Definition

A liner $(X, \mathcal{L})$ is regular if $\overline{A \cup L} = \overline{A\,L}$ for ev ery flat $A \subseteq X$ and line $L \subseteq X$with $L \cap A \neq \varnothing$.

Theorem (non-trivial)

Every 4-long affine liner is regular.

(Strongly) regular liners

Definition

A liner $(X, \mathcal{L})$ is strongly regular if $\overline{A \cup \{x\}} = \overline{A\,x}$ for every flat $A \subseteq X$ and point $x \in X \setminus A$, that have a common point.

Theorem

A liner is strongly regular if and only if it is projective.

Definition

A liner $(X, \mathcal{L})$ is regular if $\overline{A \cup L} = \overline{A\,L}$ for ev ery flat $A \subseteq X$ and line $L \subseteq X$with $L \cap A \neq \varnothing$.

Theorem (non-trivial)

Every 4-long affine liner is regular.

Spaces

Definition

A space is any 3-long regular liner of rank ≥ 3.

Example

Any 4-long affine liner of rank $\|X\| \geq 3$ is an affine space.

Projective completions of proaffine liners

Definition

A liner $(X, \mathcal{L}_X)$ is a **subliner** of a liner $(Y, \mathcal{L}_Y)$ if $X \subseteq Y$ and
$$\mathcal{L}_X = \{X \cap L : L \in \mathcal{L}_X \text{ and } |X \cap L| \geq 2\}.$$

Definition

A projective liner Y is a **projective completion** of a liner X if Y is 3-long, X is a subliner of Y and $\overline{Y \setminus X} \neq Y$.

Theorem (Kuiper–Dembowski)

Every proaffine space X has a projective completion (which is unique up to an isomorphism).
If $\|X\| \geq 4$, then the remainder $Y \setminus X$ is flat in Y.
If X is finite and $\|X\| = 3$, then $Y \setminus X$ is one of the following: the empty set, a singleton, a line, or a punctured line.

Projective completions of proaffine liners

Definition

A liner $(X, \mathcal{L}_X)$ is a subliner of a liner $(Y, \mathcal{L}_Y)$ if $X \subseteq Y$ and
$$\mathcal{L}_X = \{X \cap L : L \in \mathcal{L}_X \text{ and } |X \cap L| \geq 2\}.$$

Definition

A projective liner Y is a projective completion of a liner X if Y is 3-long, X is a subliner of Y and $\overline{Y \setminus X} \neq Y$.

Theorem (Kuiper–Dembowski)

Every proaffine space X has a projective completion (which is unique up to an isomorphism).
If $\|X\| \geq 4$, then the remainder $Y \setminus X$ is flat in Y.
If X is finite and $\|X\| = 3$, then $Y \setminus X$ is one of the following: the empty set, a singleton, a line, or a punctured line.

Projective completions of proaffine liners

Definition

A liner $(X, \mathcal{L}_X)$ is a subliner of a liner $(Y, \mathcal{L}_Y)$ if $X \subseteq Y$ and
$$\mathcal{L}_X = \{X \cap L : L \in \mathcal{L}_X \text{ and } |X \cap L| \geq 2\}.$$

Definition

A projective liner Y is a projective completion of a liner X if Y is 3-long, X is a subliner of Y and $\overline{Y \setminus X} \neq Y$.

Theorem (Kuiper–Dembowski)

Every proaffine space X has a projective completion (which is unique up to an isomorphism).
If $\|X\| \geq 4$, then the remainder $Y \setminus X$ is flat in Y.
If X is finite and $\|X\| = 3$, then $Y \setminus X$ is one of the following: the empty set, a singleton, a line, or a punctured line.

Projective completions of proaffine liners

Definition

A liner $(X, \mathcal{L}_X)$ is a subliner of a liner $(Y, \mathcal{L}_Y)$ if $X \subseteq Y$ and
$$\mathcal{L}_X = \{X \cap L : L \in \mathcal{L}_X \text{ and } |X \cap L| \geq 2\}.$$

Definition

A projective liner Y is a projective completion of a liner X if Y is 3-long, X is a subliner of Y and $\overline{Y \setminus X} \neq Y$.

Theorem (Kuiper–Dembowski)

Every proaffine space X has a projective completion (which is unique up to an isomorphism).
If $\|X\| \geq 4$, then the remainder $Y \setminus X$ is flat in Y.
If X is finite and $\|X\| = 3$, then $Y \setminus X$ is one of the following: the empty set, a singleton, a line, or a punctured line.

Projective completions of proaffine liners

Definition

A liner $(X, \mathcal{L}_X)$ is a subliner of a liner $(Y, \mathcal{L}_Y)$ if $X \subseteq Y$ and
$$\mathcal{L}_X = \{X \cap L : L \in \mathcal{L}_X \ \text{ and } \ |X \cap L| \ge 2\}.$$

Definition

A projective liner Y is a projective completion of a liner X if Y is 3-long, X is a subliner of Y and $\overline{Y \setminus X} \neq Y$.

Theorem (Kuiper–Dembowski)

Every proaffine space X has a projective completion (which is unique up to an isomorphism).
If $\|X\| \ge 4$, then the remainder $Y \setminus X$ is flat in Y.
If X is finite and $\|X\| = 3$, then $Y \setminus X$ is one of the following: the empty set, a singleton, a line, or a punctured line.

Part II
Desarguesian Liners

Subparallelity and parallelity of flats

Definition

Given two flats A, B in a liner $(X, \mathcal{L})$, we write

- $A \mathrel{\|\!|} B$ and say that the flat A is subparallel to the flat B if $A \subseteq \overline{B \cup \{a\}}$ for every point $a \in A$;
- $A \parallel B$ and say that the flat A is parallel to the flat B if $A \mathrel{\|\!|} B$ and $B \mathrel{\|\!|} A$.

Theorem (non-trivial)

For every flat A and point x in an affine space X there exists a unique flat $B \subseteq X$ with $x \in B \parallel A$.

Corollary (Playfair Axiom)

For every line L and point x in an affine space, there exists a unique line Λ that contains the point x and is paralell to L.

Subparallelity and parallelity of flats

Definition

Given two flats A, B in a liner $(X, \mathcal{L})$, we write

- $A \,\|\!\!\;| B$ and say that the flat A is subparallel to the flat B if $A \subseteq \overline{B \cup \{a\}}$ for every point $a \in A$;
- $A \parallel B$ and say that the flat A is parallel to the flat B if $A \,\|\!\!\;| B$ and $B \,\|\!\!\;| A$.

Theorem (non-trivial)

For every flat A and point x in an affine space X there exists a unique flat $B \subseteq X$ with $x \in B \parallel A$.

Corollary (Playfair Axiom)

For every line L and point x in an affine space, there exists a unique line Λ that contains the point x and is paralell to L.

Subparallelity and parallelity of flats

Definition

Given two flats A, B in a liner $(X, \mathcal{L})$, we write

- $A \mathbin{\|\!\!\!|} B$ and say that the flat A is subparallel to the flat B if $A \subseteq \overline{B \cup \{a\}}$ for every point $a \in A$;
- $A \parallel B$ and say that the flat A is parallel to the flat B if $A \mathbin{\|\!\!\!|} B$ and $B \mathbin{\|\!\!\!|} A$.

Theorem (non-trivial)

For every flat A and point x in an affine space X there exists a unique flat $B \subseteq X$ with $x \in B \parallel A$.

Corollary (Playfair Axiom)

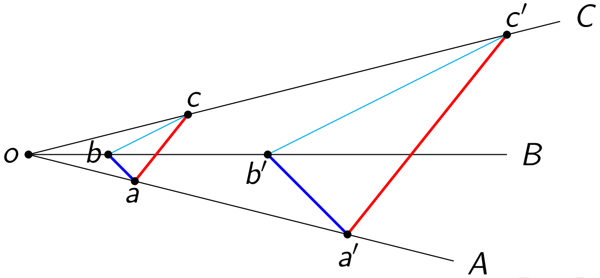
For every line L and point x in an affine space, there exists a unique line Λ that contains the point x and is paralell to L.

Desargues Theorem

Lines $L_1, \dots, L_n$ in a liner $(X, \mathcal{L})$ are called paraconcurrent if they are either pairwise parallel or have a common point.

Theorem (Desargues (1591 – 1661))

Let $(X, \mathcal{L})$ be an affine space of dimension $\dim(X) \ge 3$. For every paraconcurrent lines $A, B, C \in \mathcal{L}$ and points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$, we have $\overline{a\,c} \parallel \overline{a'\,c'}$.



Non-Desarguesian planes: the Moulton plane

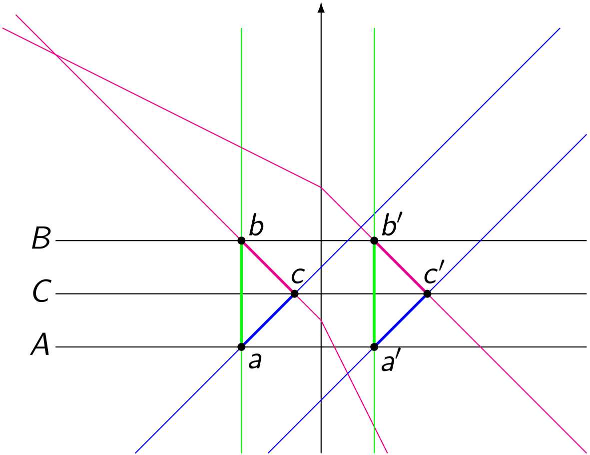
Desargues Theorem is not necessarily true in affine spaces of dimension 2. A counterexample is the Moulton plane, discovered by an american astronomer Moulton in 1902.

The Moulton plane is the liner $X := \mathbb{R} \times \mathbb{R}$ endowed with the family of lines

$$\mathcal{L} := \{L_{a,b} : a, b \in \mathbb{R}\} \cup \{L_c : c \in \mathbb{R}\}, \quad \text{where}$$

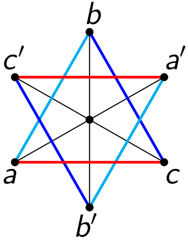
$L_c := \{(c, y) : y \in \mathbb{R}\}$,
$L_{a,b} := \{(x, ax + b) : x \in \mathbb{R}\}$ if $a \geq 0$,
$L_{a,b} := \{(x, \frac{1}{2}ax + b) : x \leq 0\} \cup \{(x, ax + b) : x \geq 0\}$, if $a < 0$.

The Moulton plane is non-Desarguesian



Definition

An affine liner $(X, \mathcal{L})$ is called Desarguesian if for any concurrent lines $A, B, C \in \mathcal{L}$ and points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$ we have $\overline{a\,c} \parallel \overline{a'\,c'}$.

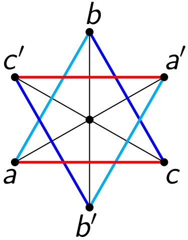


By the Desargues Theorem, every affine space of dimension ≥ 3 is Desarguesian. The Moulton plane is not Desarguesian.

Desarguesian liners

Definition

An affine liner $(X, \mathcal{L})$ is called Desarguesian if for any concurrent lines $A, B, C \in \mathcal{L}$ and points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$ we have $\overline{a\,c} \parallel \overline{a'\,c'}$.

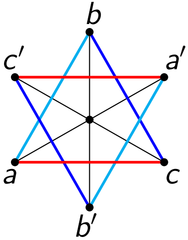


By the Desargues Theorem, every affine space of dimension ≥ 3 is Desarguesian. The Moulton plane is not Desarguesian.

Desarguesian liners

Definition

An affine liner $(X, \mathcal{L})$ is called Desarguesian if for any concurrent lines $A, B, C \in \mathcal{L}$ and points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$ we have $\overline{a\,c} \parallel \overline{a'\,c'}$.

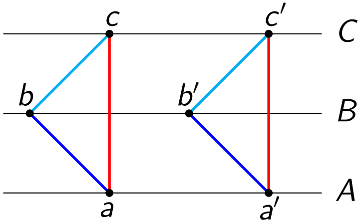


By the Desargues Theorem, every affine space of dimension ≥ 3 is Desarguesian. The Moulton plane is not Desarguesian.

Thalesian liners

Definition

An affine liner $(X, \mathcal{L})$ is called Thalesian if for any parallel lines A, B, C in X and any points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$, we have $\overline{a\,c} \parallel \overline{a'\,c'}$.

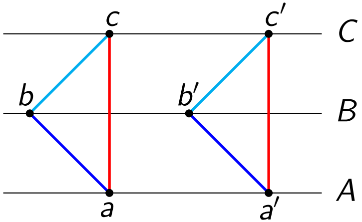


It can be shown that every Desarguesian affine space is Thalesian. The Moulton plane is not Thalesian.

Thalesian liners

Definition

An affine liner $(X, \mathcal{L})$ is called Thalesian if for any parallel lines A, B, C in X and any points $a, a' \in A \setminus (B \cup C)$, $b, b' \in B \setminus (A \cup C)$, $c, c' \in C \setminus (A \cup B)$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$, we have $\overline{a\,c} \parallel \overline{a'\,c'}$.

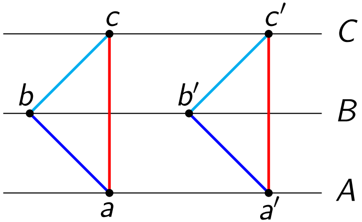


It can be shown that every Desarguesian affine space is Thalesian. The Moulton plane is not Thalesian.

Thalesian liners

Definition

An affine liner $(X, \mathcal{L})$ is called Thalesian if for any parallel lines A, B, C in X and any points $\underline{a, a' \in A \setminus (B \cup C)}$, $\underline{b, b' \in B \setminus (A \cup C)}$, $\underline{c, c' \in C \setminus (A \cup B)}$ with $\overline{a\,b} \parallel \overline{a'\,b'}$ and $\overline{b\,c} \parallel \overline{b'\,c'}$, we have $\overline{a\,c} \parallel \overline{a'\,c'}$.

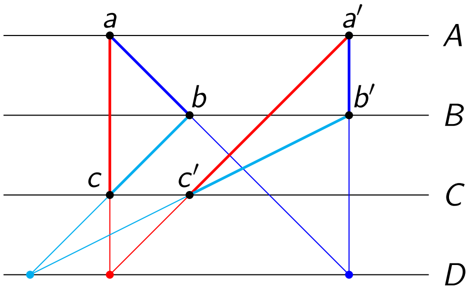


It can be shown that every Desarguesian affine space is Thalesian. The Moulton plane is not Thalesian.

Moufang liners

Definition

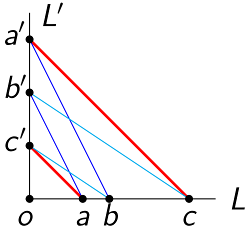
An affine liner X is Moufang if for any distinct parallel lines A, B, C, D and distinct points $a, a' \in A$, $b, b' \in B$, $c, c' \in C$ with $\varnothing \neq \overline{a\,b} \cap \overline{a'\,b'} \subseteq D$ and $\varnothing \neq \overline{b\,c} \cap \overline{b'\,c'} \subseteq D$ we have $\varnothing \neq \overline{a\,c} \cap \overline{a'\,c'} \subseteq D$.



Pappian liners

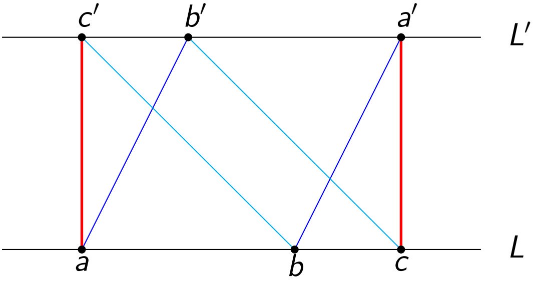
Definition

An affine liner $(X, \mathcal{L})$ is Pappian if for any concurrent lines $L, L' \in \mathcal{L}$ and any distinct points $a, b, c \in L \setminus L'$ and $a', b', c' \in L' \setminus L$ with $\overline{a\, b'} \parallel \overline{a'\, b}$ and $\overline{b\, c'} \parallel \overline{b'\, c}$, we have $\overline{a\, c'} \parallel \overline{a'\, c}$.



Definition

An affine liner $(X, \mathcal{L})$ is para-Pappian if for any parallel lines $L, L' \in \mathcal{L}$ and any distinct points $a, b, c \in L \setminus L'$ and $a', b', c' \in L' \setminus L$ with $\overline{a\,b'} \| \overline{a'\,b}$ and $\overline{b\,c'} \| \overline{b'\,c}$, we have $\overline{a\,c'} \| \overline{a'\,c}$.



Papp⇒Desargues⇒Moufang⇒Thales⇒para-Papp

For every (finite) affine space, the following implications hold:

$$\begin{array}{c} \text{Pappian} \\ \Downarrow \Big\uparrow +\textit{finite} \text{ (by Wedderburn Theorem)} \\ \text{Desarguesian} \\ \Downarrow \Big\uparrow +\textit{finite} \text{ (by Artin–Zorn Theorem)} \\ \text{Moufang} \\ \Downarrow \Big\uparrow +\textit{prime} \\ \text{Thalesian} \\ \Downarrow \Big\uparrow +\textit{prime} \\ \text{para-Pappian} \end{array}$$

Example: There exists an affine plane of order 9 which is Thalesian but not Moufang.

OpenProblem: Is every para-Pappian affine space Thalesian?

Papp⇒Desargues⇒Moufang⇒Thales⇒para-Papp

For every (finite) affine space, the following implications hold:

Pappian

$\Downarrow$ $\uparrow$ +*finite* (by Wedderburn Theorem)

Desarguesian

$\Downarrow$ $\uparrow$ +*finite* (by Artin–Zorn Theorem)

Moufang

$\Downarrow$ $\uparrow$ +*prime*

Thalesian

$\Downarrow$ $\uparrow$ +*prime*

para-Pappian

Example: There exists an affine plane of order 9 which is Thalesian but not Moufang.

OpenProblem: Is every para-Pappian affine space Thalesian?

Papp⇒Desargues⇒Moufang⇒Thales⇒para-Papp

For every (finite) affine space, the following implications hold:

Pappian

$\Downarrow$ $\uparrow$ +*finite* (by Wedderburn Theorem)

Desarguesian

$\Downarrow$ $\uparrow$ +*finite* (by Artin–Zorn Theorem)

Moufang

$\Downarrow$ $\uparrow$ +*prime*

Thalesian

$\Downarrow$ $\uparrow$ +*prime*

para-Pappian

Example: There exists an affine plane of order 9 which is Thalesian but not Moufang.
OpenProblem: Is every para-Pappian affine space Thalesian?

Algebraization of Desarguesian affine spaces

Theorem (the most probably, Hilbert, 1899)

Every Desarguesian affine space $(X, \mathcal{L})$ determines a canonical vector space $\vec{X}$ over certain corps $\mathbb{R}_X$ that acts on X so that the lines in X can be writen as $\{x + s\vec{v} : s \in \mathbb{R}_X\}$, where $x \in X$ and $\vec{v} \in \vec{X} \setminus \{\vec{0}\}$. The corps $\mathbb{R}_X$ is a field iff X is Pappian.

Question

How does the structure of a Desarguesian liner determine the structure of a vector space? What is the nature of vectors and how to define scalars and operations over vectors and scalars? Why do they have the properties, well-known from the Linear Algebra?

Algebraization of Desarguesian affine spaces

Theorem (the most probably, Hilbert, 1899)

Every Desarguesian affine space $(X, \mathcal{L})$ determines a canonical vector space $\vec{X}$ over certain corps $\mathbb{R}_X$ that acts on X so that the lines in X can be writen as $\{x + s\vec{v} : s \in \mathbb{R}_X\}$, where $x \in X$ and $\vec{v} \in \vec{X} \setminus \{\vec{0}\}$. The corps $\mathbb{R}_X$ is a field iff X is Pappian.

Question

How does the structure of a Desarguesian liner determine the structure of a vector space? What is the nature of vectors and how to define scalars and operations over vectors and scalars? Why do they have the properties, well-known from the Linear Algebra?

Part III
Vectors and Scalars

Algebraization of Desarguesian affine spaces

Theorem (the most probably, Hilbert, 1899)

Every Desarguesian affine space $(X, \mathcal{L})$ determines a canonical vector space $\vec{X}$ over certain corps $\mathbb{R}_X$ that acts on X so that the lines in X can be writen as $\{x + s\vec{v} : s \in \mathbb{R}_X\}$, where $x \in X$ and $\vec{v} \in \vec{X} \setminus \{\vec{0}\}$. The corps $\mathbb{R}_X$ is a field iff X is Pappian.

Question

How does the structure of a Desarguesian liner determine the structure of a vector space? What is the nature of vectors and how to define scalars and operations over vectors and scalars? Why do they have the properties, well-known from the Linear Algebra?

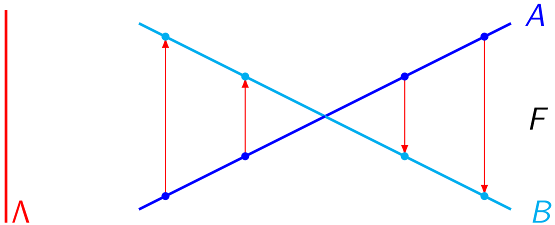
Affine transformations of lines

Definition (of parallel projection)

A bijective map $F : A \to B$ between two lines in an affine space $(X, \mathcal{L})$ is called

- a parallel projection along a line Λ, if $F = \{(a, b) \in A \times B : \overline{a\,b} \parallel \Lambda\}$;
- parallel projection if F is a parallel projection along some line Λ;
- parallel shift if $F : A \to B$ is a parallel projection between parallel lines A, B;
- parallel translation if F is a composition of finitely many parallel shifts between lines;
- affine transformation if F is a composition of finitely many parallel projections between lines.

Parallel projections between lines



Uniqueness theorem for parallel translations

Theorem (non-trivial)

Affine space $(X, \mathcal{L})$ is Thalesian if and only if for any points $x, y, z \in X$ with $x \neq y$ there exist unique line Λ and parallel translation $F : \overline{x\, y} \to \Lambda$ with $F(x) = z$.

This theorem allows to define a notion of a vector in a Thalesian affine space in the standard way as an equivalence class of ordered pairs pf points.

Vectors in Thalesian affine spaces

We say that two pairs of points $xy, uv \in X^2$ in an affine space $(X, \mathcal{L})$ are translation equivalent if $Fxy = uv$ for some parallel translation F between lines in X. The relation of translation equivalence divides the set X^2 of pairs of points into disjoint equivalence classes, called vectors.

For a pair of point $x, y \in X$, the vector $\overrightarrow{xy}$ is the class of all ordered pairs, which are translation equivalent to the pair xy.

The equivalence class $\{xx : x \in X\}$ is called the zero vector and is denoted by $\vec{0}$.

The set of all vectors in X is denoted by $\vec{X}$.

The preceding theorem implies that for every vector $\vec{v} \in \vec{X}$ and point $x \in X$ in a Thalesian affine space $(X, \mathcal{L})$ there exists a unique point $y \in X$ such that $\overrightarrow{xy} = \vec{v}$ (i.e., the vector $\vec{v}$ can be constructed from any given point x).

Operations on vectors

Theorem

For a Thalesian affine space X there exists a unique binary operation $+:\vec{X}\times\vec{X}\to\vec{X}$, turning $\vec{X}$ into a commutative group and has the proerty:

$$\forall x,y,z\in X\ \ (\overrightarrow{xy}+\overrightarrow{yz}=\overrightarrow{xz}).$$

Theorem

For any Thalesian affine space X there exists a unique action $+:\vec{X}\times X\to X$ such that:

1. $\forall\vec{v},\vec{u}\in\vec{X}\ \forall x\in X\ \ (\vec{v}+\vec{u})+x=\vec{v}+(\vec{u}+x)$;
2. $\forall x,y\in X\quad \overrightarrow{xy}+x=y$.

Uniqueness Theorem for affine transformations

Theorem (non-trivial)

An affine space $(X, \mathcal{L})$ is Desarguesian if and only if for any pairs $xy, uv \in X^2$ with $x \neq y$ and $u \neq v$ there exists a unique affine transformations $F : \overline{x\,y} \to \overline{u\,v}$ with $Fxy = uv$.

This theorem allows to define a notion of a scalar in a Desarguesian affine space as an equivalence class of ordered linear triples,
by analogy with the notion of vector.

A linear triple in a liner X is an ordered triple $xyz \in X^3$ with $y \in \overline{x\,z}$ and $x \neq z$.

Linear triples and their affine transformations

Two linear triples $xyz, uvw \in X^3$ are called affinely equivalent if $Fxyz = uvw$ for some affine transformation F between lines.

The relation of affine equivalence divides the set $\dddot{X}$ of linear triples into disjoint equivalence classes, called scalars.

For a linear triple $xzy \in \dddot{X}$, the scalar $\overrightarrow{xyz}$ is the class of all linear triples, which are affinely equivalent to the triple xyz.

The set of scalars in X is denoted by $\mathbb{R}_X$.
This set contains two distinguished elements:
$0 := \{xyz \in X^3 : x{=}y{\ne}z\}$ and $1 := \{xyz \in X^3 : x{\ne}y{=}z\}$.

The preceding theorem implies that for any scalar α and pair $xz \in X^2$ with $x \neq z$ in a Desarguesian affine space $(X, \mathcal{L})$ there exists a unique point $y \in X$ such that $xyz \in \alpha$.

Theorem

For every Desarguesian affine space X there exist unique binary operations

$$\cdot : \mathbb{R}_X \times \mathbb{R}_X \to \mathbb{R}_X \quad \text{and} \quad + : \mathbb{R}_X \times \mathbb{R}_X \to \mathbb{R}_X,$$

turning $\mathbb{R}_X$ into a corps (= division ring) and have the following properties:

- $\forall oxy, oye \in \dddot{X} \quad \overrightarrow{oxy} \cdot \overrightarrow{oye} = \overrightarrow{oxe}$;
- $\forall oxe, oye, oze \in \dddot{X} \ (\overrightarrow{ox} + \overrightarrow{oy} = \overrightarrow{oz}) \Rightarrow (\overrightarrow{oxe} + \overrightarrow{oye} = \overrightarrow{oze})$.

Multiplication of a vector by a scalar

Theorem

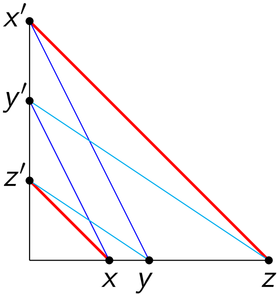
For every Desarguesian affine space X there exists a unique operation of multiplication of scalars by vectors $\cdot : \mathbb{R}_X \times \vec{X} \to \vec{X}$, $\cdot : (s, \vec{v}) \mapsto s{\cdot}\vec{v}$, satisfying the axioms:

1. $\forall s, t \in \mathbb{R}_X \; \forall \vec{v} \in \vec{X} \;\; (st) \cdot \vec{v} = s \cdot (t \cdot \vec{v})$;
2. $\forall s, t \in \mathbb{R}_X \; \forall \vec{v} \in \vec{X} \;\; (s + t) \cdot \vec{v} = s \cdot \vec{v} + t \cdot \vec{v}$;
3. $\forall s \in \mathbb{R}_X \; \forall \vec{v}, \vec{u} \in \vec{X} \;\; s \cdot (\vec{v} + \vec{u}) = s \cdot \vec{v} + s \cdot \vec{u}$.

Pappus and commutativity of the scalar corps

Theorem (Pappus of Alexandria; 270–350)

For a Desarguesian affine space X, the corps $\mathbb{R}_X$ is a field iff the Pappus Axiom holds: for any concurrent lines $L, L' \in \mathcal{L}$ and distinct points $x, y, z \in L \setminus L'$, $x', y', z' \in L' \setminus L$, if $\overline{x\,y'} \parallel \overline{x'\,y}$ and $\overline{y\,z'} \parallel \overline{y'\,z}$, then $\overline{x\,z'} \parallel \overline{x'\,z}$.



Part III: non-Desarguesian Geometry

Algebraization of affine planes

Affine spaces of dimension ≥ 3 are Desarguesian and hence admit a canonical structure of a vector space over some corps.

For affine planes (i.e., affine spaces of dimension 2), the situation is more complicated.

Nonetheless, any affine plane can be algebraized by a suitable ternary-ring, as was suggested by Marshall Hall in 1943.

Based affine planes

Definition

An affine base in an affine plane Π is any ordered triple of non-collinear points $uow \in \Pi^3$, called the unit, origin, and biunit of the affine base uow.
An affine plane endowed with an affine base is called a based affine plane.

By the Playfair Axiom (holding in affine planes), for every affine base uow in an affine plane Π, there exists a unique point $e \in \Pi$ completing the triangle uow to a parallelogram, i.e., $\overline{u\,e} \parallel \overline{o\,w}$ and $\overline{w\,e} \parallel \overline{o\,u}$.
The point e is called the diunit of the affine base uow and the line $\overline{o\,e}$ is the diagonal of the affine base uow.

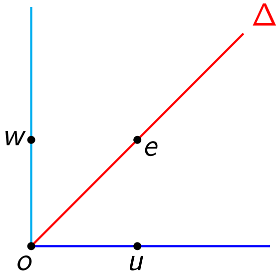
Based affine planes

Definition

An affine base in an affine plane Π is any ordered triple of non-collinear points $uow \in \Pi^3$, called the unit, origin, and biunit of the affine base uow.
An affine plane endowed with an affine base is called a based affine plane.

By the Playfair Axiom (holding in affine planes), for every affine base uow in an affine plane Π, there exists a unique point $e \in \Pi$ completing the triangle uow to a parallelogram, i.e., $\overline{u\,e} \parallel \overline{o\,w}$ and $\overline{w\,e} \parallel \overline{o\,u}$.
The point e is called the diunit of the affine base uow and the line $\overline{o\,e}$ is the diagonal of the affine base uow.

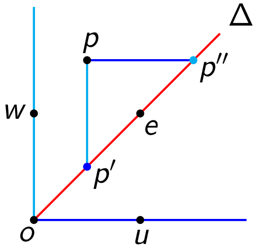
Based affine planes



o, the origin
u, the unit
w, the biunit
e, the diunit
$\overline{o\,u}$, the horizontal axis
$\overline{o\,w}$, the vertical axis
$\Delta := \overline{o\,e}$, the diagonal

Coordinates in based affine planes

Let (Π, uow) be a based affine plane. Since Π is Playfair, for every point $p \in \Pi$ there exist unique points $p', p'' \in \Delta$ (called the horizontal and vertical coordinates of p) such that $\overline{p\,p'} \parallel \overline{o\,w}$ and $\overline{p\,p''} \parallel \overline{o\,u}$.



The coordinate chart of a based affine plane

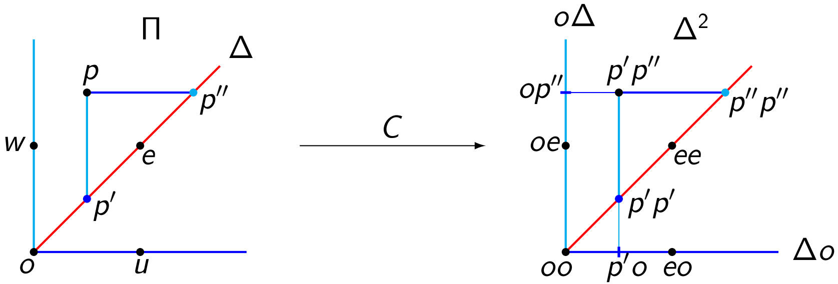
The map $C:\Pi\to\Delta^2$, $C:p\mapsto p'p''$, is a bijective map from the affine plane Π onto the square Δ^2 of the diagonal Δ of the affine base uow. This bijective map $C:\Pi\to\Delta^2$ is called the coordinate chart of the based affine plane.

The set Δ^2 endowed with the family of lines $\{C[L]:L\in\mathcal{L}\}$ is called the coordinate plane of the based affine plane (Π, uow). The coordinate plane Δ^2 carries the canonical base (eo,oo,oe).

The map $C:\Pi\to\Delta^2$, $C:p\mapsto p'p''$, is a bijective map from the affine plane Π onto the square Δ^2 of the diagonal Δ of the affine base uow. This bijective map $C:\Pi\to\Delta^2$ is called the coordinate chart of the based affine plane.

The set Δ^2 endowed with the family of lines $\{C[L]:L\in\mathcal{L}\}$ is called the coordinate plane of the based affine plane (Π,uow).

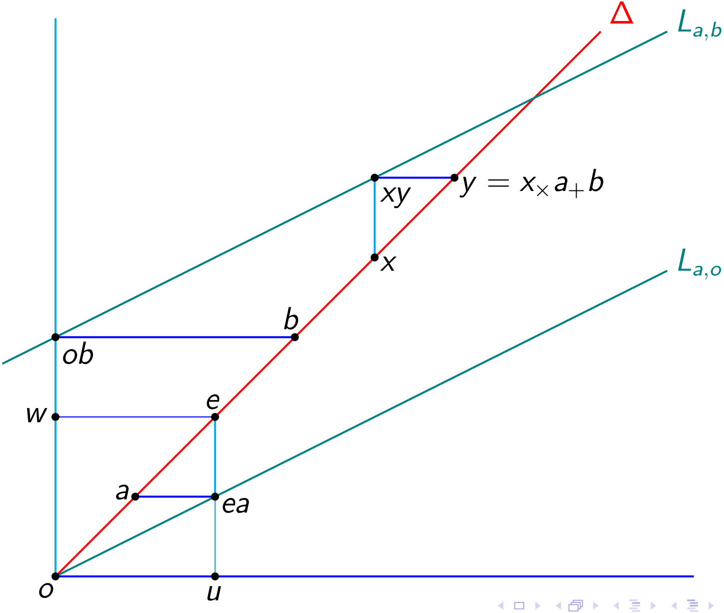
The coordinate plane Δ^2 carries the canonical base (eo,oo,oe).



Equations of lines in coordinates

The equations of non-vertical lines in based affine planes can be written down using a special ternary operation $T_{uow} : \Delta^3 \to \Delta$, which assigns to every triple $xab \in \Delta^3$ the unique point $y \in \Delta$ denoted by $x_\times a_+ b$ such that $\overline{ob\,xy} \parallel \overline{oo\,ea}$. The point y is the unique common point of the vertical line $L_x := \{p : p' = x\}$ and the unique line $L_{a,b}$ which contains the point $ob \in \overline{o\,w}$ (with coordinates o, b) and is parallel to the line $L_{o,a} := \overline{oo\,ea}$ connnecting the points o (with coordinates o, o) and the point ea (with coordinates e, a).

The ternary operation



Equations of lines $y = x_\times a_+ b$

The definition of the ternary operation T_{uow} ensures that the line $L_{a,b}$ is determined by the equation $y = x_\times a_+ b$, more precisely,

$$L_{a,b} = \{p \in \Pi : p'' = p'_\times a_+ b\} = C^{-1}[\{xy \in \Delta^2 : y = x_\times a_+ b\}].$$

Properties of the ternary-operation T_{uow}

Theorem

For every based affine plane (Π, uow), the ternary operation

$$T_{uow} : \Delta^3 \to \Delta, \quad T_{uow} : xab \mapsto x_\times a_+ b,$$

has the following four properties:

1. $x_\times o_+ b = b = o_\times x_+ b$ *and* $x_\times e_+ o = x = e_\times x_+ o$ *for every points* $x, b \in \Delta$;
2. *for every points* $a, x, y \in \Delta$, *there exists a unique point* $b \in \Delta$ *such that* $x_\times a_+ b = y$;
3. *for every points* $a, b, c, d \in \Delta$ *with* $a \neq c$, *there exists a unique point* $x \in \Delta$ *such that* $x_\times a_+ b = x_\times c_+ d$;
4. *for every points* $\check{x}, \check{y}, \hat{x}, \hat{y} \in \Delta$ *with* $\check{x} \neq \hat{x}$, *there exist unique points* $a, b \in \Delta$ *such that* $\check{x}_\times a_+ b = \check{y}$ *and* $\hat{x}_\times a_+ b = \hat{y}$.

Ternary-rings

Definition (Hall, 1943): A ternary-ring is a set R endowed with a ternary operation

$$T : R^3 \to R, \quad T : xab \mapsto x_\times a_+ b,$$

satisfying the following four axioms:

(T1) there exist distinct elements $0, 1 \in R$ such that $\forall x, b \in R$ $x_\times 0_+ b = b = 0_\times x_+ b$ and $x_\times 1_+ 0 = x = 1_\times x_+ 0$;

(T2) for every elements $a, b, c, d \in R$ with $a \neq c$, there exists a unique element $x \in R$ such that $x_\times a_+ b = x_\times c_+ d$;

(T3) for every elements $a, x, y \in R$, there exists a unique element $b \in R$ such that $x_\times a_+ b = y$;

(T4) $\forall \check{x}, \check{y}, \hat{x}, \hat{y} \in R$ with $\check{x} \neq \hat{x}$, there exist unique elements $a, b \in R$ such that $\check{x}_\times a_+ b = \check{y}$ and $\hat{x}_\times a_+ b = \hat{y}$.

The (unique) elements $0, 1$ appearing in the axiom (T1) are called the zero and unit of the ternary-ring R.

The coordinate plane of a ternary-ring

Given a ternary-ring R, consider the affine plane whose set of point is R^2 and the family of lines is

$$\mathcal{L} := \{L_c : c \in R\} \cup \{L_{a,b} : a, b \in R\},$$

where

$$L_c := \{xy \in R^2 : x = c\} \text{ and } L_{a,b} := \{xy \in R^2 : y = x_\times a_+ b\}$$

for $a, b, c \in R$. The affine plane R^2 is endowed with the canonical affine base $(10, 00, 01)$ and hence is a based affine plane, called the coordinate plane of the ternary-ring R.

Theorem (Hall, 1943)

For every ternary-ring R its coordinate plane R^2 is a based affine plane. Moreover, for every based affine plane (Π, uow), the coordinate chart $C : \Pi \to \Delta^2$ is an isomorphism between the based affine plane (Π, uow) and the coordinate plane of the ternary-ring (Δ^2, T_{uow}).

The coordinate plane of a ternary-ring

Given a ternary-ring R, consider the affine plane whose set of point is R^2 and the family of lines is

$$\mathcal{L} := \{L_c : c \in R\} \cup \{L_{a,b} : a, b \in R\},$$

where

$$L_c := \{xy \in R^2 : x = c\} \text{ and } L_{a,b} := \{xy \in R^2 : y = x_\times a_+ b\}$$

for $a, b, c \in R$. The affine plane R^2 is endowed with the canonical affine base $(10, 00, 01)$ and hence is a based affine plane, called the coordinate plane of the ternary-ring R.

Theorem (Hall, 1943)

For every ternary-ring R its coordinate plane R^2 is a based affine plane. Moreover, for every based affine plane (Π, uow), the coordinate chart $C : \Pi \to \Delta^2$ is an isomorphism between the based affine plane (Π, uow) and the coordinate plane of the ternary-ring (Δ^2, T_{uow}).

Addition and multiplication in ternary-rings

Every ternary-ring (R,T) carries two binary operations

$$+ : R\times R\to R,\quad + : (x,b)\mapsto x+b := T(x,1,b) = x_\times 1_+ b,$$
$$\cdot : R\times R\to R,\quad \cdot : (x,a)\mapsto x\cdot a := T(x,a,0) = x_\times a_+ 0,$$

called the addition and multiplication operations in R.

Definition: A ternary-ring R is linear if

$$\forall x,a,b\in R\quad x_\times a_+ b = (x\cdot a)+b := (x_\times a_+ 0)_\times 1_+ b.$$

Therefore, the ternary operation of a linear ternary-ring is uniquely determined by the addition and multiplication.

Example: Every corps R endowed with the ternary operation

$$T : R^3\to R,\quad T : (x,a,b)\mapsto (x\cdot a)+b,$$

is a linear ternary-ring.

Addition and multiplication in ternary-rings

Every ternary-ring (R, T) carries two binary operations

$$+ : R \times R \to R, \quad + : (x, b) \mapsto x + b := T(x, 1, b) = x_\times 1_+ b,$$
$$\cdot : R \times R \to R, \quad \cdot : (x, a) \mapsto x \cdot a := T(x, a, 0) = x_\times a_+ 0,$$

called the addition and multiplication operations in R.

Definition: A ternary-ring R is linear if

$$\forall x, a, b \in R \quad x_\times a_+ b = (x{\cdot}a){+}b := (x_\times a_+ 0)_\times 1_+ b.$$

Therefore, the ternary operation of a linear ternary-ring is uniquely determined by the addition and multiplication.

Example: Every corps R endowed with the ternary operation

$$T : R^3 \to R, \quad T : (x, a, b) \mapsto (x \cdot a) + b,$$

is a linear ternary-ring.

Loops and 0-loops

A magma is a set endowed with a binary operation.
A magma M is

- unital if it contains an element $1 \in M$, called the identity of M, such that $\forall x \in M \;\; 1 \cdot x = x = x \cdot 1$;
- a loop if M is a unital magma such that $\forall a, b \in M \;\; \exists! x, y \in M \;\; (x \cdot a = b \,\wedge\, a \cdot y = b)$;
- 0-magma if M contains an element $0 \in M$, called the zero of M, such that $\forall x \in M \;\; x \cdot 0 = 0 = 0 \cdot x$;
- a 0-loop if M is a unital 0-magma such that $\forall a \in M \setminus \{0\} \; \forall b \in M \; \exists! x, y \in M \;\; (x \cdot a = b \wedge a \cdot y = b)$.

Additive and multiplicative loops of a ternary-ring

Theorem

If R is a ternary-ring, then $(R, +)$ is a loop and $(R, \cdot)$ is a 0-loop such that for every $a \in R \setminus \{1\}$ and $b \in R$, the equation $x \cdot a = x + b$ has a unique solution $x \in R$.

Definition

Let R be a ternary-ring and $R^* := R \setminus \{0\}$. The loops $(R, +)$ and $(R^*, \cdot)$ are called the additive loop and the multiplicative loop of the ternary-ring R, respectively.

Additive and multiplicative loops of a ternary-ring

Theorem

If R is a ternary-ring, then $(R, +)$ is a loop and $(R, \cdot)$ is a 0-loop such that for every $a \in R \setminus \{1\}$ and $b \in R$, the equation $x \cdot a = x + b$ has a unique solution $x \in R$.

Definition

Let R be a ternary-ring and $R^* := R \setminus \{0\}$. The loops $(R, +)$ and $(R^*, \cdot)$ are called the additive loop and the multiplicative loop of the ternary-ring R, respectively.

Some properties of ternary-rings

A ternary-ring R is called

commutative-plus if $\forall x, y \in R \;\; x + y = y + x$;
commutative-dot if $\forall x, y \in R \;\; x \cdot y = y \cdot x$;
commutative if it is commutative-plus and commutative-dot;

associative-plus if $\forall x, y, z \in R \;\; x + (y + z) = (x + y) + z$;
associative-dot if $\forall x, y, z \in R \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
associative if R is associative-plus and associative-dot;

inversive-plus if $\forall x \in R \, \exists x^- \in R \, \forall y \in R \;\; (y + x) + x^- = y$;
inversive-dot if $\forall x \in R \, \exists x^{-1} \in R \, \forall y \in R \;\; (y \cdot x) \cdot x^{-1} = y$;
inversive if R is inversive-plus and inversive-dot;

left-distributive if $\forall a, x, y \in R \;\; a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
right-distributive if $\forall x, y, b \in R \;\; (x + y) \cdot b = (x \cdot b) + (y \cdot b)$;
distributive if R is left-distributive and right-distributive;

a corps if R is linear, distributive and associative;
a field if R is a commutative corps.

Some properties of ternary-rings

A ternary-ring R is called
commutative-plus if $\forall x, y \in R \;\; x + y = y + x$;
commutative-dot if $\forall x, y \in R \;\; x \cdot y = y \cdot x$;
commutative if it is commutative-plus and commutative-dot;

associative-plus if $\forall x, y, z \in R \;\; x + (y + z) = (x + y) + z$;
associative-dot if $\forall x, y, z \in R \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
associative if R is associative-plus and associative-dot;

inversive-plus if $\forall x \in R \; \exists x^- \in R \; \forall y \in R \;\; (y + x) + x^- = y$;
inversive-dot if $\forall x \in R \; \exists x^{-1} \in R \; \forall y \in R \;\; (y \cdot x) \cdot x^{-1} = y$;
inversive if R is inversive-plus and inversive-dot;

left-distributive if $\forall a, x, y \in R \;\; a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
right-distributive if $\forall x, y, b \in R \;\; (x + y) \cdot b = (x \cdot b) + (y \cdot b)$;
distributive if R is left-distributive and right-distributive;

a corps if R is linear, distributive and associative;
a field if R is a commutative corps.

Some properties of ternary-rings

A ternary-ring R is called

commutative-plus if $\forall x, y \in R \;\; x + y = y + x$;
commutative-dot if $\forall x, y \in R \;\; x \cdot y = y \cdot x$;
commutative if it is commutative-plus and commutative-dot;

associative-plus if $\forall x, y, z \in R \;\; x + (y + z) = (x + y) + z$;
associative-dot if $\forall x, y, z \in R \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
associative if R is associative-plus and associative-dot;

inversive-plus if $\forall x \in R \; \exists x^{-} \in R \; \forall y \in R \;\; (y + x) + x^{-} = y$;
inversive-dot if $\forall x \in R \; \exists x^{-1} \in R \; \forall y \in R \;\; (y \cdot x) \cdot x^{-1} = y$;
inversive if R is inversive-plus and inversive-dot;

left-distributive if $\forall a, x, y \in R \;\; a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
right-distributive if $\forall x, y, b \in R \;\; (x + y) \cdot b = (x \cdot b) + (y \cdot b)$;
distributive if R is left-distributive and right-distributive;

a corps if R is linear, distributive and associative;
a field if R is a commutative corps.

Some properties of ternary-rings

A ternary-ring R is called

commutative-plus if $\forall x, y \in R \;\; x + y = y + x$;
commutative-dot if $\forall x, y \in R \;\; x \cdot y = y \cdot x$;
commutative if it is commutative-plus and commutative-dot;

associative-plus if $\forall x, y, z \in R \;\; x + (y + z) = (x + y) + z$;
associative-dot if $\forall x, y, z \in R \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
associative if R is associative-plus and associative-dot;

inversive-plus if $\forall x \in R \; \exists x^- \in R \; \forall y \in R \;\; (y + x) + x^- = y$;
inversive-dot if $\forall x \in R \; \exists x^{-1} \in R \; \forall y \in R \;\; (y \cdot x) \cdot x^{-1} = y$;
inversive if R is inversive-plus and inversive-dot;

left-distributive if $\forall a, x, y \in R \;\; a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
right-distributive if $\forall x, y, b \in R \;\; (x + y) \cdot b = (x \cdot b) + (y \cdot b)$;
distributive if R is left-distributive and right-distributive;

a corps if R is linear, distributive and associative;
a field if R is a commutative corps.

Some properties of ternary-rings

A ternary-ring R is called

commutative-plus if $\forall x, y \in R \;\; x + y = y + x$;
commutative-dot if $\forall x, y \in R \;\; x \cdot y = y \cdot x$;
commutative if it is commutative-plus and commutative-dot;

associative-plus if $\forall x, y, z \in R \;\; x + (y + z) = (x + y) + z$;
associative-dot if $\forall x, y, z \in R \;\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
associative if R is associative-plus and associative-dot;

inversive-plus if $\forall x \in R \, \exists x^- \in R \, \forall y \in R \;\; (y + x) + x^- = y$;
inversive-dot if $\forall x \in R \, \exists x^{-1} \in R \, \forall y \in R \;\; (y \cdot x) \cdot x^{-1} = y$;
inversive if R is inversive-plus and inversive-dot;

left-distributive if $\forall a, x, y \in R \;\; a \cdot (x + y) = (a \cdot x) + (a \cdot y)$;
right-distributive if $\forall x, y, b \in R \;\; (x + y) \cdot b = (x \cdot b) + (y \cdot b)$;
distributive if R is left-distributive and right-distributive;

a corps if R is linear, distributive and associative;
a field if R is a commutative corps.

Pappian ⇔ commutative-dot

Definition: A ternary-ring R is called a ternary-ring of an affine plane Π if Π is isomorphic to the coordinate plane R^2 of the ternary-ring R.

Theorem (Klingenberg, 1955)

For an affine plane Π, the following conditions are equivalent:

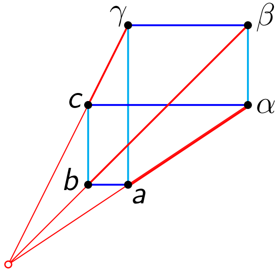
1. *Π is Pappian;*
2. *every ternary-ring of Π is commutative-dot;*
3. *every/some ternary-ring of Π is a field;*
4. *for every points $a, b, c, \alpha, \beta, \gamma \in X$ with*
$$\overline{a\,b} \parallel \overline{\alpha\,c} \parallel \overline{\beta\,\gamma} \nparallel \overline{b\,c} \parallel \overline{a\,\gamma} \parallel \overline{\alpha\,\beta},$$
the lines $\overline{a\,\alpha}, \overline{b\,\beta}, \overline{c\,\gamma}$ are paraconcurrent.

Commutative-dot affine planes

Geometrically, the last condition
(responsible for the commutativity of multiplication):
For every points $a, b, c, \alpha, \beta, \gamma \in X$ with

$$\overline{a\,b} \parallel \overline{\alpha\,c} \parallel \overline{\beta\,\gamma} \nparallel \overline{b\,c} \parallel \overline{a\,\gamma} \parallel \overline{\alpha\,\beta},$$

the lines $\overline{a\,\alpha}, \overline{b\,\beta}, \overline{c\,\gamma}$ are paraconcurrent,
looks as follows:



Desarguesian $\Leftrightarrow$ associative-dot

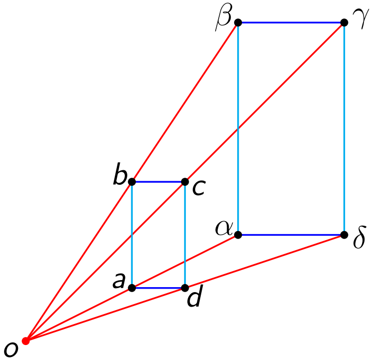
Theorem (Klingenberg, 1955)

For an affine plane Π, the following conditions are equivalent:

1. Π *is Desarguesian;*
2. *every ternary-ring of* Π *is associative-dot;*
3. *every/some ternary-ring of* Π *is a corps;*
4. *for every points* $o, a, b, c, d, \alpha, \beta, \gamma, \delta \in \Pi$ *with* $\overline{a\,b} \parallel \overline{c\,d} \parallel \overline{\alpha\,\beta} \parallel \overline{\gamma\,\delta} \nparallel \overline{a\,d} \parallel \overline{b\,c} \parallel \overline{\beta\,\gamma} \parallel \overline{\alpha\,\delta}$ *and* $o \in \overline{a\,\alpha} \cap \overline{b\,\beta} \cap \overline{c\,\gamma}$, *we have* $o \in \overline{d\,\delta}$.

Associative-dot affine planes

Geometrically, the last condition
(responsible for the associativity of multiplication):
for every points $o, a, b, c, d, \alpha, \beta, \gamma, \delta \in \Pi$ with
$\overline{a\,b} \parallel \overline{c\,d} \parallel \overline{\alpha\,\beta} \parallel \overline{\gamma\,\delta} \nparallel \overline{a\,d} \parallel \overline{b\,c} \parallel \overline{\beta\,\gamma} \parallel \overline{\alpha\,\delta}$ and
$o \in \overline{a\,\alpha} \cap \overline{b\,\beta} \cap \overline{c\,\gamma}$, we have $o \in \overline{d\,\delta}$,
looks as follows:



Moufang ⇔ inversive-dot

Theorem (Skornyakov, Saint-Soucie, 1952)

For an affine plane Π, the following conditions are equivalent:

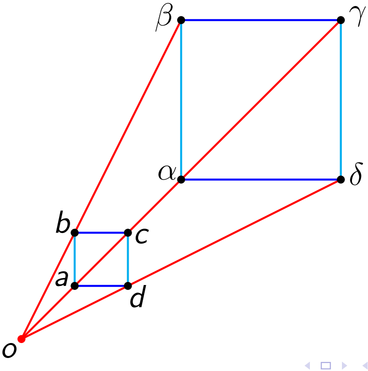
1. Π *is Moufang;*
2. *every ternary-ring of* Π *is inversive-dot;*
3. *every/some ternary-ring of* Π *is linear, distributive, inversive-dot and commutative-plus;*
4. *for every line* $D \subset X$ *and points* $o, a, c, \alpha, \gamma \in D$ *and* $b, d, \beta, \delta \in X \setminus D$ *with* $\overline{a\,b} \parallel \overline{c\,d} \parallel \overline{\alpha\,\beta} \parallel \overline{\gamma\,\delta} \nparallel \overline{a\,d} \parallel \overline{b\,c} \parallel \overline{\beta\,\gamma} \parallel \overline{\alpha\,\delta}$ *and* $o \in \overline{b\,\beta}$, *we have* $o \in \overline{d\,\delta}$.

Inversive-dot affine planes

Geometrically, the last condition
(responsible for the inversivity of multiplication):

$\forall$ line D and points $o, a, c, \alpha, \gamma \in D$ and $b, d, \beta, \delta \in X \setminus D$ with $\overline{a\,b} \parallel \overline{c\,d} \parallel \overline{\alpha\,\beta} \parallel \overline{\gamma\,\delta} \nparallel \overline{a\,d} \parallel \overline{b\,c} \parallel \overline{\beta\,\gamma} \parallel \overline{\alpha\,\delta}$ and $o \in \overline{b\,\beta}$, we have $o \in \overline{d\,\delta}$,

looks as follows:



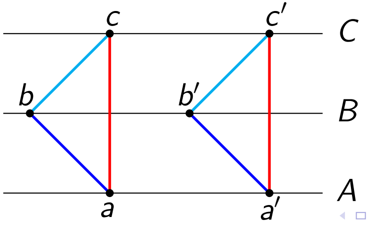
Thalesian $\Leftrightarrow$ quasi-field

Definition: A ternary-ring R is called a quasi-field if R is linear, right-distributive, and associative-plus.

Theorem (Veblen, 1916)

For an affine plane Π, the following conditions are equivalent:

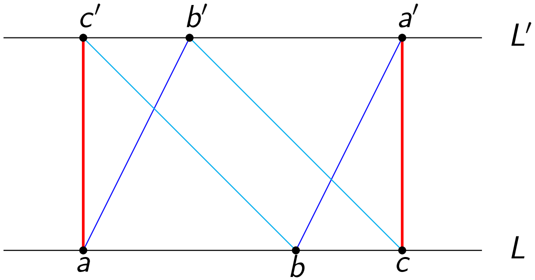
1. *Π is Thalesian;*
2. *every ternary-ring of Π is a quasi-field;*
3. *some ternary-ring of Π is a quasi-field.*



para-Pappian ⇔ commutative-plus

Theorem

An affine plane Π is para-Pappian if and only if every ternary-ring of Π is commutative-plus.



Prime affine planes

Theorem

For a prime affine plane Π, the following conditions are equivalent:

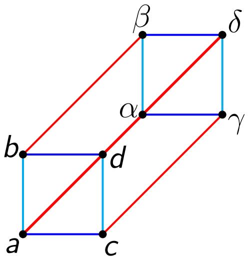
1. Π *is Pappian;*
2. Π *is Desarguesian;*
3. Π *is Moufang;*
4. Π *is Thalesian;*
5. Π *is para-Pappian;*
6. *every ternary-ring of* Π *is inversive-plus;*
7. $\forall D \in \mathcal{L}_\Pi\ \forall a, d, \alpha, \delta \in D\ \forall b, c, \beta, \gamma \in \Pi \setminus D$ *with* $\overline{a\,b} \parallel \overline{c\,d} \parallel \overline{\alpha\,\beta} \parallel \overline{\gamma\,\delta} \nparallel \overline{a\,c} \parallel \overline{b\,d} \parallel \overline{\alpha\,\gamma} \parallel \overline{\beta\,\delta}$ *and* $\overline{b\,\beta} \parallel D$*, we have* $\overline{c\,\gamma} \parallel D$.

Inversive-dot affine planes

Geometrically, the last condition
(responsible for the inversivity of the addition):

$\forall D \in \mathcal{L}_\Pi\ \forall a, d, \alpha, \delta \in D\ \forall b, c, \beta, \gamma \in \Pi \setminus D$
$(\overline{a\,b} \parallel \overline{c\,d} \parallel \overline{\alpha\,\beta} \parallel \overline{\gamma\,\delta} \nparallel \overline{a\,c} \parallel \overline{b\,d} \parallel \overline{\alpha\,\gamma} \parallel \overline{\beta\,\delta} \wedge \overline{b\,\beta} \parallel D) \Rightarrow \overline{c\,\gamma} \parallel D$

looks as follows:



Algebra versus Geometry in affine planes

<u>**Ternary-ring**</u>: <u>**Affine plane**</u>:

$$
\begin{array}{ccc}
\text{commutative-dot} & \Longleftrightarrow & \text{Pappian} \\
\Downarrow & & \Downarrow \ \big\uparrow{+\mathit{finite}} \\
\text{associative-dot} & \Longleftrightarrow & \text{Desarguesian} \\
\Downarrow & & \Downarrow \ \big\uparrow{+\mathit{finite}} \\
\text{inversive-dot} & \Longleftrightarrow & \text{Moufang} \\
\Downarrow & & \Downarrow \ \big\uparrow{+\mathit{prime}} \\
\text{quasi-field} & \Longleftrightarrow & \text{Thalesian} \\
\Downarrow & & \Downarrow \ \big\uparrow{+\mathit{prime}} \\
\text{commutative-plus} & \Longleftrightarrow & \text{para-Pappian} \\
\Downarrow \ \big\uparrow{+\mathit{prime}} & & \\
\text{inversive-plus} & &
\end{array}
$$

Part V:
Selected Open Problems

Open Problems on liners: order

An affine plane X is called prime (resp. prime power) if its order is equal to (some power of) a prime number.

Problem

Is any finite affine plane prime power?

The best existing result in the positive direction is:

Theorem (Lüneburg, 1960)

Every finite commutative-plus affine liner is prime power.

Problem

Is any finite associative-plus affine liner prime power?

Problem

Is prime affine liner of (para-)Pappian?

Open Problems on liners: order

Theorem (Bruck–Ryser, 1948)

If a finite affine liner has order $n \in \{1, 2\} + 4\mathbb{Z}$, then $n = a^2 + b^2$ for some integer numbers a, b.

Corollary

The order of an affine liner cannot be equal to 6, 14, 22,...

Theorem (Lam, Thiel, Swiercz, 1989)

The order of an affine plane cannot be equal to $10 = 9^2 + 1^2$.

Problem

Is there an affine liner of order 12?

Open Problems: embeddings

Theorem (Hall, 1943)

Every liner is a subliner of some projective plane.

Problem (Hall)

Is every finite liner a subliner of some finite projective plane?

Open Problems on liners: homogeneity

Definition

A liner X is called

- *homogeneous* if for every points $x, y \in X$ there exists an automorphism $A : X \to X$ such that $Ax = y$;
- 2-*homogeneous* if for every pairs $xy, x'y' \in X^2 \setminus \Delta$ there exists an automorphism $A : X \to X$ such that $Axy = x'y'$;
- 3-*homogeneous* if for every affine bases $uow, u'o'w' \in X^3$ there exists an automorphism $A : X \to X$ such that $Auow = u'o'w$.

Theorem

Every Moufang affine liner is 3-homogeneous.

Problem

Is every 3-homogeneous affine liner Moufang?

Open Problems: homogeneity

Theorem

Every 2-homogeneous finite affine liner is Pappian.

Problem

Is every homogeneous finite affine liner Pappian?

Theorem

There exists an affine liner with trivial automorphism group.

Problem

Is there a finite affine (or projective) liner with trivial automorphism groups?

References

T. Banakh, *Geometry of Liners*, https://www.researchgate.net/publication/383409915_Geometry_of_Liners

Thank you!