University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Zofia Grochulska

Differential and geometric properties of certain classes of homeomorphisms

PhD dissertation

Supervisor:
dr hab. Paweł Goldstein
University of Warsaw

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Author's declaration
I hereby declare that this dissertation is my own work.

11 March 2024 ....
Zofia Grochulska

Supervisor's declaration
This dissertation is ready to be reviewed.

11 March 2024 ....
dr hab. Paweł Goldstein

Summary

This thesis is devoted to studying the interplay between geometric, analytic and topological properties of homeomorphisms. We discuss homeomorphisms between subsets of $\mathbb{R}^n$ and between subsets of n-dimensional manifolds. We always assume $n \geq 2$. This research topic belongs to geometric measure and function theory. Recently, it has received more attention due to possible applications in nonlinear elasticity. We formulate new and potentially useful results about diffeomorphisms, homeomorphisms and bi-Lipschitz and almost everywhere approximately (a. e.) differentiable homeomorphisms.

There are five main, original results of this thesis—Theorems 1.1, 1.2, 1.3 contained in Chapter 3 and Theorems 1.4 and 1.5 contained in Chapters 4 and 5, respectively. They were obtained in collaboration with Paweł Goldstein (University of Warsaw) and Piotr Hajłasz (University of Pittsburgh). They are contained in two yet unpublished papers [39, 40].

The first three results are from [40] and they are very similar in spirit as each concerns gluing homeomorphisms of a certain class on an n-dimensional connected manifold $\mathcal{M}^n$ of a corresponding regularity. We show that given two disjoint families of sets $\{D_i\}_{i=1}^{\ell}$, $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, which satisfy certain regularity properties, and orientation preserving diffeomorphisms $F_i : D_i \to \mathcal{M}^n$ with $F(D_i) = D_i'$, it is possible to find a diffeomorphism $F : \mathcal{M}^n \to \mathcal{M}^n$ such that $F = F_i$ on D_i. The proof uses some classical tools from differential geometry as well as explicit constructions of diffeomorphims which we repetitively use in this thesis. Then, we show an analogous result under the assumption that F_i are orientation preserving bi-Lipschitz homeomorphisms and, eventually, that F_i are orientation preserving homeomorphisms. The proof in these two cases is almost the same, and very similar to the diffeomorphic one, yet it requires use of deep theorems from (algebraic) topology.

Theorems 1.4 and 1.5 from [39] discuss prescribing derivatives of diffeomorphisms and a. e. approximately differentiable homeomorphisms, respectively. Essentially, we show that under some mild assumptions on $T : [0,1]^n \to GL(n)^+$, where $GL(n)^+$ denotes the space of $n \times n$ matrices with positive determinant, there is a diffeomorphism Φ of the unit cube $[0,1]^n$ whose derivative equals T on an arbitrarily large (in measure) subset of the cube. Moreover, $\Phi = \mathrm{id}$ on the boundary of the cube. To construct Φ, we use in particular the already mentioned explicit constructions of diffeomorphisms and the Dacorogna–Moser theory of mappings with prescribed Jacobian.

Theorem 1.5 is the most complex theorem of this thesis. Given $T : [0,1]^n \to GL(n)$, where $GL(n)$ denotes the space of $n \times n$ invertible matrices, which satisfies some mild assumptions we construct an a. e. homeomorphism Φ of the unit cube whose approximate derivative equals T a. e. on $[0,1]^n$. Moreover, $\Phi = \mathrm{id}$ on the boundary of the cube. We show that our assumptions on T are necessary and sufficient, which provides a characterization of differentiability properties of the class of a. e. differentiable homeomorphisms. The proof employs Theorem 1.4 and a certain ingenious iteration scheme, inspired by the homeomorphic measures theorem of Oxtoby and Ulam.

Moreover, we provide corollaries to main theorems and prove a series of technical lemmata which may be of independent interest. This thesis also contains an appendix, in which certain definitions (e. g., orientation on a topological manifold) or used methods (constructing diffeomorphisms with 1-parameter groups of diffeomorphisms) are explained.

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Keywords: approximate differentiability, diffeomorphisms, homeomorphisms, bi-Lipschitz homeomorphisms, prescribed derivative, prescribed Jacobian

Streszczenie

Niniejsza rozprawa jest poświęcona badaniu związków między geometrycznymi, analitycznymi i topologicznymi własnościami homeomorfizmów. Badamy homeomorfizmy podzbiorów $\mathbb{R}^n$ oraz podzbiorów n-wymiarowych rozmaitości. Zakładamy, że $n \geq 2$. Ten obszar badań należy do geometrycznej teorii miary i przekształceń. W ostatnim czasie zwrócono większą uwagę na tę tematykę ze względu na możliwe zastosowania tej teorii w nieliniowej teorii sprężystości. Formułujemy nowe oraz użyteczne wyniki dotyczące dyfeomorfizmów, homeomorfizmów, homeomorfizmów bi-lipschitzowskich oraz prawie wszędzie (p. w.) aproksymatywnie różniczkowalnych homeomorfizmów.

Niniejsza rozprawa zawiera pięć nowych wyników. Są to Twierdzenia 1.1, 1.2, 1.3 zawarte w Rozdziale 3 oraz Twierdzenia 1.4 i 1.5 zawarte w, odpowiednio, Rozdziałach 4 i 5. Wyniki te zostały uzyskane we współpracy z Pawłem Goldsteinem (Uniwersytet Warszawski) i Piotrem Hajłaszem (University of Pittsburgh). Są one zawarte w dwóch nieopublikowanych jeszcze artykułach [39, 40].

Pierwsze trzy wyniki pochodzą z [40] i są one do siebie bardzo zbliżone. Każde z tych twierdzeń dotyczy sklejania homeomorfizmów pewnej klasy na n-wymiarowej spójnej rozmaitości $\mathcal{M}^n$ o regularności odpowiadającej regularności rozpatrywanego homeomorfizmu. Pokazujemy, że gdy mamy dane dwie rozłączne rodziny zbiorów o pewnej regularności $\{D_i\}_{i=1}^{\ell}$, $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, oraz zachowujące orientację dyfeomorfizmy $F_i : D_i \to \mathcal{M}^n$ spełniające warunek $F(D_i) = D_i'$, da się znaleźć dyfeomorfizm $F : \mathcal{M}^n \to \mathcal{M}^n$, taki że $F = F_i$ na D_i. W dowodzie korzystamy z klasycznych narzędzi geometrii różniczkowej oraz z konstrukcji konkretnych dyfeomorfizmów, wielokrotnie używanych w niniejszej rozprawie. Następnie dowodzimy analogiczny wynik przy założeniu że F_i są zachowującymi orientację bi-lipschitzowskimi homeomorfizmami oraz przy założeniu że F_i są zachowującymi orientację homeomorfizmami. Dowód w ostatnich dwóch przypadkach jest niemalże identyczny oraz bardzo podobny do dowodu w przypadku dyfeomorfizmów, jednakże wymaga on użycia głębokich twierdzeń z topologii (algebraicznej).

Twierdzenia 1.4 oraz 1.5 z [39] dotyczą zadawania pochodnej, odpowiednio, dyfeomorfizmów oraz p. w. aproksymatywnie różniczkowalnych homeomorfizmów. Pokazujemy, że przy słabych założeniach o $T : [0,1]^n \to GL(n)^+$, gdzie $GL(n)^+$ oznacza przestrzeń macierzy $n \times n$ o dodatnim wyznaczniku, istnieje dyfeomorfizm Φ kostki jednostkowej $[0,1]^n$, którego pochodna jest równa T p. w. na $[0,1]^n$. Ponadto Φ równa się identyczności na brzegu kostki. Aby skonstruować Φ używamy między innymi wspomnianych już konstrukcji dyfeomorfizmów oraz teorii Dacorogni-Mosera dotyczącej przekształceń o zadanym jakobianie.

Twierdzenie 1.5 jest najbardziej złożonym twierdzeniem niniejszej rozprawy. Mając dane $T : [0,1]^n \to GL(n)$, gdzie $GL(n)$ oznacza przestrzeń odwracalnych macierzy $n \times n$, które spełnia pewne naturalne warunki, konstruujemy p. w. różniczkowalny homeomorfizm Φ kostki jednostkowej, którego aproksymatywna pochodna równa się T p. w. na $[0,1]^n$. Ponadto Φ równa się identyczności na brzegu kostki. Co więcej, pokazujemy że nasze założenia o T są konieczne i wystarczające, co pozwala na scharakteryzowanie własności różniczkowych klasy p. w. aproksymatywnie różniczkowalnych homeomorfizmów. W dowodzie używamy Twierdzenia 1.4 oraz pewnego pomysłowego schematu iteracyjnego zainspirowanego twierdzeniem Oxtoby'ego i Ulama o homeomorficznych miarach.

Ponadto w rozprawie sformułowane są wnioski płynące z głównych twierdzeń oraz techniczne lematy, które również mogą być przydatne. Rozprawa zawiera także dodatek, w którym wyjaśnione są pewne definicje (np. orientacji na rozmaitości topologicznej) oraz użyte metody (np. konstruowanie dyfeomorfizmów za pomocą 1-parametrowych grup dyfeomorfizmów).

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Słowa kluczowe: aproksymatywna różniczkowalność, dyfeomorfizmy, homeomorfizmy, homeomorfizmy bi-lipschitzowskie, zadana pochodna, zadany jakobian

Thank you

Mathematics is a bit like tennis—it might seem like an individual sport but it really is a team effort, as my supervisor often says[1]. And the first and biggest *thank you* goes to you, Paweł. For answering tons of my questions, for asking me some others and believing I can come up with an answer. And most importantly, for helping me not to ask the worst question of all—if I am capable of doing this at all. For your wonderful ideas, expertise in not only analysis, illustrious way of explaining even the terrifying bits of algebraic topology and for almost always having cookies in your office—thank you.

By extension, I would like to thank Ania Zatorska-Goldstein, as I believe your being the absolute mastermind of organization and work-life balance helped me gain many an extra hour for discussing mathematics with Paweł. Also, thank you Ania for your support and for reminding me of the elusive truth that it suffices to be good enough, not necessarily the best.

Next, I would like to thank Piotr Hajłasz for sharing with me some of your brilliant insights as to how to construct homeomorphisms and how to construct understandable proofs. And for reminding me that mathematics is *special* and that every proof deserves the elegance of a fountain pen. Also, I would like to thank the entire Hajłasz family for your heart-warming hospitality during my stay in Pittsburgh.

A barely visible yet crucial part in this dissertation was played by my collaboration with Michał Łasica. Michał, thank you for inviting me to write with you that paper that eventually *did* get published[2].

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[1] Paweł usually credits Piotr Hajłasz with the sports metaphor but the tennis one, this is entirely on me.
[2] Fun fact: we use Brouwer's invariance of domain there, too!
[3] Not only mine, though.

Contents

Chapter 1

Introduction

This thesis is devoted to studying the interplay between geometric, analytic and topological properties of homeomorphisms. We discuss homeomorphisms between subsets of $\mathbb{R}^n$ and between subsets of n-dimensional manifolds. We always assume $n \geq 2$. This research topic belongs to geometric measure and function theory.

There are five main, original results of this thesis—Theorems 1.1, 1.2, 1.3 contained in Chapter 3 and Theorems 1.4 and 1.5 contained in Chapters 4 and 5, respectively. They were obtained in collaboration with Paweł Goldstein (University of Warsaw) and Piotr Hajłasz (University of Pittsburgh). They are contained in two yet unpublished papers [39, 40].

We shall begin with a rough description of basic definitions and general context. Then, we describe each main result and provide an overview of literature concerning the topic. At the end of this introduction, we present a detailed structure of this dissertation.

1.1 General introduction

Let Ω denote a bounded domain in $\mathbb{R}^n$ for $n \geq 2$. When writing that $F : \Omega \to \mathbb{R}^n$ is a homeomorphism, we mean that F is a homeomorphism onto its image $F(\Omega)$. We shall denote surjective mappings with $\twoheadrightarrow$, that is $F : X \twoheadrightarrow Y$ means that $F(X) = Y$. In Section 2.1, notation used throughout the thesis is collected, here we shall recall some definitions whenever we need them.

Given a homeomorphism $F : \Omega \to \mathbb{R}^n$, it is clear that for any open $U \subset \Omega$, $F(U)$ is open *in the subspace topology of* $F(\Omega)$. However, it is not clear at all why $F(U)$ (or $F(\Omega)$) should be open in *the topology of* $\mathbb{R}^n$. Indeed, the fact that $f(\Omega)$ (and hence $f(U)$) is open follows from a deep theorem of Brouwer on invariance of domain. This is specific to Euclidean spaces and, in particular, means that homeomorphisms map boundary points to boundary points and interior points to interior points. More precisely, given two bounded domains $\Omega, \Omega' \subset \mathbb{R}^n$ and a homeomorphism $F : \overline{\Omega} \twoheadrightarrow \overline{\Omega'}$, Brouwer's theorem implies that

$$F(\partial\Omega) = \partial\Omega' \text{ and } F(\Omega) = \Omega'.$$

This observation underlines all of our constructions of homeomorphisms. It allows us to easily glue mappings with one another without explicitly checking what the image of a given homeomorphism looks like as it suffices to know its behaviour only on the boundary of its domain. We elaborate further on these key topological observations in Section 2.2.

The original proof due to Brouwer used Brouwer's fixed point theorem and some arguments from algebraic topology. Nowadays, there are more proofs available, for example using the *degree theory* for continuous mappings (but it is interesting to note that the degree itself was first developed by Brouwer). Degree theory, in a sense, serves as a link between analysis and topology. Even though we will not use degree theory explicitly in the proofs of our main results, we invoke it briefly in Appendix A.4.1 to define orientation preserving homeomorphisms and indicate some of their properties.

Only one of the main results, Theorem 1.5, concerns approximate differentiability. Nonetheless, this notion plays a crucial role in this dissertation and that is why we describe the definition here.

We say that a measurable mapping $f : \Omega \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ is *approximately differentiable at a point* $x \in \Omega$ if there is a measurable set E_x and a linear mapping $L(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{r\to 0^+} \frac{|E_x \cap B(x,r)|}{|B(x,r)|} = 1 \tag{1.1}$$

and

$$\lim_{E_x \ni y \to x} \frac{|f(x) - f(y) - L(x)\,(x-y)|}{|x-y|} = 0.$$

In (1.1), $|A|$ of a measurable set A denotes the Lebesgue measure of A. Condition (1.1) means that x is a point of density 1 of the set E_x. The linear mapping $L(x)$ is the approximate derivative and we denote it with $D_{\mathrm{a}} f(x) := L(x)$. Clearly, if a measurable mapping f is classically differentiable at a point x, it is also approximately differentiable at this point. The converse is not true as portrayed by the following

Example. Let D be the planar domain with an inner cusp

$$D = \left\{(x, y) :\, -1 < x < 1,\, -1 < y < \sqrt{|x|}\right\}$$

and $f : (-1, 1)^2 \to \mathbb{R}^2$ be defined as

$$f(x) = \begin{cases} x & \text{for } x \in D, \\ (0,0) & \text{for } x \in (0,1)^2 \setminus D. \end{cases}$$

The point $(0, 0)$ is the point of density 1 of the domain D and hence f is approximately differentiable at $(0, 0)$ with $D_{\mathrm{a}} f((0, 0)) = \mathcal{I}$, the identity matrix.

The notion of approximate derivative of real-valued functions appeared in 1916 in the works of Khintchine and, independently, of Denjoy. They both studied it mainly in the context of generalizing the definition of the integral. Khintchine in [60] introduced the definition of approximate differentiability *de facto* as we have done a few paragraphs earlier and he called it *dérivée asymptotique*. However, Denjoy in [26] used an equivalent definition and he named the property *dérivée approximative*, which in the end caught on. Moreover, many authors have quoted his as the original definition. The Denjoy-like definition in modern language can be phrased in the following manner. A mapping $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is approximately differentiable at a point $x \in \Omega$ if there is a linear mapping $L(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that x is a point of density 1 of the set

$$\left\{ y \in \Omega :\, \frac{|f(y) - f(x) - L(x)(y-x)|}{|y-x|} < \varepsilon \right\}.$$

The fact that both of these definitions are equivalent has been a folklore knowledge ever since their introduction, we refer to [37, Appendix] for a precise proof.

Nowadays, approximate differentiability is mostly used in the context of the change of variables theorem proved by Federer in [30] in 1944 in, see also [31, Section 3.2]. Moreover, approximate differentiability appears naturally in theorems concerned with the so-called Lusin-type approximation. We will elaborate on both of these aspects in a farther part of this introduction as well as in Section 2.9. Nonetheless, since both Sobolev and BV mappings are a. e. approximately differentiable, studying approximate differentiability offers also insight into properties of these two very important classes of mappings.

Some interest concerning homeomorphisms with a derivative was sparked by the theory of nonlinear elasticity developed by Ball in the 1980s for example in [6]; see also [7, 9]. Nonlinear elasticity is devoted to study deformations of materials. Ball's theory is based on assuming some kind of elastic energy of the form

$$\int_\Omega W(Df(x))\,dx$$

for the deformation f, belonging to an *admissible* class and equipped with derivative Df, and a function $W : M^{n\times n} \to \mathbb{R}$, where $M^{n\times n}$ denotes the space of $n \times n$ matrices. This function is usually assumed to be C^1 and non-negative. The prevailing idea in Ball's approach is to impose on W conditions which cause the (hopefully existing) minimizers of the elastic energy to be injective and/or orientation preserving. This approach rises questions concerning many topological and geometric properties of mappings and it is only natural to ask them in the context of homeomorphisms as this is a class of mappings which describes deformations well. Ball's theory inspired many influential ideas and theorems contained for example in [77, 54, 55, 24].

In this dissertation, in Chapter 2, we develop methods for constructing diffeomorphisms with certain differential and geometric properties. To do so, we combine topological arguments (mainly to guarantee global injectivity) with standard analytical methods, for example based on ordinary differential equations. I by no means claim to have invented these arguments. We do, nonetheless, put them together in a nontrivial way to obtain results of independ interest and often providing simpler proofs than those available in the literature. Also, we show that they are useful building blocks for at least two purposes—for extending and gluing homeomorphisms (in Chapter 3) as well as for prescribing derivatives of diffeomorphisms and homeomorphisms (in Chapters 4 and 5). We also hope that these methods could be put to use in Sobolev setting and in the field of structured deformations, see e. g. [25, 19, 74].

1.2 Overview of Chapter 3

The content of Chapter 3 contains paper [40] which is currently in preparation. It is joint work with Paweł Goldstein and Piotr Hajłasz and it concerns gluing of diffeomorphisms, bi-Lipschitz homeomorphisms and homeomorphisms.

It was a straightforward question about diffeomorphisms between subsets of $\mathbb{R}^n$ that prompted this part of our research. Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{R}^n$. We asked the following

Question. Given two orientation preserving diffeomorphisms $F, G : \mathbb{B}^n \to \mathbb{R}^n$ satisfying

$$F(\overline{B}(0, 1/2)) \subset G(\mathbb{B}^n), \tag{1.2}$$

does there exist a diffeomorphism $H : \mathbb{B}^n \to \mathbb{R}^n$ such that $H = F$ on $\overline{B}(0, 1/2)$ and $H = G$ near $\partial\mathbb{B}^n$?

Clearly, condition (1.2) is necessary and it turns out that it is also sufficient. This will follow as a corollary to the following

Theorem 1.1. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented manifold of class C^k, $k \in \mathbb{N} \cup \{\infty\}$. Suppose that $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of pairwise disjoint sets and that each D_i, $i = 1, \ldots, \ell$, is a C^k-diffeomorphic closed ball. If $F_i : D_i \twoheadrightarrow D_i'$, $i = 1, 2, \ldots, \ell$, are orientation preserving C^k-diffeomorphisms, then there is a C^k-diffeomorphism $F : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $F|_{D_i} = F_i$. Moreover, if D_i and D_i' for all $i = 1, \ldots, \ell$ are contained in the interior of a C^k-diffeomorphic closed ball $K \subset \mathcal{M}^n$, F can be chosen to equal identity outside K.*

Note that we do not require balls in the family $\{D_i\}_{i=1}^{\ell}$ to be disjoint from the balls in the family $\{D_i'\}_{i=1}^{\ell}$. By a diffeomorphism F defined on a closed set, we mean a mapping which can be extended to a diffeomorphism of a neighborhood of that closed set. Even though we do know that each F_i can be extended as a diffeomorphism onto some neighborhood of D_i, it is not clear why these extensions can be glued into a global one, i. e., why an extension F should exist.

We will see in Example 3.12 that it follows from Milnor's seminal paper [69] that there is a diffeomorphism $F : A \to \mathbb{S}^7$ defined on a standard closed annulus $A \subset \mathbb{S}^7$, which cannot be extended to a diffeomorphism of $\mathbb{S}^7$. Due to the phenomenon of exotic spheres (discovered and introduced in the cited paper of Milnor), we need to assume that D_i and D_i' are *C^k-diffeomorphic closed balls*. We say that $D \subset \mathcal{M}^n$ is a C^k-diffeomorphic closed ball if there is a C^k-diffeomorphism $\Phi : \overline{\mathbb{B}}^n \twoheadrightarrow D$.

We firstly show validity of Theorem 1.1 in the Euclidean setting and, to this end, we use explicit constructions of specific diffeomorphisms described in Section 2.7. Then, a standard argument allows us to transfer the Euclidean version to manifolds. The cornerstone of the proof of Theorem 1.1 is a beautiful idea of Palais [83]. In a nutshell, it says that an orientation preserving diffeomorphism defined on a diffeomorphic closed ball $D \subset \mathcal{M}^n$ can be extended to a diffeomorphism of the entire $\mathcal{M}^n$ which equals identity *away* from D.

Let us make the following observation in the Euclidean setting: a C^k-diffeomorphism $F : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ can be easily extended radially to a C^k-diffeomorphism $\widetilde{F} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$. However, it is not that immediate to see that a C^k-diffeomorphism $F : B(0,\varrho) \to \mathbb{R}^n$ (let us stress: F might not be *onto* the ball $B(0,\varrho)$) can be extended to a C^k-diffeomorphism of the entire $\mathbb{R}^n$. Palais' trick does this and more, as it produces an extension which equals identity outside an arbitrarily small neighborhood of $F(\overline{B}(0,\varrho)) \cup \overline{B}(0,\varrho)$. Moreover, in the diffeomorphic setting it is a fairly straightforward procedure, which provides a striking contrast to the bi-Lipschitz and purely topological case, which we will discuss now.

We say that a mapping $f : X \to Y$ between metric spaces (X, d) and (Y, ϱ) is bi-Lipschitz if there is a constant $L > 0$ such that for any $p, q \in X$,

$$L^{-1}\, d(p,q) \le \varrho(f(p), f(q)) \le L\, d(p,q).$$

Clearly, any such map is a homeomorphism. Given an n-dimensional Lipschitz manifold $\mathcal{M}^n$, we say that a closed set $D \subset \mathcal{M}^n$ is a *flat bi-Lipschitz closed ball* if it is possible to find a bi-Lipschitz homeomorphism $\Phi : \overline{\mathbb{B}}^n \twoheadrightarrow D$ which can be extended as a bi-Lipschitz homeomorphism onto a neighborhood of $\overline{\mathbb{B}}^n$. It is important to note that the assumption on bi-Lipschitz flatness is restrictive since not every bi-Lipschitz closed ball is locally flat as the example of the Fox-Artin ball shows, see [35] and [65, Theorem 3.7].

All in all, we prove

Theorem 1.2. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented Lipschitz manifold. Suppose $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of pairwise disjoint flat bi-Lipschitz closed balls. If $F_i : D_i \twoheadrightarrow D_i'$, $i = 1, 2, \ldots, \ell$, are orientation preserving bi-Lipschitz homeomorphisms, then there is a bi-Lipschitz homeomorphism $F : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$*

such that $F|_{D_i} = F_i$. *Moreover, if* D_i *and* D'_i *for all* $i = 1, \dots, \ell$ *are contained in the interior of a flat bi-Lipschitz closed ball* $K \subset \mathcal{M}^n$, F *can be chosen to equal identity outside* K.

This result is significantly deeper than Theorem 1.1 as it depends on the *bi-Lipschitz stable homeomorphism conjecture*, which is a very difficult result of Sullivan [89] from 1979, see Section 3.4 for details.

We also prove an analogue of Theorems 1.1 and 1.2 for homeomorphisms. Given an n-dimensional topological manifold $\mathcal{M}^n$, we say that a closed set $D \subset \mathcal{M}^n$ is a *flat topological closed ball* if there is a homeomorphism $f : \bar{\mathbb{B}}^n \twoheadrightarrow D$ which can be extended as a homeomorphism on a neighborhood of $\bar{\mathbb{B}}^n$. Alexander's horned ball (the compact set bounded by Alexander's horned sphere) shows that not every topological closed ball is flat, see [2] for the original argument and [44, Example 2B.2] for a modern treatment.

Theorem 1.3. *Let* $\mathcal{M}^n$ *be an* n*-dimensional connected and oriented topological manifold. Suppose* $\{D_i\}_{i=1}^{\ell}$ *and* $\{D'_i\}_{i=1}^{\ell}$, $D_i, D'_i \subset \mathcal{M}^n$, *are two families of pairwise disjoint flat topological closed balls. If* $F_i : D_i \twoheadrightarrow D'_i$, $i = 1, 2, \dots, \ell$, *are orientation preserving homeomorphisms, then there is a homeomorphism* $F : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ *such that* $F|_{D_i} = F_i$. *Moreover, if* D_i *and* D'_i *for all* $i = 1, \dots, \ell$ *are contained in the interior of a flat topological closed ball* $K \subset \mathcal{M}^n$, F *can be chosen to equal identity outside* K.

The proof of Theorem 1.3 is essentially the same as the proof of Theorem 1.2; instead of the bi-Lipschitz stable homeomorphism conjecture, we invoke the stable homeomorphism conjecture. This is also a very difficult result. In dimension $n = 2$ it is due to Radó [86], in $n = 3$ due to Moise [71]. In 1969, Kirby showed it for $n > 4$ in [61]. The remaining case, $n = 4$, was solved by Quinn in [85] in 1982 (see page 1 and Theorem 2.2.2.). We recommend [45] to anyone interested in the details of the rather complicated history of the proof.

We suppose that Theorems 1.1, 1.2 and 1.3 might be useful in differential geometry and geometric function theory on manifolds. They provide a vital piece of information that diffeomorphisms or (bi-Lipschitz) homeomorphisms on *good* domains (diffeomorphic closed balls or flat (bi-Lipschitz) balls) behave *well*. Moreover, as a large part of the content of Chapter 3 hinges on deep theorems that have been known for quite a long time, I believe that these results might also draw attention to useful topological notions that have not yet been fully exploited from the point of view of analysis.

1.3 Overview of Chapters 4 and 5

A now classical result of Alberti [1, Theorem 1] states that given a measurable mapping $T : \Omega \to \mathbb{R}^n$, for any $\varepsilon > 0$, there is a function $\phi \in C^1_c(\Omega)$ and a compact set $K \subset \Omega$ such that

$$|\Omega \setminus K| < \varepsilon \;\text{ and }\; D\phi(x) = T(x) \;\text{ for all } x \in K.$$

This theorem is usually called *Lusin-type theorem for gradients* because of the resemblance to the classical Lusin theorem. Alberti's proof of this theorem uses mainly standard real analysis tools but does it in a very ingenious manner, it is constructive and hence it allows us to construct C^1 functions with prescribed derivative on an arbitrarily large set (in measure).

We were interested in learning what conditions on T suffice to construct a diffeomorphism with derivative equal T on a set of arbitrarily large measure. In other words, we wanted to know for which measurable mappings T, a diffeomorphic version of Alberti's theorem holds. The answer lies in the theorem below, where with $GL(n)^+$ we denote the space of real $n \times n$ matrices with positive determinant.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $F : \Omega \to \mathbb{R}^n$ an orientation preserving diffeomorphism onto the bounded image $F(\Omega)$. Suppose that $T : \Omega \to GL(n)^+$ is a measurable mapping such that $\int_\Omega \det T(x)\,dx \le |F(\Omega)|$. Then for any $\varepsilon > 0$, there exists a C^1-diffeomorphism $\Phi : \Omega \to F(\Omega)$ with the following properties:*

(a) $\Phi(x) = F(x)$ *near* $\partial\Omega$;
(b) there exists a compact set $K \subset \Omega$ such that for every $x \in K$, $D\Phi(x) = T(x)$ and $|\Omega \setminus K| < \varepsilon$.

Observe that the assumption $\int_\Omega \det T(x)\,dx \le |F(\Omega)|$ is necessary. Indeed, it follows from the already discussed Brouwer's invariance of domain theorem, that $\Phi(\Omega) = F(\Omega)$ and by the classical change of variables formula for any compact set $K \subset \Omega$,

$$|F(\Omega)| = |\Phi(\Omega)| = \int_\Omega \det D\Phi(x)\,dx > \int_K \det T(x)\,dx. \tag{1.3}$$

If $\int_\Omega \det T(x)\,dx > |F(\Omega)|$, then for sufficiently small $\varepsilon > 0$, we would find K such that

$$\int_K \det T(x)\,dx > |F(\Omega)|,$$

which in view of (1.3) leads to a contradiction.

Alberti's theorem has not only been an inspiration to the theorem, it also plays a key role in its proof. Nonetheless, the proof is rather involved and we also use Dacorogna-Moser theory of prescribing Jacobians, see for example [22] and [20], some topological arguments of Munkres [78] and explicit constructions of diffeomorphisms from Chapter 2.

The next main result discusses prescribing a derivative of an almost everywhere approximately differentiable homoemorphism. It is the most complex theorem of this thesis. With $GL(n)$ we denote the space of real $n \times n$ invertible matrices.

Theorem 1.5. *Let $\mathcal{Q} = [0,1]^n$. For any measurable mapping $T : \mathcal{Q} \to GL(n)$ that satisfies*

$$\int_{\mathcal{Q}} |\det T(x)|\,dx = 1, \tag{1.4}$$

there exists an a. e. approximately differentiable homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ such that $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$ and $D_{\mathrm{a}}\Phi = T$ a. e. Moreover,

(a) Φ^{-1} is approximately differentiable a. e. and $D_{\mathrm{a}}\Phi^{-1}(y) = T^{-1}(\Phi^{-1}(y))$ for almost all $y \in \mathcal{Q}$;
(b) Φ preserves the sets of measure zero, i.e., for any $A \subset \mathcal{Q}$,

$$|A| = 0 \qquad \textit{if and only if} \qquad |\Phi(A)| = 0.$$

(c) Φ is a limit of C^∞-diffeomorphisms $\Phi_k : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi_k = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$, in the uniform metric, i.e., $\|\Phi - \Phi_k\|_\infty + \|\Phi^{-1} - \Phi_k^{-1}\|_\infty \to 0$ as $k \to \infty$.

Loosely speaking, Theorem 1.5 shows that there are essentially no restrictions on derivatives of a. e. approximately differentiable homeomorphisms. Measurablility of T is clearly necessary and so is the volume constraint (1.4). This follows from the change of variables formula proved by Federer in [30], see Theorem 2.29. We say that a mapping F satisfies *the Lusin (N) condition* if F maps sets of measure zero onto sets of measure zero. This is an important property of mappings from both a physical point of view (it says that F does not 'create' matter out of nowhere) and a theoretical one. In particular, Federer's theorem states that if $F:\Omega\to\mathbb{R}^n$ is a homeomorphism which satisfies the Lusin (N) condition, then for any measurable set $E\subset\Omega$,

$$|F(E)| = \int_E |\det D_{\mathrm{a}}F(x)|\,dx.$$

Since the homeomorphism Φ in Theorem 1.5 does satisfy the Lusin (N) condition and maps the unit cube onto itself, it must satisfy (1.4).

The homeomorphism Φ from Theorem 1.5 is a uniform limit of diffeomorphisms, which is quite surprising. On the other hand, a. e. approximately differentiable mapping arise naturally as limits of C^1 mappings in another metric, which we describe now. Let us consider $m,n\in\mathbb{N}$, a measurable set $E\subset\mathbb{R}^n$ and mappings $f,g:E\to\mathbb{R}^m$ and identify those which are equal a. e. Then, we introduce the *Lusin metric* defined as

$$d_L(f,g) := |\{x\in\Omega:\ f(x)\neq g(x)\}|. \tag{1.5}$$

This indeed is a metric and the space of measurable mappings is complete with respect to it[1]. What is more, Lusin's theorem implies that continuous functions are dense in this space. If E is an open set in $\mathbb{R}^n$, then it follows from Whitney's theorem (see Lemma 2.26) that the closure of $C^1(E,\mathbb{R}^m)$ in metric d_L is the class of mappings which are approximately differentiable a. e. on E. Loosely speaking, the relation of the class of a. e. approximately differentiable mappings with respect to C^1-mappings is the same as the relation of measurable functions with respect to continuous ones.

Theorem 1.5 provides a positive answer to a conjecture of Goldstein and Hajłasz from [37]. In that same paper, in Theorem 1.4, they showed that there exists an a. e. approximately differentiable homeomorphism $\Phi:\mathcal{Q}\twoheadrightarrow\mathcal{Q}$ such that $\Phi|_{\partial\mathcal{Q}}=\mathrm{id}$ and

$$D_{\mathrm{a}}\Phi = \begin{bmatrix} 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0\\ 0 & 0 & \dots & 0 & -1\end{bmatrix} \quad \text{a. e.} \tag{1.6}$$

What is more, Φ is constructed as a limit in the uniform metric of measure preserving C^∞-diffeomorphisms $\Phi_k:\mathcal{Q}\twoheadrightarrow\mathcal{Q}$, $\Phi_k|_{\partial\mathcal{Q}}=\mathrm{id}$. The proof of this theorem is constructive and it uses some of the tools that we also employ in the proof of Theorem 1.4, most notably the Dacorogna-Moser theory of mappings with prescribed Jacobians. Nevertheless, the proof of Theorem 1.5 is fundamentally different than the proof of [37, Theorem 1.4], mainly because in Theorem 1.5 the prescribed derivative $T(x)$ varies with the point x. This was also the primary difficulty to overcome in the proof of Theorem 1.5.

A standard argument from the degree theory implies that homeomorphism Φ from (1.6) is orientation preserving[2] and yet its approximate Jacobian equals -1 a. e. This shows that there is no link whatsoever between the sign of the Jacobian of an a. e. approximately differentiable homeomorphism and the topological property of whether it preserves or reverses orientation. This example also answered one of the questions Hajłasz posed in 2001 about signs of Jacobians of a. e. approximately differentiable and Sobolev homeomorphisms, see [47, Section 5.4] and [37].

[1]We prove it in the Appendix, in Lemma A.1

[2]We comment on it in Section A.4.1 in the Appendix. The same argument also shows that Φ from Theorem 1.5 is orientation preserving.

There is a vast literature on Jacobians of Sobolev homeomorphisms and below we shall name a few results. In 2010 in [48] Hencl and Malý proved that in dimensions $n = 2, 3$ for $p \geq 1$ and for $n \geq 4$ and $p > [n/2]$, the Jacobian of Sobolev homeomorphism cannot change sign. In the proof, they used a topological concept called the linking number. On the other hand, in [50] Hencl and Vejnar showed an example of a $W^{1,1}\big((0,1)^4,(0,1)^4\big)$ homeomorphism whose Jacobian changes sign. This also showed that not every Sobolev homeomorphism can be approximated by diffeomorphisms in the Sobolev norm, which is the so-called *Ball-Evans approximation question*, see [8, p. 8] for the original question and [57, 56, 73, 49] for the interesting answers the question has already raised. The construction of Hencl and Vejnar was later improved in [16] and [17] to provide further counterexamples. All three of these papers relied on an ingenious use of homeomorphisms mapping certain Cantor sets onto another Cantor sets. The question of whether Jacobian of a Sobolev homeomorphism can change sign is still open in the case when $n \geq 4$ and $p = [n/2]$, see [38] for some partial results in this direction.

One should also be aware of the existence of such pathological examples as a homeomorphism $f \in W^{1,p}\left((0,1)^n,(0,1)^n\right)$ for $1 \leq p < n$ with $J_f = 0$ a. e. [46]. An even more surprising example was published in [27]: a homeomorphism $f \in W^{1,1}\left((0,1)^n,(0,1)^n\right)$ for $n \geq 3$ with zero Jacobian a. e. and whose inverse f^{-1} is in $W^{1,1}$ as well and also has zero Jacobian a. e.

Even though the quoted results in the Sobolev case bear little resemblance to Theorems 1.4 and 1.5, they provide great insight into the interplay between topological an geometric properties and weaker than classical notions of differentiability. It is specific to the area to mix some topological tools (like degree theory) with analytical ones and to perform iterative constructions. Also, these results show how complex the behavior of homeomorphisms might be and, consequently, that it is somewhat surprising that such a general result as Theorem 1.5 holds.

To the best of my knowledge, Theorem 1.5 is the first result in which the whole derivative is prescribed. However, there have been many papers discussing diffeomorphisms and homeomorphisms with prescribed Jacobians, beginning with a seminal Oxtoby and Ulam paper [81]. Theorem 2 of that paper says that[3]

Theorem. *Let μ be a Borel measure μ on the cube $\mathcal{Q} = [0,1]^n$ such that $\mu(\mathcal{Q}) = 1$, $\mu(\{x\}) = 0$ for all $x \in \mathcal{Q}$, $\mu(\partial\mathcal{Q}) = 0$ and $\mu(U) > 0$ for any non-empty open set $U \subset \mathcal{Q}$. Then there exists a homeomorphism $h : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $h|_{\partial\mathcal{Q}} = \mathrm{id}$, such that for any Borel set $E \subset \mathcal{Q}$,*

$$\mu(E) = |h(E)|. \tag{1.7}$$

Moreover, if such a homeomorphism h should exist, then μ needs to satisfy the conditions above.

This theorem was later generalized to cover the case of the Hilbert cube $\Pi_{i=1}^{\infty}[0,1]$ in [80] or σ-compact manifolds in [11], see also [3, Section A2.2] or [36, Chapter 7] for a modern treatment and references therein and [43] for a nice explanation of what is so surprising about Oxtoby and Ulam's result.

In [76, 22] Moser and Dacorogna and Moser proved a series of theorems about existence of diffeomorphisms with prescribed Jacobian, which we have already mentioned in this Introduction and we will discuss in more detail in Chapter 4. One of their proofs is the so-called *flow method*, a very elegant approach based on ordinary differential equations.

[3]The theorem was first stated and proved by John von Neumann, but the proof remains unpublished, see [3, p. 188]

We recommend [20, Chapter 10] for an excellent modern overview of the theory. Let us also quote [15, 68], which show the sharpness of assumptions appearing in the theorems of Dacorogna and Moser. Other notable contributions to the area include [10, 100, 87] and a recent paper [41]. The main result of [41] shows that, in general, given $f \in L^p$, $p > 1$, one cannot hope for a solution to $\det D\Phi = f$ for a Sobolev homeomorphism $\Phi \in W^{1,np}$. Nonetheless, by weakening the regularity of Φ, Theorem 1.5 allows us to find a homeomorphism with a prescribed full derivative, not only its determinant.

The proof of Theorem 1.5 is long and constructive. We use Theorem 1.4 and a certain iteration scheme which is inspired by the quoted Oxtoby-Ulam theorem from [81] and provides a considerable modification of their methods. Also, we employ the homeomorphism (1.6) from [37] to reduce the proof of Theorem 1.5 to its version with the additional assumption that $\det T > 0$ a. e.

Since approximating arbitrary mappings with smooth ones is ubiquitous in analysis, we believe that Theorem 1.4 might be useful in constructing other interesting homeomorphisms, possibly also in Sobolev setting. On the other hand, Theorem 1.5 provides a characterization of the class of a. e. approximately differentiable homeomorphisms. Supposedly, it gives little hope for using this class in nonlinear elasticity as its differential properties seem too wild. However, this is a very elegant result from theoretical point of view.

1.4 Structure of the thesis

Whenever I write that a given excerpt of the thesis is from either [39] or [40], I mean that it is copied from given article with possible adjustments for a better fit to the rest of the dissertation. I use the convention that well-known theorems quoted outside Chapter 2 or introductions to any of the Chapters 3, 4 or 5 are quoted as lemmata or propositions.

Chapter 2 contains explanation of notation and preliminary results. We collect there facts which are considered folklore but it is difficult to find a suitable reference for them. In Section 2.7, we develop methods for constructing diffeomorphisms used repetitively in the sequel. Some parts of this chapter, in particular Section 2.7, are from [39].

In Chapter 3, Theorems 1.1, 1.2 and 1.3 are proved. We follow [40].

In Chapter 4, we provide the proof of Theorem 1.4 from [39]. Moreover, we add two corollaries of Theorem 1.4, which are not included in the quoted preprint as the paper is already long. They are stated in Section 4.1.

In Chapter 5, we provide the proof of Theorem 1.5 from [39]. Similarly, as in case of Theorem 1.4, we add a few corollaries for which there was no place in [39]. They are stated in Sections 5.1 and 5.5. Also, in Section 5.1, we state two interesting open questions.

At the end of the thesis, we include the Appendix, in which we supply some additional lemmata and proofs. In particular, in Section A.2 we explain the notion of 1-parameter groups of diffeomorphisms and in Section A.4.1, we clarify the notion of orientation preserving homeomorphisms which appears in Chapter 3.

1.5 Collaboration statement

This thesis contains results obtained in collaboration with my supervisor, Paweł Goldstein (University of Warsaw) and Piotr Hajłasz (University of Pittsburgh). They are contained in two yet unpublished papers [39] and [40]. This was very much a *joint* work and I took an active part in discussing ideas, checking if they work and in the process of writing them down. Nonetheless, each of us had some more independent contributions and below I single out mine.

In particular, I posed the question (1.2), which prompted research contained in Chapter 3 ([40]). I was responsible for collecting and understanding the available literature on stable homeomorphisms and annulus theorems (both in the locally bi-Lipschitz and topological setting) and for observing and checking that the bi-Lipschitz and topological cases (i. e., Theorems 1.2 and 1.3) can be treated with the same methods.

My contributions included also stating and proving Theorem 1.4 on the basis of Piotr's idea of how to use Alberti's theorem. Moreover, I found an important component of the iteration scheme in the proof of Theorem 1.5 and proved a series of lemmata adapting a certain idea of Paweł to work in our setting (this became a part of Section 5.2 of this thesis). Also, I observed that property (a) in Theorem 1.5 holds.

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Chapter 2

Preliminaries and introductory results

In this chapter, we collect notation used throughout the thesis, present well-known theorems and prove technical results we will use in the sequel. Moreover, we state a few related results that are folklore knowledge but are hard to find in the literature.

Sections 2.4–2.7 are taken from Sections 3.1–3.4 from [39] with the exception of [39, Lemma 3.11] and [39, Proposition 3.14] as they are specific to the proof of Theorem 1.4 (they appear here in Section 4.2). Section 2.7 contains an additional Lemma 2.17. Section 2.8 is Section 3.6 from [39]. Section 2.9 contains a part of [39, Section 6] (the rest appears here in Section 5.3).

Sections 2.2 and 2.3 are new and added to keep the dissertation self-contained.

2.1 General notation

In this section, notation used throughout the thesis is explained. For the convenience of the reader, we shall recall certain notions later or introduce them only when they are needed.

Throughout the thesis we assume that $n \geq 2$. Should there be an exception to this rule, it will be duly noted.

$\mathbb{N}$ and $\mathbb{Z}$ denote the sets of positive and all integers, respectively.

$\mathcal{M}^n$ stands for an n-dimensional topological oriented manifold.

Ω always denotes an open subset of $\mathbb{R}^n$ or of an n-dimensional manifold $\mathcal{M}^n$. By a domain we mean an open and connected set.

Open balls in $\mathbb{R}^n$ are denoted with $B(x,r)$. Sometimes we write $B^n(x,r)$ to stress the dimension. The unit ball and sphere in $\mathbb{R}^n$ are denoted with $\mathbb{B}^n := B^n(0,1)$ and $\mathbb{S}^{n-1} := \partial\mathbb{B}^n$, respectively.

$\mathcal{Q}$ denotes the closed unit cube $\mathcal{Q} = [0,1]^n$. If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define $\|x\|_\infty = \max_i |x_i|$. $\mathring{Q}(p,r) = \{x : \|x-p\|_\infty < r\}$ and $Q(p,r) = \{x : \|x-p\|_\infty \leq r\}$ denote the open and the closed cube centered at p of side-length $2r$. We denote with $\mathsf{L}(Q)$ the side-length of cube Q.

Given topological spaces X, Y, by a homeomorphism $\Phi : X \to Y$ we always mean a homeomorphism onto the image, i. e., between X and $\Phi(X)$. Some authors call such mappings embeddings.

Surjective mappings are denoted by $F : X \twoheadrightarrow Y$, i. e., $F(X) = Y$.

If $E \subset \mathbb{R}^n$, we call $f : E \to \mathbb{R}$ a function and $f : E \to \mathbb{R}^m$ for $m = 2, 3, \dots$ a mapping.

Let $k \in \mathbb{N} \cup \{\infty\}$. If $f : \Omega \to \mathbb{R}$ is a k-times continuously differentiable function, we write $f \in C^k(\Omega)$. If $f : \Omega \to \mathbb{R}^n$ is a k-times continuously differentiable mapping, we write $f \in C^k(\Omega, \mathbb{R}^n)$. If $f : \Omega \to \mathbb{R}$ is a k-times continuously differentiable function with compact support, we write $f \in C^k_c(\Omega)$. If $f : \Omega \to \mathbb{R}^n$ is a k-times continuously differentiable mapping with compact support, we write $f \in C^k_c(\Omega, \mathbb{R}^n)$.

If Ω is an open subset of $\mathcal{M}^n$ (where $\mathcal{M}^n$ is of class C^k, $k \in \mathbb{N} \cup \{\infty\}$), we use $C^k(\Omega, \mathcal{M}^n)$, $C^k_c(\Omega, \mathcal{M}^n)$ for $\mathcal{M}^n$ in place of $\mathbb{R}^n$.

If $\Omega \subset \mathbb{R}^n$ or $\Omega \subset \mathcal{M}^n$ is open, then by a diffeomorphism $\Phi : \Omega \to \mathbb{R}^n$ (or $\Phi : \Omega \to \mathcal{M}^n$) we mean a diffeomorphism onto the image, i.e., diffeomorphism between Ω and $\Phi(\Omega)$. We say that $\Phi : \Omega \to \mathbb{R}^n$ (or $\Phi : \Omega \to \mathcal{M}^n$) is a C^k-diffeomorphism provided that Φ is a diffeomorphism and $\Phi, \Phi^{-1} \in C^k(\Omega, \mathbb{R}^n)$ (or when $\Phi, \Phi^{-1} \in C^k(\Omega, \mathcal{M}^n)$).

A diffeomorphism of a closed set A is a mapping that extends to a diffeomorphism of a neighborhood of A.

$C(\Omega)$ denotes the space of continuous functions on Ω and $C(\mathcal{Q}, \mathcal{Q})$ denotes the space of continuous mappings defined on the unit cube $\mathcal{Q}$ with values in $\mathcal{Q}$.

Approximate derivative is denoted by $D_{\mathrm{a}} f(x)$. Given a mapping $\Phi : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, we denote its Jacobian by $\det D\Phi(x)$ or by $J_\Phi(x)$.

We say that a mapping $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, satisfies the *Lusin (N) condition* if it maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. Lipschitz mappings satisfy the Lusin (N) condition.

Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$.

The space of homeomorphisms of the unit cube $\mathcal{Q}$ is equipped with the *uniform metric*

$$d(\Phi, \Psi) = \sup_{x \in \mathcal{Q}} |\Phi(x) - \Psi(x)| + \sup_{x \in \mathcal{Q}} |\Phi^{-1}(x) - \Psi^{-1}(x)|,$$

see Section 2.8 for more details.

Let $m, n \in \mathbb{N}$. The space of measurable mappings $f, g : E \to \mathbb{R}^n$ defined on a measurable set $E \subset \mathbb{R}^m$ is equipped with the *Lusin metric* defined as

$$d_L(f, g) := |\{x \in \Omega : \ f(x) \neq g(x)\}|. \tag{2.1}$$

This becomes a metric if we identify mappings which are equal a. e.

If $E \subset \mathbb{R}^n$ is measurable, we say that $x \in \mathbb{R}^n$ is a *density point* of E provided that $|B(x, r) \cap E| / |B(x, r)| \to 1$ as $r \to 0^+$. According to the Lebesgue differentiation theorem, almost all points $x \in E$ are density points of E.

The interior and the closure of a set A is denoted by $\mathring{A}$ and $\overline{A}$. Boundary of the set is denoted by ∂A. We write $A \Subset B$ if $\overline{A}$ is a compact subset of $\mathring{B}$.

Given a set $E \subset \mathbb{R}^n$, $\operatorname{diam} E$ denotes the diameter of E, i. e., $\operatorname{diam} E = \sup_{x,y \in E} |x - y|$.

Symmetric difference of sets A and B is $A \vartriangle B = (A \setminus B) \cup (B \setminus A)$.

The space of real $n \times n$ matrices, invertible matrices and matrices with positive determinant are denoted by $M^{n \times n}$, $GL(n)$, and $GL(n)^+$, respectively. The identity matrix is denoted by $\mathcal{I}$. The operator and the Hilbert-Schmidt norms of $A \in M^{n \times n}$ are denoted by $\|A\|$ and $|A|$, respectively. It is easy to see that $\|A\| \leq |A|$.

The tensor product of vectors $u, v \in \mathbb{R}^n$ is the matrix $u \otimes v = [u_i v_j]_{i,j=1}^n \in M^{n \times n}$.

2.2 Brouwer's theorem

Theorem 2.1 (Invariance of domain)**.** *Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^n$ be a continuous, injective mapping. Then $f(U)$ is an open set.*

As we discussed in the Introduction, this theorem is of paramount importance in the sequel. For the proof, see e.g. [32, Theorem 3.30]. If one assumes f to be a diffeomorphism, the proof that $f(U)$ is open is significantly easier and it essentially follows from inverse function theorem, see [93, Theorem 22.3].

Corollary 2.2. *Let Ω be a domain in $\mathbb{R}^n$. If $f : \overline{\Omega} \to \mathbb{R}^n$ is a homeomorphism, then, for any closed set $K \subset \Omega$,*

$$f(\overset{\circ}{K}) = \overset{\circ}{f(K)} \text{ and } f(\partial K) = \partial f(K). \tag{2.2}$$

Proof. By continuity of f, $f^{-1}(\overset{\circ}{f(K)})$ is open and therefore it must be contained in $\overset{\circ}{K}$. This implies that $\overset{\circ}{f(K)} \subset f(\overset{\circ}{K})$. On the other hand, Theorem 2.1 implies that $f(\overset{\circ}{K})$ is open and hence $f(\overset{\circ}{K}) \subset \overset{\circ}{f(K)}$. This yields the first equality in (2.2).

Consequently,

$$f(K) = f(\partial K) \cup f(\overset{\circ}{K}) = f(\partial K) \cup \overset{\circ}{f(K)},$$

from which it follows that $f(\partial K) = f(K) \setminus \overset{\circ}{f(K)}$, which finishes the proof of this corollary. □

In Chapter 3, where we deal with manifolds, we shall need also a version of invariance of domain theorem for mappings between Euclidean sets and manifolds.

Corollary 2.3. *Let $\mathcal{M}^n$ be an n-dimensional topological manifold and let $\Phi : \overline{\mathbb{B}}^n \twoheadrightarrow D$, $D \subset \mathcal{M}^n$, be a homeomorphism. Then*

$$\Phi(\mathbb{B}^n) = \overset{\circ}{D} \quad \textit{and} \quad \Phi(\partial \mathbb{B}^n) = \partial D, \tag{2.3}$$

where ∂D and $\overset{\circ}{D}$ denote the boundary and interior of D, respectively, in the topology of $\mathcal{M}^n$. In particular, $\Phi(\mathbb{B}^n)$ is open in the topology of $\mathcal{M}^n$.

Proof. It suffices to show that $\Phi(\mathbb{B}^n)$ is open in the topology of $\mathcal{M}^n$. Indeed, it then follows from a similar reasoning as in Corollary 2.2 that (2.3) holds.

Take any $x \in \Phi(\mathbb{B}^n)$, we shall show that there is an open (in the topology of $\mathcal{M}^n$) neighborhood of x contained in $\Phi(\mathbb{B}^n)$. There is an open (in $\mathcal{M}^n$) neighborhood U containing x which is homeomorphic to $\mathbb{R}^n$, i. e., a chart (U, ψ). Then the set $U \cap \Phi(\mathbb{B}^n)$ is open in the subspace topology of $\Phi(\mathbb{B}^n)$. Since Φ is continuous, $\Phi^{-1}(U \cap \Phi(\mathbb{B}^n))$ is open in $\mathbb{R}^n$. The mapping

$$\psi \circ \Phi : \Phi^{-1}(U \cap \Phi(\mathbb{B}^n)) \to \mathbb{R}^n$$

is continuous and injective. By Theorem 2.1, $\psi(U \cap \Phi(\mathbb{B}^n))$ is an open set in $\mathbb{R}^n$ which contains $\psi(x)$. We choose a neighborhood V of $\psi(x)$ which is contained in $\psi(U \cap \Phi(\mathbb{B}^n))$. Then the set $\psi^{-1}(V)$ is an open (in the topology of $\mathcal{M}^n$) neighborhood of x contained in $\Phi(\mathbb{B}^n)$. This concludes the proof. □

A similar argument allows one to formulate a version of Theorem 2.1 for homeomorphisms between manifolds, see [44, Corollary 2B.4]. Also, note that D as in Corollary 2.3 is closed in the topology of $\mathcal{M}^n$. It is a continuous image of a compact set and hence compact. Since a compact set in a Hausdorff space (and $\mathcal{M}^n$ is Hausdorff) is closed, D is closed.

2.3 Measurability

We note here two basic observations about homeomorphisms and measurable sets. We shall use them repetitively in the sequel and by stating and proving them, I want to stress their importance.

Lemma 2.4. *Let $U \subset \mathbb{R}^n$ be open and let $\Phi : U \to \mathbb{R}^n$ be a homeomorphism. Then Φ preserves Borel sets, i. e., $E \subset U$ is Borel if and only if $\Phi(E)$ is Borel. Moreover, Φ maps measurable sets onto measurable sets if and only if Φ satisfies the Lusin (N) condition.*

Proof. For any sets $A, E \subset U$, $\Phi(A \setminus E) = \Phi(A) \setminus \Phi(E)$ and for any countable family of sets $A_i \subset U$, $\Phi(\bigcup_i A_i) = \bigcup_i \Phi(A_i)$. Since the Borel σ-field is generated by open sets under these set operations and a homeomorphic image of an open set is open by Theorem 2.1, Φ preserves Borel sets.

Any measurable set is a sum of a Borel set and a set of measure zero. Therefore, if Φ satisfies the Lusin (N) condition, then it maps the Borel set onto the Borel one and the set of measure zero onto a set of measure zero. Thus, its sum is measurable. On the other hand, if Φ does not satisfy the Lusin (N) condition, there is a set Z of measure zero, $Z \subset U$, which is mapped onto a set of positive measure. It is well known that any set of positive measure contains a non-measurable set as its subset[1]. Therefore, there is a non-measurable subset $W \subset \Phi(Z)$, whose preimage $\Phi^{-1}(W)$ is contained in a set of measure zero and hence is measurable but its image is not. This concludes the proof. $\square$

2.4 Linear algebra

Lemma 2.5. *If $A \in GL(n)$ and $\|A - B\| < \|A^{-1}\|^{-1}$, then $B \in GL(n)$.*

Proof. Under the given assumptions $\|A^{-1}B - \mathcal{I}\| < 1$ and hence $A^{-1}B$ is invertible (the inverse can be written as an absolutely convergent power series). $\square$

Lemma 2.6. *Assume $T \in GL(n)^+$. Then for any $\varepsilon > 0$, there exists a finite family of matrices $\{A_i\}_{i=1}^{M_\varepsilon} \subset GL(n)^+$ such that $\|A_i - \mathcal{I}\| < \varepsilon$ for each $i = 1, \dots, M_\varepsilon$, and*

$$T = A_1 \cdot \ldots \cdot A_{M_\varepsilon}.$$

Proof. $GL(n)^+$ is a Lie group with respect to the matrix multiplication. According to [96, Theorem 3.68], $GL(n)^+$ is connected. Then the result follows from [96, Proposition 3.18] which says that if U is a neighborhood of the identity element in a connected Lie group G, then any element $g \in G$ can be represented as $g = u_1 \cdot \ldots \cdot u_k$ for some k and $u_1, \dots, u_k \in U$. $\square$

Remark 2.7. Recall that the tensor product of vectors $u, v \in \mathbb{R}^n$ is the matrix $u \otimes v = [u_i v_j]_{i,j=1}^n \in M^{n\times n}$. It is easy to see that $\|u \otimes v\| = |u \otimes v| = |u|\,|v|$. Note that if $U \subset \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^n$, $\eta : U \to \mathbb{R}$ are differentiable, then

$$D(\eta F) = F \otimes D\eta + \eta DF, \quad \text{so} \quad \|D(\eta F)\| \le |D\eta|\,|F| + |\eta|\,\|DF\|. \tag{2.4}$$

[1] For a proof, see e.g. [97, Section 3.6]

2.5 Local to global homeomorphisms

In this section, we prove a very useful lemma, Lemma 2.9, stating conditions which guarantee that a local homeomorphism is injective and hence a homeomorphism. We will for example use it to glue certain diffeomorphisms together in Section 2.7.

We say that a map is *proper* if preimages of compact sets are compact. A *local homeomorphism* $f : X \to Y$ is a map that is a homeomorphism in a neighborhood of each point $x \in X$. The following result is due to Ho [52, 51], see also [84, Chapter 4, Section 2.4].

Lemma 2.8. *Suppose that X and Y are path-connected Hausdorff spaces, where Y is simply connected. Then a local homeomorphism $f : X \to Y$ is a global homeomorphism of X onto Y if and only if f is a proper map.*

Note that in general, a local homeomorphism need not be surjective, and surjectivity is a part of the lemma. The main idea of the proof is to show that a proper local homeomorphism between path connected Hausdorff spaces is a covering map and then the result follows from general facts about covering spaces, see also [59, Lemma 3.1].

If $f : \partial\mathbb{B}^n \to \mathbb{R}^n$ is a homeomorphism (let us stress here: onto its image), then according to the Jordan-Brouwer separation theorem $f(\partial\mathbb{B}^n)$ separates $\mathbb{R}^n$ into two domains, bounded and unbounded, and $f(\partial\mathbb{B}^n)$ is their common boundary. The example of the inward Alexander horned sphere[2] shows that in general, the bounded component of $\mathbb{R}^n \setminus f(\partial\mathbb{B}^n)$ need not be simply connected.

The next result is a version of Corollary 8.2 from [78], but our proof is different and more elementary. Nonetheless, it still requires the Jordan-Brouwer separation theorem and Lemma 2.8.

Lemma 2.9. *Let $f : \overline{\mathbb{B}}^n \to \mathbb{R}^n$ be a continuous mapping such that $f|_{\partial\mathbb{B}^n}$ is one-to-one, $f|_{\mathbb{B}^n}$ is a local homeomorphism, and the bounded component of $\mathbb{R}^n \setminus f(\partial\mathbb{B}^n)$ is simply connected. Then $f : \overline{\mathbb{B}}^n \to \mathbb{R}^n$ is a homeomorphism of $\overline{\mathbb{B}}^n$ onto $f(\overline{\mathbb{B}}^n)$.*

Remark 2.10. The above result is true even if we do not assume that the bounded component of $\mathbb{R}^n \setminus f(\partial\mathbb{B}^n)$ is simply connected, but the proof requires the theory of Eilenberg-MacLane spaces from algebraic topology, see [59, Theorem 1.2].

Proof. Denote the bounded and the unbounded components of $\mathbb{R}^n \setminus f(\partial\mathbb{B}^n)$ by D and U, respectively. According to our assumptions, D is simply connected.

Since $f(\overline{\mathbb{B}}^n)$ is compact, it follows that $\partial f(\overline{\mathbb{B}}^n) \subset f(\overline{\mathbb{B}}^n)$. On the other hand, it follows from Theorem 2.1 that $f(\mathbb{B}^n)$ is open and hence $f(\mathbb{B}^n) \cap \partial f(\overline{\mathbb{B}}^n) = \varnothing$, so $\partial f(\overline{\mathbb{B}}^n) \subset f(\partial\mathbb{B}^n) = \partial U = \partial D$.

We claim that $f(\overline{\mathbb{B}}^n) \subset \overline{D}$. Suppose to the contrary that $f(x) \in U$ for some $x \in \overline{\mathbb{B}}^n$. Since U is unbounded and connected, there is a curve connecting $f(x)$ to infinity inside U. Since $f(\overline{\mathbb{B}}^n)$ is bounded, the curve must intersect with the boundary of that set and hence a point in U belongs to $\partial f(\overline{\mathbb{B}}^n)$, which is a contradiction, because $\partial f(\overline{\mathbb{B}}^n) \subset \partial U$.

Since $f(\mathbb{B}^n) \subset \overline{D}$ is an open subset of $\mathbb{R}^n$, it follows that $f(\mathbb{B}^n) \subset D$. We claim that the mapping $f : \mathbb{B}^n \to D$ is proper. Indeed, if $K \subset D$ is compact, then K is a closed subset of $\mathbb{R}^n$, so $f^{-1}(K) \cap \overline{\mathbb{B}}^n$ is closed and hence compact. On the other hand, $f(\partial\mathbb{B}^n) \cap D = \varnothing$, which means that $f^{-1}(K) \cap \partial\mathbb{B}^n = \varnothing$ and that $f^{-1}(K) \cap \overline{\mathbb{B}}^n = f^{-1}(K) \cap \mathbb{B}^n$. That is, $f^{-1}(K)$ is a compact subset of $\mathbb{B}^n$. This proves that $f : \mathbb{B}^n \to D$ is proper. Now, Lemma 2.8

[2] We briefely describe Alexander's horned sphere in Section 3.1

yields that $f : \mathbb{B}^n \to D$ is a homeomorphism onto D. That also implies that f is one-to-one on $\overline{\mathbb{B}}^n$ because $f(\partial\mathbb{B}^n) \cap D = \varnothing$. Since $\overline{\mathbb{B}}^n$ is compact, it follows that $f : \overline{\mathbb{B}}^n \to \mathbb{R}^n$ is a homeomorphism onto its image. □

2.6 Gluing homeomorphisms together

Throughout the dissertation we repeatedly use the following observation (c.f. [37, Lemma 3.7]) and its corollary. These two lemmata underline almost all further constructions of homeomorphisms, as these observations make gluing of homeomorphisms so easy.

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $F, G : \Omega \to \mathbb{R}^n$ be homeomorphisms onto their respective images. Assume moreover that $F = G$ near $\partial\Omega$. Then $F(\Omega) = G(\Omega)$.*

Proof. Assume otherwise; without loss of generality we may assume that there is $y \in \Omega$ such that $G(y) \notin F(\Omega)$. We can find subdomains $\Omega_1 \Subset \Omega_2 \Subset \Omega$ such that $F = G$ on $\Omega \setminus \Omega_1$. This means that y needs to lie in $\Omega_1 \Subset \Omega_2$ and $G(y) \notin F(\Omega_2)$. However, as $F = G$ on $\Omega_2 \setminus \Omega_1$, there is an $x \in \Omega_2$ with $G(x) = F(x)$.

Since Ω_2 is path-connected, we may connect x and y with a curve γ lying entirely in Ω_2. Recall that by Corollary 2.2, homeomorphisms F and G map interior points to interior points and boundary points to boundary points. The curve $G(\gamma)$ connects $G(x) = F(x)$, which is an interior point of $F(\Omega_2)$ with $G(y)$, which lies outside $F(\Omega_2)$ and thus $G(\gamma)$ must intersect $\partial F(\Omega_2) = F(\partial\Omega_2)$. Since $F = G$ near $\partial\Omega_2$, $G(\gamma)$ intersects $G(\partial\Omega_2)$. This leads to a contradiction, because $\gamma \subset \Omega_2$. □

Corollary 2.12. *Assume $\Omega_2 \subset \Omega_1 \subset \mathbb{R}^n$ are bounded domains and let $F : \Omega_1 \to \mathbb{R}^n$, $G : \Omega_2 \to \mathbb{R}^n$ be homeomorphisms onto their respective images. Assume moreover that for all $x \in \Omega_2$ in some neighborhood of $\partial\Omega_2$ we have $F(x) = G(x)$. Then $\widetilde{F} : \Omega_1 \to \mathbb{R}^n$ given by*

$$\widetilde{F}(x) = \begin{cases} F(x) & \text{for } x \in \Omega_1 \setminus \Omega_2, \\ G(x) & \text{for } x \in \Omega_2 \end{cases}$$

is a homeomorphism and $F(\Omega_1) = \widetilde{F}(\Omega_1)$.

Proof. $\widetilde{F}$ is a local homeomorphism since F and G are homeomorphisms and $F = G$ near $\partial\Omega_2$. Since by Lemma 2.11 $\widetilde{F}$ is injective and $F(\Omega_1) = \widetilde{F}(\Omega_1)$, it follows that $\widetilde{F}$ is a homeomorphism of Ω_1 onto $F(\Omega_1)$. □

2.7 Gluing diffeomorphisms together

This subsection is, in a sense, a forerunner of Chapter 3, as we develop here techniques for gluing diffeomorphisms. We will also use make extensive use of these results in Chapter 4, in the proof of Theorem 1.4.

We begin by showing that it is possible to glue a diffeomorphism and its tangent mapping on a sufficiently small ball. Lemma 2.13 below is similar to Lemma 3.8 in [37]. However, the proof in [37] required the diffeomorphism to be at least of class C^2. We managed to prove the results for C^1-diffeomorphisms by using a topological argument due to Munkres [78] (Lemma 2.9 above). He used it in a similar context. We also note the paper [99], which used a similar argument albeit in a slightly different way.

Lemma 2.13. *Suppose that $\Phi : U \to \mathbb{R}^n$ is a C^k-diffeomorphism, $k \in \mathbb{N} \cup \{\infty\}$, defined on an open set $U \subset \mathbb{R}^n$ and $\lambda \in (0,1)$ is given. Then for any $x_o \in U$ there is $r_{x_o} > 0$ such that $B(x_o, r_{x_o}) \Subset U$ and that for any $r \in (0, r_{x_o}]$ it is possible to find diffeomorphisms $H_1, H_2 : U \to \mathbb{R}^n$ of class C^k satisfying*

$$H_1(x) = \begin{cases} \Phi(x_o) + D\Phi(x_o)(x - x_o) & \text{for } x \in \overline{B}(x_o, \lambda r), \\ \Phi(x) & \text{for } x \in U \setminus B(x_o, r), \end{cases} \tag{2.5}$$

$$H_2(x) = \begin{cases} \Phi(x) & \text{for } x \in \overline{B}(x_o, \lambda r), \\ \Phi(x_o) + D\Phi(x_o)(x - x_o) & \text{for } x \in U \setminus B(x_o, r). \end{cases} \tag{2.6}$$

Proof. Let $S(x) := \Phi(x_o) + D\Phi(x_o)(x - x_o)$ and let $\beta = \|(D\Phi(x_o))^{-1}\|^{-1}$. Since Φ is continuously differentiable, there is $r_{x_o} > 0$ such that $B(x_o, r_{x_o}) \Subset U$,

$$\|D\Phi(x_o) - D\Phi(x)\| < \frac{\beta}{4} \quad \text{and} \quad \frac{|\Phi(x) - S(x)|}{|x - x_o|} < \frac{(1-\lambda)\beta}{4} \quad \text{for } x \in \overline{B}(x_o, r_{x_o}).$$

For $r \in (0, r_{x_o}]$, let $\eta \in C^\infty(\mathbb{R}^n)$, $0 \le \eta \le 1$, be a cut-off function such that

$$\eta = \begin{cases} 1 & \text{on } B(x_o, \lambda r), \\ 0 & \text{on } \mathbb{R}^n \setminus B(x_o, r), \end{cases} \quad \text{and} \quad \|D\eta\|_\infty \le \frac{2}{r(1-\lambda)}\,.$$

Here $\|D\eta\|_\infty$ stands for the supremum norm of $|D\eta|$.

Now, we define

$$H_1(x) := \Phi(x) + \eta(x)(S(x) - \Phi(x)) \quad \text{and} \quad H_2(x) := S(x) + \eta(x)(\Phi(x) - S(x)).$$

Clearly, $H_{1,2} \in C^k(U; \mathbb{R}^n)$ satisfy (2.5) and (2.6). It remains to show that $H_{1,2}$ are diffeomorphisms. To this end, it suffices to show that $DH_{1,2}(x)$ is invertible for all $x \in U$ and that $H_{1,2}$ are homeomorphisms.

The matrices $DH_{1,2}(x)$ are invertible for $x \in U \setminus \overline{B}(x_o, r)$, because $H_{1,2}$ are diffeomorphisms in $U \setminus \overline{B}(x_o, r)$. To show invertibility of $DH_{1,2}(x)$ for $x \in \overline{B}(x_o, r)$, it suffices to show that (see Lemma 2.5):

$$\|DH_{1,2}(x) - D\Phi(x_o)\| < \beta \quad \text{for } x \in \overline{B}(x_o, r). \tag{2.7}$$

Note that $|\eta| \le 1$ and hence (2.4) yields $\|D(\eta(S - \Phi))\| \le |D\eta|\,|S - \Phi| + \|DS - D\Phi\|$. Bearing in mind that $DS(x) = D\Phi(x_o)$, we have

$$\begin{aligned} &\|DH_1(x) - D\Phi(x_o)\| \\ &\le \|D\Phi(x) - D\Phi(x_o)\| + |D\eta(x)|\,|S(x) - \Phi(x)| + \|DS(x) - D\Phi(x)\| \\ &= |D\eta(x)|\,|\Phi(x) - S(x)| + 2\|D\Phi(x) - D\Phi(x_o)\|. \end{aligned} \tag{2.8}$$

Similarly,

$$\|DH_2(x) - D\Phi(x_o)\| \le |D\eta(x)|\,|\Phi(x) - S(x)| + \|D\Phi(x) - D\Phi(x_o)\|. \tag{2.9}$$

For $x \in \overline{B}(x_o, r)$ we have $|x - x_o| \le r < r_{x_o}$ and hence

$$|D\eta(x)|\,|\Phi(x) - S(x)| + 2\|D\Phi(x) - D\Phi(x_o)\| \le \frac{2}{1-\lambda}\frac{|\Phi(x) - S(x)|}{|x - x_o|} + 2 \cdot \frac{\beta}{4} < \beta,$$

which together with (2.8) and (2.9) proves (2.7).

It remains to show that $H_{1,2} : U \to \mathbb{R}^n$ are homeomorphisms of U onto their respective images.

Since the surfaces $H_1(\partial B(x_o, r)) = \Phi(\partial B(x_o, r)) = \partial\Phi(B(x_o, r))$ and $H_2(\partial B(x_o, r)) = \partial S(B(x_o, r))$ bound simply connected domains $\Phi(B(x_o, r))$ and $S(B(x_o, r))$, and $H_{1,2}$ are local homeomorphisms (because $\det DH_{1,2} \neq 0$), it follows from Lemma 2.9 that H_1 and H_2 are homeomorphisms of $B(x_o, r)$ onto $\Phi(B(x_o, r))$ and $S(B(x_o, r))$, respectively, and hence H_1 and H_2 are homeomorphisms of U onto $\Phi(U)$ and $S(U)$, respectively. The proof is complete. □

The next result shows how to connect linear maps in $GL(n)^+$ in a diffeomorphic way. We will use it in this section, as well as in Chapters 3 and 4. However, its importance lies not only in its usefulness. When looking at Proposition 2.14, one is prompted to ask if a similar results holds if A_1, A_2 are any orientation preserving diffeomorphisms satisfying (2.10), not necessarily linear maps. We answer this question in the positive in Corollary 3.2 in Chapter 3.

Proposition 2.14. *Fix $r > 0$ and $\theta \in (0, 1)$, and let $A_1, A_2 \in GL(n)^+$. If*

$$A_1 (B(0, \theta r)) \Subset A_2 (B(0, r)) , \tag{2.10}$$

then there exists a C^∞-diffeomorphism $H : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ which coincides with $x \mapsto A_1 x$ on $B(0, \theta r)$ and with $x \mapsto A_2 x$ on $\mathbb{R}^n \setminus B(0, r)$.

In the proof we will need the following special case of the result (c.f. [78, Lemma 8.1]):

Lemma 2.15. *Let $A \in GL(n)^+$. Then for any $r > 0$ there is $\varrho \in (0, r)$ and a C^∞-diffeomorphism $H : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that*

$$H(x) = \begin{cases} Ax & \text{for } x \in B(0, \varrho), \\ x & \text{for } x \in \mathbb{R}^n \setminus B(0, r). \end{cases} \tag{2.11}$$

Proof. Let $\eta \in C_c^\infty(B(0, 1))$, $\eta = 1$ on $B(0, 1/2)$, and let $M := \|\eta\|_\infty + \|D\eta\|_\infty$. For $r > 0$ and $L \in M^{n\times n}$ define $f(x) := \eta(x/r)Lx$. We have (see (2.4)):

$$Df(x) = \frac{1}{r}(Lx) \otimes D\eta\left(\frac{x}{r}\right) + \eta\left(\frac{x}{r}\right) L,$$

and hence

$$|Df(x)| \leq \frac{1}{r}|Lx| \left|D\eta\left(\frac{x}{r}\right)\right| + \left|\eta\left(\frac{x}{r}\right)\right| \, |L| \leq \frac{1}{r}|L|\,|x|M\chi_{B(0,r)}(x) + M|L| \leq 2M|L|.$$

In particular,

$$|f(x) - f(y)| \leq |x - y| \int_0^1 |Df(y + t(x - y))| \, dt \leq 2M|L|\,|x - y|.$$

First we will prove the lemma under the assumption that $|A - \mathcal{I}| < (2M)^{-1}$. Let $L := A - \mathcal{I}$ so that $f(x) = \eta(x/r)(A - \mathcal{I})x$ and define

$$H_A(x) := x + \eta\left(\frac{x}{r}\right)(A - \mathcal{I})x = x + f(x).$$

We have

$$\|DH_A(x) - \mathcal{I}\| \leq |DH_A(x) - \mathcal{I}| = |Df(x)| \leq 2M|A - \mathcal{I}| < 1,$$

so $DH_A(x)$ is invertible by Lemma 2.5. H_A is also one-to-one, because for $x \neq y$ we have

$$|H_A(x) - H_A(y)| \geq |x-y| - |f(x) - f(y)| \geq |x-y| - 2M|A - \mathcal{I}|\,|x-y| > 0.$$

Therefore, H_A is a diffeomorphism and (2.11) is true with $\varrho = r/2$.

Now assume that $A \in GL(n)^+$ is an arbitrary matrix. According to Lemma 2.6, we can write $A = A_1 \cdot \ldots \cdot A_k$, where $|A_i - \mathcal{I}| < (2M)^{-1}$. Then

$$H := H_{A_1} \circ \ldots \circ H_{A_k} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$$

is a diffeomorphism that satisfies $H(x) = x$ for $|x| \geq r$. Using simple induction and the fact that the diffeomorphisms H_{A_i} satisfy (2.11) with A_i and $\varrho = r/2$, one can easily check that

$$H(x) = A_1(A_2(\ldots(A_k(x)\ldots)) = Ax$$

if x belongs to the set

$$B(0,\tfrac{r}{2}) \cap A_k^{-1}(B(0,\tfrac{r}{2})) \cap (A_{k-1} \circ A_k)^{-1}(B(0,\tfrac{r}{2})) \cap \ldots \cap (A_2 \circ \ldots \circ A_k)^{-1}(B(0,\tfrac{r}{2})).$$

Since this set is an open neighborhood of 0, it contains a ball $B(0,\varrho)$ for some $0 < \varrho < r$ and hence H satisfies (2.11). □

Proof of Proposition 2.14. By composing with A_2^{-1} and scaling if necessary, we can assume that $A_2 = \mathrm{id}$ and $r = 1$. Using Lemma 2.15 we can construct a diffeomorphism h_2 of $\mathbb{R}^n$ such that $h_2(x) = A_1 x$ on $B(0,\varrho)$ for some $\varrho \in (0,1)$ and $h_2(x) = x$ on $\mathbb{R}^n \setminus B(0,1)$. This diffeomorphism would have desired properties if we could take $\varrho = \theta$, but it may happen that ϱ is smaller than θ. Thus assume that $\varrho < \theta$.

To correct h_2 we take a radial diffeomorphism h_1 of $\mathbb{R}^n$, which equals $h_1(x) = \varrho\,\theta^{-1}x$ on $B(0,\theta)$ and is identity outside $B(0,1)$. As a result, $h_2(h_1(x)) = \varrho\,\theta^{-1}A_1x$ on $B(0,\theta)$ and $h_2(h_1(x)) = x$ on $\mathbb{R}^n \setminus B(0,1)$. This diffeomorphism has all the required properties, except that it equals $\varrho\,\theta^{-1}A_1x$ on $B(0,\theta)$ instead of required A_1x. To correct it, we take a radial diffeomorphism h_3, which equals $h_3(x) = \theta\,\varrho^{-1}x$ on $A_1(B(0,\varrho))$ and $h_3(x) = x$ on $\mathbb{R}^n \setminus B(0,1)$. Such a diffeomorphism exists, because $A_1(B(0,\varrho)) \Subset B(0,1)$ and because $A_1(B(0,\varrho))$ under $x \mapsto \theta\varrho^{-1}x$ is mapped onto $A_1(B(0,\theta)) \Subset B(0,1)$. Eventually, the mapping $H = h_3 \circ h_2 \circ h_1$ is the desired diffeomorphism. □

It is a known fact that given any two points p, q lying in the interior of a smooth, connected n-dimensional manifold $\mathcal{M}^n$, one can find a diffeomorphism of $\mathcal{M}^n$ onto itself that carries p into q and is isotopic to identity, see [70, Chapter 4]. We need a slightly stronger folklore result, stating that given a finite family of points in a Euclidean domain, we can rearrange them in a diffeomorphic manner so that neighborhoods of these points are mapped by translations.

We shall use this result repetitively in Chapter 3 and in Chapter 4 in the proof of Proposition 4.6. In Chapter 4, it will be crucial to know that H maps the balls by translations, in Chapter 3, it will not. In the proof of Lemma 2.16, we use a 1-parameter group of diffeomorphisms generated by a certain vector field. In the Appendix, in Theorem A.2 we state a theorem about existence and basic properties of 1-parameter groups of diffeomorphisms.

Lemma 2.16. *Let $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$ be given points in U, a domain in $\mathbb{R}^n$, with $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$. Then there exists an $\varepsilon > 0$ and a C^∞-diffeomorphism $H : U \twoheadrightarrow U$, identity near ∂U, such that*

$$H(x) = x + (q_i - p_i) \text{ for } x \in B(p_i, \varepsilon),$$

i.e., H maps by translation each ball $B(p_i,\varepsilon)$ onto $B(q_i,\varepsilon)$ with $H(p_i) = q_i$.

Proof. Firstly, let us consider the case $N = 1$ and assume for simplicity that $p := p_1$ and $q := q_1$ can be connected by a segment γ contained in U. We choose $\varepsilon > 0$ so that the 2ε-neighborhood of γ is contained in U and find a smooth vector field X, satisfying

$$X = \begin{cases} 0 & \text{on the set } \{x : \operatorname{dist}(x, \gamma) \geq 2\varepsilon\}, \\ q - p & \text{on the set } \{x : \operatorname{dist}(x, \gamma) \leq \varepsilon\}. \end{cases}$$

If $\Phi_t : U \twoheadrightarrow U$ is the one-parameter family of diffeomorphisms generated by X, then $H^\varepsilon_{p,q} := \Phi_1$ is a diffeomorphism which acts as the translation by $q-p$ on the ball $B(p, \varepsilon)$ and maps it onto the ball $B(q, \varepsilon)$. Moreover, $H^\varepsilon_{p,q}$ equals identity outside the 2ε-neighborhood of γ. For details, see Corollary A.4 and Remark A.5 in the Appendix.

In view of path-connectedness of Euclidean domains, any two points p, q can be connected with a piecewise linear curve $\gamma \subset U$ with vertices $a_0 = p, a_1, \dots, a_m = q$. Choose $\varepsilon > 0$ so that the 2ε-neighborhood of γ is contained in U and apply the construction from previous paragraph to each pair of points a_i, a_{i+1} for $i = 0, \dots, m-1$ to construct diffeomorphisms $H^\varepsilon_{a_i,a_{i+1}}$, identity outside the 2ε-neighborhood of γ, such that

$$H^\varepsilon_{a_i,a_{i+1}}(x) = x + (a_{i+1} - a_i) \text{ for } x \in B(a_i, \varepsilon).$$

Then $H^\varepsilon_{p,q} := H^\varepsilon_{a_{m-1},q} \circ \dots \circ H^\varepsilon_{p,a_1}$ is the desired diffeomorphism when $N = 1$.

For $N > 1$, consider firstly the case when $\{p_i\}_{i=1}^N \cap \{q_i\}_{i=1}^N = \varnothing$. We can then find N pairwise disjoint piecewise linear curves γ_i, $i = 1, \dots, N$, connecting p_i with q_i, and an $\varepsilon > 0$ so that the 2ε-neighborhoods of γ_i are pairwise disjoint and contained in U and construct diffeomorphisms $H^\varepsilon_{p_i,q_i}$ from the previous paragraph. Diffeomorphism $H = H^\varepsilon_{p_N,q_N} \circ \dots \circ H^\varepsilon_{p_1,q_1}$ is the desired map.

If $\{p_i\}_{i=1}^N \cap \{q_i\}_{i=1}^N \neq \varnothing$, then we find a set of distinct points $\{s_i\}_{i=1}^N \subset U$, such that $\{p_i\}_{i=1}^N \cap \{s_i\}_{i=1}^N = \varnothing$ and $\{s_i\}_{i=1}^N \cap \{q_i\}_{i=1}^N = \varnothing$. From what we already proved, there is a diffeomorphism H_1 that translates neighborhoods of p_i's onto neighborhoods of s_i's and a diffeomorphism H_2 that translates neighborhoods of s_i's onto neighborhoods of q_i's. Then $H = H_2 \circ H_1$ satisfies the claim of the lemma. □

In the Appendix, in Lemma A.7, we present a construction of a diffeomorphism as in Lemma 2.16 with the additional property of being measure preserving. It is a very nice construction based on [81, Chapter 5] (see also [3, Chapter 2.2]) which generalizes these results, as the cited authors were interested only in homeomorphisms. Below, we add a modification of Lemma 2.16 that will be useful in Chapter 3.

Lemma 2.17. *Let U be a domain in $\mathbb{R}^n$ and $\overline{B}(p, r)$ and $\overline{B}(q, \varrho)$ be two disjoint closed balls contained in U. Then there exists an orientation preserving C^∞-diffeomorphism $H : U \twoheadrightarrow U$ such that*

$$H(\overline{B}(p, r)) \subset B(q, \varrho) \quad \text{and} \quad H = \mathrm{id} \;\text{ near } \partial U.$$

Proof. Without loss of generality, we can assume that U contains the origin and that $p = 0$. There is $\eta > 0$ for which $B(0, r + \eta) \Subset U$ and such that $\overline{B}(0, r + \eta)$ is disjoint from $\overline{B}(q, \varrho)$. For the pair of points $\{0, q\}$ we choose $\varepsilon \in (0, \min\{r, \varrho\})$ from Lemma 2.16 and set $\phi : [0, \infty) \to \mathbb{R}$ to be a non-decreasing C^∞-smooth function which satisfies

$$\phi(t) = \begin{cases} \frac{\varepsilon}{r} & \text{if } t \leq r, \\ 1 & \text{if } t > r + \eta. \end{cases}$$

Then $\Phi(x) := \phi(|x|)x$ is an orientation preserving C^∞-diffeomorphism of $\mathbb{R}^n$ which acts like scaling by a factor ε/r on $B(0, r)$ and equals identity outside $B(0, r+\eta)$. In particular,

$\Phi(B(0,r)) = B(0,\varepsilon)$ and $\Phi = \mathrm{id}$ near ∂U. Lemma 2.16 yields a C^∞-diffeomorphism $H_\varepsilon : U \twoheadrightarrow U$ which maps $B(0,\varepsilon)$ onto $B(q,\varepsilon) \Subset B(q,\varrho)$ and such that $H_\varepsilon = \mathrm{id}$ near ∂U. Note that H_ε acts as a translation on $B(0,\varepsilon)$ and hence it is orientation preserving. Eventually,

$$H := H_\varepsilon \circ \Phi|_U$$

is the desired C^∞-diffeomorphism of U. Let us stress that it is orientation preserving as a composition of such mappings. □

Remark 2.18. It is clear that we can modify H slightly so that it satisfies $H(\overline{B}(p,r)) = B(q,\varrho)$ and $H = \mathrm{id}$ near ∂U.

2.8 Uniform metric

Let us denote by

$$d(\Phi,\Psi) := \sup_{x\in\mathcal{Q}} |\Phi(x) - \Psi(x)| + \sup_{x\in\mathcal{Q}} |\Phi^{-1}(x) - \Psi^{-1}(x)|$$

the *uniform metric* in the space of homeomorphisms of the unit cube $\mathcal{Q} = [0,1]^n$ onto itself. It is known that the space of homeomorphisms is a complete metric space with respect to the metric d. More precisely, we have:

Lemma 2.19. *Let $\Phi_k : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $k = 1, 2, \ldots$ be a Cauchy sequence of surjective homeomorphisms in the uniform metric d. Then Φ_k converges uniformly to a homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, and Φ_k^{-1} converges uniformly, and the limit is equal to Φ^{-1}.*

Proof. Obviously Φ_k and Φ_k^{-1} are Cauchy sequences in the space of continuous mappings $C(\mathcal{Q},\mathcal{Q})$, thus they converge (uniformly) to some Φ and $\Psi \in C(\mathcal{Q},\mathcal{Q})$, respectively. To see that $\Psi = \Phi^{-1}$, fix a point $x \in \mathcal{Q}$ and pass with k to the limit in the equality $\Phi_k(\Phi_k^{-1}(x)) = x$ to prove that $\Phi(\Psi(x)) = x$. We show that $\Psi(\Phi(x)) = x$ in an analogous way. □

Lemma 2.20. *Assume that $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ is a C^1-diffeomorphism such that $\Phi = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$. Then Φ can be approximated in the uniform metric d by a sequence of C^∞-diffeomorphisms $\Phi_k \xrightarrow{d} \Phi$ such that $\Phi_k = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$.*

Proof. Approximating Φ by convolution with a standard symmetric mollifier ψ_ε we obtain smooth maps $\Phi_\varepsilon = \Phi * \psi_\varepsilon$ that are identity near $\partial\mathcal{Q}$ and converge uniformly to Φ on $\mathcal{Q}$. Since $\det D\Phi_\varepsilon \to \det D\Phi$ uniformly, we see that $\det D\Phi_\varepsilon > 0$ in $\mathcal{Q}$, provided $\varepsilon > 0$ is sufficiently small. This implies that Φ_ε is a local diffeomorphism and according to Lemma 2.9 it is a global diffeomorphism of $\mathcal{Q}$ onto itself. It easily follows that $\Phi_\varepsilon \to \Phi$ in the uniform metric d. □

We add an interesting lemma, shown in [88, p. 104] which shows how natural the uniform metric is for studying sequences of homeomorphisms of the cube. It also implies a nice corollary about sequences of measure preserving homeomorphisms.

Lemma 2.21. *Let $\Phi, \Phi_k : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ be homeomorphisms. If Φ_k converge to Φ uniformly on $\mathcal{Q}$, then Φ_k^{-1} converge uniformly to Φ^{-1} as well and so Φ_k converge to Φ in the uniform metric d.*

Proof. We have

$$\sup_{y\in\Omega}\|\Phi_k^{-1}(y)-\Phi^{-1}(y)\| = \sup_{x\in\Omega}\|\Phi_k^{-1}(\Phi_k(x))-\Phi^{-1}(\Phi_k(x))\| = \sup_{x\in\Omega}\|x-\Phi^{-1}(\Phi_k(x))\|.$$

Since Φ^{-1} is uniformly continuous on Ω, $\Phi^{-1}\circ\Phi_k$ converge uniformly to identity and the quantity above converges to zero. Therefore, Φ_k^{-1} converge uniformly to Φ^{-1}. $\square$

We say that a homeomorphism $\Phi:\Omega\twoheadrightarrow\Omega$ is measure preserving if for any measurable set $E\subset\Omega$, $\Phi(E)$ is also measurable and $|\Phi(E)|=|E|$. We have already checked in Lemma 2.4 that a homeomorphism maps measurable sets onto measurable sets if and only if it satisfies the Lusin (N) condition. Clearly, a measure preserving homeomorphism must map sets of zero measure onto sets of zero measure.

Corollary 2.22. *Let $\Phi_k:\Omega\twoheadrightarrow\Omega$ be measure preserving homeomorphisms converging uniformly to a homeomorphism $\Phi:\Omega\twoheadrightarrow\Omega$. Then Φ is also measure preserving.*

Proof. By Lemma 2.21, we know that Φ_k converge to Φ in the uniform metric d. The limit in metric d of a sequence of measure preserving homeomorphisms is measure preserving, see [37, Lemma 1.2]. Therefore, Φ is measure preserving. $\square$

2.9 Approximate differentiability

In this section, we discuss the basics of approximate differentiability. For brevity, we state the definitions and most of the lemmata for scalar functions. Applying those definitions and theorems componentwise yields corresponding facts for mappings.

Definition 2.23 (Classical definition)**.** Let $f:E\to\mathbb{R}$ be a measurable function defined on a measurable set $E\subset\mathbb{R}^n$. We say that f is *approximately differentiable* at $x\in E$ if there is a linear function $L:\mathbb{R}^n\to\mathbb{R}$ such that for any $\varepsilon>0$ the set

$$\left\{y\in E:\ \frac{|f(y)-f(x)-L(y-x)|}{|y-x|}<\varepsilon\right\}$$

has x as a density point.

The next result provides a useful characterization of approximate differentiability, for the proof see [37, Proposition 5.2].

Lemma 2.24. *A measurable function $f:E\to\mathbb{R}$ defined in a measurable set $E\subset\mathbb{R}^n$ is approximately differentiable at $x\in E$ if and only if there is a measurable set $E_x\subset E$ and a linear function $L:\mathbb{R}^n\to\mathbb{R}$ such that x is a density point of E_x and*

$$\lim_{E_x\ni y\to x}\frac{|f(y)-f(x)-L(y-x)|}{|y-x|}=0.$$

If a function f is approximately differentiable at x, L is unique, and we call it approximate derivative of f at x. For the proof of uniqueness of approximate derivative, see e.g. [29, Theorem 3, Section 6.1] or [31, Section 3.1.2]. The approximate derivative will be denoted by $D_{\mathrm{a}}f(x)$ or simply by $Df(x)$. If f is measurable, so is $D_{\mathrm{a}}f$, see Theorem 3.1.4 in [31].

Lemma 2.25. *Assume that $f, g : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, are given measurable functions and $E \subset U$ is a measurable set. If f is approximately differentiable a. e. and $f = g$ in E, then g is approximately differentiable a. e. in E and*

$$D_{\mathrm{a}} g(x) = D_{\mathrm{a}} f(x) \quad \textit{for almost all } x \in E. \tag{2.12}$$

Proof. It easily follows from Lemma 2.24 that (2.12) is satisfied whenever $x \in E$ is a density point of E and a point of approximate differentiability for f. Indeed, for any such x it follows from Lemma 2.24 that for any $y \to x$, $y \in E$,

$$\lim_{y \to x} \frac{|g(y) - g(x) - D_{\mathrm{a}} f(x)(y - x)|}{|y - x|} = \lim_{y \to x} \frac{|f(y) - f(x) - D_{\mathrm{a}} f(x)(y - x)|}{|y - x|} = 0.$$

□

The next result was proved by Whitney [98]. It says that approximately differentiable functions coincide with C^1 mappings on sets which are large in measure (but possibly very twisted and with very complicated geometry).

Lemma 2.26 (Whitney)**.** *Let $U \subset \mathbb{R}^n$ be open. Then a function $f : U \to \mathbb{R}$ is approximately differentiable a. e. if and only if for every $\varepsilon > 0$ there is a function $f_\varepsilon \in C^1(\mathbb{R}^n)$ such that $|\{x \in U : f(x) \neq f_\varepsilon(x)\}| < \varepsilon$.*

Remark 2.27. By Lemma 2.25, it is easy to see that if $x \in \{f = f_\varepsilon\}$ is a density point of the set $\{f = f_\varepsilon\}$, then f is approximately differentiable at x and $D_{\mathrm{a}} f(x) = Df_\varepsilon(x)$.

Remark 2.28. Let U be an open subset of $\mathbb{R}^n$. Lemma 2.26 also implies that the closure of $C^1(U)$ in the Lusin metric d_L (see (2.1) for the definition) is the class of a. e. approximately differentiable functions. Indeed, if $f : U \to \mathbb{R}$ is the limit in d_L of a sequence of functions $f_k \in C^1(U)$, then for any $\varepsilon > 0$ there is $\ell \in \mathbb{N}$ such that

$$d_L(f, f_\ell) = |\{x \in U : f_\ell(x) \neq f(x)\}| < \varepsilon.$$

This, by Lemma 2.26, implies that f is approximately differentiable a. e. on U.

We now state Federer's change of variables theorem proved in [30], see also [31, Section 3.2]. The exact statement is taken from [42]. By $N(\Phi, E, y)$ we denote the number of points (cardinality) of the preimage $\Phi^{-1}(y) \cap E$.

Theorem 2.29 (Federer)**.** *Let $\Phi : \Omega \to \mathbb{R}^n$ be a measurable mapping defined on an open set $\Omega \subset \mathbb{R}^n$. Assume that it is approximately differentiable a. e. If Φ satisfies the Lusin (N) condition, then for any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ we have*

$$\int_\Omega (f \circ \Phi)(x) |\det D_{\mathrm{a}} \Phi(x)| \, dx = \int_{\Phi(\Omega)} f(y) N(\Phi, \Omega, y) \, dy. \tag{2.13}$$

If Φ does not satisfy the Lusin (N) condition, then we can redefine Φ on a Borel set of measure zero so that after the redefinition, Φ satisfies the Lusin (N) condition and (2.13).

To be more precise, (2.13) means that the function on the left hand side is integrable if and only if the function on the right hand side is integrable and then we have equality.

Remark 2.30. If Φ does not satisfy the condition (N), we redefine it using Lemma 2.26. By Lemma 2.26, for any $k \in \mathbb{N}$, we can find a closed set $E_k \subset \Omega$ and a mapping $F_k \in C^1(\Omega, \mathbb{R}^n)$ such that

$$|\Omega \setminus E_k| < 1/k \quad \text{and} \quad \Phi = F_k \text{ on } E_k.$$

This means that Φ satisfies the Lusin (N) condition on each E_k, $k \in \mathbb{N}$. Consequently, Φ satisfies the Lusin (N) condition on $E := \bigcup_{k=1}^{\infty} E_k$ and E is a Borel set. For every $k \in \mathbb{N}$,

$$|\Omega \setminus E| \leq |\bigcap_{k=1}^{\infty} \Omega \setminus E_k| \leq |\Omega \setminus E_k| < 1/k \text{ for every } k \in \mathbb{N},$$

which implies that $|\Omega \setminus E| = 0$. We choose $x_o \in \Phi(\Omega)$ and set for $x \in \Omega$

$$\Psi(x) := \begin{cases} \Phi(x) & \text{for } x \in E, \\ x_o & \text{for } x \in \Omega \setminus E. \end{cases}$$

This is the required redefinition of Φ as it clearly satisfies the Lusin (N) condition. Moreover, $\Psi(\Omega) \subset \Phi(\Omega)$ and $\Omega \setminus E$ is Borel.

Note that if Φ is continuous or injective, then Ψ need not be so. In particular, if Φ is a homeomorphism, then Ψ need not be one.

Corollary 2.31. *Let $\Phi : \Omega \to \mathbb{R}^n$ be an a. e. approximately differentiable homeomorphism. Then for any Borel set $E \subset \Omega$,*

$$\int_E |\det D_{\mathrm{a}}\Phi(x)|\, dx \leq |\Phi(E)|. \tag{2.14}$$

Moreover, for any measurable set $A \subset \Omega$ such that Φ satisfies the Lusin (N) condition on A,

$$\int_A |\det D_{\mathrm{a}}\Phi(x)|\, dx = |\Phi(A)|. \tag{2.15}$$

Proof. By Theorem 2.29 and Remark 2.30, there exists a mapping $\Psi : \Omega \to \mathbb{R}^n$ and a Borel set of measure zero $Z \subset \Omega$ such that $\Psi = \Phi$ on $\Omega \setminus Z$ and Φ satisfies the Lusin (N) condition on $\Omega \setminus Z$ and Ψ satisfies the Lusin (N) condition on Ω. In particular, $\Psi(Z)$ has measure zero. Also, for any Borel set $E \subset \Omega$, $E \setminus Z$ is Borel and since homeomorphisms map Borel sets onto Borel sets, $\Phi(E \setminus Z) = \Psi(E \setminus Z)$ is Borel.

By Lemma 2.25, $D_{\mathrm{a}}\Psi = D_{\mathrm{a}}\Phi$ a. e. on $\Omega \setminus Z$ and therefore for any Borel set $E \subset \Omega$

$$\int_E |\det D_{\mathrm{a}}\Phi(x)|\, dx = \int_{E \setminus Z} |\det D_{\mathrm{a}}\Phi(x)|\, dx = \int_{E \setminus Z} |\det D_{\mathrm{a}}\Psi(x)|\, dx. \tag{2.16}$$

Using (2.13) from Theorem 2.29 for the measurable function $f = \chi_{\Psi(E \setminus Z)}$ yields

$$\int_{E \setminus Z} |\det D_{\mathrm{a}}\Psi(x)|\, dx = \int_{\Psi(E \setminus Z)} N(\Psi, \Omega, y)\, dy. \tag{2.17}$$

Now, note that since $\Psi = \Phi$ on $\Omega \setminus Z$ and Φ is injective, for any $y \in \Psi(E \setminus Z) \setminus \Psi(Z)$, there is exactly one $x \in \Omega$ such that $\Psi(x) = \Phi(x) = y$. Since $\Psi(Z)$ is a set of measure zero, $N(\Psi, \Omega, y) = 1$ a. e. on $\Psi(E \setminus Z)$.

Therefore, by (2.16) and (2.17), we get

$$\int_E |\det D_{\mathrm{a}}\Phi(x)|\, dx = \int_{\Psi(E \setminus Z)} N(\Psi, \Omega, y)\, dy = |\Psi(E \setminus Z)| = |\Phi(E \setminus Z)| \leq |\Phi(E)|, \tag{2.18}$$

which proves (2.14).

If A is a measurable subset of Ω and Φ satisfies the Lusin (N) condition on A, then $\Phi(A)$ is measurable and so is $A \setminus Z$ and $\Phi(A \setminus Z) = \Psi(A \setminus Z)$. Therefore, $f = \chi_{\Psi(A \setminus Z)}$ is measurable. We can repeat (2.16), (2.17) with A in place of E and get as in (2.18)

$$\int_A |\det D_{\mathrm{a}}\Phi(x)|\, dx = |\Phi(A \setminus Z)| = |\Phi(A)|.$$

Indeed, the last equality is true, since $\Phi(A) = \Phi(A \setminus Z) \cup \Phi(A \cap Z)$ and $\Phi(A \cap Z)$ is a set of measure zero. This proves (2.15) and finishes the proof. $\square$

At the end of this section, we quote without proof the lemma stating that BV functions are approximately differentiable a. e. By $BV_{loc}(\mathbb{R}^n)$ we denote the class of functions of bounded variation $f : \mathbb{R}^n \to \mathbb{R}$, i. e., functions f whose distributional derivative is a Radon measure.

Lemma 2.32. *Let $f \in BV_{loc}(\mathbb{R}^n)$. Then f is approximately differentiable a. e. on $\mathbb{R}^n$.*

For the proof, see Theorem 4 on p. 233–234 of [29]. As $W^{1,p}_{loc}(\mathbb{R}^n) \subset BV_{loc}(\mathbb{R}^n)$ for $p \in [1, \infty]$, this lemma implies that Sobolev functions are a. e. approximately differentiable. For the proof in the Sobolev case only, see [47, Theorem A.31].

We shall farther discuss approximate differentiability in Section 5.3. We also recommend [29, Chapter 6], [47, Appendix A.7], [36, Section 4.4] and Sections 3.1 and 3.2 in [31] for more information or another treatment of approximate differentiability.

Chapter 3

Gluing diffeomorphisms, bi-Lipschitz homeomorphisms and homeomorphisms

3.1 Introduction

In this chapter, we prove Theorems 1.1, 1.2 and 1.3. These are results from [40], which is a joint work with Paweł Goldstein and Piotr Hajłasz. We shall recall their statements in due course in this introduction. The three theorems concern orientation preserving homeomorphisms of different regularity: diffeomorphisms, bi-Lipschitz homeomorphisms and homeomorphisms, respectively.

Let $\mathcal{M}^n$ and $\mathcal{N}^n$ be two n-dimensional topological manifolds and let $\Omega \subset \mathcal{M}^n$ be open. By a homeomorphism $f : \Omega \to \mathcal{N}^n$ we mean a homeomorphism onto the image, i. e., a homeomorphism between Ω and $f(\Omega)$. Let us recall that $n \geq 2$ and that we write $F : X \twoheadrightarrow Y$ if F is a surjective mapping onto Y. For a comprehensive list of used symbols, see Section 2.1.

If $\mathcal{M}^n$ is a manifold of class C^k, $k \in \mathbb{N} \cup \{\infty\}$, and $A \subset \mathcal{M}^n$ is closed, then by a C^k-diffeomorphism $F : A \to \mathcal{N}^n$ we mean a diffeomorphism that extends to a C^k-diffeomorphism on an open neighborhood of A. We say that a diffeomorphism $F : A \to \mathcal{N}^n$ is orientation preserving if its Jacobian is positive, see [93, Proposition 21.8]. Finally, a closed set $D \subset \mathcal{M}^n$ is a C^k-*diffeomorphic closed ball* if there is a C^k-diffeomorphism $F : \bar{\mathbb{B}}^n \twoheadrightarrow D \subset \mathcal{M}^n$. In other words, if there is a C^k-diffeomorphism $F : B^n(0, 1+\varepsilon) \to \mathcal{M}^n$ such that $F(\bar{B}^n(0, 1)) = D$. We can (and we shall) assume that F is orientation preserving.

We are now ready to state the first main result of this chapter.

Theorem 1.1. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented manifold of class C^k, $k \in \mathbb{N} \cup \{\infty\}$. Suppose that $\{D_i\}_{i=1}^{\ell}$ and $\{D'_i\}_{i=1}^{\ell}$, $D_i, D'_i \subset \mathcal{M}^n$, are two families of pairwise disjoint sets and that each D_i, $i = 1, \ldots, \ell$, is a C^k-diffeomorphic closed ball. If $F_i : D_i \twoheadrightarrow D'_i$, $i = 1, 2, \ldots, \ell$, are orientation preserving C^k-diffeomorphisms, then there is a C^k-diffeomorphism $F : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $F|_{D_i} = F_i$. Moreover, if D_i and D'_i for all $i = 1, \ldots, \ell$ are contained in the interior of a C^k-diffeomorphic closed ball $K \subset \mathcal{M}^n$, F can be chosen to equal identity outside K.*

Note that we do not assume that the sets in the family $\{D_i\}_{i=1}^{\ell}$ are disjoint from the sets in the family $\{D'_i\}_{i=1}^{\ell}$. Among many possible corollaries based on Theorem 1.1, below we state two in Euclidean setting, which are useful due to their elementary statements.

Corollary 3.1. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let D_1 and D_2, $D_2 \subset \mathring{D}_1 \subset \mathbb{R}^n$, be C^k-diffeomorphic closed balls, $k \in \mathbb{N} \cup \{\infty\}$. Let $F : \mathbb{R}^n \setminus \mathring{D}_1 \to \mathbb{R}^n$ be an orientation preserving C^k-diffeomorphism that can be extended to a C^k-diffeomorphism of the entire $\mathbb{R}^n$. Suppose that $G : D_2 \to \mathbb{R}^n$ is a C^k-diffeomorphism. If the diffeomorphisms have disjoint images, i. e., $G(D_2) \cap F(\mathbb{R}^n \setminus \mathring{D}_1) = \varnothing$, then there is a C^k-diffeomorphism $H : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ that agrees with F on $\mathbb{R}^n \setminus \mathring{D}_1$ and with G on D_2.*

Observe that the assumption about the existence of a diffeomorphic extension of F is necessary. Indeed, if we only assumed existence of a diffeomorphic extension of F to $\mathbb{R}^n \setminus \{0\}$, then Milnor's theorem [69, Theorem 5] would yield a counterexample when $n = 7$ (see Example 3.12).

Corollary 3.2. *Let $k \in \mathbb{N} \cup \{\infty\}$ and let D_1 and D_2, $D_1, D_2 \subset \mathbb{R}^n$, be two C^k-diffeomorphic closed balls and $F, G : \mathring{D}_2 \to \mathbb{R}^n$ be two C^k-diffeomorphisms. Suppose that*

$$D_1 \subset \mathring{D}_2 \quad \text{and} \quad F(D_1) \subset G(\mathring{D}_2).$$

Then there is a C^k-diffeomorphism $H : \mathring{D}_2 \to \mathbb{R}^n$ such that $H = F$ on D_1 and $H = G$ near ∂D_2.

As mentioned in Section 1.2, the question if Corollary 3.2 is true prompted this part of research. It is a good moment to observe that in Section 2.7, we already proved a few results about gluing diffeomorphisms in $\mathbb{R}^n$. We can glue a diffeomorphism and its tangent mapping on a sufficiently small ball (Lemma 2.13) and a linear mapping $A \in GL(n)^+$ with identity on a sufficiently small ball as well (Lemma 2.15). In the case of linear $A_1, A_2 \in GL(n)^+$ mappings, which behave well under scaling and re-scaling, it is then easy to show that if only for some $r > 0$, $A_1(B(0,r)) \Subset A_2(B(0,1))$, then there is a C^∞-diffeomorphism H of $\mathbb{R}^n$, which coincides with $x \mapsto A_1 x$ on the smaller ball and with $x \mapsto A_2 x$ outside the bigger ball. This is Proposition 2.14. Corollary 3.2 shows that a similar result can be obtained for diffeomorphisms.

In the proof of Theorem 1.1 we use techniques developed in the classical papers in geometric topology [78, 83, 82]. The heart of the proof (and the whole chapter) is the trick of Palais from [83]. Essentially, it says that an orientation preserving diffeomorphism defined on a diffeomorphic closed ball $D \subset \mathcal{M}^n$ can be extended to a diffeomorphism of the entire $\mathcal{M}^n$ which equals identity *away* from D. We shall also use some constructions from Chapter 2.

We now leave the comfortable world of C^k-regularity to tackle a version of Theorem 1.1 in bi-Lipschitz and, then, purely topological setting. The lack of smoothness implies for example that it is more difficult to define orientation on a manifold and what it means for a homeomorphism to be orientation preserving. Here, we assume the reader is familiar with these notions, if not, we refer to the short explanation of these definitions in the Appendix in Section A.4.1.

Theorem 1.2. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented Lipschitz manifold. Suppose $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of pairwise disjoint flat bi-Lipschitz closed balls. If $F_i : D_i \twoheadrightarrow D_i'$, $i = 1, 2, \ldots, \ell$, are orientation preserving bi-Lipschitz homeomorphisms, then there is a bi-Lipschitz homeomorphism $F : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $F|_{D_i} = F_i$. Moreover, if D_i and D_i' for all $i = 1, \ldots, \ell$ are contained in the interior of a flat bi-Lipschitz closed ball $K \subset \mathcal{M}^n$, F can be chosen to equal identity outside K.*

We say that D is a *flat bi-Lipschitz closed ball* if there is a bi-Lipschitz homeomorphism $F : \bar{\mathbb{B}}^n \twoheadrightarrow D$ that can be extended to a bi-Lipschitz mapping on a neighborhood of $\bar{\mathbb{B}}^n$. Not every bi-Lipschitz image of a closed ball is flat. One counterexample is the Fox-Artin ball, as originally observed by Gehring in [35] and proved by Martin in [65, Theorem 3.7]. This is a thickening of a standard Fox-Artin arc in $\mathbb{R}^3$, see [33]. The Fox-Artin ball is a bi-Lipschitz image of a closed ball, i. e., it equals $F(\bar{\mathbb{B}}^3)$ for some bi-Lipschitz homeomorphism F. However, there is no homeomorphic extension of F onto a neighborhood of $\bar{\mathbb{B}}^3$ and hence, clearly, no bi-Lipschitz extension. This follows from a lack of sufficiently good simple-connectedness properties of the complement of the Fox-Artin ball.

On the other hand, it follows from the Schönflies theorem for Lipschitz maps [95, Theorem A] that on the plane every bi-Lipschitz closed ball is flat, and hence the assumption on bi-Lipschitz flatness is superfluous for $n = 2$. To an interested reader, we recommend papers [23, 63] for recent related results.

Remark 3.3. In the last sentence of Theorem 1.2, it suffices to assume that K is a bi-Lipschitz image of a closed ball and not necessarily a flat bi-Lipschitz ball. Indeed, if D_i, D_i' for $i = 1, \ldots, \ell$ are contained in the interior of a bi-Lipschitz image of a closed ball K', then there is a flat bi-Lipschitz ball K such that $D_i, D_i' \subset \mathring{K} \subset K'$. This follows from Corollary 2.3.

The general outline of the proof of Theorem 1.2 is similar to that for Theorem 1.1. However, the result is much deeper as it requires the *bi-Lipschitz stable homeomorphisms theorem* proved by Sullivan [89]. The proof demands from the reader expertise in algebraic topology, not only because of how involved the topic is but also because of the style in which the cited article is written. In 1981, Tukia and Väisälä proved rigorously most of the theses from Sullivan's paper in [94]. The only result they took for granted from [89] is the existence of the *Sullivan groups*[1]. We also need the so called *annulus theorem*, which is closely related to the stable homeomorphisms theorem. In the bi-Lipschitz case it was proved in [94, Theorem 3.12].

We now state the version of Theorem 1.1 in the purely topological setting.

Theorem 1.3. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented topological manifold. Suppose $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of pairwise disjoint flat topological closed balls. If $F_i : D_i \twoheadrightarrow D_i'$, $i = 1, 2, \ldots, \ell$, are orientation preserving homeomorphisms, then there is a homeomorphism $F : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $F|_{D_i} = F_i$. Moreover, if D_i and D_i' for all $i = 1, \ldots, \ell$ are contained in the interior of a flat topological closed ball $K \subset \mathcal{M}^n$, F can be chosen to equal identity outside K.*

We say that D is a *flat topological closed ball* if there is a homeomorphism $F : \bar{\mathbb{B}}^n \twoheadrightarrow D$ that can be extended to a homeomorphism on a neighborhood of $\bar{\mathbb{B}}^n$. As before, we can (and we shall) assume that F is orientation preserving. Note that not every homeomorphic image of a closed ball is flat. The classical example is the Alexander's horned ball, see [2] for the original argument and [44, Example 2B.2] for a modern treatment of the construction. The complement of Alexander's horned ball is not simply connected since in any neighborhood of Alexander's horned ball there are non-contractible loops, which rules out the possibility of a homeomorphic extension.

Nonetheless, it is interesting to note that the flat topological closed balls in $\mathbb{R}^n$ have a simple characterization that follows from generalized Schönflies theorem due to Mazur [67, 66], Morse [75] and Brown [12]: *A topological closed ball D is flat if and only if ∂D is a topological submanifold, meaning that for every point $x \in \partial D$ there is a neighborhood $U \subset \mathbb{R}^n$ and a homeomorphism $H : U \twoheadrightarrow \mathbb{R}^n$ such that $H(U \cap \partial D) = \mathbb{R}^{n-1} \times \{0\}$.*

[1] For the precise definition of the Sullivan groups, see [94, Section 2.9]. They are explicitly used in the construction in [94, Lemma 3.3].

Remark 3.4. As explained in Remark 3.3, in the last sentence of Theorem 1.3, it suffices to assume that for every $i = 1, \dots, \ell$, D_i, D_i' is contained in the interior of a homeomorphic image of a closed ball, i. e., in a homeomorphic image of an open ball.

In the proof of Theorem 1.3, we employ the topological stable homeomorphisms and annulus theorems. We provide bibliographical details about proofs of these theorems in Section 3.4, when we discuss their statements.

While the above results seem very natural and possibly useful, quite surprisingly, we could not find such results in the literature. As the proof of Theorem 1.1 is straightforward and portrays well the use of Palais' trick, we present it separately and only later prove Theorems 1.2 and 1.3 invoking often nearly the same arguments as in the proof of Theorem 1.1.

In Section 3.2, we present Palais' trick for C^k-diffeomorphisms in $\mathbb{R}^n$ and we then prove Theorem 1.1 and Corollaries 3.1 and 3.2 in Section 3.3. The rest of this chapter concerns bi-Lipschitz homeomorphisms and homeomorphisms and, for brevity, we treat those two cases together since the arguments are the same. In Section 3.4, we discuss the stable homeomorphisms and annulus theorems and prove Lemma 3.18, which is central to the use of Palais' trick in the homeomorphic setting. In Section 3.5, we prove Theorems 1.2 and 1.3.

Acknowledgement. We would like to thank Igor Belegradek for directing us to the paper [83] of Palais.

3.2 Palais' trick

The next result says that any C^k-diffeomorphism can be linearized to identity on a sufficiently small ball.

Lemma 3.5. *Suppose that $H : \overline{B}(0,\varrho) \to \mathbb{R}^n$ is an orientation preserving C^k-diffeomorphism with $H(0) = 0$. Then there is a $\delta \in (0, \varrho/2)$ and a C^k-diffeomorphism $H_1 : \overline{B}(0,\varrho) \to \mathbb{R}^n$ such that*

$$H_1(x) = \begin{cases} H(x) & \text{for } x \in \overline{B}(0,\varrho) \setminus B(0,\varrho/2), \\ x & \text{for } x \in B(0,\delta). \end{cases} \tag{3.1}$$

Proof. The mapping H_1 is constructed with the help of a standard convex isotopy between H and $DH(0)$ and an isotopy between $DH(0)$ and identity.

By (2.5) in Lemma 2.13, we construct a C^k-diffeomorphism $G_1 : \overline{B}(0,\varrho) \to \mathbb{R}^n$, which equals $x \mapsto DH(0)x$ on $B(0, 3\varrho/8)$ and H on $\overline{B}(0,\varrho) \setminus B(0,\varrho/2)$. It is then possible to find a $\delta \in (0, \varrho/8)$ for which

$$B(0,\delta) \Subset DH(0)(B(0,\varrho/8)).$$

By Proposition 2.14, we find a C^∞-diffeomorphism $G_2 : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $G_2(x) = x$ for $x \in B(0,\delta)$ and $G_2(x) = DH(0)x$ for $x \in \mathbb{R}^n \setminus B(0,\varrho/8)$. We then set

$$H_1(x) = \begin{cases} G_2(x) & \text{for } x \in B(0,\varrho/4), \\ G_1(x) & \text{for } x \in \overline{B}(0,\varrho) \setminus B(0,\varrho/4). \end{cases}$$

Thus defined H_1 is a C^k-diffeomorphism because $G_2(x) = DH(0)x = G_1(x)$ near $\partial B(0,\varrho/4)$. Clearly, H_1 satisfies (3.1). $\square$

The next quite surprising result was proved by Palais [83] in the C^∞ case. The proof is short, elementary and beautiful.

Lemma 3.6. *Suppose that $H : \overline{B}(0,\varrho) \to \mathbb{R}^n$ is an orientation preserving C^k-diffeomorphism with $H(0) = 0$. Then for any $\varepsilon > 0$, there is a C^k-diffeomorphism $\widetilde{H} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that*

$$\widetilde{H}(x) = \begin{cases} H(x) & \text{if } x \in \overline{B}(0,\rho), \\ x & \text{if } \operatorname{dist}(x,A) \geq \varepsilon, \end{cases} \tag{3.2}$$

where $A = \overline{B}(0,\rho) \cup H(\overline{B}(0,\rho))$.

Remark 3.7. If $\widetilde{H} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is a diffeomorphism such that $\widetilde{H}(0) = 0$, but $\widetilde{H}(x) \neq x$ for $0 \neq x \in \overline{B}(0,\rho)$, then clearly $\widetilde{H}(x) \neq x$ for $0 \neq x \in A$ and hence $\widetilde{H}(x) \neq x$ in a neighborhood of ∂A. Therefore, the condition $\widetilde{H}(x) = x$ if $\operatorname{dist}(x,A) \geq \varepsilon$ in (3.2) is sharp.

Proof. Let $\varepsilon > 0$ be given. By assumption, H extends to a C^k-diffeomorphism on $B(0,\varrho + 3\tau)$ for some $\tau > 0$; we denote this extension by H as well. By decreasing $\tau > 0$ if necessary, we may assume that

$$B(0,\rho + 3\tau) \cup H(B(0,\rho + 3\tau)) \subset \{x :\, \operatorname{dist}(x,A) < \varepsilon\}.$$

By Lemma 3.5, there is a $\delta \in (0,\rho/2)$ and a C^k-diffeomorphism $H_1 : B(0,\rho + 3\tau) \to \mathbb{R}^n$ such that

$$H_1(x) = \begin{cases} H(x) & \text{if } x \in B(0,\rho + 3\tau) \setminus B(0,\rho/2), \\ x & \text{if } x \in B(0,\delta). \end{cases}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, smooth function satisfying

$$\phi(t) = \begin{cases} 1 & \text{if } t > \rho + 2\tau, \\ \delta(\rho+\tau)^{-1} & \text{if } t < \rho + \tau \end{cases}$$

and define $\Phi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ by $\Phi(x) := \phi(|x|)x$. The mapping Φ is obviously a smooth diffeomorphism of $\mathbb{R}^n$. Note also that $\Phi(x)$ on $B(0,\rho + \tau)$ acts as scaling by a factor of $\delta(\rho+\tau)^{-1}$, $\Phi(B(0,\rho+\tau)) = B(0,\delta)$, while $\Phi(x) = x$ when $|x| > \rho + 2\tau$. Since $B(0,\varrho + 3\tau) \subset \{\operatorname{dist}(x,A) < \varepsilon\}$, $\Phi = \mathrm{id}$ in $\{\operatorname{dist}(x,A) \geq \varepsilon\}$.

Consider the C^k-diffeomorphism

$$H_1 \circ \Phi^{-1} \circ H_1^{-1} : H_1(B(0,\rho + 3\tau)) \to \mathbb{R}^n. \tag{3.3}$$

It is a well defined diffeomorphism, because Φ maps the ball $B(0,\rho+3\tau)$ onto itself. The diffeomorphism defined in (3.3) is identity near the boundary of $H_1(B(0,\rho+3\tau))$, because

$$H_1 \circ \Phi^{-1} \circ H_1^{-1}(x) = x \quad \text{for} \quad x \in H_1\big(B(0,\rho+3\tau) \setminus B(0,\rho+2\tau)\big). \tag{3.4}$$

Indeed, Φ^{-1} is the identity on $B(0,\rho+3\tau) \setminus B(0,\rho+2\tau)$, so for x as in (3.4), we have $\Phi^{-1}(H_1^{-1}(x)) = H_1^{-1}(x)$. Therefore, the C^k-diffeomorphism (3.3) has the extension to a C^k-diffeomorphism of $\mathbb{R}^n$ by identity:

$$\widetilde{H_1 \circ \Phi^{-1} \circ H_1^{-1}} = \begin{cases} H_1 \circ \Phi^{-1} \circ H_1^{-1} & \text{in } H_1(B(0,\rho+3\tau)), \\ \mathrm{id} & \text{in } \mathbb{R}^n \setminus H_1(B(0,\rho+3\tau)). \end{cases} \tag{3.5}$$

Moreover, the mapping defined in (3.5) is identity in $\{x :\, \operatorname{dist}(x,A) \geq \varepsilon\}$, because $H_1(B(0,\rho+3\tau)) = H(B(0,\rho+3\tau)) \subset \{\operatorname{dist}(x,A) < \varepsilon\}$.

Now, we define the C^k-diffeomorphism

$$H_2 = \left(\widetilde{H_1 \circ \Phi^{-1} \circ H_1^{-1}}\right) \circ \Phi : \mathbb{R}^n \to \mathbb{R}^n. \tag{3.6}$$

Note that since Φ and the diffeomorphism defined in (3.5) are both equal identity in the set $\{\mathrm{dist}(x, A) \geq \varepsilon\}$, so does H_2.

If $x \in B(0, \rho + \tau)$, then $\Phi(x) = \delta(\rho + \tau)^{-1}x$ and therefore $\Phi(x) \in B(0, \delta)$. It follows from the definition of H_1 that

$$B(0, \delta) \subset H_1(B(0, \varrho + 3\tau)).$$

Thus by (3.5) for $x \in B(0, \varrho + \tau)$, we have

$$H_2(x) = H_1 \circ \Phi^{-1} \circ H_1^{-1}(\Phi(x)) = H_1 \circ \Phi^{-1}(\Phi(x)) = H_1(x),$$

because $H_1^{-1} = \mathrm{id}$ on $B(0, \delta)$.

Finally, since $H_2 = H_1 = H$ near $\partial B(0, \rho)$, we can set

$$\widetilde{H} := \begin{cases} H_2 & \text{on } \mathbb{R}^n \setminus B(0, \rho), \\ H & \text{on } B(0, \rho), \end{cases}$$

and clearly, $\widetilde{H}$ is a C^k-diffeomorphism that satisfies (3.2). □

Corollary 3.8. *Let $G : \overline{B}(a, r) \twoheadrightarrow \overline{B}(a, r)$ be an orientation preserving C^k-diffeomorphism, $k \in \mathbb{N} \cup \{\infty\}$. Then for any $\varepsilon > 0$, there is a C^k-diffeomorphism $\widetilde{G} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that*

$$\widetilde{G}(x) = \begin{cases} G(x) & \text{for } x \in \bar{B}(a, r) \\ x & \text{for } x \notin B(a, r + \varepsilon). \end{cases}$$

Proof. If $G(a) = a$, it is a straightforward consequence of Lemma 3.5. If not, we construct a C^∞-diffeomorphism $F : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $F(G(a)) = a$ and $F = \mathrm{id}$ in $\mathbb{R}^n \setminus B(a, r/2)$. One can construct F through a 1-parameter group of diffeomorphisms generated by a compactly supported vector field, see Corollary A.4 in the Appendix for a proof.

Let $T(x) = x - a$ and $G_1 = T \circ F \circ G \circ T^{-1}$. Then G_1 is an orientation preserving C^k-diffeomorphism of $\overline{B}(0, r)$ onto itself, $G_1(0) = 0$. For any $\varepsilon > 0$, we may thus apply Lemma 3.6 to G_1, obtaining a diffeomorphism $\widetilde{G}_1 : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that

$$\widetilde{G}_1(x) = \begin{cases} G_1(x) & \text{for } x \in \overline{B}(0, r), \\ x & \text{for } x \notin B(0, r + \varepsilon) \end{cases}$$

(note that $G_1(\overline{B}(0, r)) = \overline{B}(0, r)$). Now, $\widetilde{G} := F^{-1} \circ T^{-1} \circ \widetilde{G}_1 \circ T$ satisfies the claim of the lemma. □

3.3 Proof of Theorem 1.1

Lemma 3.9. *Let $\{D_i\}_{i=1}^{\ell}$, $D_i \subset \mathbb{R}^n$, be a family of pairwise disjoint C^k-diffeomorphic closed balls and let $p_i \in \mathring{D}_i$, $i = 1, \ldots, \ell$, be given. There is $\varepsilon_o > 0$ such that for any $\varepsilon \in (0, \varepsilon_o)$ there is a diffeomorphism $F_\varepsilon : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $F_\varepsilon(\overline{B}(p_i, \varepsilon)) = D_i$ for $i = 1, \ldots, \ell$ and*

$$F_\varepsilon(x) = x \text{ if } \mathrm{dist}\left(x, \bigcup_{i=1}^{\ell} D_i\right) \geq \varepsilon.$$

Remark 3.10. Clearly, the balls $\{\overline{B}(p_i,\varepsilon)\}_{i=1}^{\ell}$ are pairwise disjoint and

$$\overline{B}(p_i,\varepsilon)\subset\{x:\ \mathrm{dist}(x,\bigcup\nolimits_{i=1}^{\ell}D_i)\leq\varepsilon\}.$$

Proof. Set $\varepsilon_o:=\frac{1}{4}\min_{i,j}\mathrm{dist}(D_i,D_j)$ (obviously, $\varepsilon_o>0$) and let $\varepsilon\in(0,\varepsilon_o)$. By assumption, there are C^k-diffeomorphisms $H_i:\overline{B}(p_i,\varepsilon)\twoheadrightarrow D_i$. We can additionally assume that $H_i(p_i)=p_i$. Indeed, if $H_i(p_i)\neq p_i$, we compose H_i with a diffeomorphism of D_i which is identity near ∂D_i and maps $H_i(p_i)$ to p_i (cf. the proof of Corollary 3.8). Note that for each $i=1,\ldots,\ell$

$$A_i:=\overline{B}(p_i,\varepsilon)\cup H_i(\overline{B}(p_i,\varepsilon))$$

is a compact subset of the open set $\{x:\ \mathrm{dist}(x,D_i)<\varepsilon\}$ and hence there is $\tilde{\varepsilon}_i>0$ such that

$$\{x:\ \mathrm{dist}(x,D_i)\geq\varepsilon\}\subset\{x:\ \mathrm{dist}(x,A_i)\geq\tilde{\varepsilon}_i\}.$$

It follows from Lemma 3.6 with ε replaced by $\tilde{\varepsilon}_i$ that for each $i=1,\ldots,\ell$, there is a diffeomorphism $\widetilde{H}_i:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that

$$\widetilde{H}_i(x)=\begin{cases}H_i(x) & \text{if } x\in\overline{B}(p_i,\varepsilon),\\ x & \text{if } \mathrm{dist}(x,D_i)\geq\varepsilon.\end{cases}$$

Since $\varepsilon<\varepsilon_o$, the sets $\{\mathrm{dist}(x,D_i)<\varepsilon\}$ are pairwise disjoint and hence we can glue the diffeomorphisms $\widetilde{H}_i$ to a diffeomorphism $F_\varepsilon:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that $F_\varepsilon=\widetilde{H}_i$ on $\overline{B}(p_i,\varepsilon)$. $\square$

In the next lemma, we shall use Lemma 2.16. Actually, we do not need to know that the diffeomorphism H from Lemma 2.16 acts like translation on $\overline{B}(p_i,\varepsilon)$, it is enough to know that the ball $B(p_i,\varepsilon)$ is mapped onto the corresponding ball $B(q_i,\varepsilon)$. Combining Lemmata 3.9 and 2.16 we obtain the following result, which is a Euclidean variant of Theorem 1.1.

Lemma 3.11. *Let $U\subset\mathbb{R}^n$ be a domain. Suppose that $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i,D_i'\subset U$, are two families of pairwise disjoint C^k-diffeomorphic closed balls. If $F_i:D_i\twoheadrightarrow D_i'$, $i=1,2,\ldots,\ell$, are orientation preserving diffeomorphisms, then there is a diffeomorphism $F:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that $F|_{D_i}=F_i$ and $F=\mathrm{id}$ in $\mathbb{R}^n\setminus U$.*

Proof. Let $p_i\in\mathring{D}_i$ and $q_i\in\mathring{D}_i'$. There is $\tilde{\varepsilon}_1>0$ such that for any $\varepsilon\in(0,\tilde{\varepsilon}_1)$ the compact sets

$$\left\{x:\ \mathrm{dist}\left(x,\bigcup\nolimits_{i=1}^{\ell}D_i\right)\leq\varepsilon\right\}\quad\text{and}\quad\left\{x:\ \mathrm{dist}\left(x,\bigcup\nolimits_{i=1}^{\ell}D_i'\right)\leq\varepsilon\right\}\tag{3.7}$$

are contained in U. By Lemma 3.9, there is $\varepsilon_1\in(0,\tilde{\varepsilon}_1)$ such that for any $\varepsilon\in(0,\varepsilon_1)$ there are diffeomorphisms $A_1,\ A_2:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ which satisfy

$$A_1(\overline{B}(p_i,\varepsilon))=D_i,\ \text{and}\ A_2(\overline{B}(q_i,\varepsilon))=D_i'\qquad\text{for } i=1,\ldots,\ell$$

and that are equal to identity outside compact sets given in (3.7), respectively. Therefore, $A_1=A_2=\mathrm{id}$ in $\mathbb{R}^n\setminus U$. Let $\varepsilon_o\in(0,\varepsilon_1/2)$ be as in Lemma 2.16. Hence, for $\varepsilon\in(0,\varepsilon_o)$ there is a diffeomorphism $H:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ which equals identity in $\mathbb{R}^n\setminus U$ and maps each $\overline{B}(q_i,\varepsilon)$ to $\overline{B}(p_i,\varepsilon)$. Note also that $B(p_i,2\varepsilon)\subset U$.

Now, for any $i=1,\ldots,\ell$,

$$G_i:=H\circ A_2^{-1}\circ F_i\circ A_1:\overline{B}(p_i,\varepsilon)\twoheadrightarrow\overline{B}(p_i,\varepsilon)$$

is an orientation preserving diffeomorphism, thus by Corollary 3.8 there is a diffeomorphism $\widetilde{G}_i : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $\widetilde{G}_i|_{\overline{B}(p_i,\varepsilon)} = G_i$ and $\widetilde{G}_i$ equals identity outside $B(p_i, \frac{3}{2}\varepsilon)$. Setting

$$G(x) = \begin{cases} \widetilde{G}_i(x) & \text{if } x \in B(p_i, 2\varepsilon), \\ x & \text{otherwise,} \end{cases}$$

we get a diffeomorphism of $\mathbb{R}^n$ equal to identity outside $\bigcup_{i=1}^{\ell} B(p_i, \frac{3}{2}\varepsilon)$ and hence outside U. Finally, one easily checks that the composition $F = A_2 \circ H^{-1} \circ G \circ A_1^{-1}$ is a diffeomorphism of $\mathbb{R}^n$ which equals identity in $\mathbb{R}^n \setminus U$ (as a composition of diffeomorphisms of $\mathbb{R}^n$ with this property) and coincides with F_i on D_i for $i = 1, \dots, \ell$. Indeed, if $x \in D_i$, then $A_1^{-1}(x) \in \bar{B}(p_i, \varepsilon)$, thus

$$\begin{aligned} F(x) &= (A_2 \circ H^{-1} \circ G \circ A_1^{-1})(x) = (A_2 \circ H^{-1} \circ G_i \circ A_1^{-1})(x) \\ &= (A_2 \circ H^{-1} \circ \underbrace{H \circ A_2^{-1} \circ F_i \circ A_1}_{G_i} \circ A_1^{-1})(x) = F_i(x) \end{aligned}$$

□

Proof of Theorem 1.1. We prove the result by reducing it to the Euclidean setting, i.e., to the case treated in Lemma 3.11.

Let K' be a C^k-diffeomorphic closed ball contained in $\mathcal{M}^n$ which is disjoint from $\bigcup_{i=1}^{\ell}(D_i \cup D_i')$. We can find an orientation preserving C^k-diffeomorphism $H : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $H(D_i)$, $H(D_i')$ are contained in the interior of K'. This can be done by adapting in a standard way the Euclidean method from Lemma 2.17 to the case of a manifold. We prove it in Lemma A.18 in the Appendix and here present a sketch of the idea.

Let us for now assume that the family $\{D_i\}_{i=1}^{\ell}$ is disjoint from the family $\{D_i'\}_{i=1}^{\ell}$. For each ball D_i (and D_i') we create a finite chain of connected coordinate systems that connect D_i (and D_i') to K' in a way that consecutive systems have nonempty overlapping. Moreover, we assume that the first coordinate system in the chain contains a neighborhood of D_i (D_i'). Then we construct a diffeomorphism H_i (and H_i') as a composition of diffeomorphisms defined in the local coordinate systems that move the ball D_i from one coordinate system to the next one. We can guarantee that on the set where H_i or H_i' differs from the identity, all other diffeomorphisms H_j and H_j' are equal identity. Finally we define H as a composition of all diffeomorphisms H_i and H_i'. If $D_i \cap D_j'$ for some $i, j \in \{1, \dots, \ell\}$, one has to slightly modify the construction of $H_i = H_j'$ so that the sum $D_i \cup D_j'$ is mapped into $\mathring{K}'$ without spoiling the construction of other diffeomorphisms.

Let $\Phi : \overline{\mathbb{B}}^n \twoheadrightarrow K'$ be the C^k-diffeomorphism between the Euclidean ball and K'. Set $E_i := \Phi^{-1}(H(D_i))$ and $E_i' := \Phi^{-1}(H(D_i'))$ and $\widehat{F}_i := \Phi^{-1} \circ H \circ F_i \circ H^{-1} \circ \Phi|_{E_i}$. The map $\widehat{F}_i$ is an orientation preserving C^k-diffeomorphism, E_i and E_i' are C^k-diffeomorphic closed balls and $\widehat{F}_i$ maps E_i onto E_i'. Moreover, it follows from Corollary 2.3 that E_i, E_i' are contained in $\mathbb{B}^n$. Consequently, there is a $\varrho \in (0,1)$ for which $E_i, E_i' \subset B(0,\varrho)$ for $i = 1, \dots, \ell$. By Lemma 3.11, there is a C^k-diffeomorphism $\widehat{F} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ which satisfies

$$\widehat{F}|_{E_i} = \widehat{F}_i \quad \text{and} \quad \widehat{F} = \mathrm{id} \ \text{ in } \mathbb{R}^n \setminus B(0,\varrho).$$

Then the composition $\Phi \circ \widehat{F} \circ \Phi^{-1} : K' \twoheadrightarrow K'$ is a C^k-diffeomorphism, which equals identity near $\partial K'$ and thus can be extended by identity to a C^k-diffeomorphism $G : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$. Eventually, set

$$F := H^{-1} \circ G \circ H.$$

This is a C^k-diffeomorphism of $\mathcal{M}^n$ such that $F|_{D_i} = F_i$, which can be easily checked in the same manner as in Lemma 3.11.

If all the sets D_i and D_i' for $i = 1, \ldots, \ell$ are contained in a C^k-diffeomorphic closed ball $K \subset \mathcal{M}^n$, then K' can be chosen to equal K and thus $H = \mathrm{id}$ and $F = G = \mathrm{id}$ outside K. □

Example 3.12. In his celebrated article [69] introducing exotic 7-spheres, Milnor proved the existence of an orientation preserving C^∞-diffeomorphism $f : \mathbb{S}^6 \twoheadrightarrow \mathbb{S}^6$ which is not C^∞-isotopic to identity ([69, Theorem 5]). This diffeomorphism can be extended radially to a C^∞-diffeomorphism $\tilde{f} : \mathbb{R}^7 \setminus B(0,r) \to \mathbb{R}^7$ for any $r > 0$. However, it cannot be extended to a C^∞-diffeomorphism of the entire $\mathbb{R}^7$.

Assume the contrary, i. e., that there is an orientation preserving C^∞-diffeomorphism $\widetilde{G} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ with $G = f$ on $\partial\mathbb{B}^7$. Then by Lemma 3.5, it is possible to find a C^∞-diffeomorphism G such that $G = f$ near $\partial\mathbb{B}^7$ and $G = \mathrm{id}$ on $B^7(0,r)$ for some $r \in (0,1)$. This, in turn, provides a C^∞-diffeomorphism

$$F : \mathbb{S}^7 \times [1,r] \twoheadrightarrow \mathbb{S}^7 \times [1,r], \quad F(x,1) = f(x) \quad \text{and} \quad F(x,r) = x.$$

Such a diffeomorphism need not be an isotopy (the latter must preserve the sets $\mathbb{S}^7 \times \{t\}$ for $t \in [1,r]$) but it is a pseudoisotopy between f and the identity. Since we discuss diffeomorphisms of $\mathbb{S}^6$, by Cerf's pseudoisotopy-to-isotopy theorem ([18], see also [58, Theorem 2.71]) this implies that f is isotopic to the identity, which is a contradiction.

Also, if we consider $\mathbb{S}^6$ to be embedded into $\mathbb{S}^7$, we see that the same f can be extended to a C^∞-diffeomorphism on an annulus $A \subset \mathbb{S}^7$ but cannot be extended to a C^∞-diffeomorphism of $\mathbb{S}^7$.

Proof of Corollary 3.1. Let $\widetilde{F}$ denote the assumed extension of F onto $\mathbb{R}^n$, i. e., $\widetilde{F} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ and $\widetilde{F} = F$ on $\mathbb{R}^n \setminus \mathring{D}_1$. By assumption, $G(D_2) \subset \widetilde{F}(\mathring{D}_1)$ and hence $\widetilde{F}^{-1}(G(D_2)) \subset \mathring{D}_1$. By Lemma 3.11, we can find a C^k-diffeomorphism $\widetilde{H} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $\widetilde{H} = \widetilde{F}^{-1} \circ G$ on D_2 and $\widetilde{H} = \mathrm{id}$ outside $\mathring{D}_1$. We set

$$H := \widetilde{F} \circ \widetilde{H}.$$

It is clear that H is the required C^k-diffeomorphism. □

The proof of Corollary 3.2 is analogous to the proof of Corollary 3.1, we provide it for completeness.

Proof of Corollary 3.2. Note that $D_1' := G^{-1} \circ F(D_1) \subset \mathring{D}_2$. The mapping $G^{-1} \circ F$ is a C^k-diffeomorphism, which maps D_1 onto D_1'. Both D_1 and D_1' are C^k-diffeomorphic closed balls and are contained in $\mathring{D}_2$. Consequently, $D_1, D_2 \subset U$ for some domain $U \Subset \mathring{D}_2$.

By Lemma 3.11, we find a diffeomorphism $\widetilde{H} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $\widetilde{H} = G^{-1} \circ F$ on D_1 and $\widetilde{H} = \mathrm{id}$ in $\mathbb{R}^n \setminus U$. Clearly, $\widetilde{H}(\mathring{D}_2) = \mathring{D}_2$ and therefore the mapping

$$H = G \circ \widetilde{H}|_{\mathring{D}_2}$$

is a well-defined C^k-diffeomorphism. Moreover, H equals F on D_1 and G on $\mathring{D}_2 \setminus U$, i. e., $F = G$ near ∂D_2. □

3.4 Topological local linearization

The main result of this section is Lemma 3.21, which is a homeomorphic version of Lemma 3.5. The homeomorphic version is much more difficult as it requires stable homeomorphisms and annulus theorems, which we discuss here.

Definition 3.13. Let $\varrho > 0$. We say that a homeomorphism $F : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ (or $F : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$) is *stable* if

$$F = f_1 \circ \ldots \circ f_k \tag{3.8}$$

for some $k \in \mathbb{N}$ and homeomorphisms $f_i : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ ($f_i : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$) such that $f_i|_{U_i} = \mathrm{id}$ for some nonempty open set $U_i \subset \mathbb{R}^n$ ($U_i \subset B(0,\varrho)$).

We say that a homeomorphism $F : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ (or $F : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$) is *locally bi-Lipschitz stable* if F can be written as in (3.8) for some $k \in \mathbb{N}$ and for locally bi-Lipschitz homeomorphisms $f_i : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ ($f_i : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$) such that $f_i|_{U_i} = \mathrm{id}$ for some nonempty open set $U_i \subset \mathbb{R}^n$ ($U_i \subset B(0,\varrho)$).

Remark 3.14. Without loss of generality, we assume that the U_i in the definition above are pairwise disjoint balls with the same radius, compactly contained in $B(0,\varrho)$.

The stable homeomorphism theorem in the topological case in dimensions $n = 2, 3$ is a classical result due to Radó [86] for $n = 2$ and Moise [71] for $n = 3$. For $n > 4$, it was proved by Kirby in [61] and for $n = 4$ by Quinn in [85] (see page 1 and Theorem 2.2.2). For a short explanation of the intricacies of the proof in the topological case, we refer to [45]. In the bi-Lipschitz case, the stable homeomorphism theorem was derived in [94, Theorem 3.12] from the existence of groups known as Sullivan groups, whose existence was claimed by Sullivan in [89].

Lemma 3.15 (Stable homeomorphisms theorem)**.** *Any orientation preserving homeomorphism $F : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is stable. If F is additionally locally bi-Lipschitz, then F is locally bi-Lipschitz stable.*

Corollary 3.16. *An orientation preserving homeomorphism $F : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ is stable. If, additionally, F is locally bi-Lipschitz, then F is also locally bi-Lipschitz stable.*

Proof. Let $\varphi : \mathbb{R}^n \twoheadrightarrow B(0,\varrho)$ be an orientation preserving homeomorphism. Then $\widetilde{F} = \varphi^{-1} \circ F \circ \varphi$ is an orientation preserving homeomorphism of the entire $\mathbb{R}^n$, so by Lemma 3.15 it can be decomposed as in (3.8) as

$$\widetilde{F} = \tilde{f}_1 \circ \ldots \circ \tilde{f}_k, \quad \tilde{f}_i = \mathrm{id} \ \text{ on } U_i.$$

for some open nonempty sets $U_i \subset \mathbb{R}^n$. We then set $f_i := \varphi \circ \tilde{f}_i \circ \varphi^{-1} : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$. Clearly, $f_i = \mathrm{id}$ on the open set $\varphi(U_i)$ and $F = f_1 \circ \ldots \circ f_k$.

We can choose φ to be locally bi-Lipschitz and then $\widetilde{F}$ is locally bi-Lipschitz given a locally bi-Lipschitz F. By Lemma 3.15, each $\tilde{f}_i$ is locally bi-Lipschitz and, consequently, so is f_i. This implies that F is locally bi-Lipschitz stable and finishes the proof. $\square$

We shall show in Lemma 3.18 that it is possible to locally linearize a homeomorphism or a bi-Lipschitz homeomorphism F of a ball $B(0,\varrho)$ onto itself. The assumption that $F(B(0,\varrho)) = B(0,\varrho)$ is important as it allows us to use (bi-Lipschitz) stable homeomorphism theorem. Before that, we recall a certain well-known fact.

Remark 3.17. If $f : \mathbb{B}^n \twoheadrightarrow \mathbb{B}^n$ is a homeomorphism, then for any sequence $\{x_k\}_{k=1}^{\infty} \subset \mathbb{B}^n$,

$$\lim_{k\to\infty} |x_k| = 1 \implies \lim_{k\to\infty} |f(x_k)| = 1. \tag{3.9}$$

Indeed, fix $\varepsilon \in (0,1)$. The set $f^{-1}(\overline{B}(0, 1-\varepsilon))$ is compact and hence it is contained in a ball $B(0,\varrho)$ for some $\varrho \in (0,1)$. For $x_k \in \mathbb{B}^n$ with $|x_k| \to 1$, there is a k_o such that for all $k \geq k_o$, $|x_k| > \varrho$. Consequently, $x_k \notin f^{-1}(\overline{B}(0, 1-\varepsilon))$. Therefore, for such x_k, $|f(x_k)| \geq 1-\varepsilon$. The claim follows from arbitrariness of ε.

Lemma 3.18. *Let $F : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ be an orientation preserving homeomorphism. Then for any $\delta \in (0,\varrho)$, there is a homeomorphism $\widehat{F} : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ such that*

$$\widehat{F}(x) = \begin{cases} F(x) & \text{for } x \text{ near } \partial B(0,\varrho), \\ x & \text{for } x \in B(0,\delta). \end{cases} \tag{3.10}$$

If F is locally bi-Lipschitz, then $\widehat{F}$ is locally bi-Lipschitz as well.

Proof. By Corollary 3.16, there are homeomorphisms $f_i : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$, $i = 1,\ldots,k$ for $k \in \mathbb{N}$, and balls $U_i \subset B(0,\varrho)$ of the same radius such that

$$F = f_1 \circ \ldots \circ f_k, \quad f_i|_{U_i} = \mathrm{id}\,. \tag{3.11}$$

We can find $\tau \in (0,1)$ for which $B(0,\tau)$ is disjoint from U_i for every $i = 1,\ldots,k$. It follows from Remark 2.18 that there is a diffeomorphism $\psi_1 : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ which maps $B(0,\tau)$ onto U_1 and equals identity near $\partial B(0,\varrho)$. Then $\widetilde{F}_1 := \psi_1^{-1} \circ f_1 \circ \psi_1$ is a homeomorphism of $B(0,\varrho)$ s. t. $\widetilde{F}_1(x) = x$ for $x \in B(0,\tau)$ and $\widetilde{F}_1(x) = f_1(x)$ for x near $\partial B(0,\varrho)$. Indeed, as recalled in Remark 3.17, given any $\varepsilon > 0$, we can find $\eta \in (0, \rho - \eta)$ such that if $\varrho - \eta < |x| < \varrho$, then $\varrho - \varepsilon < |f_1(x)| < \varrho$ so that both x and $f_1(x)$ lie in the set in which $\psi = \mathrm{id}$. If f_1 is locally bi-Lipschitz, so is $\widetilde{F}_1$.

Assume that we have found $\widetilde{F}_{j-1}$ for $j = 1,\ldots,k$ such that $\widetilde{F}_{j-1} = f_1 \circ \ldots \circ f_{j-1}$ near $\partial B(0,\varrho)$ and $\widetilde{F}_{j-1} = \mathrm{id}$ on $B(0,\tau)$. Then, we find a diffeomorphism $\psi_j : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$, which maps $B(0,\tau)$ onto U_j and equals identity near $\partial B(0,\varrho)$. Then $\widetilde{F}_j := \psi_j^{-1} \circ \widetilde{F}_{j-1} \circ f_j \circ \psi_j$ is a homeomorphism of $B(0,\varrho)$, which equals $f_1 \ldots \circ f_j$ near $\partial B(0,\varrho)$ and identity on $B(0,\tau)$. If $\widetilde{F}_{j-1}$ is locally bi-Lipschitz, so is $\widetilde{F}_j$.

Finally, set $\widetilde{F} := \widetilde{F}_k$ so that $F = \mathrm{id}$ on $B(0,\delta)$ and by (3.11), $\widetilde{F} = F$ near $\partial B(0,\varrho)$. If F is locally bi-Lipschitz, then by Corollary 3.16, each f_i is locally bi-Lipschitz and hence $\widetilde{F}_j$ for all $j = 1,\ldots,k$ is locally bi-Lipschitz as well. In particular, $\widetilde{F}$ is locally bi-Lipschitz.

Note that $\widetilde{F}$ satisfies (3.10) for $\delta = \tau$. To correct it, we find a diffeomorphism $\Phi : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$, $\Phi = \mathrm{id}$ near $\partial B(0,\varrho)$ which acts like scaling by a factor $\tau\delta^{-1}$ on $B(0,\delta)$ (cf. proof of Lemma 3.6). Then $\widehat{F} := \Phi^{-1} \circ \widetilde{F} \circ \Phi$ is the required mapping. If $\widetilde{F}$ is locally bi-Lipschitz, so is $\widehat{F}$. □

Albeit we will not use it, we note an interesting corollary stating that, according to the phrase coined in this section, a homeomorphism can be locally topologically linearized if and only if it is stable. By stable homeomorphisms theorem, it is equivalent to being orientation preserving.

Corollary 3.19. *Let $F : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ be a (locally bi-Lipschitz) homeomorphism. Then for any $\delta \in (0,\varrho)$, there exists a (locally bi-Lipschitz) homeomorphism $\widetilde{F} : B(0,\varrho) \twoheadrightarrow B(0,\varrho)$ satisfying (3.10) if and only if F is stable (or F is locally bi-Lipschitz stable).*

Proof. Indeed, the proof of Lemma 3.18 implies that if F is (locally bi-Lipschitz) stable, then there exists such $\widetilde{F}$. To prove the reverse implication, let $r \in (\delta, \varrho)$ be such that $\widetilde{F} = F$ on $B(0,\varrho) \setminus B(0,r)$. Then set

$$H(x) := \begin{cases} x & \text{for } B(0,\varrho) \setminus \widetilde{F}(B(0,r)), \\ F \circ \widetilde{F}^{-1}(x) & \text{for } x \in \widetilde{F}(B(0,r)). \end{cases}$$

Since $\widetilde{F} = F$ on $B(0,\varrho)\setminus B(0,r)$, H indeed is a homeomorphism of $B(0,\varrho)$[2]. Then $F = H \circ \widetilde{F}$ and since both H and $\widetilde{F}$ are equal identity on a non-empty open set in $\mathbb{R}^n$, F is stable. If F and $\widetilde{F}$ are locally bi-Lipschitz, then so is H and hence F is locally bi-Lipschitz stable. $\square$

We say that a set $S \subset \mathbb{R}^n$ is a *flat topological (or flat bi-Lipschitz) sphere* if there is a (bi-Lipschitz) homeomorphism $F : \mathbb{S}^{n-1} \twoheadrightarrow S$, which extends as a (bi-Lipschitz) homeomorphism onto some neighborhood of $\mathbb{S}^{n-1}$.

Lemma 3.20 (Annulus theorem)**.** *Let $S_1, S_2 \subset \mathbb{R}^n$ be two disjoint flat topological (or flat bi-Lipschitz) spheres. Then the compact set bounded by S_1 and S_2 is (bi-Lipschitz) homeomorphic to the annulus $\overline{\mathbb{B}}^n \setminus B^n(0,1/2)$.*

In the bi-Lipschitz case, the annulus theorem was proved in [94, Theorem 3.12], following [89]. Earlier, Brown and Gluck considered the topological case and showed in [13] that if the stable homeomorphisms theorem is true (it was not known then), so is the annulus theorem. It follows from Corollary on page 8 and Theorem 3.5 (i). In fact, this yields Theorem 1.3 for $\ell = 1$. It is expected that by modification of the methods of Brown and Gluck, one could prove Theorem 1.3 for $\ell > 1$. We shall show it using the trick of Palais, to which end we need the local linearization Lemma 3.21 below. In the bi-Lipschitz case this lemma follows from [94, Theorem 3.16]; since we present the proof for topological case we include the bi-Lipschitz one as well, as the proof is the same.

By a radial extension of a homeomorphism $f : \partial B(0,\varrho) \twoheadrightarrow \partial B(0,\varrho)$ we mean a mapping $\tilde{f} : \mathbb{B}^n \twoheadrightarrow \mathbb{B}^n$ defined as $\tilde{f}(x) = |x| f(x/|x|)$ for $x \neq 0$ and $\tilde{f}(0) = 0$. It is easy to check that $\tilde{f}$ is a homeomorphism and that if f is bi-Lipschitz, then so is $\tilde{f}$.

Lemma 3.21. *Let $F : \overline{B}(0,\varrho) \to \mathbb{R}^n$, $F(0) = 0$, be an orientation preserving (bi-Lipschitz) homeomorphism which can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of $\overline{B}(0,\varrho)$. There is a $\delta \in (0,\varrho/2)$ and a (bi-Lipschitz) homeomorphism $\widetilde{F} : \overline{B}(0,\varrho) \to \mathbb{R}^n$ such that $\widetilde{F} = F$ near $\partial B(0,\varrho)$ and $F = \mathrm{id}$ on $\overline{B}(0,\delta)$.*

Let us stress the difference between Lemma 3.18, where we dealt with homeomorphisms *of* a given ball, and Lemma 3.21. In Lemma 3.21, we linearize homeomorphisms which might not be *onto* a ball. That is why we need annulus theorem.

Proof. Firstly, note that since $F(0) = 0$ and homeomorphisms of subsets of $\mathbb{R}^n$ map interior points to interior points, there is a $\delta \in (0,\varrho/2)$ for which $\overline{B}(0,2\delta) \subset F(B(0,\varrho))$. Let $r \in (2\delta, \varrho)$.

The set $\partial F(B(0,r)) = F(\partial B(0,r))$ is a flat topological sphere; let A denote the compact region between $\partial F(B(0,r))$ and $\partial B(0,2\delta)$. By Lemma 3.20, there is a homeomorphism $H : \overline{B}(0,r) \setminus B(0,2\delta) \twoheadrightarrow A$ and therefore

$$H^{-1} \circ F : \partial B(0,r) \twoheadrightarrow \partial B(0,r) \text{ and } H : \partial B(0,2\delta) \twoheadrightarrow \partial B(0,2\delta).$$

[2]We showed it in details in Corollary 2.12

We can find radial extensions of $H^{-1}\circ F$ and H to homeomorphisms $\Phi : \overline{B}(0,r) \twoheadrightarrow \overline{B}(0,r)$ and $\Psi : B(0,2\delta) \twoheadrightarrow B(0,2\delta)$, respectively. By Lemma 3.18, we find homeomorphisms $\widetilde{\Phi} : \overline{B}(0,r) \twoheadrightarrow \overline{B}(0,r)$ and $\widetilde{\Psi} : B(0,2\delta) \twoheadrightarrow B(0,2\delta)$, such that $\widetilde{\Phi} = \Phi$ near $\partial B(0,r)$ and $\widetilde{\Phi} = \mathrm{id}$ on $B(0,2\delta)$ and $\widetilde{\Psi} = \Psi$ near $\partial B(0,2\delta)$ and $\widetilde{\Psi} = \mathrm{id}$ on $B(0,\delta)$. Eventually, we set

$$\widetilde{F} = \begin{cases} F(x) & \text{for } x \in \overline{B}(0,\varrho) \setminus \overline{B}(0,r), \\ H(\widetilde{\Phi}(x)) & \text{for } x \in \overline{B}(0,r) \setminus \overline{B}(0,2\delta), \\ \widetilde{\Psi} & \text{for } x \in \overline{B}(0,2\delta). \end{cases}$$

Since $\widetilde{F}(\overline{B}(0,r) \setminus B(0,2\delta)) = A$ and $H \circ \widetilde{\Phi}$ agrees with F on $\partial B(0,r)$ and $\widetilde{\Psi} = H$ on $\partial B(0,2\delta)$, $\widetilde{F}$ is indeed a homeomorphism. It can be easily checked that it satisfies the required properties.

If F is additionally assumed to be a bi-Lipschitz homeomorphism which can be extended as a bi-Lipschitz homeomorphism on a neighborhood of $\overline{B}(0,\varrho)$, then by Lemma 3.20 H is a bi-Lipschitz homeomorphism and so are the radial extensions Φ and Ψ. Therefore, with the help of Lemma 3.18, $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are bi-Lipschitz and hence so is $\widetilde{F}$. □

Note that in the statement of Lemma 3.21 we can replace 'there is a $\delta \in (0,\varrho/2)$' with 'for any $\delta \in (0,\varrho/2)$'. Indeed, at the end of the proof it suffices to use the scaling diffeomorphism Φ exactly like in the proof of Lemma 3.18.

3.5 Proof of Theorems 1.2 and 1.3

Lemma 3.22. *Suppose that $H : \overline{B}(0,\varrho) \to \mathbb{R}^n$, $H(0) = 0$, is an orientation preserving (bi-Lipschitz) homeomorphism which can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of $\overline{B}(0,\varrho)$. Then for any $\varepsilon > 0$, there is a (bi-Lipschitz) homeomorphism $\widetilde{H} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that*

$$\widetilde{H}(x) = \begin{cases} H(x) & \text{if } x \in \overline{B}(0,\rho), \\ x & \text{if } \operatorname{dist}(x,A) \geq \varepsilon, \end{cases} \tag{3.12}$$

where $A = \overline{B}(0,\rho) \cup H(\overline{B}(0,\rho))$.

Proof. Let $\varepsilon > 0$ be given. By Lemma 3.21, there is a (bi-Lipschitz) homeomorphism $H_1 : \overline{B}(0,\varrho) \to \mathbb{R}^n$ and $\delta \in (0,\varrho/2)$ such that $H_1 = \mathrm{id}$ on $B(0,\delta)$ and $H_1 = H$ near $\partial B(0,\varrho)$. By assumption, H extends as a (bi-Lipschitz) homeomorphism on $B(0,\varrho+3\tau)$ for some $\tau > 0$ and we may assume that

$$B(0,\rho+3\tau) \cup H(B(0,\rho+3\tau)) \subset \{x : \operatorname{dist}(x,A) < \varepsilon\}.$$

As $H_1 = H$ near $\partial B(0,\varrho)$, H_1 extends in the same way; we denote this extension with H_1.

We define $\phi : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ as in the proof of Lemma 3.6. Note that Φ is a diffeomorphism equal identity outside a compact set, hence it is bi-Lipschitz. The mapping Φ acts as scaling by $\delta(\varrho+\tau)^{-1}$, $\Phi(B(0,\varrho+\tau)) = B(0,\delta)$ and $\Phi(x) = x$ when $|x| > \varrho + 2\tau$.

The mapping

$$H_1 \circ \Phi^{-1} \circ H_1^{-1} : H_1(B(0,\rho+3\tau)) \to \mathbb{R}^n. \tag{3.13}$$

is a well defined homeomorphism, because Φ maps the ball $B(0,\rho+3\tau)$ onto itself. As explained in the proof of Lemma 3.6, the homeomorphism defined in (3.13) is identity near the boundary of $H_1(B(0,\rho+3\tau))$ and can be extended to a (bi-Lipschitz) homeomorphism of $\mathbb{R}^n$ by identity:

$$\widetilde{H_1\circ\Phi^{-1}\circ H_1^{-1}} = \begin{cases} H_1\circ\Phi^{-1}\circ H_1^{-1} & \text{in } H_1(B(0,\rho+3\tau)),\\ \mathrm{id} & \text{in } \mathbb{R}^n\setminus H_1(B(0,\rho+3\tau)). \end{cases} \tag{3.14}$$

Now we define the (bi-Lipschitz) homeomorphism

$$H_2 = \big(\widetilde{H_1\circ\Phi^{-1}\circ H_1^{-1}}\big)\circ\Phi : \mathbb{R}^n\to\mathbb{R}^n.$$

It can be checked exactly like in the proof of Lemma 3.6 that $H_2 = H_1$ on $B(0,\varrho+\tau)$ and that $H_2(x) = x$ if $\operatorname{dist}(x,A)\geq\varepsilon$. Eventually, since $H_2 = H_1$ on $B(0,\varrho+\tau)$ and $H_1 = H$ near $\partial B(0,\varrho)$, we have $H_2 = H$ near $\partial B(0,\varrho)$ and we can set

$$\widetilde{H} := \begin{cases} H_2 & \text{on } \mathbb{R}^n\setminus B(0,\rho),\\ H & \text{on } B(0,\rho). \end{cases}$$

Then $\widetilde{H}$ is a homeomorphism that satisfies (3.12). If H is bi-Lipschitz, so is H_2 and so is $\widetilde{H}$. □

Following the proof of Corollary 3.8 yields

Corollary 3.23. *Let $G:\overline{B}(a,r)\twoheadrightarrow\overline{B}(a,r)$ be an orientation preserving (bi-Lipschitz) homeomorphism which can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of $\overline{B}(a,r)$. Then for any $\varepsilon>0$, there is a (bi-Lipschitz) homeomorphism $\widetilde{G}:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that $\widetilde{G}=G$ on $\overline{B}(a,r)$ and $\widetilde{G}=\mathrm{id}$ outside $B(a,r+\varepsilon)$.*

Next, we proceed with the proof of Theorems 1.2 and 1.3 following closely that of Theorem 1.1.

Lemma 3.24. *Let $\{D_i\}_{i=1}^{\ell}$, $D_i\subset\mathbb{R}^n$, be a family of pairwise disjoint flat topological (or flat bi-Lipschitz) closed balls and let $p_i\in\mathring{D}_i$, $i=1,\dots,\ell$ be given. Then there is an $\varepsilon_o>0$ such that for any $\varepsilon\in(0,\varepsilon_o)$ there is a (bi-Lipschitz) homeomorphism $F_\varepsilon:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that $F_\varepsilon(\overline{B}(p_i,\varepsilon))=D_i$ for $i=1,\dots,\ell$ and*

$$F_\varepsilon(x) = x \text{ if } \operatorname{dist}\Big(x,\bigcup\nolimits_{i=1}^{\ell} D_i\Big)\geq\varepsilon.$$

Proof. Set $\varepsilon_o := \frac14\min_{i,j}\operatorname{dist}(D_i,D_j)$ and choose $\varepsilon\in(0,\varepsilon_o)$. By assumption, for any $i=1,\dots,\ell$, there is a (bi-Lipschitz) homeomorphism $H_i:\overline{B}(p_i,\varepsilon)\twoheadrightarrow D_i$ which can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of $\overline{B}(p_i,\varepsilon)$. As explained in the proof of Lemma 3.9, we can assume that $H_i(p_i)=p_i$. Note that for each $i=1,\dots,\ell$

$$A_i := \overline{B}(p_i,\varepsilon)\cup H_i(\overline{B}(p_i,\varepsilon))$$

is a compact subset of the open set $\{\operatorname{dist}(x,D_i)<\varepsilon\}$ and hence there is $\tilde\varepsilon_i>0$ such that

$$\{x:\ \operatorname{dist}(x,D_i)\geq\varepsilon\}\subset\{x:\ \operatorname{dist}(x,A_i)\geq\tilde\varepsilon_i\}.$$

It follows from Lemma 3.22 with ε replaced by $\tilde\varepsilon_i$ that there are (bi-Lipschitz) homeomorphisms $\widetilde{H}_i:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that

$$\widetilde{H}_i(x) = \begin{cases} H_i(x) & \text{if } x\in\overline{B}(p_i,\varepsilon),\\ x & \text{if } \operatorname{dist}(x,D_i)\geq\varepsilon. \end{cases}$$

Since $\varepsilon<\varepsilon_o$, the sets $\{\operatorname{dist}(x,D_i)<\varepsilon\}$ are pairwise disjoint and hence we can glue the (bi-Lipschitz) homeomorphisms $\widetilde{H}_i$ to a (bi-Lipschitz) homeomorphism $F_\varepsilon:\mathbb{R}^n\twoheadrightarrow\mathbb{R}^n$ such that $F_\varepsilon=\widetilde{H}_i$ on $\overline{B}(p_i,\varepsilon)$. □

As in Section 3.3, combining Lemma 2.16 with Lemma 3.24 yields the Euclidean version of Theorems 1.2 and 1.3. However, before we proceed, we note that each (bi-Lipschitz) homeomorphism F_i, $i = 1, \ldots, \ell$ in Theorems 1.2 and 1.3 can in fact be extended as a (bi-Lipschitz) homeomorphism on some neighborhood of D_i.

Remark 3.25. Let $\mathcal{M}^n$ be an oriented and connected n-dimensional topological (or bi-Lipschitz) manifold and let D, $D \subset \mathcal{M}^n$, be a flat topological (or flat bi-Lipschitz) closed ball. Then the (bi-Lipschitz) homeomorphism $F : D \twoheadrightarrow D'$ can be extended as a (bi-Lipschitz) homeomorphism on some neighborhood of D if and only if D' is a flat topological (or flat bi-Lipschitz) closed ball.

Indeed, if F can be extended as a (bi-Lipschitz) homeomorphism, then D' is a flat topological (or flat bi-Lipschitz) closed ball by definition. On the other hand, if both D and D' are flat topological (or flat bi-Lipschitz) closed balls, then there is $\varepsilon > 0$ and (bi-Lipschitz) homeomorphisms Φ, $\Psi : B^n(0, 1+\varepsilon) \to \mathcal{M}^n$ such that $\Phi(\bar{\mathbb{B}}^n) = D$ and $\Psi(\bar{\mathbb{B}}^n) = D'$. Then $\widetilde{G} := \Psi^{-1} \circ F \circ \Phi|_{\bar{\mathbb{B}}^n}$ is a (bi-Lipschitz) homeomorphism of $\bar{\mathbb{B}}^n$. It is standard to check that a radial extension yields a (bi-Lipschitz) homeomorphism G on $B^n(0, 1+\varepsilon)$. Then $\Psi \circ G \circ \Phi^{-1}$ is the required (bi-Lipschitz) homeomorphism which extends F onto $\Phi(B^n(0, 1+\varepsilon))$, which is a neighborhood of D.

Lemma 3.26. *Let $U \subset \mathbb{R}^n$ be a domain. Suppose that $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset U$, are two families of pairwise disjoint flat topological (or flat bi-Lipschitz) closed balls. If $F_i : D_i \twoheadrightarrow D_i'$, $i = 1, 2, \ldots, \ell$, are orientation preserving (bi-Lipschitz) homeomorphisms, then there is a (bi-Lipschitz) homeomorphism $F : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $F|_{D_i} = F_i$ and $F = \mathrm{id}$ in $\mathbb{R}^n \setminus U$.*

Proof. Let $p_i \in \mathring{D}_i$ and $q_i \in \mathring{D}_i'$. There is $\tilde{\varepsilon}_1 > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_1)$ the compact sets
$$\Big\{x : \operatorname{dist}\Big(x, \bigcup_{i=1}^{\ell} D_i\Big) \le \varepsilon\Big\} \quad \text{and} \quad \Big\{x : \operatorname{dist}\Big(x, \bigcup_{i=1}^{\ell} D_i'\Big) \le \varepsilon\Big\} \tag{3.15}$$
are contained in U. According to Lemma 3.24, there is $\varepsilon_1 \in (0, \tilde{\varepsilon}_1)$ such that for any $\varepsilon \in (0, \varepsilon_1)$, there are (bi-Lipschitz) homeomorphisms A_1, $A_2 : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ which satisfy
$$A_1(\overline{B}(p_i, \varepsilon)) = D_i, \text{ and } A_2(\overline{B}(q_i, \varepsilon)) = D_i' \qquad \text{for } i = 1, \ldots, \ell$$
and which are equal to identity outside compact sets given in (3.15), respectively. Therefore, $A_1 = A_2 = \mathrm{id}$ in $\mathbb{R}^n \setminus U$. Let $\varepsilon_o \in (0, \varepsilon_1)$ be as in Lemma 2.16. Hence, for $\varepsilon \in (0, \varepsilon_o)$ there is a (bi-Lipschitz) homeomorphism $H : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ which equals identity in $\mathbb{R}^n \setminus U$ and maps each $\overline{B}(q_i, \varepsilon)$ to $\overline{B}(p_i, \varepsilon)$. Note also that $B(p_i, 2\varepsilon) \subset U$.

By Remark 3.25, F_i can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of D_i. Hence, for any $i = 1, \ldots, \ell$,
$$G_i := H \circ A_2^{-1} \circ F_i \circ A_1 : \overline{B}(p_i, \varepsilon) \twoheadrightarrow \overline{B}(p_i, \varepsilon)$$
is an orientation preserving (bi-Lipschitz) homeomorphism which can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of $\overline{B}(p_i, \varepsilon)$. Then by Corollary 3.23 there is a (bi-Lipschitz) homeomorphism $\widetilde{G}_i : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $\widetilde{G}_i|_{\overline{B}(p_i, \varepsilon)} = G_i$ and $\widetilde{G}_i$ equals identity outside $B(p_i, \frac{3}{2}\varepsilon)$. Setting
$$G(x) = \begin{cases} \widetilde{G}_i(x) & \text{if } x \in B(p_i, 2\varepsilon), \\ x & \text{otherwise}, \end{cases}$$
we get a homeomorphism of $\mathbb{R}^n$ equal to identity outside $\bigcup_{i=1}^{\ell} B(p_i, \frac{3}{2}\varepsilon)$. Finally, we define $F = A_2 \circ H^{-1} \circ G \circ A_1^{-1}$, which is a (bi-Lipschitz) homeomorphism equal identity in $\mathbb{R}^n \setminus U$ (as it is a composition of diffeomorphisms of $\mathbb{R}^n$ with this property). It can be checked as in the proof of Lemma 3.11 that $F = F_i$ on D_i, as required. $\square$

Proof of Theorems 1.2 and 1.3. Like in the case of Theorem 1.1, we prove these results by reducing them to the Euclidean setting.

Let K' be a flat topological (or flat bi-Lipschitz) closed ball contained in $\mathcal{M}^n$ which does not intersect $\bigcup_{i=1}^{\ell}(D_i \cup D_i')$. We can find an orientation preserving (bi-Lipschitz) homeomorphism $H : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $H(D_i)$, $H(D_i')$ are contained in the interior of K'. This is stated in Lemma A.19 and A.20, respectively, and can be done in a standard way by constructing a chain of connected coordinate systems, see the proof of Theorem 1.1 for the overview of this idea. In Section A.4.2, we present the proof of this construction in the C^k-smooth setting which can be easily transferred to the Lipschitz and purely topological case.

Let $\Phi : \bar{\mathbb{B}}^n \twoheadrightarrow K'$ be the (bi-Lipschitz) homeomorphism between the Euclidean closed unit ball and K', which can be extended as a (bi-Lipschitz) homeomorphism on a neighborhood of $\bar{\mathbb{B}}^n$. Set $E_i := \Phi^{-1}(H(D_i))$ and $E_i' := \Phi^{-1}(H(D_i'))$ and

$$\widehat{F}_i := \Phi^{-1} \circ H \circ F_i \circ H^{-1} \circ \Phi|_{E_i}.$$

Sets E_i and E_i' are flat topological (or flat bi-Lipschitz) closed balls. Since $H \circ F_i$ is orientation preserving, so is $\widetilde{F}_i$ (see Remark A.16 for details). Moreover, $\widehat{F}_i$ maps each E_i onto E_i'. By Remark 3.25, $\widehat{F}_i$ can be extended as a (bi-Lipschitz) homeomorphism onto a neighborhood of E_i. As discussed in the proof of Theorem 1.1, E_i and E_i' are contained in $B(0, \varrho)$ for $\varrho \in (0, 1)$.

By Lemma 3.26, there is a (bi-Lipschitz) homeomorphism $\widehat{F} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$, which satisfies

$$\widehat{F}|_{E_i} = \widehat{F}_i \quad \text{and} \quad \widehat{F} = \mathrm{id} \;\text{ on } \mathbb{R}^n \setminus B(0, \varrho).$$

Then the composition $\Phi \circ \widehat{F} \circ \Phi^{-1} : K' \twoheadrightarrow K'$ is a (bi-Lipschitz) homeomorphism, which equals identity near $\partial K'$ and thus can be extended by identity to a (bi-Lipschitz) homeomorphism $G : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$. Eventually, set

$$F := H^{-1} \circ G \circ H.$$

This is a (bi-Lipschitz) homeomorphism of $\mathcal{M}^n$ such that $F|_{D_i} = F_i$, which can be easily checked in the same manner as in Lemma 3.11.

If all the sets D_i and D_i' for $i = 1, \ldots, \ell$ are contained in a flat topological (or flat bi-Lipschitz) closed ball $K \subset \mathcal{M}^n$, then K' can be chosen to equal K and thus $H = \mathrm{id}$ and $F = G = \mathrm{id}$ outside K. $\square$

Chapter 4

Constructing diffeomorphisms with prescribed derivative

4.1 Introduction

This chapter is devoted to proving Theorem 1.4, which we copy below for the convenience of the reader. This is Theorem 1.2 from [39].

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $F : \Omega \to \mathbb{R}^n$ an orientation preserving diffeomorphism onto the bounded image $F(\Omega)$. Suppose that $T : \Omega \to GL(n)^+$ is a measurable mapping such that $\int_\Omega \det T(x)\, dx \leq |F(\Omega)|$. Then for any $\varepsilon > 0$, there exists a C^1-diffeomorphism $\Phi : \Omega \to F(\Omega)$ with the following properties:*

(a) $\Phi(x) = F(x)$ near $\partial\Omega$;

(b) there exists a compact set $K \subset \Omega$ such that for every $x \in K$, $D\Phi(x) = T(x)$ and $|\Omega \setminus K| < \varepsilon$.

The proof of Theorem 1.4 follows [39], in particular this chapter contains a few technical lemmata from [39] (Lemma 3.11, Proposition 3.14, Lemma 5.12, Section 3.5) as well as the proof from Section 4 from [39]. Let us recall that in Section 1.3 it was shown that the assumption $\int_\Omega |\det T(x)| \leq |\Omega|$ is necessary. In Section 4.4, we include two corollaries of Theorem 1.4, which do not appear in [39] and will be stated later in this introduction.

In Section 1.3, we have already mentioned Alberti's theorem [1, Theorem 1], which we state and discuss below.

Theorem 4.1 (Alberti)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $T : \Omega \to \mathbb{R}^n$ be a measurable function. Then for every $\varepsilon > 0$, there is a function $\phi \in C^1_c(\Omega)$ and a compact set $K \subset \Omega$ such that $|\Omega \setminus K| < \varepsilon$ and $D\phi(x) = T(x)$ for all $x \in K$.*

Actually, Alberti proved more in his Theorem 1 as he also included estimates on L^p-norm of $D\phi$ for $p \in [1, \infty]$. In that same paper, in Theorem 3, he also proved a version of Theorem 4.1 for T in L^1, in which he constructed $\phi \in BV(\mathbb{R}^n)$, whose absolutely continuous part of the derivative coincides with $f\,\mathcal{L}^n$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

Moonens and Pfeffer in [72] showed that for any measurable function $T : \Omega \to \mathbb{R}^n$, there exists a continuous function $\phi \in C(\mathbb{R}^n)$, which is differentiable a. e. on Ω with

$$D\phi = T \text{ a. e. on } \Omega \quad \text{and} \quad D\phi(x) = 0 \text{ for } x \in \mathbb{R}^n \setminus \Omega. \tag{4.1}$$

The main ingredient of the proof of this theorem is iteration of Theorem 4.1. Later on, in [34] Alberti's and Moonens and Pfeffer's results were generalized for higher-order derivatives.

Given a measurable $T : \Omega \to GL(n)$ as in Theorem 1.4, we can apply Theorem 4.1 componentwise to get a continuously differentiable mapping $\mathcal{A} \in C^1_c(\Omega, \mathbb{R}^n)$ and a *large* compact set $K \subset \Omega$ such that $D\mathcal{A} = T$ on K. Since on K we have $\det D\mathcal{A} = \det T \neq 0$, $\mathcal{A}$ is a local diffeomorphism in a neighborhood of every point of K. However, we do not know anything about its injectivity on a global scale nor even about its image. In particular, $\mathcal{A}(\mathcal{Q})$ not need to be contained in the unit cube $\mathcal{Q}$. But thanks to the fact that $\mathcal{A}$ is a *local* diffeomorphism, we can construct a *global* diffeomorphism Φ from Theorem 1.4 with the desired derivative on a large set. The transition from from $\mathcal{A}$ to Φ is the most difficult part of the proof. To make this transition possible, we construct a diffeomorphism Ψ which coincides with F near $\partial\Omega$ and whose Jacobian is *close* to $\det T$. This can be done thanks to the volume constraint $\int_\Omega \det T \leq |F(\Omega)|$ and the Dacorogna-Moser theorem from Section 4.2.1. In the proof of Theorem 1.4, we will also use a few constructions from Section 2.7 to glue the global diffeomorphism Ψ with the local diffeomorphism $\mathcal{A}$.

We will use a special case of the Dacorogna-Moser theorem, see [22, Theorem 5]. The problem of finding a diffeomorphism with a prescribed Jacobian was originally studied by Moser in [76] in the language of differential forms. In that paper, he did not consider any boundary conditions and introduced his seminal *flow method*, which uses ordinary differential equations, see Section A.2 for some details. Later on, Dacorogna and Moser [22] used a different approach which employed Schauder estimates for elliptic equations. We say that $f \in C^{k,\alpha}(\overline{\Omega})$ for $k \in \mathbb{N} \cup \{0\}$ if $f \in C^k(\overline{\Omega})$ (or $f \in C(\overline{\Omega})$ for $k = 0$) and if the derivatives of f up to k-th order (or f itself for $k = 0$) are α-Hölder continuous, see [20, Section 16.1] for a precise definition. In [22] it was shown that for sufficiently regular domains Ω, given a function $f : \overline{\Omega} \to \mathbb{R}$, $f > 0$ on $\overline{\Omega}$ and $f \in C^{k,\alpha}(\overline{\Omega})$, it is possible to find a diffeomorphism Φ of $\overline{\Omega}$ of class $C^{k+1,\alpha}$ with $\det D\Phi = f$ in Ω, that is a diffeomorphism with the prescribed Jacobian and the optimal regularity. Moreover, $\Phi(x) = x$ for $x \in \partial\Omega$.

In the excellent book [20], authors proved Theorem 10.11, in which they constructed a diffeomorphism with no gain in regularity, that is of class $C^{k,\alpha}$ given $f \in C^{k,\alpha}$, but with additional control of support, i.e., if $\operatorname{supp}(f-1) \subset \Omega$, then $\operatorname{supp}(\Phi - \mathrm{id}) \subset \Omega$. They used a modified version of the Moser's flow method. Eventually, Teixeira in [90, 91] showed, building on [5] and the already cited works, how to construct a diffeomorphism of the optimal regularity and with the control of support, see also [62]. Other important contributions to this area include [10, 15, 68, 100, 87, 41]. Let us stress the importance of [15, 68] which independently showed that the Hölder regularity of f for $k = 0$ is necessary. Indeed, by [15, 68] for any $\varepsilon > 0$ there is a continuous function $f : [0,1]^2 \to [1, 1+\varepsilon]$ for which there is no bi-Lipschitz homeomorphism whose Jacobian equals f.

Section 4.4 contains two corollaries to Theorem 1.4. The juxtaposition of Theorem 4.1 and Moonens and Pfeffer's result (4.1) shows that by demanding less regularity from the function ϕ and its derivative, one can prescribe it on a larger set (even a set of full measure). However, neither of these results says anything about injectivity of thus constructed functions. We shall see that with the help of Theorem 1.4, one can arrive at a homeomorphic version of Alberti's theorem, Theorem 4.2, in which approximate instead of classical derivative is used.

Theorem 4.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $T : \Omega \to GL(n)$ is measurable and such that $\int_\Omega |\det T(x)|\, dx \leq |\Omega|$. Then for any $\varepsilon > 0$, there exists an a.e. approximately differentiable homeomorphism $\Phi : \Omega \twoheadrightarrow \Omega$ with the following properties*

(a) $\Phi(x) = x$ near $\partial\Omega$,
(b) Φ satisfies the Lusin (N) condition,

(c) *there exists a compact set $K \subset \Omega$ such that for almost every $x \in K$, $D_{\mathrm{a}}\Phi(x) = T(x)$ and $|\Omega \setminus K| < \varepsilon$.*

In the proof we also employ [37, Theorem 1.4], see Proposition 4.12, which we also discussed in Section 1.3.

To obtain the next result, we iterate Theorem 1.4 to get an a. e. injective mapping F which is a. e. approximately differentiable with $D_{\mathrm{a}}F = T$ when $\det T > 0$. More precisely,

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $T : \Omega \to GL(n)^+$ be measurable and such that $\int_\Omega |\det T(x)|\,dx = |\Omega|$. Then there exists an a. e. approximately differentiable mapping $F : \overline{\Omega} \to \overline{\Omega}$ such that*

(a) *$F(x) = x$ for $x \in \partial\Omega$,*
(b) *F satisfies the Lusin (N) condition,*
(c) *F is injective a. e. on Ω, i. e., there is a set E with $|E| = |\Omega|$ such that $F|_E$ is injective,*
(d) *$D_{\mathrm{a}}F = T$ a. e. on Ω.*

Note that we do not claim that $F(\overline{\Omega}) = \overline{\Omega}$. Nonetheless, again at a cost of regularity of the mapping and its derivative, we gain some injectivity properties.

Section 4.4 is in a sense an introduction to Chapter 5, where we will describe a much more involved iteration procedure which allows to construct an a. e. approximately differentiable homeomorphism from Theorem 1.5. Even though Theorem 1.5 generalizes Theorems 4.2 and 4.3, I believe that eventually the their statements and proofs enhance one's understanding of Theorem 1.5.

In Section 4.2, we present a few preliminary results needed in Section 4.3, where we prove Theorem 1.4. In Section 4.4 we present proofs of Theorems 1.1 and 1.2.

4.2 Preliminaries for Theorem 1.4

The main results of this section are Proposition 4.6 and Corollary 4.10. We will use them in the proof of Theorem 1.4. For the proof of Proposition 4.6 we need two technical lemmata, Lemma 4.4 (below) and Lemma 2.16 (which has already been proved in Chapter 2). With $\mathsf{L}(Q)$ we denote the side-length of a cube $Q \subset \mathbb{R}^n$.

Lemma 4.4 (Storage lemma)**.** *Fix $\ell \in \mathbb{Z}$. Assume that $\mathcal{V}$ is a finite family of closed cubes $V \subset \mathbb{R}^n$ with pairwise disjoint interiors, and each cube $V \in \mathcal{V}$ has the same side-length $\mathsf{L}(V) = 2^{-\ell}$. Assume that $\mathcal{W}$ is another finite family of closed cubes $W \subset \mathbb{R}^n$ such that*

$$\sum_{W \in \mathcal{W}} |W| \le \sum_{V \in \mathcal{V}} |V| \tag{4.2}$$

and that for each $W \in \mathcal{W}$, $\mathsf{L}(W) = 2^{-k}$ for some $k \in \mathbb{Z}$, $k \ge \ell$. Then for each $W \in \mathcal{W}$, there is an isometric closed cube $\widetilde{W}$, i. e., $\mathsf{L}(\widetilde{W}) = \mathsf{L}(W)$, such that the cubes $\{\widetilde{W}\}_{W\in\mathcal{W}}$ have pairwise disjoint interiors, and

$$\bigcup\nolimits_{W\in\mathcal{W}} \widetilde{W} \subset \bigcup\nolimits_{V\in\mathcal{V}} V.$$

Remark 4.5. The lemma has a practical interpretation. You can place dyadic boxes $W \in \mathcal{W}$ in the storage containers $V \in \mathcal{V}$ (identical and dyadic) if and only if the total volume of the boxes W does not exceed the total volume of the storage, and no box W is larger than a storage container.

Proof. Let $\mathcal{V}_\ell := \mathcal{V}$. Divide the family $\mathcal{W}$ into subfamilies according to the side-length: $\mathcal{W} = \bigcup_{i=\ell}^N \mathcal{W}_i$, where $\mathcal{W}_i = \{W \in \mathcal{W} : \mathsf{L}(W) = 2^{-i}\}$. Clearly, (4.2) can be rewritten as

$$\sum_{i=\ell}^{N} \sum_{W \in \mathcal{W}_i} |W| \leq \sum_{V \in \mathcal{V}_\ell} |V|. \tag{4.3}$$

It follows that the number of cubes in $\mathcal{W}_\ell$ is less than or equal to the number of cubes in $\mathcal{V}_\ell$. Thus, for each $W \in \mathcal{W}_\ell$, we can find $\widetilde{W} \in \mathcal{V}_\ell$ so that the cubes $\widetilde{W}$ have pairwise disjoint interiors. Clearly, W and $\widetilde{W}$ are isometric.

Divide each of the cubes in the remaining family

$$\mathcal{V}_\ell \setminus \{\widetilde{W} : W \in W_\ell\} \tag{4.4}$$

into 2^n dyadic closed cubes of side-length $2^{-(\ell+1)}$. Denote the resulting family of cubes by $\mathcal{V}_{\ell+1}$. That is, each of the cubes in $\mathcal{V}_{\ell+1}$ has side-length $2^{-(\ell+1)}$ and the number of cubes in $\mathcal{V}_{\ell+1}$ equals 2^n times the number of the cubes in (4.4). Clearly, (4.3) implies that

$$\sum_{i=\ell+1}^{N} \sum_{W \in \mathcal{W}_i} |W| \leq \sum_{V \in \mathcal{V}_{\ell+1}} |V|,$$

because by removing cubes $W \in \mathcal{W}_\ell$ from $\mathcal{W}$ and cubes $\widetilde{W} \in \mathcal{V}_\ell$, from $\mathcal{V}_\ell$, we removed equal volumes from both sides of (4.3). Now, we can repeat the procedure described above and match each $W \in \mathcal{W}_{\ell+1}$ with a suitable cube $\widetilde{W} \in \mathcal{V}_{\ell+1}$. We repeat the procedure by induction. The required family $\{\widetilde{W}\}_{W\in\mathcal{W}}$ will be constructed after a finite number of steps. □

The next result shows in particular that if $A_1, A_2 \in GL(n)^+$ and $\det A_1 = \det A_2$, then it is possible to find a diffeomorphism of a ball B onto $A_2(B)$ which on a large part of B acts like a piecewise affine map with A_1 as its linear part.

Proposition 4.6. *Let $G \subset B$ be a measurable subset of an open ball $B \subset \mathbb{R}^n$ centered at the origin, and let $r : G \to (0, \infty)$ be any function. Let $A_1, A_2 \in GL(n)^+$ satisfy*

$$\det A_2 > \beta \det A_1 \textit{ for some } \beta \in (0, 1). \tag{4.5}$$

Then there is a finite family of pairwise disjoint closed balls $\overline{B}(p_j, r_j) \subset B$ such that

$$p_j \in G, \qquad r_j < r(p_j), \qquad \Big|G \cap \bigcup\nolimits_j \overline{B}(p_j, r_j)\Big| > \beta|G|, \tag{4.6}$$

and a diffeomorphism $F : B \to A_2(B)$ which agrees with A_2 in a neighborhood of ∂B and

$$F(x) = A_1 x + v_j \textit{ for all } x \in \overline{B}(p_j, r_j) \textit{ and some } v_j \in \mathbb{R}^n. \tag{4.7}$$

Remark 4.7. If $B = B(p, R)$ is a ball not necessarily centered at the origin, and $q \in \mathbb{R}^n$, but all other assumptions remain the same, then we can find balls $\overline{B}(p_j, r_j) \subset B$ satisfying (4.6) and a diffeomorphism $F : B \to \mathbb{R}^n$ such that $F(x) = A_2(x - p) + q$ in a neighborhood of ∂B and F satisfies (4.7) in each of the balls $\overline{B}(p_j, r_j)$.

Indeed, such a diffeomorphism is obtained from Proposition 4.6 by composing with the translations $x \mapsto x - p$ in the domain and $y \mapsto y + q$ in the target. Then in each of the balls $\overline{B}(p_j, r_j)$ we have $F(x) = A_1(x - p) + v_j + q = A_1 x + w_j$, for some $w_j \in \mathbb{R}^n$.

Proof. Assume first that $A_1 = \mathrm{id}$. Let $U = A_2(B)$.

Before proceeding to details, let us describe the main idea of the proof. We begin by finding a finite family of disjoint closed cubes $Q_i \subset B$ satisfying $|G \cap \bigcup_i Q_i| > \beta|G|$. Working with cubes allows us to apply Lemma 4.4 and to find translated cubes $Q_i + w_i \subset A_2(B)$ with pairwise disjoint interiors. Then for each i we find a finite family of pairwise disjoint balls $\overline{B}(p_{ik}, r_{ik}) \subset \mathring{Q}_i$. Clearly, the balls $\overline{B}(p_{ik}, r_{ik}) + w_i \subset A_2(B)$ are pairwise disjoint. After re-enumeration, we can write

$$\{\overline{B}(p_j, r_j)\}_j := \{\overline{B}(p_{ik}, r_{ik})\}_{i,k}, \quad v_j := w_i \ \text{ if } \ p_j = p_{ik}.$$

Choosing the balls carefully, we can guarantee (4.6). Note that the balls $\overline{B}(p_j, r_j) + v_j \subset A_2(B)$ are pairwise disjoint. Then we construct a diffeomorphism F that equals A_2 near ∂B and satisfies $F(x) = x + v_j$ for $x \in \overline{B}(p_j, r_j)$, which is (4.7) in the case when $A_1 = \mathrm{id}$.

Step 1. Finding cubes. Since $|U| > \beta|B|$ by (4.5), we also have $|U| > \alpha|B|$ for some $\alpha \in (\beta, 1)$. Let $\mathcal{F}_\ell$ be the family of all closed dyadic cubes (i. e., with vertices at points of $2^{-\ell}\mathbb{Z}^n$) of side-length $2^{-\ell}$ that are contained in U. Clearly, the family $\mathcal{F}_\ell$ is finite. We choose ℓ large enough to guarantee that

$$\sum_{Q \in \mathcal{F}_\ell} |Q| > \frac{\beta}{\alpha}|U|. \tag{4.8}$$

Let $V \subset \mathbb{R}^n$ be an open set such that

$$G \subset V \subset B \quad \text{and} \quad |V \setminus G| < \frac{1}{2}\beta(\alpha^{-1}|U| - |G|)$$

(note that $\alpha^{-1}|U| - |G| > 0$).

For each $q \in G$ consider the family $\mathcal{G}_q$ of all closed cubes $Q = Q(q, 2^{-k})$, $k \in \mathbb{Z}$, centered at q, that satisfy

$$Q \subset V, \quad \mathsf{L}(Q) \le 2^{-\ell}, \quad |Q| < \frac{1}{2}\beta(\alpha^{-1}|U| - |G|). \tag{4.9}$$

The family $\widetilde{\mathcal{G}} := \bigcup_{q \in G} \mathcal{G}_q$ is a Vitali covering of G and Vitali's covering theorem yields a finite sub-family $\mathcal{G}' = \{Q_i\}_{i=1}^{N'} \subset \widetilde{\mathcal{G}}$ of pairwise disjoint cubes such that

$$\sum_{i=1}^{N'} |Q_i| \ge \sum_{i=1}^{N'} |Q_i \cap G| > \beta|G|. \tag{4.10}$$

By removing some cubes from the family $\mathcal{G}'$, we can obtain a family $\mathcal{G} = \{Q_i\}_{i=1}^N$ (so $N \le N'$) such that

$$\frac{\beta}{\alpha}|U| \ge \sum_{i=1}^{N} |Q_i| \ge \sum_{i=1}^{N} |Q_i \cap G| > \beta|G|. \tag{4.11}$$

Indeed, to show this, it suffices to prove for any m the implication

$$\sum_{i=1}^{m+1} |Q_i| > \frac{\beta}{\alpha}|U| \quad \Longrightarrow \quad \sum_{i=1}^{m} |Q_i \cap G| > \beta|G|. \tag{4.12}$$

Note that

$$\sum_{i=1}^{m} |Q_i| - \sum_{i=1}^{m} |Q_i \cap G| = \sum_{i=1}^{m} |Q_i \setminus G| \le |V \setminus G| < \frac{1}{2}\beta(\alpha^{-1}|U| - |G|).$$

On the other hand, the hypothesis in (4.12) and the upper estimate for $|Q|$ in (4.9) yield

$$\sum_{i=1}^{m}|Q_i| > \frac{\beta}{\alpha}|U| - |Q_{m+1}| > \frac{\beta}{\alpha}|U| - \frac{1}{2}\beta(\alpha^{-1}|U|-|G|) = \beta|G| + \frac{1}{2}\beta(\alpha^{-1}|U|-|G|),$$

and hence

$$\sum_{i=1}^{m}|Q_i\cap G| = \sum_{i=1}^{m}|Q_i| - \Big(\sum_{i=1}^{m}|Q_i| - \sum_{i=1}^{m}|Q_i\cap G|\Big) > \beta|G|.$$

This proves the implication (4.12) and hence proves the existence of $\mathcal{G}=\{Q_i\}_{i=1}^N$ satisfying (4.11).

Now, (4.8) and (4.11) yield

$$\sum_{Q\in\mathcal{F}_\ell}|Q| \geq \sum_{i=1}^{N}|Q_i|.$$

Since $\mathsf{L}(Q_i)=2^{-k}$, $k\geq \ell$, it follows from Lemma 4.4 that there are vectors $w_i\in\mathbb{R}^n$ such that the cubes $\widetilde{Q}_i := Q_i + w_i$ have pairwise disjoint interiors and

$$\bigcup_{i=1}^{N}\widetilde{Q}_i \subset \bigcup\nolimits_{Q\in\mathcal{F}_\ell} Q\subset U.$$

The cubes Q_i are pairwise disjoint, but the cubes $\widetilde{Q}_i$ need not be.

To summarize, we constructed a family of pairwise disjoint closed cubes $\{Q_i\}_{i=1}^N$, $Q_i\subset B$, such that

$$\Big|G\cap \bigcup\nolimits_i \mathring{Q}_i\Big| = \Big|G\cap\bigcup\nolimits_i Q_i\Big| > \beta|G|, \tag{4.13}$$

and we constructed vectors $\{w_i\}_{i=1}^N$ such that the cubes $\widetilde{Q}_i = Q_i+w_i\subset U = A_2(B)$ have pairwise disjoint interiors.

<u>Step 2. Finding balls.</u> We will now find a finite family of closed, pairwise disjoint balls $\overline{B}(p_j,r_j)$, satisfying (4.6). For each $i=1,2,\ldots,N$, let

$$\mathcal{B}_i = \Big\{\overline{B}(p,r) : \, p\in G\cap\mathring{Q}_i,\ \overline{B}(p,r)\subset\mathring{Q}_i,\ r<r(p)\Big\}.$$

Vitali's covering theorem and (4.13) yield a finite sub-family of disjoint closed balls $\overline{B}(p_{ik},r_{ik})$, $k=1,\ldots,N_i$, so that

$$p_{ik}\in G,\qquad r_{ik}<r(p_{ik}),\qquad \overline{B}(p_{ik},r_{ik})\subset\mathring{Q}_i,$$

and

$$\Big|G\cap\bigcup\nolimits_{ik}\overline{B}(p_{ik},r_{ik})\Big| > \beta|G|.$$

For each i, the balls $\overline{B}(p_{ik},r_{ik})+w_i$ are pairwise disjoint and contained in the interior of $\widetilde{Q}_i$. Since the interiors of the cubes $\widetilde{Q}_i$ are pairwise disjoint, the balls in the family $\{\overline{B}(p_{ik},r_{ik})+w_i\}_{i,k}$ are pairwise disjoint as well. After re-enumerating, we get a family of balls $\overline{B}_j := \overline{B}(p_j,r_j)$, $j=1,2,\ldots,M$ that satisfy (4.6) and vectors $v_j\in\mathbb{R}^n$ such that the balls $\overline{B}'_j := \overline{B}(p_j,r_j)+v_j\subset U=A_2(B)$ are pairwise disjoint.

<u>Step 3. Finding the diffeomorphism F.</u> To complete the proof in the case $A_1=\mathrm{id}$, it remains to construct a diffeomorphism $F: B\to\mathbb{R}^n$ which equals A_2 near ∂B and $F(x)=x+v_j$ for $x\in\overline{B}_j$ for $j=1,2,\ldots,M$.

Fix $R_1 > 0$ such that $B_{R_1} := B(0, R_1) \Subset B \cap A_2(B)$. Let $B_{R_2} := B(0, R_2)$ be a ball such that

$$\bigcup_{j=1}^{M} \overline{B}_j \subset B_{R_2} \Subset B.$$

Let $H_1 : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ be a radial diffeomorphism such that $H_1(x) = x$ near ∂B and $H_1(x) = R_1 R_2^{-1} x$ on B_{R_2}. Proposition 2.14 yields a diffeomorphism $H_2 : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $H_2 = A_2$ near ∂B and $H_2 = \mathrm{id}$ on $\overline{B}_{R_1}$. Then $H_2 \circ H_1 : B \twoheadrightarrow A_2(B)$ equals A_2 near ∂B and maps the balls $\overline{B}_j$ by scaling (with factor $R_1 R_2^{-1}$) onto balls in $B_{R_1} \Subset A_2(B)$.

Let $a_j := H_2(H_1(p_j))$ be the centers of the balls $H_2(H_1(\overline{B}_j))$ and let $b_j := p_j + v_j$ be the centers of the balls $\overline{B}'_j = \overline{B}_i + v_j$. Both families $\{a_j\}_{j=1}^M$ and $\{b_j\}_{j=1}^M$ are contained in $U = A_2(B)$, and Lemma 2.16 gives $\varepsilon > 0$ and a diffeomorphism $\Theta : U \twoheadrightarrow U$ that equals identity near ∂U and satisfies

$$\Theta(x) = x + b_j - a_j \text{ for } x \in B(a_j, \varepsilon).$$

Clearly, we can assume that

$$\varepsilon < \min_j R_1 R_2^{-1} r_j.$$

Since the balls

$$H_2(H_1(\overline{B}_j)) = \overline{B}(a_j, R_1 R_2^{-1} r_j) \subset U \tag{4.14}$$

are pairwise disjoint, there is $\delta > 0$ such that the balls

$$\overline{B}(a_j, R_1 R_2^{-1} r_j + \delta) \subset B \tag{4.15}$$

are pairwise disjoint.

For each $j = 1, 2, \ldots, M$, we find a diffeomorphism $H_1^j : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ that is similar to H_1. It is a radial diffeomorphism centered at a_j, it is identity outside the ball (4.15) and it maps the ball (4.14) onto $\overline{B}(a_j, \varepsilon)$ by scaling centered at a_j with factor $\varepsilon R_1^{-1} R_2 r_j^{-1} < 1$. Clearly, the diffeomorphism

$$H_3 := H_1^1 \circ H_1^2 \circ \ldots \circ H_1^M : U \twoheadrightarrow U$$

is identity near ∂U and maps each of the balls (4.14) onto $\overline{B}(a_j, \varepsilon)$ by scaling (centered at a_j). Now $\Theta \circ H_3 \circ H_2 \circ H_1 : B \twoheadrightarrow U$ equals A_2 near ∂U and maps the balls $B_j = B_j(p_j, r_j)$ onto the balls $\overline{B}(b_j, \varepsilon)$ by affine maps whose linear part is scaling by factor εr_j^{-1}.

Finally, if $H_4 : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is a diffeomorphism similar to H_3 that equals identity near ∂U and expands the balls $\overline{B}(b_j, \varepsilon)$ to $\overline{B}(b_j, r_j) = \overline{B}'_j$ by scaling, then the diffeomorphism

$$F : H_4 \circ \Theta \circ H_3 \circ H_2 \circ H_1 : B \twoheadrightarrow A_2(B)$$

is the required diffeomorphism satisfying (4.7) for $A_1 = \mathrm{id}$.

Step 4. The general case. Finally, suppose that A_1 and A_2 are arbitrary $GL(n)^+$ mappings satisfying (4.5). Then $\widetilde{A}_1 := \mathrm{id}$ and $\widetilde{A}_2 := A_1^{-1} \circ A_2$ satisfy $\det \widetilde{A}_2 > \beta \det \widetilde{A}_1$ and the construction from Step 3 yields a family of balls $\overline{B}_j$ satisfying (4.6) and a diffeomorphism $\widetilde{F} : B \to \widetilde{A}_2(B)$ such that

$$\widetilde{F}(x) = x + \widetilde{v}_j \text{ for all } x \in \overline{B}_j \text{ and some } \widetilde{v}_j \in \mathbb{R}^n.$$

Setting $F = A_1 \circ \widetilde{F}$ yields the desired diffeomorphism satisfying (4.7). □

4.2.1 The Dacorogna-Moser theorem

The aim of this section is to prove Corollary 4.10, which will be used in the sequel. To this end, we use the following lemma, a special case of a theorem of Dacorogna and Moser [22, Theorem 5]. This exact statement appeared also as [37, Theorem 3.1].

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C^\infty(\Omega)$ be a positive function equal 1 in a neighborhood of $\partial\Omega$ such that*

$$\int_\Omega f(x)\,dx = |\Omega|.$$

Then there exists a C^∞-diffeomorphism Ψ of Ω onto itself that is identity on a neighborhood of $\partial\Omega$ and satisfies

$$J_\Psi(x) = f(x) \quad \text{for all } x \in \Omega.$$

Actually, the proofs in [20] and [22] explicitly cover only the case of $f \in C^k(\Omega)$ for some $k \in \mathbb{N}$ but they do work also for $k = \infty$. In [21, Appendix, Lemma 2.3] (in the first edition of the book), one can find a proof based on Moser's flow method for $k = \infty$.

The next result shows that if f is only integrable and positive a. e., then on a large set we can uniformly approximate f by the Jacobian of a smooth diffeomorphism.

Lemma 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in L^1(\Omega)$, $f > 0$ a. e., be such that*

$$\int_\Omega f(x)\,dx = |\Omega|.$$

Then for any $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ with $|\Omega \setminus K| < \varepsilon$, and a C^∞-diffeomorphism Ψ of Ω onto itself that is identity on a neighborhood of $\partial\Omega$ and satisfies

$$|J_\Psi(x) - f(x)| < \varepsilon f(x) \qquad \text{for all } x \in K. \tag{4.16}$$

Proof. By Lusin's theorem, we can find a compact set $K \subset \Omega$, such that $|\Omega \setminus K| < \varepsilon$ and f is continuous and strictly positive in K. Let

$$m := \inf_K f \qquad \text{and} \qquad M := \sup_K f\,.$$

Clearly,

$$\int_K f(x)\,dx = |\Omega| - 2M\delta \quad \text{for some } \delta > 0.$$

Let Ω' and Ω'' be open sets such that

$$K \subset \Omega' \Subset \Omega'' \Subset \Omega, \quad \text{and} \quad |\Omega' \setminus K| < \delta,$$

We can extend f from K to a continuous function $0 \le f_1 \le M$ that is compactly supported in Ω', so

$$\int_\Omega f_1(x)\,dx = \int_{\Omega'\setminus K} f_1(x)\,dx + \int_K f(x)\,dx < |\Omega' \setminus K| \cdot M + (|\Omega| - 2M\delta) < |\Omega| - M\delta.$$

Using a standard approximation of f_1 by convolution, we find $f_2 \in C_c^\infty(\Omega')$, $f_2 \ge 0$, such that

$$|f(x) - f_2(x)| = |f_1(x) - f_2(x)| < \frac{\varepsilon m}{2} \quad \text{for all } x \in K. \tag{4.17}$$

Since the approximation by convolution preserves the L^1 norm of a non-negative function (by Fubini's theorem), we have

$$\int_\Omega f_2(x)\,dx = \int_\Omega f_1(x)\,dx < |\Omega| - M\delta. \tag{4.18}$$

It is easy to see that there is $f_3 \in C^\infty(\Omega)$ that it is strictly positive in Ω, $f_3 = 1$ in a neighborhood of $\partial\Omega$, $f_3 < \varepsilon m/2$ in K, and

$$\int_\Omega f_3(x)\,dx < M\delta. \tag{4.19}$$

Integrals in (4.18) and (4.19) add to a number less than $|\Omega|$ and we can find a function $f_4 \in C_c^\infty(\Omega'' \setminus \bar{\Omega}')$, $f_4 \geq 0$, such that

$$\int_\Omega f_2(x) + f_3(x) + f_4(x)\,dx = |\Omega|.$$

Observe that the function $f_\varepsilon := f_2 + f_3 + f_4 \in C^\infty(\Omega)$ equals $f_3 = 1$ in a neighborhood of $\partial\Omega$ and equals $f_2 + f_3$ in K, so

$$|f(x) - f_\varepsilon(x)| \leq |f(x) - f_2(x)| + f_3(x) < \varepsilon m \leq \varepsilon f(x) \quad \text{for all } x \in K.$$

These conditions and Lemma 4.8 imply the existence of a smooth diffeomorphism $\Psi : \Omega \twoheadrightarrow \Omega$ that is identity near the boundary and satisfies $J_\Psi = f_\varepsilon$, so (4.16) is satisfied. □

Corollary 4.10. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded domains and let $F : \Omega \twoheadrightarrow \Omega' = F(\Omega)$ be an orientation preserving diffeomorphism. Suppose that $f \in L^1(\Omega)$, $f > 0$ a. e. and*

$$\int_\Omega f(x)\,dx = |\Omega'|.$$

Then for any $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ with $|\Omega \setminus K| < \varepsilon$, and a C^∞-diffeomorphism F' of Ω onto Ω', that equals F on a neighborhood of $\partial\Omega$ and satisfies

$$|J_{F'}(x) - f(x)| < \varepsilon f(x) \qquad \text{for all } x \in K. \tag{4.20}$$

Remark 4.11. The condition that F is orientation preserving is necessary. Indeed, in view of (4.20), $J_{F'}(x) > 0$ for $x \in K$, so $J_{F'} > 0$ on Ω, and hence $J_F > 0$ on Ω, because $F = F'$ on an open set.

Proof. Let

$$g(y) = \frac{f(F^{-1}(y))}{J_F(F^{-1}(y))}, \qquad \text{so} \qquad g(F(x))J_F(x) = f(x).$$

It follows from the classical change of variables formula that $g \in L^1(\Omega')$, $g > 0$ a.e, and

$$\int_{\Omega'} g(y)\,dy = \int_\Omega f(x)\,dx = |\Omega'|.$$

Therefore, for any $\varepsilon' > 0$, Lemma 4.9 yields a compact set $K' \subset \Omega'$, $|\Omega' \setminus K'| < \varepsilon'$, and a diffeomorphism G of Ω' onto itself that is identity in a neighborhood of $\partial\Omega'$ and satisfies

$$|J_G(y) - g(y)| < \varepsilon' g(y) \quad \text{for all } y \in K'. \tag{4.21}$$

Let $K = F^{-1}(K')$. By taking $\varepsilon' \leq \varepsilon$ sufficiently small, we can guarantee that $|\Omega \setminus K| < \varepsilon$. Now the diffeomorphism $F' = G \circ F : \Omega \twoheadrightarrow \Omega'$ satisfies the claim of the corollary. Indeed, $F' = F$ near $\partial\Omega$ and (4.21) yields

$$|J_{F'}(x) - f(x)| = |J_G(F(x))J_F(x) - g(F(x))J_F(x)| < \varepsilon' g(F(x))J_F(x) = \varepsilon' f(x) \leq \varepsilon f(x),$$

whenever $x \in K$. The proof is complete. □

4.3 Proof of Theorem 1.4

Proof of Theorem 1.4. First, we prove the theorem under the assumption that

$$\int_\Omega \det T(x)\, dx = |F(\Omega)|. \tag{4.22}$$

The general case will then easily follow from this one.

Let $0 < \varepsilon < |\Omega|$ be given and fix β such that $(1-\varepsilon|\Omega|^{-1})^{1/8} < \beta < 1$.

Corollary 4.10 yields a C^∞-diffeomorphism $\Psi : \Omega \twoheadrightarrow F(\Omega)$ and a compact set $K_1 \subset \Omega$ with $|\Omega \setminus K_1| < \frac{1}{2}(1-\beta)|\Omega|$ such that $\Psi = F$ near $\partial\Omega$ and

$$|\det D\Psi(x) - \det T(x)| < (1-\beta)\det T(x) \text{ for } x \in K_1. \tag{4.23}$$

On the other hand, Theorem 4.1 gives us a mapping $\mathcal{A} \in C^1_c(\Omega, \mathbb{R}^n)$ and a compact set $K_2 \subset \Omega$ with $|\Omega \setminus K_2| < \frac{1}{2}(1-\beta)|\Omega|$ such that

$$D\mathcal{A}(x) = T(x) \text{ for } x \in K_2. \tag{4.24}$$

Let $G \subset K_1 \cap K_2$ be the set of density points of $K_1 \cap K_2$ that belong to $K_1 \cap K_2$ and observe that

$$|G| = |K_1 \cap K_2| > \beta|\Omega|.$$

It follows from (4.23) and (4.24) that

$$D\mathcal{A}(x) = T(x) \quad \text{and} \quad \det D\Psi(x) > \beta \det D\mathcal{A}(x) > 0 \text{ for all } x \in G. \tag{4.25}$$

Let us interrupt the proof for a moment and explain its main idea. The idea is to find a finite family of balls — in the proof it will be the family $\{B(p_{ij}, \beta^{2/n} r_{ij})\}_{i,j}$ — such that we can replace Ψ with $\mathcal{A}(x) + \tau_{ij}$ for some $\tau_{ij} \in \mathbb{R}^n$ on each of the balls and the resulting map Φ will be a diffeomorphism. Then $\Phi = \Psi = F$ near $\partial\Omega$ and

$$D\Phi(x) = D\mathcal{A}(x) = T(x) \text{ for } x \in G \cap \bigcup\nolimits_{ij} B(p_{ij}, \beta^{2/n} r_{ij}). \tag{4.26}$$

Moreover, the family of balls will be constructed in such a way that the measure of the complement of the set in (4.26) (reproduced in the proof as (4.34)) will be less than ε. This will complete the proof. We will replace Ψ with $\mathcal{A}(x) + \tau_{ij}$ by a sequence of diffeomorphic gluing. We will glue Ψ with its affine approximation, then we will glue the affine approximation of Ψ with the affine approximation of $\mathcal{A}(x) + \tau_{ij}$, which we will glue with $\mathcal{A}(x) + \tau_{ij}$. To this end we will use (2.5) and (2.6) in Lemma 2.13, Proposition 2.14 and Proposition 4.6. Now, we shall return to the proof.

For any $x_o \in G$, there is $r_{x_o} > 0$ such that for all $r \leq r_{x_o}$, the following conditions hold

(a) $B(x_o, r) \Subset \Omega$;
(b) $|B(x_o, r) \cap G| \geq \beta |B(x_o, r)|$;
(c) $\mathcal{A}$ is a diffeomorphism on $B(x_o, r)$;
(d) $D\mathcal{A}(x)$ is close to $D\mathcal{A}(x_o)$ for $x \in B(x_o, r)$ in the sense that

$$\sup_{x \in B(x_o, r)} \|D\mathcal{A}(x) - D\mathcal{A}(x_o)\| < \left(\beta^{-1/(2n)} - 1\right) \|(D\mathcal{A}(x_o))^{-1}\|^{-1}; \tag{4.27}$$

(e) there exists a diffeomorphism $\Psi'_{x_o,r} : \Omega \twoheadrightarrow F(\Omega)$ such that

$$\Psi'_{x_o,r}(x) = \begin{cases} \Psi(x_o) + D\Psi(x_o)(x - x_o) & \text{for } x \in \overline{B}(x_o, \beta^{1/n} r), \\ \Psi(x) & \text{for } x \in \Omega \setminus B(x_o, r). \end{cases} \tag{4.28}$$

Property (b) follows from the fact that x_o is a density point of G. Property (c) follows from (4.25). Property (d) is a consequence of continuity of $D\mathcal{A}$. Finally, (e) follows from Lemma 2.13.

The family of balls

$$\mathfrak{B} = \{\overline{B}(x_o, r) : x_o \in G,\, r \leq r_{x_o}\}$$

is a Vitali covering of G, so by Vitali's covering theorem, we can choose a finite subfamily of pairwise disjoint balls $\{\overline{B}_i\}_{i=1}^N$ so that the measure of these balls satisfies

$$\Big|\bigcup_{i=1}^N \overline{B}_i\Big| \geq \beta|G| > \beta^2|\Omega|. \tag{4.29}$$

We replace Ψ with Ψ'_{x_i,r_i} in each of the balls $\overline{B}_i = \overline{B}(x_i, r_i)$. The resulting diffeomorphism satisfies

$$\Psi'(x) = \begin{cases} \Psi(x) & \text{on } \ \Omega \setminus \bigcup_{i=1}^N B_i, \\ \Psi(x_i) + D\Psi(x_i)(x - x_i) & \text{on } \ B'_i := B(x_i, \beta^{1/n} r_i). \end{cases}$$

In particular, $\Psi' = F$ near $\partial\Omega$.

Let $r : G \to (0, \infty)$ be such that for any $p \in G$ and any $0 < r < r(p)$,

$$|G \cap B(p, r)| > \beta|B(p, r)| \tag{4.30}$$

and there is a diffeomorphism on $\mathbb{R}^n$ that equals

$$\Upsilon(x) = \begin{cases} \mathcal{A}(x) & \text{on } \overline{B}(p, \beta^{2/n} r), \\ \mathcal{A}(p) + D\mathcal{A}(p)(x - p) & \text{on } \mathbb{R}^n \setminus \overline{B}(p, \beta^{3/(2n)} r). \end{cases} \tag{4.31}$$

Existence of such a function r is guaranteed by the fact that G consists of density points and by formula (2.6) in Lemma 2.13 (note that $\mathcal{A}$ is a diffeomorphism in a neighborhood of p, by (c) above).

Since $x_i \in G$, (4.25) yields that $D\Psi(x_i), D\mathcal{A}(x_i) \in GL(n)^+$ and

$$\det D\Psi(x_i) > \beta \det D\mathcal{A}(x_i).$$

This is condition (4.5) from Proposition 4.6 which we want to use to modify Ψ' on each of the balls B'_i. Proposition 4.6 and Remark 4.7 give a finite family of pairwise disjoint closed balls $\overline{B}_{ij} = \overline{B}(p_{ij}, r_{ij}) \subset B'_i$ such that

$$p_{ij} \in G \cap B'_i, \quad r_{ij} < r(p_{ij}), \quad \Big|G \cap \bigcup_j \overline{B}_{ij}\Big| > \beta|G \cap B'_i| > \beta^2|B'_i| \tag{4.32}$$

(the last inequality follows from (b)), and a diffeomorphism $F_i : B'_i \to \mathbb{R}^n$ such that

$$F_i(x) = \Psi(x_i) + D\Psi(x_i)(x - x_i) \quad \text{in a neighborhood of } \partial B'_i,$$

and

$$F_i(x) = D\mathcal{A}(x_i)x + v_{ij} \text{ for } x \in \overline{B}_{ij} \text{ and some } v_{ij} \in \mathbb{R}^n.$$

Note that $F_i = \Psi'$ in a neighborhood of $\partial B'_i$. Hence, if we replace Ψ' with F_i on each of the balls B'_i, we will obtain a diffeomorphism $F' : \Omega \twoheadrightarrow F(\Omega)$ which agrees with F near $\partial\Omega$ and satisfies

$$F'(x) = D\mathcal{A}(x_i)x + v_{ij} \text{ for } x \in \overline{B}_{ij} \text{ and some } v_{ij} \in \mathbb{R}^n. \tag{4.33}$$

We will now replace the affine map (4.33) with a diffeomorphism $\mathcal{A}(x) + \tau_{ij}$, $\tau_{ij} \in \mathbb{R}^n$, in two steps. In the first step, we will replace (4.33) with an affine map $D\mathcal{A}(p_{ij})x + w_{ij}$

(on a smaller ball) using Proposition 2.14. Then we will replace this new affine map (on an even smaller ball) with a diffeomorphism $\mathcal{A}(x)+\tau_{ij}$ using formula (2.6) from Lemma 2.13.

Let us fix $A_1 := D\mathcal{A}(p_{ij})$ and $A_2 := D\mathcal{A}(x_i)$ and observe that in view of (4.27),

$$\|A_1 - A_2\| < (\beta^{-1/(2n)} - 1)\|A_2^{-1}\|^{-1},$$

which by triangle inequality and sublinearity of operator norm implies that

$$\|A_2^{-1}A_1\| = \|A_2^{-1}(A_1 - A_2) + \mathcal{I}\| \leq \|A_2^{-1}\|\,\|A_1 - A_2\| + 1 < \beta^{-1/(2n)}.$$

Therefore,

$$A_2^{-1}A_1(B(0,1)) \Subset B(0,\beta^{-1/(2n)}) \quad \text{so} \quad A_1(B(0,\beta^{1/n}r_{ij})) \Subset A_2(B(0,\beta^{1/(2n)}r_{ij})).$$

Applying Proposition 2.14 yields a diffeomorphism $\Theta_{ij} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$, which equals A_1 on $B(0,\beta^{1/n}r_{ij})$ and A_2 on $\mathbb{R}^n \setminus B(0,\beta^{1/(2n)}r_{ij})$. Set

$$H_{ij}(x) := \Theta_{ij}(x - p_{ij}) + A_2 p_{ij} + v_{ij},$$

where v_{ij} were defined in (4.33). We have

$$H_{ij}(x) = \begin{cases} D\mathcal{A}(p_{ij})x \underbrace{- D\mathcal{A}(p_{ij})p_{ij} + D\mathcal{A}(x_i)p_{ij} + v_{ij}}_{\omega_{ij}} & \text{on } B(p_{ij},\beta^{1/n}r_{ij}), \\ D\mathcal{A}(x_i)x + v_{ij} & \text{on } \mathbb{R}^n \setminus B(p_{ij},\beta^{1/(2n)}r_{ij}). \end{cases}$$

Observe that according to the definition of F' in (4.33),

$$H_{ij} = F' \text{ on } B(p_{ij}, r_{ij}) \setminus B(p_{ij},\beta^{1/(2n)}r_{ij}), \text{ for all } i \text{ and } j.$$

Hence, if we replace F' with H_{ij} on each of the balls $\overline{B}_{ij}$, we will obtain a diffeomorphism $F'' : \Omega \twoheadrightarrow F(\Omega)$ which agrees with F near $\partial\Omega$ and satisfies

$$F''(x) = D\mathcal{A}(p_{ij})x + \omega_{ij} \text{ for } x \in \overline{B}(p_{ij},\beta^{1/n}r_{ij}) \text{ and some } \omega_{ij} \in \mathbb{R}^n.$$

Since $r_{ij} < r(p_{ij})$, (4.31) gives a diffeomorphism Υ_{ij} such that

$$\Upsilon_{ij}(x) = \begin{cases} \mathcal{A}(x) & \text{for } x \in \overline{B}(p_{ij},\beta^{2/n}r_{ij}), \\ \mathcal{A}(p_{ij}) + D\mathcal{A}(p_{ij})(x - p_{ij}) & \text{for } x \in \mathbb{R}^n \setminus \overline{B}(p_{ij},\beta^{3/(2n)}r_{ij}). \end{cases}$$

The translated diffeomorphism

$$\Upsilon'_{ij}(x) := \Upsilon_{ij}(x) + \omega_{ij} - \mathcal{A}(p_{ij}) + D\mathcal{A}(p_{ij})p_{ij} =: \Upsilon_{ij}(x) + \tau_{ij}$$

satisfies

$$\Upsilon'_{ij}(x) = \begin{cases} \mathcal{A}(x) + \tau_{ij} & \text{for } x \in \overline{B}(p_{ij},\beta^{2/n}r_{ij}), \\ D\mathcal{A}(p_{ij})x + \omega_{ij} & \text{for } x \in \mathbb{R}^n \setminus \overline{B}(p_{ij},\beta^{3/(2n)}r_{ij}). \end{cases}$$

Observe that

$$\Upsilon'_{ij} = F'' \text{ on } B(p_{ij},\beta^{1/n}r_{ij}) \setminus B(p_{ij},\beta^{3/(2n)}r_{ij}).$$

Hence if we replace F'' with Υ'_{ij} on $\overline{B}(p_{ij},\beta^{1/n}r_{ij})$, we obtain a diffeomorphism $\Phi : \Omega \twoheadrightarrow F(\Omega)$ which agrees with F near $\partial\Omega$ and satisfies

$$\Phi(x) = \mathcal{A}(x) + \tau_{ij} \text{ for } x \in \overline{B}(p_{ij},\beta^{2/n}r_{ij}) \text{ and some } \tau_{ij} \in \mathbb{R}^n.$$

Clearly, $D\Phi = D\mathcal{A}$ on $B(p_{ij}, \beta^{2/n} r_{ij})$. Since $D\mathcal{A} = T$ on G by (4.25), we have that

$$D\Phi(x) = T(x) \text{ for } x \in G \cap \bigcup_{i,j} B(p_{ij}, \beta^{2/n} r_{ij}) \tag{4.34}$$

and it remains to show that the complement of this set has measure less that ε.

Since $p_{ij} \in G$ and $r_{ij} < r(p_{ij})$, (4.30) implies that

$$|G \cap B(p_{ij}, \beta^{2/n} r_{ij})| > \beta |B(p_{ij}, \beta^{2/n} r_{ij})| = \beta^3 |B_{ij}|.$$

Therefore, the fact that the balls B_{ij} are pairwise disjoint, (4.32) and (4.29) give

$$\Big| G \cap \bigcup_{ij} B(p_{ij}, \beta^{2/n} r_{ij}) \Big| > \beta^3 \sum_{i,j} |B_{ij}| > \beta^5 \sum_i |B'_i| = \beta^6 \sum_i |B_i| > \beta^8 |\Omega|.$$

Clearly, we can find a compact set K contained in the set (4.34) so that $|K| > \beta^8 |\Omega|$ and hence $|\Omega \setminus K| < (1 - \beta^8)|\Omega| < \varepsilon$. This completes the proof under the assumption (4.22).

Now we will prove the result in the general case, when $\int_\Omega \det T \le |F(\Omega)|$. It is easy to see that there is a measurable map $\widetilde{T} : \Omega \to GL(n)^+$ such that $\int_\Omega \det \widetilde{T} = |F(\Omega)|$ and $|\{\widetilde{T} \ne T\}| < \varepsilon/2$. We proved that in this situation there is a C^1-diffeomorphism $\Phi : \Omega \to F(\Omega)$ that agrees with F near $\partial\Omega$ and satisfies $|\{D\Phi \ne \widetilde{T}\}| < \varepsilon/2$. Clearly, $|\{D\Phi \ne T\}| < \varepsilon$ and we can find a compact set $K \subset \Omega$ such that $D\Phi = T$ on K and $|\Omega \setminus K| < \varepsilon$. The proof is complete. □

4.4 Corollaries

To prove Theorem 4.2, we need the following main result from [37].

Proposition 4.12. *There is a homeomorphism $\Phi : \mathcal{Q} = [0,1]^n \twoheadrightarrow \mathcal{Q}$, which satisfies the Lusin (N) condition, $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$, such that*

$$D_{\mathrm{a}}\Phi = \mathcal{R} := \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix} \quad \text{a. e.} \tag{4.35}$$

and Φ is a limit of measure preserving C^∞-diffeomorphisms $\Phi_k : \mathcal{Q} \to \mathcal{Q}$, $\Phi_k = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$, in the uniform metric.[1]

In fact, the use of Proposition 4.12 here is similar to what we are going to do in Chapter 5 in Theorem 5.2. Here, we will construct an a. e. approximately differentiable homeomorphism whose derivative equals $\mathcal{R}$ on a large part of a given set and $\mathcal{I}$ on a large part of its complement. In Theorem 5.2, we will construct an a. e. approximately differentiable homeomorphism whose derivative equals $\mathcal{R}$ a. e. on a given set and $\mathcal{I}$ a. e. on its complement.

Moreover, we need the following technical lemmata about approximate differentiability of a composition of an a. e. approximately differentiable mapping and a bi-Lipschitz homeomorphim. This will be generalized in Lemma 5.17 in Section 5.3 with a slightly different method.

[1] The statement of [37, Theorem 1.4] says $\Phi_k|_{\partial\mathcal{Q}} = \mathrm{id}$, but the proof shows that $\Phi_k = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$.

Lemma 4.13. *Let $\Psi : \Omega \twoheadrightarrow \Omega$ be a bi-Lipschitz homeomorphism and $P : \Omega \twoheadrightarrow \Omega$ a mapping which is approximately differentiable a. e. on Ω. Then at a. e. point $x \in \Omega$, $\Phi = P \circ \Psi$ is approximately differentiable and at any such point*

$$D_{\mathrm{a}}\Phi(x) = D_{\mathrm{a}}P(\Psi(x))\, D\Psi(x). \tag{4.36}$$

Proof. We claim that (4.36) holds for any x such that Ψ is differentiable at x and P is approximately differentiable at $\Psi(x)$. By our assumptions, Rademacher's theorem and the fact that Ψ^{-1} satisfies the Lusin (N) condition, the set of such points is of full measure.

Given any x for which P is approximately differentiable at $\Psi(x)$, there is a set $E_{\Psi(x)}$, with $\Psi(x)$ being its density point, such that

$$\lim_{E_{\Psi(x)} \ni z \to \Psi(x)} \frac{|P(\Psi(x)) - P(z) - D_{\mathrm{a}}P(\Psi(x))(\Psi(x) - z)|}{|\Psi(x) - z|}. \tag{4.37}$$

Since Ψ is bi-Lipschitz, x is a density point of the set $E_x := \Psi^{-1}(E_{\Psi(x)})$, as bi-Lipschitz homeomorphisms map density points to density points, see [14] or [47, Lemma A.32]. Therefore, for any $y \in E_x$, $y \to x$, (4.37) holds for $z = \Psi(y)$.

Eventually, by triangle inequality,

$$\begin{aligned}
&\lim_{E_x \ni y \to x} \frac{|P(\Psi(x)) - P(\Psi(y)) - D_{\mathrm{a}}P(\Psi(x))\, D\Psi(x)(x - y)|}{|x - y|} \leq \\
&\lim_{E_x \ni y \to x} \|D_{\mathrm{a}}P(\Psi(x))\| \frac{|\Psi(x) - \Psi(y) - D\Psi(x)(x - y)|}{|x - y|} \\
&\qquad + \frac{|\Psi(x) - \Psi(y)|}{|x - y|} \frac{|P(\Psi(x)) - P(\Psi(y)) - D_{\mathrm{a}}P(\Psi(x))(P(\Psi(x)) - P(\Psi(y))|}{|\Psi(x) - \Psi(y)|} \\
&\qquad = 0.
\end{aligned}$$

Indeed, the first term above converges to zero because Ψ is differentiable at x and the second because P is approximately differentiable at $\Psi(x)$, as explained in the paragraph above. Note that $|\Psi(x) - \Psi(y)|/|x - y|$ is bounded, because Ψ is Lipschitz. This finishes the proof of this lemma. □

Proof of Theorem 4.2. Let

$$\hat{T}(x) = \begin{cases} T(x) & \text{if } \det T(x) > 0, \\ \mathcal{R}T(x) & \text{if } \det T(x) < 0, \end{cases}$$

where $\mathcal{R}$ is the reflection matrix defined in (4.35). Theorem 1.4 yields a diffeomorphism $\Psi : \Omega \twoheadrightarrow \Omega$, $\Psi = \mathrm{id}$ near $\partial\Omega$, and a compact set $C \subset \Omega$ with $|\Omega \setminus C| < \varepsilon/4$ such that for any $x \in C$, $D\Psi(x) = \hat{T}(x)$.

Set

$$C_- = \{x \in C : \det T(x) < 0\} \text{ and } C_+ = \{x \in C : \det T(x) > 0\}.$$

The derivative on C_+ is as required. Now, the idea is to approximate the set C_- with cubes and, in each cube, to use a version of the homeomorphism from Proposition 4.12 to correct the prescribed derivative. In doing so, we will spoil the already prescribed derivative on a small part of C_+. We now choose these cubes. For any $\eta > 0$, it is possible to find a finite family of closed cubes $\{Q_i^\eta\}_{i=1}^{m(\eta)}$, $Q_i^\eta \subset \Omega$, with pairwise disjoint interiors such that

$$\left| \Psi(C_-) \triangle \bigcup_{i=1}^{m(\eta)} Q_i^\eta \right| < \eta. \tag{4.38}$$

Let $E^\eta := \Psi^{-1}\left(\bigcup_{i=1}^{m(\eta)} Q_i^\eta\right)$, we choose η so that

$$|C_- \vartriangle E^\eta| < \varepsilon/4, \tag{4.39}$$

which is always possible due to the classical change of variables formula and absolute continuity of integral. From now on, we will write E, Q_i, m instead of E^η, Q_i^η, $m(\eta)$ for thus chosen η.

For each $i = 1, \dots, m$, we can construct $P_i : Q_i \twoheadrightarrow Q_i$, a properly rescaled and translated copy of the homeomorphism from Proposition 4.12. That is, $P_i = \mathrm{id}$ on ∂Q_i, P_i satisfies the Lusin (N) condition and P_i is a. e. approximately differentiable with $D_\mathrm{a} P_i = \mathcal{R}$ a. e. on Q_i. We then set

$$P(x) = \begin{cases} P_i(x) & \text{for } x \in Q_i, \\ x & \text{for } x \in \Omega \setminus \bigcup_{i=1}^m Q_i. \end{cases}$$

Since $P_i = \mathrm{id}$ on ∂Q_i, P is indeed a homeomorphism of Ω. Since $\bigcup_{i=1}^m Q_i$ is compactly contained in Ω, P equals identity near $\partial\Omega$. Moreover, P satisfies the Lusin (N) condition, is approximately differentiable a. e. on Ω and for a. e. $x \in E$, $D_\mathrm{a} P(\Psi(x)) = \mathcal{R}$.

We claim that

$$\Phi := P \circ \Psi$$

is the required mapping, which has the required derivative at almost every point of the set $(C_- \cap E) \cup (C_+ \setminus E)$. This is not yet the set K from the statement of the theorem (it may be not compact) and before we go on showing properties of Φ, let us check that

$$\begin{aligned} \Omega \setminus [(C_- \cap E) \cup (C_+ \setminus E)] &= [(\Omega \setminus C_-) \cup (C_- \setminus E)] \cap [(\Omega \setminus C_+) \cup (C_+ \cap E)] \\ &\subset [(\Omega \setminus C_-) \cap (\Omega \setminus C_+)] \cup (C_+ \cap E) \cup (C_- \setminus E) \\ &\subset (\Omega \setminus C) \cup (E \setminus C_-) \cup (C_- \setminus E). \end{aligned}$$

In the last inclusion, we used the fact that C_- and C_+ are disjoint and that therefore $C_+ \cap E \subset E \setminus C_-$. Consequently, by (4.39)

$$|\Omega \setminus [(C_- \cap E) \cup (C_+ \setminus E)]| \le |\Omega \setminus C| + |C_- \vartriangle E| \le \varepsilon/2.$$

Let us choose compact sets $K_1 \subset C_- \cap E$, $K_2 \subset C_+ \setminus E$ so that

$$|(C_- \cap E) \setminus K_1| < \varepsilon/4 \quad \text{and} \quad |(C_+ \setminus E) \setminus K_2| < \varepsilon/4.$$

Then $K := K_1 \cup K_2$ is a compact set which satisfies

$$|\Omega \setminus K| < \varepsilon. \tag{4.40}$$

By definition, $\Phi : \Omega \twoheadrightarrow \Omega$ is an a. e. approximately differentiable homeomorphism which satisfies the Lusin (N) condition and equals identity near $\partial\Omega$, i. e., satisfies (a) and (b).

For almost any $x \in C_- \cap E$, $D\Psi(x) = \hat{T}(x) = \mathcal{R}T(x)$ and $D_\mathrm{a} P(\Psi(x)) = \mathcal{R}$. Note that Ψ is a diffeomorphism of Ω, which equals identity near $\partial\Omega$ and hence is bi-Lipschitz on Ω. By Lemma 4.13, for almost every point x of $C_- \cap E$,

$$D_\mathrm{a}\Phi(x) = DP_\mathrm{a}(\Psi(x))\, D\Psi(x) = \mathcal{R}\mathcal{R}T(x) = T(x).$$

Therefore, for almost every point x of the set $K_1 \subset C_- \cap E$, $D_\mathrm{a}\Phi(x) = T(x)$. Next, we check that for almost every point x of $C_+ \setminus E$, $D\Psi(x) = T(x)$ and $D_\mathrm{a} P(\Psi(x)) = \mathcal{I}$. Again by Lemma 4.13, we see that $D_\mathrm{a}\Phi(x) = T(x)$. Since $K_2 \subset C_+ \setminus E$, $D_\mathrm{a}\Phi = T$ a. e. on K_2. All in all we have shown that $D_\mathrm{a}\Phi = T$ on K which satisfies (4.40) and hence (c) holds. $\square$

Before we proceed with the proof of Theorem 4.3, we state and prove a technical lemma concerning connectedness of complements of compact sets. We will also use it in Chapter 5.

Lemma 4.14. *Let E be a measurable subset of a domain Ω in $\mathbb{R}^n$. Then for any $\varepsilon > 0$ there is a compact set $K \subset E$ such that $|E \setminus K| < \varepsilon$ and $\Omega \setminus K$ is connected.*

Proof. First, let $K_1 \subset E$ be a compact set such that $|E \setminus K_1| \leq \varepsilon/2$.

Next, let $K_2 \subset \Omega$ denote a finite sum of pairwise disjoint closed cubes, $K_2 = \bigcup_{i=1}^N P_i$, such that $|K_1 \setminus K_2| < \varepsilon/4$. In each of the cubes P_i, let C_i denote the standard Cantor set of positive measure, such that $|P_i \setminus C_i| < \varepsilon/(4N)$. Denote $C = \bigcup_{i=1}^N C_i$. Finally, set $K = K_1 \cap C$.

Then K is obviously compact,

$$E \setminus K \subset (E \setminus K_1) \cup (K_1 \setminus K_2) \cup (K_2 \setminus K)$$

and $K_2 \setminus K \subset \bigcup_{i=1}^N (P_i \setminus C_i)$, thus

$$|E \setminus K| \leq |E \setminus K_1| + |K_1 \setminus K_2| + \sum_{i=1}^N |P_i \setminus C_i| < \varepsilon/2 + \varepsilon/4 + N\varepsilon/(4N) = \varepsilon.$$

The fact that $\Omega \setminus K$ is connected follows from construction of the Cantor sets C_i. Indeed, in each of the cubes P_i the complement of the set C_i is path-connected and contains ∂P_i. It follows that any point p in $\Omega \setminus C$ can be connected to a given point q in C by a path which intersects C in q only. Therefore, the complement of any subset of C (in particular, of K) is path connected. (The fact that $\Omega \setminus K$ is connected follows also from more involved results in topological dimension theory [53, p. 22 and p. 48].) □

Proof of Theorem 4.3. The proof is inductive. We construct a family of diffeomorphisms $\Phi_k : \Omega \twoheadrightarrow \Omega$ and compact sets $E_k \subset \Omega$ so that $D\Phi_k = T$ on E_k. The sets E_k form an increasing family so that $\bigcup_k E_k$ has full measure, i. e. $|\bigcup_k E_k| = |\Omega|$. Diffeomorphisms Φ_k do not converge in the uniform metric d but in the Lusin metric d_L, see Section 2.1 to recall the definitions. In the end, the limit has the required derivative T a. e. on Ω.

We begin with Φ_1. By Theorem 1.4, we find a diffeomorphism $\Phi_1 : \Omega \twoheadrightarrow \Omega$, $\Phi_1 = \mathrm{id}$ near $\partial\Omega$, and a compact set $E_1' \subset \Omega$ with $|\Omega \setminus E_1'| < 1/4$ such that $D\Phi_1 = T$ on E_1'. By Lemma 4.14, we can replace E_1' with a compact set $E_1 \subset E_1'$ so that $|\Omega \setminus E_1| < 1/2$ and $\Omega \setminus E_1$ is a domain. We shall soon comment on why $\Omega \setminus E_1$ needs to be connected.

Assume now that for $i = 1, \ldots, k$ we have found diffeomorphisms $\Phi_i : \Omega \twoheadrightarrow \Omega$, $\Phi_i = \mathrm{id}$ near $\partial\Omega$, and an increasing family of compact sets $E_i \subset \Omega$ such that

$$|\Omega \setminus E_i| < 2^{-i}, \quad \Omega \setminus E_i \text{ is a domain}, \quad D\Phi_i = T \text{ on } E_i. \tag{4.41}$$

Moreover, for $i = 2, \ldots, k$,

$$\Phi_i = \Phi_{i-1} \text{ near } E_{i-1}. \tag{4.42}$$

We shall now show that it is possible to construct a diffeomorphism Φ_{k+1} and a compact set $E_{k+1} \subset \Omega$, $E_k \subset E_{k+1}$ which satisfies (4.41) and (4.42) for $i = k + 1$.

Observe that using the assumption, (4.41), and the classical change of variables formula, we get

$$
\begin{aligned}
\int_{\Omega\setminus E_k} |\det T(x)|\,dx &= \int_\Omega |\det T(x)|\,dx - \int_{E_k} |\det T(x)|\,dx \\
&= |\Omega| - \int_{E_k} |\det D\Phi_k(x)|\,dx = |\Phi_k(\Omega)| - |\Phi_k(E_k)| \\
&= |\Phi_k(\Omega\setminus E_k)|.
\end{aligned}
\tag{4.43}
$$

This is the moment when we need $\Omega\setminus E_k$ to be connected. Since (4.43) holds and $\Omega\setminus E_k$ is a domain, Theorem 1.4 yields a diffeomorphism $\widetilde{\Phi}_{k+1} : \Omega\setminus E_k \to \mathbb{R}^n$, $\widetilde{\Phi}_{k+1} = \Phi_k$ near $\partial(\Omega\setminus E_k)$ and a compact set $K'_{k+1} \subset \Omega\setminus E_k$ such that

$$
|(\Omega\setminus E_k)\setminus K'_{k+1}| < 2^{-(k+2)}, \quad D\widetilde{\Phi}_{k+1} = T \text{ on } K'_{k+1}.
$$

Then, by Lemma 4.14, we replace K'_{k+1} with its compact subset $K_{k+1} \subset K'_{k+1}$ so that $|K'_{k+1}\setminus K_{k+1}| < 2^{-(k+2)}$ and $\Omega\setminus(E_k\cup K_{k+1})$ is a domain.

We set $E_{k+1} := E_k \cup K_{k+1}$ and

$$
\Phi_{k+1}(x) := \begin{cases} \widetilde{\Phi}_{k+1}(x) & \text{for } x\in\Omega\setminus E_k, \\ \Phi_k(x) & \text{for } x\in E_k. \end{cases}
$$

Since $\widetilde{\Phi}_{k+1} = \Phi_k$ near ∂E_k, Φ_{k+1} is indeed a diffeomorphism. Moreover, Φ_{k+1} satisfies (4.42) for $i = k+1$ and $D\Phi_{k+1} = T$ on E_{k+1}. We have already checked that $\Omega\setminus E_{k+1}$ is a domain, it remains to see that the left-most condition for $i = k+1$ from (4.41) holds. Indeed,

$$
|\Omega\setminus E_{k+1}| = |(\Omega\setminus E_k)\setminus K'_{k+1}| + |K'_{k+1}\setminus K_{k+1}| < 2\cdot 2^{-(k+2)} = 2^{-(k+1)}.
$$

We have constructed inductively a sequence of diffeomorphisms $\Phi_k : \Omega \twoheadrightarrow \Omega$ and an increasing family $\{E_k\}_{k=1}^\infty$ of compact sets, which satisfies $|\Omega\setminus\bigcup_{k=1}^\infty E_k| = 0$ by (4.41). Since $\Phi_{k+1} = \Phi_k$ near E_k, the sequence Φ_k converges in the Lusin metric d_L. However, this limit is only defined up to a set of measure zero and we want to have a mapping defined everywhere on $\overline{\Omega}$. Therefore, we set $E := \bigcup_{k=1}^\infty E_k$, choose $x_o \in \Omega\setminus E$ and define

$$
F(x) = \begin{cases} \Phi_k(x) & \text{for } x\in E_k, \\ x & \text{for } x\in\partial\Omega, \\ x_o & \text{for } x\in\Omega\setminus E. \end{cases}
$$

The mapping F is well defined everywhere on $\overline{\Omega}$. Indeed, as $\{E_k\}_k$ is an increasing family of sets and (4.42) holds, F is well defined on E. For every $k = 1,2,\dots$, we have $E_k \subset \Omega$ and therefore $E\cap\partial\Omega = \varnothing$. Moreover, $F(\overline{\Omega}) \subseteq \Omega$ and clearly $F(x) = x$ for $x\in\partial\Omega$, i. e., (a) holds.

As $F = \Phi_k$ on E_k and Φ_k is a diffeomorphism, for any density point x of the set E_k, F is approximately differentiable at x with $D_{\mathrm{a}}F(x) = \Phi_k(x)$, i. e., $D_{\mathrm{a}}F(x) = T(x)$ a. e. on E_k. As $E = \bigcup_{k=1}^\infty E_k$ is a set of full measure, this means that F is approximately differentiable a. e. on Ω and $D_{\mathrm{a}}F = T$ a. e. on Ω, i. e., (d) holds.

We claim that $F|_E$ is injective. Indeed, given any two distinct points $x, y \in E$, we clearly have $F(x) \neq F(y)$ if $x, y \in E_k$ for some $k\in\mathbb{N}$. If $x\in E_k$ and $y\in E_\ell$ for $\ell > k$, by (4.42), $\Phi_\ell = \Phi_k$ on E_k and hence $\Phi_k(x) = \Phi_\ell(x) \neq \Phi_\ell(y)$, because Φ_ℓ is injective. Therefore, property (c) is true.

Eventually, we check property (b). Let $Z \subset \bar{\Omega}$ be a set of measure zero. Clearly, then $F(Z \cap (\Omega \setminus E)) \subset \{x_o\}$, which is a set of measure zero. On the other hand, $F(\partial\Omega \cap Z) = \partial\Omega \cap Z$, which also is a set of measure zero. Consequently,

$$|F(Z \cap E)| = |\bigcup\nolimits_{k=1}^{\infty} \Phi_k(Z \cap E_k)| \leq \sum_{k=1}^{\infty} |\Phi_k(Z \cap E_k)| = 0.$$

All in all, an image under F of a set of measure zero is a set of measure zero, which shows that (b) holds and finishes the proof. $\square$

We cannot claim that F is classically differentiable a. e. on E. Even though $E_k \subset E_{k+1}$ and $\Phi_{k+1} = \Phi_k$ near E_k, it may happen that no neighborhood of E_k is contained in E_{k+1}. In such a situation, we do not know if $F = \Phi_k$ *near* E_k and hence we cannot say anything about classical derivative.

Observe that such a straightforward method of iteration cannot guarantee the limit mapping to be a homeomorphism. The problem of how to perform the iteration to construct a homeomorphism with prescribed derivative will be the main topic of the next chapter.

Chapter 5

Constructing homeomorphisms with prescribed approximate derivative

5.1 Introduction

In this chapter, we prove

Theorem 1.5. *Let $\mathcal{Q} = [0,1]^n$. For any measurable mapping $T : \mathcal{Q} \to GL(n)$ that satisfies*

$$\int_{\mathcal{Q}} |\det T(x)|\, dx = 1, \tag{1.4}$$

there exists an a. e. approximately differentiable homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ such that $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$ and $D_{\mathrm{a}}\Phi = T$ a. e. Moreover,

(a) Φ^{-1} is approximately differentiable a. e. and $D_{\mathrm{a}}\Phi^{-1}(y) = T^{-1}(\Phi^{-1}(y))$ for almost all $y \in \mathcal{Q}$;

(b) Φ preserves the sets of measure zero, i.e., for any $A \subset \mathcal{Q}$,

$$|A| = 0 \qquad \text{if and only if} \qquad |\Phi(A)| = 0.$$

(c) Φ is a limit of C^∞-diffeomorphisms $\Phi_k : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi_k = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$, in the uniform metric, i.e., $\|\Phi - \Phi_k\|_\infty + \|\Phi^{-1} - \Phi_k^{-1}\|_\infty \to 0$ as $k \to \infty$.

This is Theorem 1.4 from [39]. The proof follows [39], in particular this chapter contains the entire Sections 5 and 7 and a large part of Section 6 (the rest appears in Chapter 2) from [39].

Let us recapitulate here why a. e. approximately differentiable homeomorphisms are important, as discussed in Section 1.3. Firstly, they are the class for which the change of variables formula holds, see Theorem 2.29. Secondly, they appear naturally as limits of diffeomorphisms in the Lusin metric, see Lemma A.1. It is also worthwhile to investigate their properties because Sobolev and BV mappings are a. e. approximately differentiable, see Lemma 2.32. In Section 5.3, we show a few technical lemmata about approximately differentiable mappings used in the sequel. In particular, in Lemma 5.17 we prove a useful version of chain rule for approximate derivative.

Let us note an important observation concerning Φ from Theorem 1.5. It follows from degree theory (see Remark A.11 for details) that if at $x_o \in \mathcal{Q}$ we have $\det T(x_o) < 0$, then Φ cannot be classically differentiable at x_o. However, there is no immediate obstruction as to why, if $\det T(x_o) > 0$, Φ could not be classically differentiable at x_o. This motivates the following

Question 5.1. Let $\mathcal{Q} = [0,1]^n$ and $T : \mathcal{Q} \to GL(n)^+$ be a measurable mapping satisfying (1.4). Does there exist a homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$ which is differentiable a. e. on $\mathcal{Q}$ with $D\Phi = T$ a. e. on $\mathcal{Q}$?

Our construction of Φ in Theorem 1.5 does not yield classical differentiability at points in which $\det T > 0$. To answer Question 5.1 in the positive with such an approach, one would have to substantially modify it as to take better care of what happens *in a neighborhood of* each point and not only *at* a given point.

The proof of Theorem 1.5 is long and constructive and we present here its key steps. The first step reduces the problem to the case when $\det T > 0$ with the help of Theorem 5.2 below. Let $\mathcal{I}$ be the identity matrix and

$$\mathcal{R} := \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix}. \tag{5.1}$$

In Section 5.3.1, we prove the following

Theorem 5.2. *Let $\mathcal{Q} = [0,1]^n$ and let $E \subset \mathcal{Q}$ be a measurable set. Then there exists an almost everywhere approximately differentiable homeomorphism Φ of the cube $\mathcal{Q}$ onto itself, such that $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$ and*

$$D_{\mathrm{a}}\Phi(x) = \begin{cases} \mathcal{R} & \text{for almost all } x \in E, \\ \mathcal{I} & \text{for almost all } x \in \mathcal{Q} \setminus E. \end{cases} \tag{5.2}$$

Moreover, Φ is a limit, in the uniform metric d, of measure preserving C^∞-diffeomorphisms that are identity in a neighborhood of $\partial\mathcal{Q}$.

This is also a new result, which generalizes the main result from [37], see Proposition 4.12 for the precise statement.

We inductively show that there exists a family of orientation preserving C^1-diffeomorphisms Φ_k of $\mathcal{Q}$ and Borel sets $C_k \subset \mathcal{Q}$ with the following properties for $k \geq 1$:

(i) $\Phi_k = \mathrm{id}$ near $\partial\mathcal{Q}$;
(ii) $\Phi_{k+1} = \Phi_k$ on C_k;
(iii) $D\Phi_k = T$ on C_k;
(iv) C_k is an increasing family of sets, $C_1 \subset C_2 \subset \cdots$, with $\lim_{k\to\infty} |C_k| = 1$;
(v) $d(\Phi_k, \Phi_{k+1}) < 2^{-(k-1)}$ for $k \geq 2$.

From this, it easily follows that the limit $\Phi := \lim_{k\to\infty} \Phi_k$ is the desired mapping, see Section 5.4.2 for detailed explanation. However, it is not easy to construct such a family. Below we give the flavour of the idea.

We have seen in Section 4.4, in the proof of Theorem 4.3, that a naive iteration scheme used there is far from suitable since it fails to guarantee uniform convergence. To overcome this difficulty, we use a much more complex iteration scheme. At each step k of the iteration, we construct a diffeomorphism Φ_k and a partition $\mathcal{P}_k$ of the unit cube, which is a refinement of the partition $\mathcal{P}_{k-1}$ from the previous step. Thanks to the construction of the partitions, we can construct Φ_k by a local modification of Φ_{k-1}, which guarantees convergence of the entire sequence in the uniform metric.

The construction of Φ_k given Φ_{k-1} consists of two steps. Firstly, we correct the way Φ_{k-1} distributes the measure (i. e., we correct its Jacobian) and then we repetitively apply

Theorem 1.4 to prescribe the derivative on a larger set. This first step is necessary to guarantee that the assumption of Theorem 1.4 holds. We develop the technique for this in Section 5.2. It is a substantial modification of the construction of the mapping in the proof of the homeomorphic measures theorem by Oxtoby and Ulam [81, Theorem 2]. Their construction inspired ours.

In Section 5.5, we include a few corollaries to Theorem 1.5. Firstly, in Corollary 5.21, we show that given $T:\mathcal{Q}\to GL(n)$ satisfying the weaker assumption $\int_{\mathcal{Q}}|\det T|<\infty$ instead of (1.4), it is also possible to find an a. e. approximately differentiable homeomorphism $\Phi:\mathcal{Q}\to\mathbb{R}^n$. Naturally, the image $\Phi(\mathcal{Q})$ cannot equal $\mathcal{Q}$ unless (1.4) holds and hence if (1.4) does not hold, we cannot require Φ to fix the boundary of $\mathcal{Q}$.

Also, a direct application of Theorem 1.5 yields

Corollary 5.3. *Let $\mathcal{Q}=[0,1]^n$ and let $f:\mathcal{Q}\to\mathbb{R}$ be measurable with $f\neq 0$ a. e. on $\mathcal{Q}$ and $\int_{\mathcal{Q}}|f(x)|\,dx=1$. Then there exists an a. e. approximately differentiable homeomorphism $\Phi:\mathcal{Q}\twoheadrightarrow\mathcal{Q}$, $\Phi_{\partial\mathcal{Q}}=\mathrm{id}$ which satisfies the Lusin (N) condition and such that for any measurable set $E\subset\mathcal{Q}$,*

$$|\Phi(E)|=\int_E|f(x)|\,dx. \tag{5.3}$$

This is a generalization of the homeomorphic measures theorem by Oxtoby and Ulam proved in [81, Theorem 2] quoted in (1.7) in Section 1.3. Indeed, let μ_f be a measure defined on $\mathcal{Q}$ with the formula

$$\mu_f(E)=\int_E|f(x)|\,dx \text{ for a measurable set } E\subset\mathcal{Q}.$$

Such a measure must be equivalent to the n-dimensional Lebesgue measure (i. e., $\mu_f(E)=0$ if and only if $|E|=0$). Corollary 5.3 says that given such a measure μ_f, we can find a homeomorphism as in Oxtoby and Ulam's theorem which, additionally, is a. e. approximately differentiable. This also gives rise to an interesting

Question 5.4. Let $\mathcal{Q}=[0,1]^n$ and $f:\mathcal{Q}\to\mathbb{R}$ be measurable with $\int_{\mathcal{Q}}|f(x)|\,dx=1$. Does there always exist an a. e. approximately differentiable homeomorphism $\Phi:\mathcal{Q}\twoheadrightarrow\mathcal{Q}$, $\Phi_{\partial Q}=\mathrm{id}$, such that for any Borel set $E\subset\mathcal{Q}$, (5.3) holds?

This would yield a generalization of Oxtoby and Ulam's theorem for a measure μ which is only absolutely continuous w. r. t. Lebesgue measure and not equivalent to it. As discussed in Section 1.3, there is even a bi-Sobolev homeomorphism with vanishing Jacobian a. e. [27] and hence the condition that $f\neq 0$ is not necessary for a construction of an a. e. approximately differentiable homeomorphism whose Jacobian equals f.

If $\Phi_k:\mathcal{Q}\twoheadrightarrow\mathcal{Q}$ is a sequence of orientation preserving C^1-diffeomorphisms which converge to a homeomorphism $\Phi:\mathcal{Q}\twoheadrightarrow\mathcal{Q}$ in the Lusin metric, then Φ is approximately differentiable a. e. and

$$\det D_{\mathrm{a}}\Phi(x)>0 \text{ a. e.} \quad \text{and} \quad \int_{\mathcal{Q}}\det D_{\mathrm{a}}\Phi(x)\,dx\leq 1,$$

see Lemma 5.22 for details. One can then ask, what other conditions must $D_{\mathrm{a}}\Phi$ satisfy? It turns out that basically none.

Theorem 5.5. *Let $\mathcal{Q}=[0,1]^n$. For any measurable mapping $T:\mathcal{Q}\to GL(n)^+$ such that*

$$\int_{\mathcal{Q}}\det T(x)\,dx=1,$$

there exists a sequence of C^1-diffeomorphisms $\Phi_k : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi_k = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$, that converges both in the uniform metric d and the Lusin metric d_L to a homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$, that is a. e. approximately differentiable and satisfies $D_a\Phi = T$ a. e.

It is interesting to look at Theorems 1.5 and 5.5 from the point of view of nonlinear elasticity with the following question in mind: could the class of a. e. approximately differentiable homeomorphisms be useful in studying deformations? Since their derivatives can be so general and bear no information as to the properties of the mapping itself, the answer seems to be *no*. Nonetheless, the moral of these two theorems is that one should keep in mind how complicated even uniform limits of diffeomorphisms can be.

Sections 5.2 and 5.3 develop tools needed in the proof of Theorem 1.5. In Section 5.4, we prove Theorem 1.5 in a series of steps for better readability. Section 5.5 consits of a few corollaries to the main theorem, as described above.

5.2 Preliminaries for Theorem 1.5

This section consists of a series of similar lemmata of increasing complexity, whose aim is to prove Proposition 5.16. In this proposition we construct a particular partition of a diffeomorphic closed cube (i. e., diffeomorphic image of a closed cube), which is instrumental to carry out the iteration in the proof of Theorem 1.5.

Essentially, the results in this section are far-reaching modifications of the construction of the mapping in the proof of the homeomorphic measures theorem by Oxtoby and Ulam, see [81, Theorem 2] for the original paper or [3, Section A2.2], [36, Chapter 7] for a concise treatment.

The original construction of Oxtoby and Ulam, [81, 3, 36], does not lead to any differentiability properties of the homeomorphism, even if the measure is absolutely continuous with respect to the Lebesgue measure (which is the case considered by us). Therefore, in order to prove the a. e. approximate differentiability claimed in Theorem 1.5, we need essential modifications of the argument of Oxtoby and Ulam.

Before we begin, let us recollect and introduce some notation. For a comprehensive list of used symbols see Section 2.1.

Notation. $\mathcal{Q}$ denotes the unit cube in $\mathbb{R}^n$, $\mathcal{Q} := [0,1]^n$.

The space of homeomorphisms of the unit cube $\mathcal{Q}$ is equipped with the *uniform metric*

$$d(\Phi, \Psi) = \sup_{x\in\mathcal{Q}} |\Phi(x) - \Psi(x)| + \sup_{x\in\mathcal{Q}} |\Phi^{-1}(x) - \Psi^{-1}(x)|,$$

see Section 2.8 for more details.

Let $m, n \in \mathbb{N}$. The space of measurable mappings $f, g : E \to \mathbb{R}^n$ defined on a measurable set $E \subset \mathbb{R}^m$ is equipped with the *Lusin metric* defined as

$$d_L(f, g) := |\{x \in \Omega : \ f(x) \neq g(x)\}|.$$

A set P is a *diffeomorphic closed cube* if there is a diffeomorphism Θ defined on a neighborhood of P such that $\Theta(P) = \mathcal{Q}$. We say that $\mathcal{P} = \{P_i\}_{i=1}^N$ is *a partition of* P if $P = \bigcup_{i=1}^N P_i$ and P_i are diffeomorphic closed cubes with pairwise disjoint interiors.

Important examples of partitions of $\mathcal{Q}$ are the *dyadic partitions* into 2^{nk} identical cubes of edge length 2^{-k}, for $k = 0, 1, 2, \ldots$

A partition $\mathcal{P} = \{P_i\}_{i=1}^N$ of P is a *diffeomorphic dyadic partition* if there is a diffeomorphism Θ defined in a neighborhood of P and such that $\{\Theta(P_i)\}_{i=1}^N$ forms a dyadic partition of $\mathcal{Q}$. More generally, a partition $\mathcal{P}$ of P is *diffeomorphic* to a partition $\mathcal{P}'$ of P' if $\mathcal{P}' = \{\Theta(P_i) \ : \ P_i \in \mathcal{P}\}$ for some diffeomorphism Θ defined in a neighborhood of P.

In the lemma below, we will construct the the desired diffeomorphism using a 1-parameter group of diffeomorphisms, see Theorem A.2 for details about existence and properties of thus defined mappings.

Lemma 5.6. *Let P be a rectangular box*

$$P = P_1 \cup \ldots \cup P_k = [0, a] \times [0, 1]^{n-1}, \quad k \geq 2,$$

represented as the union of adjacent boxes

$$P_i = [a_{i-1}, a_i] \times [0, 1]^{n-1}, \quad 0 = a_0 < a_1 < \ldots < a_k = a.$$

If functions $f, g \in L^1(P)$, $f, g > 0$ a. e., are such that

$$\int_P f(x)\, dx = \int_P g(x)\, dx, \tag{5.4}$$

then there is a diffeomorphism $\Phi : P \twoheadrightarrow P$ that is identity in a neighborhood of ∂P, and such that

$$\int_{P_i} f(x)\, dx = \int_{\Phi(P_i)} g(x)\, dx \quad \textit{for } i = 1, 2, \ldots, k.$$

Remark 5.7. The lemma has a simple geometric interpretation. The functions f and g are densities of absolutely continuous measures μ_f and μ_g, and (5.4) means that $\mu_f(P) = \mu_g(P)$. The lemma says that given the partition $P_1, \ldots, P_k$ of P, we can find a diffeomorphic partition $\Phi(P_1), \ldots, \Phi(P_k)$ of P, such that the corresponding cells have equal measures $\mu_f(P_i) = \mu_g(\Phi(P_i))$.

Proof of Lemma 5.6. The proof will be by induction with respect to k. Thus, first assume that $k = 2$. Since $\int_{P_1} f + \int_{P_2} f = \int_{P_1} g + \int_{P_2} g$, without loss of generality, we may assume that $\int_{P_1} f \geq \int_{P_1} g$. Let K be a compact rectangular box in the interior of P, with edges parallel to the coordinate axes, such that

$$\int_{P_1} f(x)\, dx < \int_K g(x)\, dx.$$

By taking K sufficiently large, we may assume that the common face of P_1 and P_2 intersects K. Let X be a smooth vector field parallel to the x_1 coordinate axis, non-zero in a neighborhood of K, zero in a neighborhood of ∂P, and such that X points in the positive direction of the x_1-axis whenever $X \neq 0$. If Φ_t is the one-parameter family of diffeomorphisms generated by X, then $\Phi_0(P_1) = P_1$, so

$$\int_{P_1} f(x)\, dx \geq \int_{\Phi_0(P_1)} g(x)\, dx. \tag{5.5}$$

Since the common face of P_1 and P_2 intersects K, X is non-zero in a neighborhood of that intersection. Now, from a standard compactness and open covering argument recalled below, we see that there is t_o such that $K \subset \Phi_t(P_1)$ for all $t > t_o$, so

$$\int_{P_1} f(x)\, dx < \int_K g(x)\, dx \leq \int_{\Phi_t(P_1)} g(x)\, dx \quad \text{for all } t > t_o. \tag{5.6}$$

Indeed, since Φ_t depends continuously on the parameter t, for any $x \in K$, there is a $t_x > 0$ and $\varepsilon_x > 0$ such that for $t > t_x$, $\Phi_{-t}(B(x, \varepsilon_x)) \subset P_1$. Balls $B(x, \varepsilon_x)$ for $x \in K$ form an open covering of K and by compactness of K, we can choose a finite subcovering of K, balls $B(x_i, \varepsilon_{x_i})$ for $i = 1, \ldots, N$. Setting $t_o := \max_i t_{x_i}$, we see that for $t > t_o$ for all $x \in K$, $\Phi_{-t}(x) \in P_1$, i.e., $x \in \Phi_t(P_1)$. This implies that $K \subset \Phi_t(P_1)$ for $t > t_o$.

Since the function $t \mapsto \int_{\Phi_t(P_1)} g$ is continuous, it follows from (5.5) and (5.6) that there is $t \in [0, t_o]$, such that $\Phi := \Phi_t$ satisfies

$$\int_{P_1} f(x)\,dx = \int_{\Phi(P_1)} g(x)\,dx \quad \text{and hence} \quad \int_{P_2} f(x)\,dx = \int_{\Phi(P_2)} g(x)\,dx.$$

Observe that the diffeomorphism Φ equals identity in a neighborhood of ∂P, on the set where $X = 0$.

We completed the proof in the case of $k = 2$. Suppose now that the claim is true for all integers in $\{2, \ldots, k\}$ and we will prove it for $k + 1$.

Let us write

$$P = \underbrace{P_1 \cup \ldots \cup P_k}_{\widetilde{P}} \cup P_{k+1} = \widetilde{P} \cup P_{k+1}.$$

Applying the claim for two boxes, we can find a diffeomorphism $\Phi_1 : P \twoheadrightarrow P$, that is identity in a neighborhood of ∂P and such that

$$\int_{\widetilde{P}} f(x)\,dx = \int_{\Phi_1(\widetilde{P})} g(x)\,dx \quad \text{and} \quad \int_{P_{k+1}} f(x)\,dx = \int_{\Phi_1(P_{k+1})} g(x)\,dx. \tag{5.7}$$

Note that the second equality in (5.7) is as desired and our diffeomorphism Φ will be equal Φ_1 in P_{k+1}. However, we have to modify it in $\widetilde{P}$. The diffeomorphism Φ_1 is orientation preserving and hence its Jacobian $J_{\Phi_1} = \det D\Phi_1$ is positive in $\widetilde{P}$. Let

$$\tilde{g}(x) = (g \circ \Phi_1)(x) J_{\Phi_1}(x) \quad \text{for } x \in \widetilde{P}.$$

The change of variables formula yields

$$\int_{\widetilde{P}} \tilde{g}(x)\,dx = \int_{\widetilde{P}} (g \circ \Phi_1)(x) J_{\Phi_1}(x)\,dx = \int_{\Phi_1(\widetilde{P})} g(x)\,dx = \int_{\widetilde{P}} f(x)\,dx,$$

and hence the pair of functions f and $\tilde{g}$ satisfies the assumption (5.4) on $\widetilde{P} = P_1 \cup \ldots \cup P_k$. Thus, the induction hypothesis yields a diffeomorphism $\Phi_2 : \widetilde{P} \twoheadrightarrow \widetilde{P}$ that is identity near $\partial\widetilde{P}$ and such that for $i = 1, 2, \ldots, k$, we have

$$\int_{P_i} f(x)\,dx = \int_{\Phi_2(P_i)} \tilde{g}(x)\,dx = \int_{\Phi_2(P_i)} (g \circ \Phi_1)(x) J_{\Phi_1}(x)\,dx = \int_{(\Phi_1 \circ \Phi_2)(P_i)} g(x)\,dx.$$

Therefore, the diffeomorphism $\Phi : P \twoheadrightarrow P$ defined by

$$\begin{cases} \Phi_1 \circ \Phi_2(x) & \text{if } x \in \widetilde{P}, \\ \Phi_1(x) & \text{if } x \in P_{k+1}, \end{cases}$$

satisfies the claim for $k + 1$. To see that Φ is a well defined diffeomorphism, note that Φ_2 is identity near $\partial\widetilde{P}$ and hence $\Phi_1 \circ \Phi_2 = \Phi_1$ near the common face of the boxes $\widetilde{P}$ and P_{k+1}. Also, Φ is identity near the boundary of P. The proof is complete. □

The following corollary shows that Lemma 5.6 can be applied to diffeomorphic closed cubes and their partitions that are diffeomorphic to the partition in Lemma 5.6.

Corollary 5.8. *Let P and its partition be as in Lemma 5.6. Let $\Theta : P \to \mathbb{R}^n$ be a diffeomorphism. Denote by $\widetilde{P}$ and $\widetilde{P}_i$ the images of P and P_i under Θ. If functions $f, g \in L^1(\widetilde{P})$, $f, g > 0$ a. e., are such that*

$$\int_{\widetilde{P}} f(x)\,dx = \int_{\widetilde{P}} g(x)\,dx,$$

then there is a diffeomorphism $\Phi : \widetilde{P} \twoheadrightarrow \widetilde{P}$, that is identity in a neighborhood of $\partial\widetilde{P}$, and such that

$$\int_{\widetilde{P}_i} f(x)\,dx = \int_{\Phi(\widetilde{P}_i)} g(x)\,dx \quad \text{for } i = 1, 2, \ldots, k.$$

Proof. Using Θ as a change of variables we can reduce the problem to Lemma 5.6. The induced functions on P will be

$$(f \circ \Theta)(x)|J_\Theta(x)| \qquad \text{and} \qquad (g \circ \Theta)(x)|J_\Theta(x)|.$$

Actually, we used a very similar argument in the proof of Lemma 5.6 and we leave easy details to the reader. □

Naturally, given a compact set K in a rectangular box P, it cannot be expected that $\overline{P \setminus K}$ is diffeomorphic to a cube. However, in Lemma 5.9 we show that K can be replaced by another compact set K', with small measure of the symmetric difference $|K \vartriangle K'|$, so that $\overline{P \setminus K'}$ is a diffeomorphic closed cube. We do it by approximating K with small balls and smoothly connecting them with thin tubes which start from one face of P, see Figure 5.1. We will need it in Lemma 5.10 to construct a diffeomorphism as in Lemma 5.6 which additionally is identity on a large part of a given compact set.

Lemma 5.9. *Let $a > 0$, $P = [0, a] \times [0, 1]^{n-1}$ be a rectangular box and $F = (0, a) \times (0, 1)^{n-2} \times \{0\}$ a fixed open face of P. If K is a compact subset of P, then for any $\varepsilon > 0$, it is possible to find a compact set $K' \subset P$ and a diffeomorphism $\Psi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that*

$$K' \cap (\partial P \setminus F) = \varnothing, \quad |K \vartriangle K'| < \varepsilon, \quad \Psi(P) = \overline{P \setminus K'},$$

and $\Psi = \mathrm{id}$ outside an arbitrarily small neighborhood of K'.

Proof. We find a finite number of disjoint closed balls B_j contained in $\mathring{P}$ so that $|K \vartriangle \bigcup_j B_j| < \varepsilon/2$ (easy exercise) and a smooth curve γ without self-intersections which starts at a point of the open face F and connects all balls: the curve enters each ball once and leaves it at another point, except the last ball that it does not leave. We can easily guarantee that γ intersects ∂P only at the starting point in F.

We thicken slightly the curve γ to get a tube γ_ε so that the measure of γ_ε does not exceed $\varepsilon/2$. In doing so, we can guarantee that γ_ε connects the balls and touches ∂P at the face F only, in a smooth manner, see Figure 5.1. The desired compact set K' consists of $\bigcup_j B_j$ and γ_ε. Clearly, $|K \vartriangle K'| < \varepsilon$.

Since K' is diffeomorphic to a ball connected smoothly with a thin tube to the face F of P, the set $\overline{P \setminus K'}$ is diffeomorphic to a rectangular box. Intuitively speaking, diffeomorphism $\Psi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$, which maps P onto $\overline{P \setminus K'}$ pushes a smooth cylinder glued to F outside P inside this rectangular box, transforming P into $\overline{P \setminus K'}$. It can be guaranteed that Ψ fixes points outside an arbitrarily small neighborhood of K'. □

The next lemma is an enhanced version of Lemma 5.6. We find a diffeomorphism of a rectangular box onto itself which transforms measures of cells accordingly and, additionally, equals identity on a large part of a given compact subset K of P.

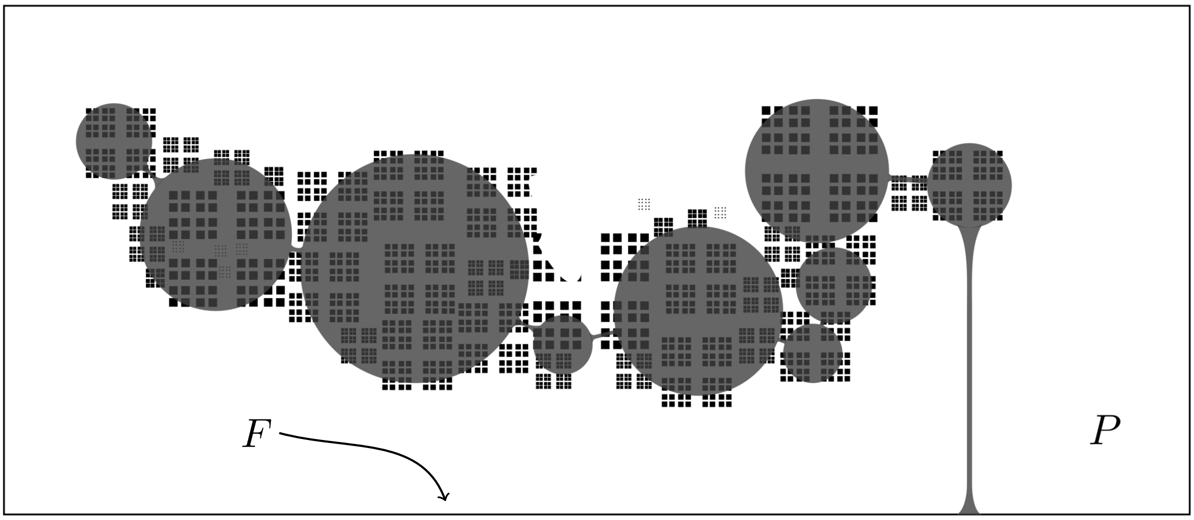

Figure 5.1: The compact set $K \subset P$ (black) is approximated by the set K' (grey) which consists of a finite number of balls, connected smoothly by thin tubes to the side F of P. The proof of Lemma 4.14 explains the shape of the set K.

Lemma 5.10. *Let P, P_i, f and g be as in Lemma 5.6. Assume that K is a compact subset of P and that the functions f and g satisfy the additional condition that*

$$f = g \text{ a. e. on } K. \tag{5.8}$$

Then for any $\varepsilon > 0$, there is a diffeomorphism $\Phi : P \twoheadrightarrow P$ and a compact set $\widetilde{K} \subset K$ such that Φ equals identity in a neighborhood of $\widetilde{K} \cup \partial P$, $|K \setminus \widetilde{K}| < \varepsilon$ and

$$\int_{P_i} f(x)\, dx = \int_{\Phi(P_i)} g(x)\, dx \quad \text{for } i = 1, 2, \ldots, k.$$

Proof. The main idea of the proof is applying Lemma 5.9 to each P_i to eventually remove a set including a large part of set K and to obtain a diffeomorphic closed cube. We can then use Corollary 5.8 to construct the desired diffeomorphism which transforms the measures of cells and equals identity near the boundary.

By removing from K a subset A with $|A| < \varepsilon/2$ we can assume that $\hat{K} := K \setminus A$ is compact and $K_i := \hat{K} \cap P_i \neq P_i$ for all $i = 1, 2, \ldots, k$.

Choose a sufficiently small $0 < \eta < \varepsilon/(2k)$ so that

$$|E| < \eta \quad \Longrightarrow \quad \int_E g(x)\, dx < \min_i \int_{P_i \setminus K_i} f(x)\, dx. \tag{5.9}$$

(Since $K_i \neq P_i$, the minimum in (5.9) is positive.) Lemma 5.9 applied to each P_i, the compact sets K_i and the chosen η yields diffeomorphisms $\Psi_i : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ and compact sets K_i' for $i = 1, 2, \ldots, k$ such that $\Psi_i(P_i) = \overline{P_i \setminus K_i'}$, $|K_i \vartriangle K_i'| < \eta$, and $\Psi_i = \text{id}$ outside an arbitrarily small neighborhood U_i of K_i'. Also, the sets K_i' do not intersect any of the common faces of the partition of P and thus they are pairwise disjoint.

Since the sets K_i' are compact and pairwise disjoint, we may guarantee that the neighborhoods U_i are pairwise disjoint and hence the diffeomorphisms Ψ_i can be glued together to a diffeomorphism $\Psi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that $\Psi = \Psi_i$ on each P_i.

Consequently, if $K' = \bigcup_{i=1}^{k} K_i'$, then $\overline{P \setminus K'}$ is diffeomorphic to P by this diffeomorphism Ψ and hence it satisfies the assumptions of Corollary 5.8 for

$$\widetilde{P} = \overline{P \setminus K'}, \quad \widetilde{P}_i = \overline{P_i \setminus K_i'}.$$

At this point, observe that $|P \setminus K'| = |\overline{P \setminus K'}| = |\widetilde{P}|$, because the boundary of K' is piecewise smooth. Also, it is obvious that

$$P = \widetilde{P} \cup K' \quad \text{and} \quad P_i = \widetilde{P}_i \cup K_i' \quad \text{for } i = 1, \dots, k. \tag{5.10}$$

We still need to define suitable functions to use in place of f and g in Corollary 5.8. Observe that $|K_i \vartriangle K_i'| < \eta$ for each $i = 1, \dots, k$, yields

$$\begin{aligned}\int_{K_i'} g(x)\,dx &\le \int_{K_i} g(x)\,dx + \int_{K_i' \setminus K_i} g(x)\,dx \\ &\overset{(5.9)}{<} \int_{K_i} g(x)\,dx + \int_{P_i \setminus K_i} f(x)\,dx \overset{(5.8)}{=} \int_{P_i} f(x)\,dx.\end{aligned}$$

Consequently, the function

$$\hat{f}(x) = \sum_{i=1}^{k} \left(\int_{P_i} f - \int_{K_i'} g \right) \frac{\chi_{\widetilde{P}_i}(x)}{|\widetilde{P}_i|} \tag{5.11}$$

is positive a. e. on $\widetilde{P}$ and satisfies

$$\int_{\widetilde{P}} \hat{f}(x)\,dx = \int_{\widetilde{P}} g(x)\,dx. \tag{5.12}$$

Indeed, since $|P \setminus K'| = |\widetilde{P}|$, (5.12) follows from

$$\int_{\widetilde{P}} \hat{f}(x)\,dx = \sum_{i=1}^{k} \Big(\int_{P_i} f - \int_{K_i'} g \Big) = \int_{P} f(x)\,dx - \int_{K'} g(x)\,dx \overset{(5.4)}{=} \int_{P \setminus K'} g(x)\,dx.$$

All in all, we have checked the assumptions of Corollary 5.8 for functions $\hat{f}$ and g. This corollary provides us with a diffeomorphism $\Phi' : \widetilde{P} \twoheadrightarrow \widetilde{P}$, $\Phi' = \mathrm{id}$ in a neighborhood of $\partial\widetilde{P}$, such that

$$\int_{\widetilde{P}_i} \hat{f}(x)\,dx = \int_{\Phi'(\widetilde{P}_i)} g(x)\,dx. \tag{5.13}$$

Set $\widetilde{K} = K \cap K'$ and

$$\Phi(x) = \begin{cases} x & \text{for } x \in P \setminus \widetilde{P}, \\ \Phi'(x) & \text{for } x \in \widetilde{P}. \end{cases}$$

The set $\widetilde{K}$ is clearly compact whereas Φ is a well defined diffeomorphism of P onto itself since $\Phi'(x) = x$ near $\partial\widetilde{P}$. We shall now check that Φ and $\widetilde{K}$ satisfy all the desired properties. Writing $\partial P = (\partial P \setminus \partial\widetilde{P}) \cup (\partial P \cap \partial\widetilde{P})$, we see that $\Phi = \mathrm{id}$ near ∂P. It follows immediately from the definition of Φ and $\overline{P \setminus \widetilde{P}} = K'$, that $\Phi = \mathrm{id}$ on K'. We can say even more: since $\partial K' \subset \partial P \cup \partial\widetilde{P}$, $\Phi = \mathrm{id}$ in a neighborhood of K' and, consequently, in a neighborhood of $\widetilde{K}$.

Since $\Phi = \Phi'$ on $\widetilde{P}_i$, (5.13) yields

$$\int_{\widetilde{P}_i} \hat{f}(x)\,dx = \int_{\Phi(\widetilde{P}_i)} g(x)\,dx. \tag{5.14}$$

By (5.10), $\Phi(P_i) = \Phi(\widetilde{P}_i) \cup \Phi(K_i')$ and the two sets $\Phi(\widetilde{P}_i)$ and $\Phi(K_i')$ overlap on a set of measure zero (the image of a part of the piecewise smooth boundary of K_i'). Hence (5.14) yields

$$\int_{\Phi(P_i)} g(x)\,dx = \int_{\Phi(\widetilde{P}_i)} g(x)\,dx + \int_{\Phi(K_i')} g(x)\,dx = \int_{\widetilde{P}_i} \hat{f}(x)\,dx + \int_{K_i'} g(x)\,dx$$
$$\overset{(5.11)}{=} \int_{P_i} f(x)\,dx - \int_{K_i'} g(x)\,dx + \int_{K_i'} g(x)\,dx = \int_{P_i} f(x)\,dx,$$

as required. At last, we compute

$$|K \setminus \widetilde{K}| = |K \setminus K'| = |A \setminus K'| + |\widehat{K} \setminus K'| \leq |A| + \sum_{i=1}^{k} |K_i \vartriangle K_i'| < \frac{\varepsilon}{2} + k\eta < \varepsilon$$

which finishes the proof. □

Lemma 5.11. *Let $\mathcal{Q} = \bigcup_{j=1}^{2^{nk}} Q_j$ be the dyadic partition of the unit cube $\mathcal{Q} = [0,1]^n$ into cubes Q_j of side-length 2^{-k}. Let K be a compact subset of $\mathcal{Q}$ and $f, g \in L^1(\mathcal{Q})$, $f, g > 0$ a. e. be such that*

$$\int_{\mathcal{Q}} f(x)\,dx = \int_{\mathcal{Q}} g(x)\,dx \text{ and } f(x) = g(x) \text{ for a. e. } x \in K.$$

Then for any $\varepsilon > 0$, there is a diffeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ and a compact set $\widetilde{K} \subset K$ such that $\Phi = \mathrm{id}$ in a neighborhood of $\widetilde{K} \cup \partial\mathcal{Q}$, $|K \setminus \widetilde{K}| < \varepsilon$ and

$$\int_{Q_j} f(x)\,dx = \int_{\Phi(Q_j)} g(x)\,dx \quad \text{for } j = 1, 2, \ldots, 2^{nk}.$$

Remark 5.12. The statement of the lemma is very similar to that of Lemma 5.10. The main difference is that the boxes in Lemma 5.10 were arranged into a single line. This arrangement played an important role in the proof and it is not obvious how to modify the proof of Lemma 5.10 to cover the situation described in Lemma 5.11.

Proof of Lemma 5.11. For $1 \leq \ell \leq n$ and $j_1, \ldots, j_\ell \in \{1, \ldots, 2^k\}$ we shall denote

$$P_{j_1 \ldots j_\ell} = [(j_1 - 1)2^{-k}, j_1 2^{-k}] \times [(j_2 - 1)2^{-k}, j_2 2^{-k}] \times \cdots \times [(j_\ell - 1)2^{-k}, j_\ell 2^{-k}],$$
$$L_{j_1 \ldots j_\ell} = P_{j_1 \ldots j_\ell} \times [0,1]^{n-\ell}.$$

The sets $P_{j_1 \ldots j_\ell}$ are cubes in the dyadic partition of $[0,1]^{n-\ell}$ and $L_{j_1 \ldots j_\ell}$ are "towers" over cubes $P_{j_1 \ldots j_\ell}$ covering $[0,1]^n$. In particular $L_1, \ldots, L_{2^k}$ result from slicing of $\mathcal{Q}$ along the first coordinate like a toast bread into 2^k sandwiches. If $\ell = n$, $L_{j_1 \ldots j_n} = P_{j_1 \ldots j_n}$ are exactly the dyadic cubes Q_j.

Our aim is to prove (by finite induction on ℓ) the following claim:

For any $\ell \in \{1, 2, \ldots, n\}$ there exists a diffeomorphism $\Psi_\ell : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ and a compact set $K_\ell \subset K$ such that

$$\Psi_\ell = \mathrm{id} \text{ in a neighborhood of } K_\ell \cup \partial\mathcal{Q}, \tag{5.15}$$

$$\int_{L_{j_1 \ldots j_\ell}} f(x)dx = \int_{\Psi_\ell(L_{j_1 \ldots j_\ell})} g(x)dx \quad \text{for all } j_1, \ldots, j_\ell \in \{1, \ldots, 2^k\}, \tag{5.16}$$

$$|K \setminus K_\ell| < 2^{\ell - n}\varepsilon. \tag{5.17}$$

Note that for $\ell = n$ we obtain $\Phi := \Psi_n$ and $\widetilde{K} := K_n$ that satisfy the conditions of the lemma (because $L_{j_1\dots j_n}$ are exactly the cubes Q_j), so proving the above claim suffices to prove the lemma.

Let us first consider the case $\ell = 1$. Then

$$\mathcal{Q} = L_1 \cup L_2 \cup \dots \cup L_{2^k} = \left(\left[0, \frac{1}{2^k}\right] \cup \left[\frac{1}{2^k}, \frac{2}{2^k}\right] \cup \dots \cup \left[\frac{2^k - 1}{2^k}, 1\right] \right) \times [0,1]^{n-1}$$

is a decomposition of a rectangular box into a union of adjacent boxes, exactly as in Lemma 5.6. By Lemma 5.10 we can find a diffeomorphism $\Psi_1 : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ and a compact set $K_1 \subset K$ such that $\Psi_1 = \mathrm{id}$ in a neighborhood of $K_1 \cup \partial\mathcal{Q}$,

$$|K \setminus K_1| < 2^{1-n}\varepsilon \text{ and } \int_{L_j} f(x)dx = \int_{\Psi_1(L_j)} g(x)dx \quad \text{for all } j \in \{1, \dots, 2^k\}.$$

Assume now that the claim holds for some ℓ, $1 \le \ell < n$, so that there exists a diffeomorphism $\Psi_\ell : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ and a compact set $K_\ell \subset K$ satisfying (5.15), (5.16), (5.17).

For any $j_1, \dots, j_\ell$

$$\int_{L_{j_1\dots j_\ell}} f(x)dx = \int_{\Psi_\ell(L_{j_1\dots j_\ell})} g(x)dx = \int_{L_{j_1\dots j_\ell}} (g \circ \Psi_\ell)(x) J_{\Psi_\ell}(x)dx = \int_{L_{j_1\dots j_\ell}} \tilde{g}_\ell(x)dx$$

for $\tilde{g}_\ell(x) = (g \circ \Psi_\ell)(x) J_{\Psi_\ell}(x)$. Since $\Psi_\ell = \mathrm{id}$ near $\partial\mathcal{Q}$, the diffeomorphism Ψ_ℓ is orientation preserving and hence $J_{\Psi_\ell} > 0$. Since $\Psi_\ell = \mathrm{id}$ near K_ℓ, $D\Psi_\ell = \mathcal{I}$ on K_ℓ, and hence $f(x) = g(x) = \tilde{g}_\ell(x)$ for a. e. $x \in K_\ell$.

Now, let us fix $j_1 \dots j_\ell$. Note that

$$\begin{aligned} L_{j_1\dots j_\ell} &= L_{j_1\dots j_\ell\,1} \cup L_{j_1\dots j_\ell\,2} \cup \dots \cup L_{j_1\dots j_\ell\,2^k} \\ &= P_{j_1\dots j_\ell} \times \left(\left[0, \frac{1}{2^k}\right] \cup \left[\frac{1}{2^k}, \frac{2}{2^k}\right] \cup \dots \cup \left[\frac{2^k - 1}{2^k}, 1\right] \right) \times [0,1]^{n-\ell-1} \end{aligned}$$

is again a decomposition of the rectangular box $L_{j_1\dots j_\ell}$ into a union of adjacent boxes isometric to that in Lemma 5.6, satisfying assumptions of Lemma 5.10 for the functions f, $\tilde{g}_\ell$, and the compact set $K_\ell \cap L_{j_1\dots j_\ell}$. Therefore, we can find a diffeomorphism $\Upsilon_{j_1\dots j_\ell} : L_{j_1\dots j_\ell} \twoheadrightarrow L_{j_1\dots j_\ell}$ and a compact set $\widetilde{K}_{j_1\dots j_\ell} \subset K_\ell \cap L_{j_1\dots j_\ell}$ such that $\Upsilon_{j_1\dots j_\ell} = \mathrm{id}$ in a neighborhood of $\widetilde{K}_{j_1\dots j_\ell} \cup \partial L_{j_1\dots j_\ell}$,

$$\int_{L_{j_1\dots j_\ell\,j_{\ell+1}}} f(x)dx = \int_{\Upsilon_{j_1\dots j_\ell}(L_{j_1\dots j_\ell\,j_{\ell+1}})} \tilde{g}_\ell(x)dx \quad \text{for all } j_{\ell+1} \in \{1, \dots, 2^k\}$$

and

$$|\,(K_\ell \cap L_{j_1\dots j_\ell}) \setminus \widetilde{K}_{j_1\dots j_\ell}| < 2^{-k\ell}\, 2^{\ell-n}\, \varepsilon. \tag{5.18}$$

Since the diffeomorphisms $\Upsilon_{j_1\dots j_\ell}$ are identity near $\partial L_{j_1\dots j_\ell}$, they agree near the boundary of adjacent boxes $L_{j_1\dots j_\ell}$ and thus we can glue them to a diffeomorphism $\Upsilon : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, identity near $\partial\mathcal{Q}$. Setting $K_{\ell+1} = \bigcup_{j_1\dots j_\ell} \widetilde{K}_{j_1\dots j_\ell}$, we see that $K_{\ell+1} \subset K_\ell$, that $\Upsilon(x) = x$ in a neighborhood of $K_{\ell+1}$ and that for any $j_1, \dots, j_{\ell+1}$

$$\begin{aligned} \int_{L_{j_1\dots j_\ell\,j_{\ell+1}}} f(x)dx &= \int_{\Upsilon(L_{j_1\dots j_\ell\,j_{\ell+1}})} \tilde{g}_\ell(x)dx = \int_{\Upsilon(L_{j_1\dots j_\ell\,j_{\ell+1}})} (g \circ \Psi_\ell)(x) J_{\Psi_\ell}(x)dx \\ &= \int_{(\Psi_\ell \circ \Upsilon)(L_{j_1\dots j_\ell\,j_{\ell+1}})} g(x)dx. \end{aligned}$$

Moreover, by (5.18) we can see that

$$|K_\ell \setminus K_{\ell+1}| = \sum_{j_1\dots j_\ell} |\,(K_\ell \cap L_{j_1\dots j_\ell}) \setminus \widetilde{K}_{j_1\dots j_\ell}| < 2^{\ell-n}\varepsilon,$$

which implies that

$$|K \setminus K_{\ell+1}| = |K \setminus K_\ell| + |K_\ell \setminus K_{\ell+1}| < 2^{\ell+1-n}\varepsilon.$$

Therefore, we set $\Psi_{\ell+1} = \Psi_\ell \circ \Upsilon$, which is identity in a neighborhood of $K_{\ell+1} \cup \partial\mathcal{Q}$ and consequently satisfies the claim for $\ell+1$ in place of ℓ, which completes the inductive step and the proof. □

Moreover, we immediately see that the diameters of the cubes Q_j can be made arbitrarily small by taking large k. However, we have no control over the diameters of $\Phi(Q_j)$. The next proposition corrects that.

Proposition 5.13. *Let $\mathcal{Q} = [0,1]^n$ and K be a compact subset of $\mathcal{Q}$. Assume that $f, g \in L^1(\mathcal{Q})$, $f, g > 0$ a. e. satisfy*

$$\int_{\mathcal{Q}} f(x)\,dx = \int_{\mathcal{Q}} g(x)\,dx \text{ and } f(x) = g(x) \text{ for a. e. } x \in K. \tag{5.19}$$

Then for any $\varepsilon > 0$, $\eta > 0$, there exist a diffeomorphic dyadic partition $\mathcal{Q} = \bigcup_{j=1}^{2^{nN}} P_j$, a diffeomorphism $\Psi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ and a compact set $\widetilde{K} \subset K$, with $|K \setminus \widetilde{K}| < \eta$ such that $\Psi = \mathrm{id}$ in a neighborhood of $\widetilde{K} \cup \partial\mathcal{Q}$, $\operatorname{diam} P_j < \varepsilon$, $\operatorname{diam} \Psi(P_j) < \varepsilon$, and

$$\int_{P_j} f(x)\,dx = \int_{\Psi(P_j)} g(x)\,dx, \quad \text{for } j = 1, 2, \dots, 2^{nN}.$$

Remark 5.14. Recall that here, the diffeomorphic dyadic partition means that there is a diffeomorphism $\Theta : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ such that $\Theta(Q_j) = P_j$, where $\mathcal{Q} = \bigcup_{j=1}^{2^{nN}} Q_j$ is the standard dyadic partition of $\mathcal{Q}$ into 2^{nN} identical cubes of side-length 2^{-N}. In fact, the diffeomorphism Θ constructed in the proof will have the additional property that $\Theta = \mathrm{id}$ in a neighborhood of $\partial\mathcal{Q}$.

Remark 5.15. The idea is to take the diffeomorphism Φ from Lemma 5.11 and then apply a version of Lemma 5.11 for a diffeomorphic dyadic partition to each of the sets $\Phi(Q_j)$ and the inverse diffeomorphism $\Phi^{-1} : \Phi(Q_j) \twoheadrightarrow Q_j$. While we do not have control of the diameters of the sets $\Phi(Q_j)$, after the construction described here, we will partition $\Phi(Q_j)$ into sets of as small diameters as we wish.

Proof. Choose $k \in \mathbb{N}$ such that $2^{-k}\sqrt{n} < \varepsilon$ and let $\mathcal{Q} = \bigcup_{j=1}^{2^{nk}} Q_j$ be the dyadic decomposition into 2^{nk} identical cubes of side-length 2^{-k}, so $\operatorname{diam} Q_j < \varepsilon$. Let Φ be the diffeomorphism and $K_1 \subset K$ the compact set provided by Lemma 5.11 so that $\Phi = \mathrm{id}$ in a neighborhood of $K_1 \cup \partial\mathcal{Q}$, $|K \setminus K_1| < \eta/2$ and

$$\int_{Q_j} f(x)\,dx = \int_{\Phi(Q_j)} g(x)\,dx \quad \text{for } j = 1, 2, \dots, 2^{nk}.$$

The diffeomorphism Φ is uniformly continuous in $\mathcal{Q}$; let $\delta > 0$ be such that

$$|\Phi(x) - \Phi(y)| < \varepsilon \text{ whenever } |x - y| < \delta. \tag{5.20}$$

Let $\ell \in \mathbb{N}$ satisfy $2^{-(k+\ell)}\sqrt{n} < \delta$ and consider the dyadic partition $\mathcal{Q} = \bigcup_{j=1}^{2^{nk}} \bigcup_{i=1}^{2^{n\ell}} \widetilde{P}_{ij}$ into identical cubes of side-length $2^{-(\ell+k)}$, so that each cube Q_j is partitioned into $2^{n\ell}$ identical cubes $\widetilde{P}_{ij}$. Clearly, $\operatorname{diam} \widetilde{P}_{ij} < \delta$.

For any j we have, bearing in mind that $J_\Phi > 0$,

$$\int_{Q_j} f(x)\,dx = \int_{\Phi(Q_j)} g(x)\,dx = \int_{Q_j} (g \circ \Phi)(x) J_\Phi(x)\,dx,$$

so if we denote $\tilde{g}(x) := f(x)$, $\tilde{f}(x) := (g \circ \Phi)(x) J_\Phi(x)$,[1] we have

$$\int_{Q_j} \tilde{f}(x)\,dx = \int_{Q_j} \tilde{g}(x)\,dx.$$

Observe that for a. e. $x \in K_1$, $\tilde{f}(x) = \tilde{g}(x)$. Indeed,

$$\tilde{f}(x) = (g \circ \Phi)(x) J_\Phi(x) = g(x) = f(x) = \tilde{g}(x).$$

Applying Lemma 5.11 with Q_j in place of $\mathcal{Q}$, $\tilde{f}$ for f and $\tilde{g}$ for g, partition $Q_j = \bigcup_{i=1}^{2^{n\ell}} \widetilde{P}_{ij}$ and the compact set $K_1 \cap Q_j$ yields a diffeomorphism $\Theta_j : Q_j \twoheadrightarrow Q_j$ and a compact set $K_{2j} \subset K_1 \cap Q_j$ such that $\Theta_j = \mathrm{id}$ in a neighborhood of $K_{2j} \cup \partial Q_j$,

$$|(K_1 \cap Q_j) \setminus K_{2j}| < \tfrac{\eta}{2}\, 2^{-nk} \text{ and } \int_{\widetilde{P}_{ij}} \tilde{f}(x)\,dx = \int_{\Theta_j(\widetilde{P}_{ij})} \tilde{g}(x)\,dx. \tag{5.21}$$

Since Θ_j are identity near ∂Q_j, they glue together to a diffeomorphism $\Theta : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, identity near $\partial\mathcal{Q}$. Set

$$\widetilde{K} := \bigcup_j K_{2j} \subset K_1 \subset K.$$

By (5.21), $|K_1 \setminus \widetilde{K}| < \eta/2$ and consequently,

$$|K \setminus \widetilde{K}| = |K \setminus K_1| + |K_1 \setminus \widetilde{K}| < \eta. \tag{5.22}$$

Moreover, $\Theta = \mathrm{id}$ in a neighborhood of $\widetilde{K}$. Let $P_{ij} = \Theta(\widetilde{P}_{ij})$.

Then we check that

$$\begin{aligned}\int_{P_{ij}} f(x)\,dx &= \int_{\Theta(\widetilde{P}_{ij})} \tilde{g}(x)\,dx \overset{(5.21)}{=} \int_{\widetilde{P}_{ij}} \tilde{f}(x)\,dx \\ &= \int_{\widetilde{P}_{ij}} (g \circ \Phi)(x) J_\Phi(x)\,dx \\ &= \int_{\Phi(\widetilde{P}_{ij})} g(x)\,dx = \int_{\Phi(\Theta^{-1}(P_{ij}))} g(x)\,dx,\end{aligned}$$

so setting $\Psi = \Phi \circ \Theta^{-1}$ we get

$$\int_{P_{ij}} f(x)\,dx = \int_{\Psi(P_{ij})} g(x)\,dx.$$

Since $\Theta^{-1} = \mathrm{id}$ and $\Phi = \mathrm{id}$ in a neighborhood of $\widetilde{K} \cup \partial\mathcal{Q}$, $\Psi = \mathrm{id}$ there, as well. We have already checked in (5.22) that $|K \setminus \widetilde{K}| < \eta$. Finally, $P_{ij} \subset Q_j$, so $\operatorname{diam} P_{ij} \le \operatorname{diam} Q_j < \varepsilon$; also $\operatorname{diam} \widetilde{P}_{ij} = 2^{-(k+\ell)}\sqrt{n} < \delta$, so $\operatorname{diam} \Psi(P_{ij}) = \operatorname{diam} \Phi(\widetilde{P}_{ij}) < \varepsilon$ by (5.20).

Note also that the partition of $\mathcal{Q}$ into the $2^{n(\ell+k)}$ sets P_{ij} is diffeomorphic to the dyadic partition $\mathcal{Q} = \bigcup_{j=1}^{2^{nk}} \bigcup_{i=1}^{2^{n\ell}} \widetilde{P}_{ij}$ by the diffeomorphism $\Theta : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$. $\square$

[1] $\tilde{g} = f$ is not a typo; we reverse notation of f and g for a reason.

Eventually, we will need an analogue of the previous proposition in terms of diffeomorphic images of cubes, which is again a consequence of the change of variables theorem.

Proposition 5.16. *Let $\mathcal{Q} = [0,1]^n$ and $\Theta : \mathcal{Q} \to \mathbb{R}^n$ be a diffeomorphism. Let $\widetilde{\mathcal{Q}} = \Theta(\mathcal{Q})$ and K be a compact subset of $\widetilde{\mathcal{Q}}$. Suppose that $f, g \in L^1(\widetilde{\mathcal{Q}})$, $f, g > 0$ a. e. satisfy*

$$\int_{\widetilde{\mathcal{Q}}} f(x)\, dx = \int_{\widetilde{\mathcal{Q}}} g(x)\, dx \text{ and } f(x) = g(x) \text{ for a. e. } x \in K. \tag{5.23}$$

Then for any $\varepsilon, \eta > 0$ there exists a diffeomorphic dyadic partition $\widetilde{\mathcal{Q}} = \bigcup_{j=1}^{2^{nN}} \widetilde{P}_j$, a diffeomorphism $\Psi : \widetilde{\mathcal{Q}} \twoheadrightarrow \widetilde{\mathcal{Q}}$ and a compact set $\widetilde{K} \subset K$, with $|K \setminus \widetilde{K}| < \eta$ such that $\Psi = \mathrm{id}$ in a neighborhood of $\widetilde{K} \cup \partial\widetilde{\mathcal{Q}}$, $\operatorname{diam} \widetilde{P}_j < \varepsilon$, $\operatorname{diam} \Psi(\widetilde{P}_j) < \varepsilon$ and

$$\int_{\widetilde{P}_j} f(x)\, dx = \int_{\Psi(\widetilde{P}_j)} g(x)\, dx \quad \text{for } j = 1, 2, \ldots, 2^{nN}.$$

5.3 Approximately differentiable mappings

In this section, we prove a few lemmata about a. e. approximately differentiable mappings. The main result of this section is Theorem 5.22 stated in the introduction to this chapter.

The first lemma is a chain rule, which generalizes the one proved in Lemma 4.13. To the best of my knowledge, [39] is the first place where it appears in the literature. Its proof is similar to that of Lemma 13 in [38].

Lemma 5.17. *Let $U \subset \mathbb{R}^n$ be open. Assume that $f : U \to \mathbb{R}^n$ is approximately differentiable a. e. and $\det D_{\mathrm{a}} f(x) \neq 0$ a. e. Assume that $V \subset \mathbb{R}^n$ is an open set such that $f(U) \subset V$ and $g : V \to \mathbb{R}^m$ is approximately differentiable a. e. Then $g \circ f : U \to \mathbb{R}^m$ is approximately differentiable a. e. and*

$$D_{\mathrm{a}}(g \circ f)(x) = D_{\mathrm{a}} g(f(x)) \cdot D_{\mathrm{a}} f(x) \quad \text{for almost all } x \in U. \tag{5.24}$$

Proof. According to Lemma 2.26, there is a sequence of functions $f_k \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ and a sequence of closed sets $E_k \subset \{f_k = f\}$ such that E_k is contained in the set of density points of the set $\{f_k = f\}$, $|U \setminus E_k| < 1/k$, and $\det D_{\mathrm{a}} f(x) \neq 0$ for $x \in E_k$. Note that if $x \in E_k$, then $D_{\mathrm{a}} f(x) = D f_k(x)$ and hence $\det D f_k \neq 0$ in E_k. It suffices to prove that (5.24) holds at almost all points $x \in E_k$ for all k.

The mapping f_k is a diffeomorphism in a neighborhood of every point of E_k and hence we can decompose $E_k = \bigcup_{i=1}^{\infty} W_i$ into countably many compact sets W_i, such that $f = f_k$ is bi-Lipschitz on W_i. Clearly, it suffices to prove that (5.24) is satisfied at almost all points of W_i.

According to Lemma 2.26, there is a sequence of functions $g_\ell \in C^1(\mathbb{R}^n; \mathbb{R}^m)$, and a sequence of closed sets $F_\ell \subset \{g_\ell = g\}$ such that F_ℓ is contained in the set of density points of the set $\{g_\ell = g\}$, and $|V \setminus F_\ell| < 1/\ell$. Clearly, $D_{\mathrm{a}} g(y) = D g_\ell(y)$ for $y \in F_\ell$.

Let $M_{i\ell} := f(W_i) \cap F_\ell$. Since f is bi-Lipschitz on W_i,

$$\Big| f(W_i) \setminus \bigcup_{\ell=1}^{\infty} M_{i\ell} \Big| = 0 \quad \text{implies that} \quad \Big| W_i \setminus \bigcup_{\ell=1}^{\infty} f^{-1}(M_{i\ell}) \Big| = 0.$$

Therefore, it suffices to prove that (5.24) is satisfied at almost all points of each of the sets $Z_{i\ell} := W_i \cap f^{-1}(M_{i\ell})$. This is, however, obvious because for $x \in Z_{i\ell}$, $f(x) = f_k(x)$, $D_\mathrm{a} f(x) = Df_k(x)$, $g(f(x)) = g_\ell(f(x))$, $D_\mathrm{a} g(f(x)) = Dg_\ell(f(x))$ and hence

$$D(g_\ell \circ f_k)(x) = Dg_\ell(f_k(x))Df_k(x) = D_\mathrm{a} g(f(x)) \cdot D_\mathrm{a} f(x).$$

It remains to observe that since $g_\ell \circ f_k \in C^1$, and $g_\ell \circ f_k = g \circ f$ on $Z_{i\ell}$, we have that $D(g_\ell \circ f_k)(x) = D_\mathrm{a}(g \circ f)(x)$ at all density points of $Z_{i\ell}$ and hence almost everywhere in $Z_{i\ell}$. □

The next two lemmata will not be needed in the construction of the homeomorphism Φ in Theorem 1.5. Nonetheless, we will need them to check that the constructed homeomorphism Φ satisfies (a) and (b). The first lemma below boils down to using Federer's change of variables, i. e., Theorem 2.29. This is Lemma 6.6 from [39], we present a differently phrased proof for the coherence of the thesis.

Lemma 5.18. *Let $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ be an a. e. approximately differentiable homeomorphism of $\mathcal{Q}$ such that $\int_\mathcal{Q} |\det D_\mathrm{a}\Phi| = 1$ and $\det D_\mathrm{a}\Phi \neq 0$ a. e. Then Φ preserves the sets of zero measure, i. e., both Φ and Φ^{-1} satisfy the Lusin (N) condition.*

Proof. Let $N \subset \mathcal{Q}$ be a set of measure zero. We claim that $\Phi(N)$ has measure zero as well. It follows from the definition of measurable sets that each measurable set is contained in a Borel one of the same measure. Therefore, there is a Borel set of measure zero $N' \subset \mathcal{Q}$ such that $N \subset N'$. Consequently, $\mathring{\mathcal{Q}} \setminus N'$ is Borel and so is $\Phi(\mathring{\mathcal{Q}} \setminus N')$ and $\Phi(N')$, because homeomorphisms preserve Borel sets.

By the assumptions and (2.14) in Corollary 2.31,

$$1 = \int_\mathcal{Q} |\det D_\mathrm{a}\Phi(x)|\, dx = \int_{\mathring{\mathcal{Q}}\setminus N'} |\det D_\mathrm{a}\Phi(x)|\, dx \leq |\Phi(\mathring{\mathcal{Q}} \setminus N')| \leq |Q| = 1.$$

Therefore, $|\Phi(\mathring{\mathcal{Q}} \setminus N')| = |\Phi(\mathring{\mathcal{Q}})| = 1$, which readily implies that $|\Phi(N')| = 0$. Since $\Phi(N) \subset \Phi(N')$, $\Phi(N)$ is measurable and is of measure zero. This shows that Φ satisfies the Lusin (N) condition.

To prove that Φ^{-1} satisfies the Lusin (N) condition, assume $A \subset \mathcal{Q}$ is a set of zero measure. We need to show that $\Phi^{-1}(A)$ has measure zero. *A priori* it need not be even measurable.

Let A' be a Borel set of measure zero such that $A \subset A' \subset \mathcal{Q}$. Then $\Phi^{-1}(A') \cap \mathring{\mathcal{Q}}$ is Borel. Again by (2.14) from Corollary 2.31,

$$0 \leq \int_{\Phi^{-1}(A')\cap\mathring{\mathcal{Q}}} |\det D_\mathrm{a}\Phi(x)|\, dx \leq |A' \cap \Phi(\mathring{\mathcal{Q}})| = 0.$$

Since $\det D_\mathrm{a}\Phi \neq 0$ a. e., the set $\Phi^{-1}(A') \cap \mathring{\mathcal{Q}}$ has measure zero, and so does the set $\Phi^{-1}(A')$. Therefore, its subset, $\Phi^{-1}(A)$, is measurable and has measure zero. This concludes the proof that Φ preserves sets of measure zero. □

Next, we show that the inverse of an a. e. approximately differentiable homeomorphism which satisfies the Lusin (N) condition is also a. e. approximately differentiable. A somewhat different proof of this lemma can be also found in [28, Corollary 5.1], we provide it here for convenience of the reader.

Lemma 5.19. *Let $U \subset \mathbb{R}^n$ be open. Assume that $\Phi : U \to \mathbb{R}^n$ is an a. e. approximately differentiable homeomorphism satisfying the Lusin (N) condition. Then Φ^{-1} is also a. e. approximately differentiable and*

$$D_{\mathrm{a}}\Phi^{-1}(y) = (D_{\mathrm{a}}\Phi)^{-1}(\Phi^{-1}(y)) \quad \text{for a. e. } y \in \Phi(U). \tag{5.25}$$

Proof. Since it suffices to prove (5.25) on every subdomain $U' \Subset U$, and clearly,

$$\int_{U'} |\det D_{\mathrm{a}}\Phi(x)|\, dx = |\Phi(U')| < \infty,$$

we can assume that U is bounded and $\det D_{\mathrm{a}}\Phi \in L^1(U)$ is integrable.

Fix $\varepsilon > 0$ and choose η so that for any $|E| < \eta$, $\int_E |\det D_{\mathrm{a}}\Phi| < \varepsilon$. By Lemma 2.26 and approximation of measurable sets with compact ones, we find $f_\eta \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a compact set K_η such that

$$f_\eta = \Phi \text{ and } Df_\eta = D_{\mathrm{a}}\Phi \text{ on } K_\eta \quad \text{and} \quad |U \setminus K_\eta| < \eta. \tag{5.26}$$

Let

$$Z := \{x \in U : \Phi \text{ is approximately differentiable at } x \text{ and } \det D_{\mathrm{a}}\Phi(x) = 0\}. \tag{5.27}$$

The set Z is measurable. Indeed, the function $x \mapsto \det D_{\mathrm{a}}\Phi(x)$ is measurable (as a composition of a measurable mapping $D_{\mathrm{a}}\Phi$ and a continuous one). The set Z is therefore a preimage of a Borel set $\{0\}$ under a measurable mapping and hence it is measurable.

We aim to show that Φ^{-1} is approximately differentiable at density points of the set $\Phi(K_\eta \setminus Z)$ and that (5.25) holds there. This suffices to prove the lemma. Indeed, by (2.15) in Corollary 2.31, $\Phi(Z) = 0$ and thus, by the choice of η,

$$|\Phi(U) \setminus \Phi(K_\eta \setminus Z)| \leq |\Phi(U \setminus K_\eta)| + |\Phi(Z)| < \varepsilon.$$

Consequently, approximate differentiability of Φ^{-1} a. e. on $\Phi(U)$ follows from arbitrariness of ε.

Observe that for any $y \in \Phi(K_\eta \setminus Z)$, $f_\eta(\Phi^{-1}(y)) = y$, $\det Df_\eta(\Phi^{-1}(y)) \neq 0$, and the inverse function theorem implies that f_η is a diffeomorphism on some neighborhood of $\Phi^{-1}(y)$. Denote the inverse diffeomorphism of f_η restricted to a neighborhood of $\Phi^{-1}(y)$ by g_y so we may assume that g_y is defined in $B_y := B(y, r)$ for some $r > 0$ and $f_\eta \circ g_y = \mathrm{id}$ on B_y. In particular,

$$Dg_y(z) = (Df_\eta)^{-1}(g_y(z)) \text{ for } z \in B_y \quad \text{and} \quad g_y = \Phi^{-1} \text{ on } \Phi(K_\eta \setminus Z) \cap B_y.$$

Take a density point y of the set $\Phi(K_\eta \setminus Z)$ and set

$$L := (D_{\mathrm{a}}\Phi)^{-1}(\Phi^{-1}(y)) \overset{(5.26)}{=} (Df_\eta)^{-1}(g_y(y)) = Dg_y(y).$$

Since g_y is differentiable at y,

$$\lim_{\substack{z \to y,\\ z \in \Phi(K_\eta \setminus Z)}} \frac{|\Phi^{-1}(z) - \Phi^{-1}(y) - L(z-y)|}{|z-y|} = \lim_{\substack{z \to y,\\ z \in \Phi(K_\eta \setminus Z)}} \frac{|g_y(z) - g_y(y) - L(z-y)|}{|z-y|} = 0,$$

as for z sufficiently close to y, $z \in B(y, r)$, where g_y is well-defined. By Lemma 2.24, this shows that Φ^{-1} is approximately differentiable at density points of $\Phi(K_\eta \setminus Z)$ and that (5.25) holds. As explained earlier, this finishes the proof. $\square$

Note that the difficulty in the proof of (5.25) in Lemma 5.19 lies in the situation when the set Z defined in (5.27) has positive measure. If Z is a set of measure zero, (5.24) in Lemma 5.17 applied for Φ and Φ^{-1} (after we check that Φ^{-1} is a. e. approximately differentiable) yields the claim.

5.3.1 Reflection on a measurable set

We are now ready to prove Theorem 5.2. It is a generalization of the main result in [37], which was recalled as Proposition 4.12 in Section 4.4 and which we will also use here.

Proof of Theorem 5.2. Denote the function on the right hand side of (5.2) by λ, so (5.2) reads as $D_{\mathrm{a}}\Phi = \lambda$ a. e. Note that

$$\mathcal{R}\lambda = \begin{cases} \mathcal{I} & \text{for almost all } x \in E, \\ \mathcal{R} & \text{for almost all } x \in \mathcal{Q} \setminus E. \end{cases}$$

The homeomorphism Φ will be constructed as a limit of a sequence of measure preserving homeomorphisms Φ_j, where at almost every point the approximate derivative of Φ_j will be equal $\mathcal{I}$ or $\mathcal{R}$; in other words, it will be equal $\lambda(x)$ of $\mathcal{R}\lambda(x)$. The main idea behind the construction of the sequence is as follows. If $D_{\mathrm{a}}\Phi_j(x) = \lambda(x)$, then x is a 'good' point. Otherwise we have a 'bad' point, where $D_{\mathrm{a}}\Phi_j(x) = \mathcal{R}\lambda(x)$. We want to modify Φ_j in a way that bad points will became good.

The main step in the construction of Φ_{j+1} from Φ_j is based on the following idea. Let $B \subset \mathcal{Q}$ be the set of bad points, i.e., $D_{\mathrm{a}}\Phi_j = \mathcal{R}\lambda$ on B, and let $G = \mathcal{Q} \setminus B$, so $D_{\mathrm{a}}\Phi_j = \lambda$ a. e. in G, i.e., almost all points of G are good.

Suppose $K \subset \mathcal{Q}$ is a closed cube and assume that $|K \cap \Phi_j(B)| = (1-\varepsilon)|K|$. Clearly, $|K \cap \Phi_j(G)| = \varepsilon|K|$. That is, most of the cube K is covered by the image of bad points.

Note that $D_{\mathrm{a}}\Phi_j = \mathcal{R}\lambda$ in $\Phi_j^{-1}(K) \cap B$, and $D_{\mathrm{a}}\Phi_j = \lambda$ a. e. in $\Phi_j^{-1}(K) \cap G$. We want to change the derivative of Φ_j on the bad set $\Phi_j^{-1}(K) \cap B$ from $\mathcal{R}\lambda$ to λ.

Denote by $\Psi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ the homeomorphism Φ from Proposition 4.12. Let $\kappa : K \twoheadrightarrow \mathcal{Q}$ be the standard similarity, and let

$$\Psi_K(y) := \begin{cases} \kappa^{-1} \circ \Psi \circ \kappa(y) & \text{if } y \in K \\ y & \text{if } y \in \mathcal{Q} \setminus K. \end{cases}$$

Since Ψ is the identity on the boundary of $\mathcal{Q}$, the mapping Ψ_K is a homeomorphism of $\mathcal{Q}$. Then, we define $\Phi_{j+1} = \Psi_K \circ \Phi_j$.

It follows from the chain rule, Lemma 5.17, that $D_{\mathrm{a}}\Phi_{j+1} = \lambda$ a. e. in $\Phi_j^{-1}(K) \cap B$, so the bad set becomes a good one. Unfortunately, we also have that $D_{\mathrm{a}}\Phi_{j+1} = \mathcal{R}\lambda$ a. e. in $\Phi_j^{-1}(K) \cap G$, so the good set is bad now. However,

$$|\Phi_j^{-1}(K) \cap B| = |K \cap \Phi_j(B)| = (1-\varepsilon)|K| \quad \text{and} \quad |\Phi_j^{-1}(K) \cap G| = |K \cap \Phi_j(G)| = \varepsilon|K|$$

(because the transformation Φ_j is measure preserving), so we changed the bad derivative to the good one on a set of measure $(1-\varepsilon)|K|$ which is much larger than the measure $\varepsilon|K|$ of the set where the good derivative turned bad. We iterate this procedure infinitely many times in such a way that the measure of the set of good points converges to the measure of the cube $\mathcal{Q}$.

In fact, in the actual construction, we will use not only one cube K to modify Φ_j, but a finite family of cubes that approximates well the measure of the set $\Phi_j(B)$.

By taking sufficiently small cubes we will guarantee that Φ_j is a Cauchy sequence in the uniform metric d. Thus, the sequence will converge to a homeomorphism Φ and it will follow that Φ satisfies (5.2).

Using the above idea, we will construct a sequence of measure preserving homeomorphisms $\{\Phi_j\}_{j=0}^{\infty}$, $\Phi_j : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi_j|_{\partial\mathcal{Q}} = \mathrm{id}$, that are approximately differentiable almost everywhere and for all $j \geq 1$ satisfy:

$$D_{\mathrm{a}}\Phi_j(x) \in \{\mathcal{I}, \mathcal{R}\} = \{\lambda(x), \mathcal{R}\lambda(x)\} \quad \text{a. e. in } \mathcal{Q}, \tag{5.28}$$

$$|B_j| \leq 2^{-j}, \tag{5.29}$$

$$d(\Phi_j, \Phi_{j+1}) \leq 2^{-j+1}, \quad |L_j| \leq 2^{-j+1}, \quad \Big|B_j \setminus \bigcup_{\ell=j}^{\infty} L_\ell\Big| = 0, \tag{5.30}$$

where d is the uniform metric, $L_j := \overline{\{x \in \mathcal{Q} : \Phi_j \neq \Phi_{j+1}\}}$, and the set

$$B_j := \{x \in \mathcal{Q} : D_{\mathrm{a}}\Phi_j(x) = \mathcal{R}\lambda(x)\}$$

is the set of points where the approximate derivative of Φ_j is bad.

Before we construct such a sequence, we show that a sequence satisfying the above conditions converges to a homeomorphism Φ that has all the required properties, except for being a limit of volume preserving diffeomorphisms (we will take care of it at the end of the proof).

Clearly, (5.30) implies convergence in the uniform metric to some homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ that is identity on the boundary (see Lemma 2.19 for details).

Note that by (5.30),

$$A_j := \mathcal{Q} \setminus \bigcup_{\ell=j}^{\infty} L_\ell.$$

is an increasing sequence of measurable sets that exhaust $\mathcal{Q}$ up to a set of measure zero.

We have that $\Phi = \Phi_j$ on A_j, because $\Phi_j = \Phi_{j+1} = \Phi_{j+2} = \ldots$ on A_j, and hence

$$D_{\mathrm{a}}\Phi = D_{\mathrm{a}}\Phi_j = \lambda \quad \text{almost everywhere in } A_j.$$

The first equality follows from Lemma 2.25 and the last equality follows (5.28), from the definition of B_j and the fact that $|A_j \cap B_j| = 0$ (which is a consequence of (5.30) and the definition of A_j). Since the sets A_j exhaust $\mathcal{Q}$ up to a set of measure zero, $D_{\mathrm{a}}\Phi = \lambda$ a. e. in $\mathcal{Q}$.

Therefore, it remains to construct the sequence Φ_j with the properties described above and after the construction is completed to prove that the homeomorphisms Φ_j can be approximated by volume preserving diffeomorphisms.

We will construct a sequence Φ_j by induction as a sequence of measure preserving homeomorphisms that are identity on the boundary, and satisfy properties (5.28) and (5.29). Then, the properties listed in (5.30) will be verified directly, but they are not needed to run the induction.

In the initial step we choose $\Phi_0 = \mathrm{id} : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, so obviously $B_0 = E$ (up to a set of measure zero) and conditions (5.28) and (5.29) are satisfied. Now suppose that we already constructed homeomorphisms Φ_ℓ for $\ell \leq j$; we will describe construction of Φ_{j+1} as a modification of Φ_j.

The homeomorphism Φ_j^{-1} is uniformly continuous on $\mathcal{Q}$, so let $\delta_j > 0$ be such that $|\Phi_j^{-1}(x) - \Phi_j^{-1}(y)| \leq 2^{-j}$ whenever $|x - y| \leq \delta_j$.

Next, we choose a finite family of closed cubes K_i^j, $i = 1, \dots, m_j$, with pairwise disjoint interiors, with $\operatorname{diam} K_i^j \le \min(\delta_j, 2^{-j})$, and such that $\bigcup_{i=1}^{m_j} K_i^j$ well approximates the set $\Phi_j(B_j)$ measurewise:

$$\left| \Phi_j(B_j) \bigtriangleup \bigcup_{i=1}^{m_j} K_i^j \right| \le 2^{-(j+1)}. \tag{5.31}$$

We set $L_j = \Phi_j^{-1}(\bigcup_{i=1}^{m_j} K_i^j)$; then (5.31) and the fact that Φ_j is a measure preserving homeomorphism yields $|B_j \bigtriangleup L_j| \le 2^{-(j+1)}$.

Note that with this choice

$$|L_j| \le |B_j| + |B_j \bigtriangleup L_j| \le 2^{-j} + 2^{-(j+1)} < 2^{-j+1}.$$

Let $\kappa_i^j : K_i^j \twoheadrightarrow \mathcal{Q}$ be the standard similarity maps between the cubes and define

$$\Psi_j(y) := \begin{cases} (\kappa_i^j)^{-1} \circ \Psi \circ \kappa_i^j(y) & \text{if } y \in K_i^j, \\ y & \text{if } y \in \mathcal{Q} \setminus \bigcup_{i=1}^{m_j} K_i^j. \end{cases}$$

Clearly,

$$D_{\mathrm{a}}\Psi_j(y) = \mathcal{R} \qquad \text{for almost all } y \in \bigcup_{i=1}^{m_j} K_i^j = \Phi_j(L_j). \tag{5.32}$$

Since the cubes K_i^j have pairwise disjoint interiors, it follows that Ψ_j is a measure preserving homeomorphism. Moreover, $\Psi_j|_{\partial\mathcal{Q}} = \mathrm{id}$. Then we define $\Phi_{j+1} = \Psi_j \circ \Phi_j$ and clearly Φ_{j+1} is measure preserving, too, with $\Phi_{j+1}|_{\partial\mathcal{Q}} = \mathrm{id}$. Note that Ψ_j is approximately differentiable a. e. and $\det D_{\mathrm{a}}\Psi = \pm 1$ a. e. on $\mathcal{Q}$ so by Lemma 5.17, Φ_{j+1} which is the composition of Ψ_j with Φ_j, is approximately differentiable a. e. on $\mathcal{Q}$.

The homeomorphisms Φ_j and Φ_{j+1} differ only on the set $L_j = \bigcup_i \Phi_j^{-1}(K_i^j)$. Since both $\operatorname{diam} \Phi_j^{-1}(K_i^j)$ and $\operatorname{diam} K_i^j$ are at most 2^{-j}, we have $d(\Phi_j, \Phi_{j+1}) \le 2^{-j+1}$.

Also, the definition of the set L_j and the construction of Φ_{j+1} shows that

$$L_j = \overline{\{x \in \mathcal{Q} : \Phi_j(x) \neq \Phi_{j+1}(x)\}}.$$

It follows from Lemma 5.17 and (5.32) that

$$D_{\mathrm{a}}\Phi_{j+1}(x) = D_{\mathrm{a}}\Psi_j(\Phi_j(x)) \cdot D_{\mathrm{a}}\Phi_j(x) = \mathcal{R} D_{\mathrm{a}}\Phi_j(x) \quad \text{for almost all } x \in L_j.$$

Thus, $D_{\mathrm{a}}\Phi_{j+1} = \lambda$ on $L_j \cap B_j$. Also,

$$D_{\mathrm{a}}\Phi_{j+1}(x) = D_{\mathrm{a}}\Phi_j(x) \quad \text{for almost all } x \in \mathcal{Q} \setminus L_j,$$

so $D_{\mathrm{a}}\Phi_{j+1} = \lambda$ on $(\mathcal{Q} \setminus L_j) \setminus B_j$. This means that the set B_{j+1} of points where $D_{\mathrm{a}}\Phi_{j+1} = \mathcal{R}\lambda$ is contained (modulo a set of measure zero) in the complement of the union of these two sets, which is $B_j \bigtriangleup L_j$, and hence

$$|B_{j+1}| \le |B_j \bigtriangleup L_j| \le 2^{-(j+1)}.$$

This completes the proof that Φ_{j+1} satisfies the induction hypothesis. What is left to prove is the last part of (5.30), $\left| B_j \setminus \bigcup_{\ell=j}^{\infty} L_\ell \right| = 0$.

To see that, it suffices to realize that for all $k > j$ the mappings Φ_k have bad derivative $D_{\mathrm{a}}\Phi_k = \mathcal{R}\lambda$ at almost all points of $B_j \setminus \bigcup_{\ell=j}^{\infty} L_\ell$, because Φ_k is obtained from Φ_j by a sequence of modifications which happen only in $\bigcup_{\ell=j}^{\infty} L_\ell$. Thus

$$B_j \setminus \bigcup_{\ell=j}^{\infty} L_\ell \subset \bigcap_{k \ge j} B_k \quad \text{(modulo a set of measure zero).}$$

Since $|B_k| \to 0$, it follows that $\left|B_j \setminus \bigcup_{\ell=j}^{\infty} L_\ell\right| = 0$.

Now it remains to show that the homeomorphism Φ can be approximated in the metric d by volume preserving diffeomorphisms that are identity in a neighborhood of ∂Q.

Let $\Psi_k \xrightarrow{d} \Psi$ be a sequence of volume preserving diffeomorphisms, $\Psi_k = \mathrm{id}$ near $\partial \mathcal{Q}$, from Proposition 4.12. Note that the diffeomorphisms

$$\Psi_{j,k} := \begin{cases} (\kappa_i^j)^{-1} \circ \Psi_k \circ \kappa_i^j(y) & \text{if } y \in K_i^j, \\ y & \text{if } y \in \mathcal{Q} \setminus \bigcup_{i=1}^{m_j} K_i^j \end{cases}$$

are volume preserving and $\Psi_{j,k} = \mathrm{id}$ near $\partial \mathcal{Q}$. Clearly, $\Psi_{j,k} \xrightarrow{d} \Psi_j$ as $k \to \infty$.

An easy induction on j shows that Φ_j can be approximated in the metric d by C^∞ volume preserving diffeomorphisms $\Phi_{j,k}$ that are identity near $\partial \mathcal{Q}$. If $j = 0$, then $\Phi_0 = \mathrm{id}$ and we can take $\Phi_{0,k} := \Phi_0$. If the claim is true for j, $\Phi_{j,k} \xrightarrow{d} \Phi_j$, then, it is easy to check that $\Phi_{j+1,k} := \Psi_{j,k} \circ \Phi_{j,k} \xrightarrow{d} \Psi_j \circ \Phi_j = \Phi_{j+1}$. Therefore, for each j, we can find k_j such that $d(\Phi_{j,k_j}, \Phi_j) < 2^{-j}$ and hence $\Phi_{j,k_j} \xrightarrow{d} \Phi$. □

Remark 5.20. As $\det D_{\mathrm{a}}\Phi = \pm 1$ a. e. on $\mathcal{Q}$, Lemma 5.18 implies that Φ satisfies the Lusin (N) condition. It then follows from (2.15) in Corollary 2.31 that the homeomorphism Φ is measure preserving. Moreover, by Lemma 5.19, Φ^{-1} is also approximately differentiable a. e. on $\mathcal{Q}$ and $D_{\mathrm{a}}\Phi^{-1}(y) = (D_{\mathrm{a}}\Phi)^{-1}(\Phi^{-1}(y))$ for a. e. $y \in \mathcal{Q}$. The fact that Φ is volume preserving can also be directly concluded from the fact that Φ is a limit of volume preserving diffeomorphisms in the metric d, see [37, Lemma 1.2].

5.4 Proof of Theorem 1.5

Proof of Theorem 1.5.

5.4.1 Reduction to the case $\det T > 0$

It suffices to prove Theorem 1.5 under an additional assumption that $\det T > 0$ a. e. Indeed, assume that we have already proven Theorem 1.5 under this assumption and, for a general T, let

$$\hat{T}(x) = \begin{cases} T(x) & \text{if } \det T > 0, \\ \mathcal{R}T(x) & \text{if } \det T < 0, \end{cases}$$

where $\mathcal{R}$ is the reflection matrix defined in (5.1). Then $\det \hat{T} = |\det T| > 0$ a. e. Let $\hat{\Phi}$ be the almost everywhere approximately differentiable homeomorphism provided by Theorem 1.5 with $\hat{T}$ in place of T, $D_{\mathrm{a}}\hat{\Phi} = \hat{T}$ a. e.

Let $E := \{\det T < 0\}$. Note that since $\hat{\Phi}$ satisfies the Lusin (N) condition, the set $\hat{E} := \hat{\Phi}(E)$ is measurable (because E is the union of a Borel set and a set of measure zero). Theorem 5.2 yields an a. e. approximately differentiable homeomorphism Φ' such that $\Phi'|_{\partial\mathcal{Q}} = \mathrm{id}$ and

$$D_{\mathrm{a}}\Phi'(x) = \begin{cases} \mathcal{R} & \text{for a. e. } x \in \hat{E}, \\ \mathcal{I} & \text{for a. e. } x \in \mathcal{Q} \setminus \hat{E}. \end{cases}$$

Then Lemma 5.17 implies that the composition $\Phi := \Phi' \circ \hat{\Phi}$ is a. e. approximately differentiable and satisfies the chain rule (5.24), so for a. e. $x \in \Omega$ we have

$$D_{\mathrm{a}}\Phi(x) = D_{\mathrm{a}}\Phi'(\hat{\Phi}(x))\, D_{\mathrm{a}}\hat{\Phi}(x) = \left.\begin{cases} \mathcal{R}\mathcal{R}T(x) & \text{for a. e. } x \in E, \\ \mathcal{I}\,T(x) & \text{for a. e. } x \in \Omega \setminus E \end{cases}\right\} = T(x). \tag{5.33}$$

Obviously, $\Phi|_{\partial\Omega} = \mathrm{id}$ and, since Φ' is measure preserving (see Remark 5.20) and $\hat{\Phi}$ preserves sets of measure zero, Φ preserves sets of measure zero as well.

Since by Theorem 5.2 and Theorem 1.5 the homeomorphisms Φ' and $\hat{\Phi}$ can be approximated in the uniform metric d by C^∞-diffeomorphisms Φ'_k and $\hat{\Phi}_k$, $\Phi'_k = \hat{\Phi}_k = \mathrm{id}$ near $\partial\Omega$, one can easily check that

$$\Phi'_k \circ \hat{\Phi}_k \xrightarrow{d} \Phi' \circ \hat{\Phi} = \Phi.$$

It remains to check that Φ^{-1} is approximately differentiable a. e. on Ω and $D_{\mathrm{a}}\Phi^{-1}(y) = T^{-1}(\Phi^{-1})(y)$ for a. e. $y \in \Omega$. Indeed, $\Phi^{-1} = \hat{\Phi}^{-1} \circ \Phi'^{-1}$ and both $\hat{\Phi}^{-1}$ (by Theorem 1.5) and Φ'^{-1} (see Remark 5.20) are approximately differentiable a. e. on Ω. Since we have $\det D_{\mathrm{a}}(\Phi'^{-1}) \neq 0$ a. e. (Remark 5.20), Φ^{-1} is approximately differentiable a. e. by Lemma 5.17. Then applying the chain rule (5.24) to $\Phi^{-1} \circ \Phi = \mathcal{I}$ yields that the derivative of Φ^{-1} has the required form.

This concludes the proof of Theorem 1.5 in the general case, provided we can prove it for T such that $\det T > 0$ a. e. in Ω.

Thus, from now on, we assume that $\det T > 0$ a. e.

5.4.2 General outline of the construction

The general plan of the proof is as follows: to construct a homeomorphism Φ such that $\Phi|_{\partial\Omega} = \mathrm{id}$ and $D_{\mathrm{a}}\Phi = T$ a. e., we shall iterate the construction from Theorem 1.4 on the smaller and smaller subsets of Ω on which the derivative is not yet as required.

We inductively show that there exists a family of orientation preserving C^1-diffeomorphisms Φ_k of Ω and Borel sets $C_k \subset \Omega$ with the following properties for $k \geq 1$:

(i) $\Phi_k = \mathrm{id}$ near $\partial\Omega$;
(ii) $\Phi_{k+1} = \Phi_k$ on C_k;
(iii) $D\Phi_k = T$ on C_k;
(iv) C_k is an increasing family of sets, $C_1 \subset C_2 \subset \cdots$, with $\lim_{k\to\infty} |C_k| = 1$;
(v) $d(\Phi_k, \Phi_{k+1}) < 2^{-(k-1)}$ for $k \geq 2$.

The limit map $\Phi := \lim_{k\to\infty} \Phi_k$ is the required homeomorphism. Indeed, property (v) implies that (Φ_k) is a Cauchy sequence in the uniform metric d, hence its limit is a homeomorphism as shown in Lemma 2.19. By (i), $\Phi = \mathrm{id}$ on $\partial\Omega$. Note that $\Phi = \Phi_k$ on C_k by (ii) and (iv). Therefore, Lemma 2.25 and (iii) imply that

$$D_{\mathrm{a}}\Phi = T \text{ a. e. on } \bigcup_{k=1}^{\infty} C_k,$$

and hence $D_{\mathrm{a}}\Phi = T$ a. e. on Ω, because $|C_k| \to 1$.

At this point, let us stress that Lemma 5.18 implies that Φ and Φ^{-1} satisfy the Lusin (N) condition which is part (b) of Theorem 1.5 and that property (a) follows directly from Lemma 5.19.

Finally, Φ_k are C^1-diffeomorphisms, identity near $\partial\Omega$, that converge to Φ in the uniform metric d, but according to Lemma 2.20, each Φ_k can be approximated in the metric d by C^∞-diffeomorphisms and (c) follows.

This completes the proof of Theorem 1.5 and it remains to construct diffeomorphisms Φ_k and Borel sets C_k satisfying (i)-(v).

The construction of the family Φ_k and of the sets C_k is complicated. In fact, the sets C_k are not constructed inductively, but defined only at the end, when all the steps of the inductive construction are concluded.

The actual inductive construction provides a quadruple $(\Phi_k, \mathcal{P}_k, E_k, L_k)_{k=1}^\infty$:

- a diffeomorphism Φ_k of Ω onto itself; each diffeomorphism Φ_k is constructed by a modification of Φ_{k-1},
- a partition $\mathcal{P}_k$ of the unit cube; the corrections leading from Φ_{k-1} to Φ_k are done at a small scale, i.e., within the elements of the partition $\mathcal{P}_{k-1}$.
- a large set E_k, on which $D\Phi_k = T$,
- a small set $L_k \subset E_{k-1}$ such that $\Phi_k = \Phi_{k-1}$ on $E_{k-1} \setminus L_k$ (although $D\Phi_{k-1} = T$ on E_{k-1}, for technical reasons in the construction of Φ_k from Φ_{k-1} we alter Φ_k on the subset L_k of E_{k-1}),
- the sets C_k are constructed at the very end by (5.37).

To be more precise, we have, for $k \geq 1$,

(a) a C^1-diffeomorphism $\Phi_k : \Omega \twoheadrightarrow \Omega$, $\Phi_k = \mathrm{id}$ near $\partial\Omega$;
(b) a compact set $E_k \subset \Omega$ such that $D\Phi_k = T$ on E_k;
(c) $2^{-(k+1)} < |\Omega \setminus E_k| < 2^{-k}$;
(d) a Borel set $L_k \subset E_{k-1}$ for $k \geq 2$, such that $\Phi_k = \Phi_{k-1}$ on $E_{k-1} \setminus L_k$ and $|L_k| < 2^{-k}$;
(e) a partition $\mathcal{P}_k$ of the unit cube, $\mathcal{P}_k = \{P_{ki}\}_{i=1}^{M_k}$ for $k \geq 2$, such that

$$\Phi_k(P_{k-1,i}) = \Phi_{k-1}(P_{k-1,i}) \quad \text{for } k \geq 3, \tag{5.34}$$

$$|\Phi_k(P_{ki})| = \int_{P_{ki}} \det T(x)\, dx \quad \text{for } k \geq 2, \tag{5.35}$$

$$\operatorname{diam} P_{ki} < 2^{-k}, \quad \operatorname{diam} \Phi_k(P_{ki}) < 2^{-k} \quad \text{for } k \geq 2. \tag{5.36}$$

We will show that the family of diffeomorphisms (Φ_k) with properties (a)-(e) satisfies conditions (i)-(v) for

$$C_k := \bigcap_{j=k}^{\infty} (E_j \setminus L_{j+1})\,. \tag{5.37}$$

Conditions (i) and (a) are the same. Clearly, $C_k \subset E_k$, which means that condition (iii) is satisfied. Applying (d) with $k+1$ in place of k, we get $\Phi_{k+1} = \Phi_k$ on $E_k \setminus L_{k+1}$ and since $C_k \subset E_k \setminus L_{k+1}$, condition (ii) also holds. Since the sets C_k form an increasing family of sets, in order to show (iv), it remains to check that $\lim_{k\to\infty} |C_k| = 1$.

For the sake of this calculation, set $A_j := E_j \setminus L_{j+1}$. Since $L_{j+1} \subset E_j$ and in view of (c) and (d),

$$|\Omega \setminus A_j| = |\Omega \setminus E_j| + |L_{j+1}| < 2^{-j} + 2^{-(j+1)} = 3 \cdot 2^{-(j+1)}. \tag{5.38}$$

Therefore,

$$|\mathcal{Q} \setminus C_k| = \Big|\mathcal{Q} \setminus \bigcap_{j=k}^{\infty} A_j\Big| = \Big|\bigcup_{j=k}^{\infty}(\mathcal{Q} \setminus A_j)\Big| \le \sum_{j=k}^{\infty} |\mathcal{Q} \setminus A_j| < 3\sum_{j=k}^{\infty} 2^{-(j+1)} = 3 \cdot 2^{-k}.$$

This implies that

$$|C_k| > 1 - 3 \cdot 2^{-k}$$

and since $|C_k| \le 1$, this shows that $|C_k| \to 1$ and finishes the proof of (iv). It remains to show (v).

It follows from condition (e) that for $k \ge 3$ the diffeomorphism Φ_k is a modification of Φ_{k-1} in each of the sets $P_{k-1,i}$, i.e., $\Phi_k(P_{k-1,i}) = \Phi_{k-1}(P_{k-1,i})$. Hence

$$\|\Phi_k - \Phi_{k-1}\|_\infty \le \max_i \left\{ \operatorname{diam} \Phi_{k-1}(P_{k-1,i}) \right\} < 2^{-(k-1)},$$

$$\|\Phi_k^{-1} - \Phi_{k-1}^{-1}\|_\infty \le \max_i \{\operatorname{diam} P_{k-1,i}\} < 2^{-(k-1)},$$

so $d(\Phi_k, \Phi_{k-1}) < 2 \cdot 2^{-(k-1)} = 2^{-(k-2)}$ and (v) follows. The proof of properties (i)-(v) is complete[2].

5.4.3 Construction of Φ_1 and Φ_2.

By a direct application of Theorem 1.4, we obtain a diffeomorphism Φ_1 of the unit cube, $\Phi_1 = \mathrm{id}$ near $\partial\mathcal{Q}$, and a compact set E_1 such that $D\Phi_1 = T$ on E_1 and $1/4 < |\mathcal{Q} \setminus E_1| < 1/2$. Note that Φ_1 and E_1 satisfy conditions (a)-(e), because conditions (d) and (e) do not apply to $k = 1$.

We shall now describe in detail the construction of Φ_2, which demonstrates all the crucial aspects of the construction of Φ_k based on Φ_{k-1}. The induction step for general k will be described later.

In the course of the proof we use two numbers $\alpha = 1/2$, $\beta = 3/4$. We write α, β instead of the actual fractions as we believe it makes it easier to transfer this argument to the proof for arbitrary k. Note that $|E_1| > 1 - \alpha$.

We begin with correcting the way Φ_1 distributes measure so that we are later able to apply Theorem 1.4 in sets of smaller diameter. To this end, we use Proposition 5.13 for the compact set $\Phi_1(E_1)$ and functions

$$f(y) = \det T(\Phi_1^{-1}(y)) \det D\Phi_1^{-1}(y) \quad \text{and} \quad g(y) = 1.$$

We check that (5.19) is satisfied. By change of variables,

$$\int_{\mathcal{Q}} \det T(\Phi_1^{-1}(y)) \det D\Phi_1^{-1}(y)\, dy = \int_{\Phi_1^{-1}(\mathcal{Q})} \det T(x)\, dx = \int_{\mathcal{Q}} \det T(x)\, dx = \int_{\mathcal{Q}} 1\, dx.$$

Moreover, since $D\Phi_1 = T$ on E_1, we have $f(y) = g(y)$ for all $y \in \Phi_1(E_1)$. Therefore, the assumptions of Proposition 5.13 are satisfied.

By Proposition 5.13, we get a diffeomorphism $\Psi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, a partition $\mathcal{R} = \{R_{2i}\}_{i=1}^{2^{nN_2}}$ of $\mathcal{Q}$ and a compact set $\widetilde{K} \subset \Phi_1(E_1)$ with properties described below.

[2] In the construction of Φ_2 from Φ_1, we modify Φ_1 in $\mathcal{Q}$, so in this step we use the trivial partition $\mathcal{P}_1 = \{\mathcal{Q}\}$.

Partition $\mathcal{R}$ is a diffeomorphic dyadic partition and satisfies

$$\operatorname{diam}\Psi(R_{2i}) < 1/4 \text{ and } \operatorname{diam}(\Phi_1^{-1}(R_{2i})) < 1/4 \tag{5.39}$$

(we use here the uniform continuity of Φ_1^{-1}) and

$$\int_{R_{2i}} f(y)\,dy = |\Psi(R_{2i})|,$$

which after a change of variables in the integral becomes

$$\int_{\Phi_1^{-1}(R_{2i})} \det T(x)\,dx = |\Psi(R_{2i})|. \tag{5.40}$$

We find $\eta > 0$ such that for any measurable set $A \subset \Omega$ with $|A| < \eta$, $\int_A \det D\Phi_1^{-1} < \alpha^2/2$. We choose $\widetilde{K} \subset \Phi_1(E_1)$ so that

$$|\Phi_1(E_1) \setminus \widetilde{K}| < \eta.$$

Moreover, $\Psi = \mathrm{id}$ near $\widetilde{K} \cup \partial\Omega$. In view of the inequality above, setting $K := \Phi_1^{-1}(\widetilde{K}) \subset E_1$, we have

$$|E_1 \setminus K| = |\Phi_1^{-1}(\Phi_1(E_1) \setminus \widetilde{K})| < \alpha^2/2. \tag{5.41}$$

Eventually, we set

$$\widetilde{\Phi}_2 := \Psi \circ \Phi_1 \quad \text{and} \quad \mathcal{P}_2 := \{P_{2i}\}_{i=1}^{M_2}, \text{ where } M_2 = 2^{nN_2}, \text{ and } P_{2i} := \Phi_1^{-1}(R_{2i}).$$

Then (5.39) and (5.40) become

$$\operatorname{diam}\widetilde{\Phi}_2(P_{2i}) < 1/4, \quad \operatorname{diam}(P_{2i}) < 1/4, \quad \int_{P_{2i}} \det T(x)\,dx = |\widetilde{\Phi}_2(P_{2i})|, \tag{5.42}$$

i. e., diffeomorphism $\widetilde{\Phi}_2$ satisfies (e) for $k = 2$ (condition (5.34) does not apply to $k = 2$). Observe also that $\widetilde{\Phi}_2 = \Phi_1$ near $K \cup \partial\Omega$. Consequently,

$$D\widetilde{\Phi}_2 = T \text{ on } K, \quad \widetilde{\Phi}_2 = \mathrm{id} \text{ near } \partial\Omega \quad \text{and} \quad \widetilde{\Phi}_2 = \Phi_1 \text{ on } K \subset E_1. \tag{5.43}$$

We construct Φ_2 by replacing $\widetilde{\Phi}_2$ inside each P_{2i} with a diffeomorphism Φ_{2i} which has correct derivative $D\Phi_{2i} = T$ on a larger set. To this end, we would like to keep $\widetilde{\Phi}_2$ on K unchanged and apply Theorem 1.4 to the open set $\mathring{P}_{2i} \setminus K$. Unfortunately, this set need not be connected and Theorem 1.4 cannot be applied. To overcome this difficulty we use Lemma 4.14 to find a compact set $\widetilde{E}_{2i} \subset \mathring{P}_{2i} \cap K$ so that $\mathring{P}_{2i} \setminus \widetilde{E}_{2i}$ is connected and

$$|(\mathring{P}_{2i} \cap K) \setminus \widetilde{E}_{2i}| < \alpha^2/(2M_2). \tag{5.44}$$

Set

$$\widetilde{E}_2 := \bigcup_{i=1}^{M_2} \widetilde{E}_{2i} \subset E_1 \quad \text{and} \quad L_2 := E_1 \setminus \widetilde{E}_2.$$

Clearly, $\widetilde{E}_2$ is compact and summing (5.44) over $i = 1, \dots, M_2$ yields

$$|K \setminus \widetilde{E}_2| < \alpha^2/2.$$

By (5.41), we arrive at

$$|L_2| = |E_1 \setminus \widetilde{E}_2| = |E_1 \setminus K| + |K \setminus \widetilde{E}_2| < \alpha^2. \tag{5.45}$$

Consequently,

$$|\widetilde{E}_2| = |E_1| - |L_2| > 1 - \alpha - \alpha^2. \tag{5.46}$$

Let us stress that $\widetilde{E}_2 \subset K$ so (5.43) yields

$$D\widetilde{\Phi}_2 = T \text{ on } \widetilde{E}_2 \tag{5.47}$$

and

$$\widetilde{\Phi}_2 = \Phi_1 \text{ on } \widetilde{E}_2 = E_1 \setminus L_2. \tag{5.48}$$

Although $D\Phi_1 = T$ on E_1, as explained earlier, the set L_2 is the set on which we will have spoiled the already prescribed derivative of Φ_1. This is the cost we bear in order to be able to prescribe the derivative farther. One can also think of the set $\widetilde{E}_2$ as the set of points in which the prescribed derivative *survives* the transition from Φ_1 to Φ_2.

Let us now focus on applying Theorem 1.4, i. e., correcting the derivative of $\widetilde{\Phi}_2$. The set

$$\Omega_{2i} := \mathring{P}_{2i} \setminus \widetilde{E}_{2i}$$

is open and connected. Observe that, setting $\Omega_2 := \bigcup_i \Omega_{2i}$,

$$|\Omega_2| = |\mathcal{Q} \setminus \widetilde{E}_2|,$$

since the set Ω_2 coincides with $\mathcal{Q} \setminus \widetilde{E}_2$ up to a set of measure zero which consists of boundaries of the diffeomorphic cubes P_{2i}. We check that Ω_{2i} satisfies

$$\begin{aligned} |\widetilde{\Phi}_2(\Omega_{2i})| = |\widetilde{\Phi}_2(P_{2i})| - |\widetilde{\Phi}_2(\widetilde{E}_{2i})| &\overset{(5.42)}{=} \int_{P_{2i}} \det T(x)\,dx - \int_{\widetilde{E}_{2i}} \det D\widetilde{\Phi}_2(x)\,dx \\ &\overset{(5.47)}{=} \int_{P_{2i}} \det T(x)\,dx - \int_{\widetilde{E}_{2i}} \det T(x)\,dx = \int_{\Omega_{2i}} \det T(x)\,dx. \end{aligned} \tag{5.49}$$

We are now in position to use Theorem 1.4 for the domain Ω_{2i} and the diffeomorphism $\widetilde{\Phi}_2$. We find a diffeomorphism $\Phi_{2i} : \Omega_{2i} \to \widetilde{\Phi}_2(\Omega_{2i})$ and a compact set $E'_{2i} \subset \Omega_{2i}$ such that

$$D\Phi_{2i} = T \text{ on } E'_{2i}, \quad \Phi_{2i} = \widetilde{\Phi}_2 \text{ near } \partial\Omega_{2i} \quad \text{and} \quad |E'_{2i}| > \beta|\Omega_{2i}|. \tag{5.50}$$

Let

$$E'_2 := \bigcup_{i=1}^{M_2} E'_{2i}.$$

Clearly, E'_2 is compact. By the third condition in (5.50), we have

$$|E'_2| > \beta|\Omega_2| = \beta|\mathcal{Q} \setminus \widetilde{E}_2|.$$

We replace the diffeomorphism $\widetilde{\Phi}_2$ with Φ_{2i} inside each Ω_{2i}, setting

$$\Phi_2 := \begin{cases} \Phi_{2i} & \text{on } \Omega_{2i}, \\ \widetilde{\Phi}_2 & \text{on } \mathcal{Q} \setminus \Omega_2. \end{cases}$$

Observe that thanks to the second condition in (5.50), Φ_2 is indeed a diffeomorphism. Moreover, $\Phi_2 = \widetilde{\Phi}_2$ near $\bigcup_i \partial P_{2i}$. Since $\partial\mathcal{Q} \subset \bigcup_i \partial P_{2i}$, $\Phi_2 = \mathrm{id}$ near $\partial\mathcal{Q}$, whence (a) holds.

Note that $\widetilde{E}_2 \subset \mathcal{Q} \setminus \Omega_2$ and $\Phi_2 = \widetilde{\Phi}_2$ in a neighborhood of $\widetilde{E}_2$. Hence, the first condition in (5.50) together with (5.47) imply that

$$D\Phi_2 = T \text{ on } \hat{E}_2 := E'_2 \cup \widetilde{E}_2.$$

We calculate

$$\begin{aligned}|\hat{E}_2| &= |E_2'| + |\widetilde{E}_2| > \beta|\Omega \setminus \widetilde{E}_2| + |\widetilde{E}_2| = \beta|\Omega| + (1-\beta)|\widetilde{E}_2| \\ &\overset{(5.46)}{>} \beta + (1-\beta)(1-\alpha-\alpha^2) = 1 - \alpha(1+\alpha)(1-\beta) \\ &> 1-\alpha^2,\end{aligned}$$

since $\beta > 1/(1+\alpha)$. Therefore, $|\Omega \setminus \hat{E}_2| < \alpha^2 = 1/4$. By discarding some points from $\hat{E}_2$ if necessary, we obtain a compact set $E_2 \subset \hat{E}_2$ such that $1/8 < |\Omega \setminus E_2| < 1/4$. Clearly, $D\Phi_2 = T$ on E_2.

Let us check if Φ_2 satisfies properties (a)-(e). We have already verified conditions (a), (b) and (c). Moreover, in view of (5.42), (e) holds as well, because $\Phi_2(P_{2i}) = \widetilde{\Phi}_2(P_{2i})$ ((5.34) does not apply to $k = 2$). Since $\Phi_2 = \widetilde{\Phi}_2$ on $\widetilde{E}_2 = E_1 \setminus L_2$, recalling (5.48) and (5.45), we see that (d) also holds.

5.4.4 Construction of Φ_k given Φ_{k-1}.

Let $k \geq 3$. As in the construction of Φ_2, we use $\alpha = 1/2$ and $\beta = 3/4$. We assume that we have found Φ_{k-1} satisfying properties (a)-(e) for $k-1$ instead of k and we show how to construct Φ_k. In fact, we only need the following properties from the previous step:

- $\Phi_{k-1} : \Omega \twoheadrightarrow \Omega$, $\Phi_{k-1} = \mathrm{id}$ near $\partial\Omega$,
- $D\Phi_{k-1} = T$ on a compact set E_{k-1}, $\alpha^k < |\Omega \setminus E_{k-1}| < \alpha^{k-1}$,
- $\Phi_{k-1}(P_{k-1,i}) = \int_{P_{k-1,i}} \det T(x)\, dx$.

In the construction of Φ_2 from Φ_1, we alter Φ_1 inside Ω while keeping $\Phi_2 = \Phi_1$ near $\partial\Omega$. In the construction of Φ_k from Φ_{k-1} we repeat the same construction, but we alter Φ_{k-1} inside each diffeomorphic closed cube $P_{k-1,i}$ while keeping $\Phi_k = \Phi_{k-1}$ near $\partial P_{k-1,i}$. Therefore, the crucial though technical difference between the construction for $k \geq 3$ is that we do not use Proposition 5.13 (stated for a closed cube) but Proposition 5.16 (stated for a diffeomorphic closed cube).

Define $E_{k-1,i} := E_{k-1} \cap P_{k-1,i}$ for $i = 1, \dots, M_{k-1}$. Note that $E_{k-1,i}$ is compact and that $|\bigcup_i E_{k-1,i}| = |E_{k-1}|$.

Let us firstly show that it suffices to construct for $i = 1, \dots, M_{k-1}$

- a family of diffeomorphisms $\Phi_{ki} : P_{k-1,i} \twoheadrightarrow \Phi_{k-1}(P_{k-1,i})$,
- compact sets $\hat{E}_{ki} \subset \mathring{P}_{k-1,i}$,
- Borel sets $L_{ki} \subset E_{k-1,i}$,
- a partition $\mathcal{P}_{ki} = \{P_{kij}\}_{j=1}^{2^{nN_{ki}}}$ of $P_{k-1,i}$

such that

(1) $\Phi_{ki} = \Phi_{k-1}$ near $\partial P_{k-1,i}$;
(2) $D\Phi_{ki} = T$ on $\hat{E}_{ki}$;
(3) $|\hat{E}_{ki}| > \beta|P_{k-1,i}| + (1-\beta)|E_{k-1,i} \setminus L_{ki}|$;
(4) $\Phi_{ki} = \Phi_{k-1}$ on $E_{k-1,i} \setminus L_{ki}$ and $|L_{ki}| < \alpha^k M_{k-1}^{-1}$;

(5) partition $\mathcal{P}_{ki}$ is a diffeomorphic dyadic partition and satisfies for $j = 1, \dots, 2^{nN_{ki}}$

$$|\Phi_{ki}(P_{kij})| = \int_{P_{kij}} \det T(x)\, dx$$

and

$$\operatorname{diam} P_{kij} < 2^{-k}, \quad \operatorname{diam} \Phi_{ki}(P_{kij}) < 2^{-k}.$$

We now define Φ_k by setting

$$\Phi_k := \Phi_{ki} \text{ on } P_{k-1,i}.$$

By condition (1), Φ_k is indeed a diffeomorphism and $\Phi_k = \mathrm{id}$ near $\partial\Omega$, which is (a). Next, we set

$$\hat{E}_k := \bigcup_{i=1}^{M_{k-1}} \hat{E}_{ki} \quad \text{and} \quad L_k := \bigcup_{i=1}^{M_{k-1}} L_{ki} \subset E_{k-1}.$$

Since $D\Phi_k = D\Phi_{ki} = T$ on $\hat{E}_{ki}$, we have that $D\Phi_k = T$ on $\hat{E}_k$.

Summing (4) over $i = 1, \dots, M_{k-1}$, we readily see that $\Phi_k = \Phi_{k-1}$ on $E_{k-1} \setminus L_k$ and $|L_k| < \alpha^k$, i.e., (d) holds. Moreover, summing (3) over $i = 1, \dots, M_{k-1}$ and recalling that $|E_{k-1}| > 1 - \alpha^{k-1}$, we get

$$|\hat{E}_k| > \beta|\Omega| + (1-\beta)|E_{k-1} \setminus L_k| > \beta + (1-\beta)(1 - \alpha^{k-1} - \alpha^k) > 1 - \alpha^k,$$

so $|\Omega \setminus \hat{E}_k| < \alpha^k$. By discarding some points from $\hat{E}_k$ if necessary, we can find another compact set $E_k \subset \hat{E}_k$ such that $\alpha^{k+1} < |\Omega \setminus E_k| < \alpha^k$. Clearly, $D\Phi_k = T$ on E_k i.e., conditions (b) and (c) are satisfied. It remains to verify (e).

We define the partition[3] $\mathcal{P}_k$ as the union of partitions $\mathcal{P}_{ki}$

$$\mathcal{P}_k = \bigcup_{i=1}^{M_{k-1}} \bigcup_{j=1}^{2^{nN_{ki}}} \{P_{kij}\}.$$

After re-enumeration of the diffeomorphic cubes P_{kij}, we can write

$$\mathcal{P}_k = \bigcup_{\ell=1}^{M_k} \{P_{k\ell}\}, \quad \text{where} \quad M_k = \sum_{i=1}^{M_{k-1}} 2^{nN_{ki}}.$$

Since $\Phi_k = \Phi_{ki}$ on $P_{k-1,i}$, and $\Phi_{ki} = \Phi_{k-1}$ near $\partial P_{k-1,i}$, it follows that $\Phi_k(P_{k-1,i}) = \Phi_{k-1}(P_{k-1,i})$, which is (5.34). Since $P_{k\ell} = P_{kij}$ for some i, j and $\Phi_k = \Phi_{ki}$ on $P_{k\ell} = P_{kij} \subset P_{k-1,i}$, condition (5) yields

$$|\Phi_k(P_{k\ell})| = |\Phi_{ki}(P_{kij})| = \int_{P_{kij}} \det T(x)\, dx = \int_{P_{k\ell}} \det T(x)\, dx,$$

which is (5.35). Also

$$\operatorname{diam} P_{k\ell} = \operatorname{diam} P_{kij} < 2^{-k} \quad \text{and} \quad \operatorname{diam} \Phi_k(P_{k\ell}) = \operatorname{diam} \Phi_{ki}(P_{kij}) < 2^{-k},$$

which is (5.36). This completes the proof of (e) and hence that of (a)-(e).

Fix any $i = 1, \dots, M_{k-1}$. We shall now show that given Φ_{k-1}, we can construct Φ_{ki} as described above. As before, we begin by correcting the way Φ_{k-1} distributes measure

[3] While $\mathcal{P}_{ki}$ are a diffeomorphic dyadic partitions, $\mathcal{P}_k$ need not be, because we might divide each of the diffeomorphic cubes $P_{k-1,i}$ into a different number of diffeomorphic dyadic cubes.

so that the subsequent corrections from Φ_{k-1} to Φ_k are made at a smaller scale. To this end, we use Proposition 5.16 for the diffeomorphic closed cube $\Phi_{k-1}(P_{k-1,i})$, the compact set $\Phi_{k-1}(E_{k-1,i})$ and functions

$$f(y) = \det T(\Phi_{k-1}^{-1}(y)) \det D\Phi_{k-1}^{-1}(y) \quad \text{and} \quad g(y) = 1.$$

We check that (5.23) is satisfied by change of variables and the inductive assumption (5.35) for Φ_{k-1},

$$\begin{aligned}\int_{\Phi_{k-1}(P_{k-1,i})} \det T(\Phi_{k-1}^{-1}(y)) \det D\Phi_{k-1}^{-1}(y)\, dy &= \int_{P_{k-1,i}} \det T(x)\, dx \overset{(5.35)}{=} |\Phi_{k-1}(P_{k-1,i})| \\ &= \int_{\Phi_{k-1}(P_{k-1,i})} 1\, dx.\end{aligned}$$

Moreover, since $D\Phi_{k-1} = T$ on $E_{k-1,i}$, we have $f(y) = g(y)$ for all $y \in \Phi_{k-1}(E_{k-1,i})$. Therefore, assumptions of Proposition 5.16 are satisfied.

By Proposition 5.16, we get a diffeomorphism $\Psi_i : \Phi_{k-1}(P_{k-1,i}) \twoheadrightarrow \Phi_{k-1}(P_{k-1,i})$, a partition $\mathcal{R}_{ki} = \{R_{kij}\}_{j=1}^{2^{nN_{ki}}}$ of $\Phi_{k-1}(P_{k-1,i})$ and a compact set $\widetilde{K}_i \subset \Phi_{k-1}(E_{k-1,i})$ with properties described below.

Partition $\mathcal{R}_{ki}$ is a diffeomorphic dyadic partition and satisfies an analogue of (5.39) and (5.40), namely

$$\operatorname{diam} \Psi_i(R_{kij}) < 2^{-k}, \quad \operatorname{diam}(\Phi_{k-1}^{-1}(R_{kij})) < 2^{-k}, \tag{5.51}$$

and

$$\int_{\Phi_{k-1}^{-1}(R_{kij})} \det T(x)\, dx = |\Psi_i(R_{kij})|. \tag{5.52}$$

We find $\eta > 0$ such that for any measurable set $A \subset \Omega$,

$$|A| < \eta \quad \Rightarrow \quad \int_A \det D\Phi_{k-1}^{-1} < \frac{\alpha^k}{2M_{k-1}}\,.$$

We choose a compact set $\widetilde{K}_i \subset \Phi_{k-1}(E_{k-1,i})$ so that

$$|\Phi_{k-1}(E_{k-1,i}) \setminus \widetilde{K}_i| < \eta.$$

Moreover, $\Psi_i = \mathrm{id}$ near $\widetilde{K}_i \cup \partial(\Phi_{k-1}(P_{k-1,i}))$. In view of the inequality above, setting $K_i := \Phi_{k-1}^{-1}(\widetilde{K}_i) \subset E_{k-1,i}$, we have

$$|E_{k-1,i} \setminus K_i| < \frac{\alpha^k}{2M_{k-1}}\,. \tag{5.53}$$

Eventually, we set

$$\widetilde{\Phi}_{ki} := \Psi_i \circ \Phi_{k-1} \text{ in } P_{k-1,i} \quad \text{and} \quad \mathcal{P}_{ki} := \{P_{kij}\}_{j=1}^{2^{nN_{ki}}}, \text{ where } P_{kij} := \Phi_{k-1}^{-1}(R_{kij}).$$

Then (5.51) and (5.52) become

$$\operatorname{diam} \widetilde{\Phi}_{ki}(P_{kij}) < 2^{-k}, \quad \operatorname{diam}(P_{kij}) < 2^{-k}, \quad \int_{P_{kij}} \det T(x)\, dx = |\widetilde{\Phi}_{ki}(P_{kij})|, \tag{5.54}$$

i. e., diffeomorphism $\widetilde{\Phi}_{ki}$ satisfies (5). Observe also that

$$\widetilde{\Phi}_{ki} = \Phi_{k-1} \text{ near } K_i \cup \partial P_{k-1,i} \quad \text{so} \quad D\widetilde{\Phi}_{ki} = T \text{ on } K_i \subset E_{k-1,i}. \tag{5.55}$$

We will now replace $\widetilde{\Phi}_{ki}$ inside each P_{kij} with a diffeomorphism Φ_{kij} which has correct derivative on a larger set.

As explained earlier, above inequality (5.44), we need to use Lemma 4.14 to get a compact set $\widetilde{E}_{kij} \subset \mathring{P}_{kij} \cap K_i$ such that $\mathring{P}_{kij} \setminus \widetilde{E}_{kij}$ is connected and

$$|(\mathring{P}_{kij} \cap K_i) \setminus \widetilde{E}_{kij}| < \alpha^k \, 2^{-nN_{ki}-1} M_{k-1}^{-1}. \tag{5.56}$$

Set

$$\widetilde{E}_{ki} := \bigcup_{j=1}^{2^{nN_{ki}}} \widetilde{E}_{kij} \subset K_i \quad \text{and} \quad L_{ki} := E_{k-1,i} \setminus \widetilde{E}_{ki}.$$

Clearly, $\widetilde{E}_{ki} \subset \mathring{P}_{k-1,i}$. Summing (5.56) over $j = 1, \ldots, 2^{nN_{ki}}$ yields

$$|K_i \setminus \widetilde{E}_{ki}| < \frac{\alpha^k}{2M_{k-1}} \, .$$

By (5.53), we arrive at

$$|L_{ki}| = |E_{k-1,i} \setminus K_i| + |K_i \setminus \widetilde{E}_{ki}| < \alpha^k M_{k-1}^{-1}. \tag{5.57}$$

Consequently,

$$|\widetilde{E}_{ki}| = |E_{k-1,i}| - |L_{ki}| > |E_{k-1,i}| - \alpha^k \, M_{k-1}^{-1}. \tag{5.58}$$

Let us stress that $\widetilde{E}_{ki} \subset K_i$ so (5.55) yields

$$D\widetilde{\Phi}_{ki} = T \text{ on } \widetilde{E}_{ki}. \tag{5.59}$$

and

$$\widetilde{\Phi}_{ki} = \Phi_{k-1} \text{ on } \widetilde{E}_{ki} = E_{k-1,i} \setminus L_{ki}. \tag{5.60}$$

Although $D\Phi_{k-1} = T$ on $E_{k-1,i}$, the set L_{ki} is the set on which we will have spoiled the already prescribed derivative of Φ_{k-1}. On the other hand, the set $\widetilde{E}_{ki}$ consists of points in which the prescribed derivative *survives* the transition from Φ_{k-1} to Φ_{ki} and, as we will soon see, to Φ_k.

We now correct the derivative of $\widetilde{\Phi}_{ki}$. The set

$$\Omega_{kij} := \mathring{P}_{kij} \setminus \widetilde{E}_{kij}$$

is open and connected. Observe that, setting $\Omega_{ki} := \bigcup_j \Omega_{kij}$,

$$|\Omega_{ki}| = |P_{k-1,i} \setminus \widetilde{E}_{ki}|.$$

Exactly as in (5.49), invoking (5.54) and (5.59) instead of (5.42) and (5.47), we check that Ω_{kij} satisfies

$$|\widetilde{\Phi}_{ki}(\Omega_{kij})| = \int_{\Omega_{kij}} \det T(x) \, dx.$$

We are now in the position to use Theorem 1.4 for the domain Ω_{kij} and the diffeomorphism $\widetilde{\Phi}_{ki}$. We find a diffeomorphism $\Phi_{kij} : \Omega_{kij} \twoheadrightarrow \widetilde{\Phi}_{ki}(\Omega_{kij})$ and a compact set $E'_{kij} \subset \Omega_{kij}$ such that

$$D\Phi_{kij} = T \text{ on } E'_{kij}, \quad \Phi_{kij} = \widetilde{\Phi}_{ki} \text{ near } \partial\Omega_{kij} \quad \text{and} \quad |E'_{kij}| > \beta|\Omega_{kij}|. \tag{5.61}$$

and thus $\Phi_{kij}(\Omega_{kij}) = \widetilde{\Phi}_{ki}(\Omega_{kij})$. Let

$$E'_{ki} := \bigcup_{j=1}^{2^{nN_{ki}}} E'_{kij} \subset \mathring{P}_{k-1,i}.$$

By the third condition in (5.61), we have

$$|E'_{ki}| > \beta|\Omega_{ki}| = \beta|P_{k-1,i} \setminus \widetilde{E}_{ki}|.$$

We replace the diffeomorphism $\widetilde{\Phi}_{ki}$ with Φ_{kij} inside each Ω_{kij}, setting

$$\Phi_{ki} := \begin{cases} \Phi_{kij} & \text{on } \Omega_{kij}, \\ \widetilde{\Phi}_{ki} & \text{on } P_{k-1,i} \setminus \Omega_{ki}. \end{cases}$$

Observe that thanks to the second condition in (5.61), Φ_{ki} is indeed a diffeomorphism.

Moreover, we have $\Phi_{ki} = \widetilde{\Phi}_{ki}$ near $\partial P_{k-1,i} \subset \bigcup_j \partial P_{kij}$. Since $\widetilde{\Phi}_{ki} = \Phi_{k-1}$ near $\partial P_{k-1,i}$, property (1) follows.

Since $\Phi_{ki} = \widetilde{\Phi}_{ki}$ near $\bigcup_j \partial P_{kij}$, $\Phi_{ki}(P_{kij}) = \widetilde{\Phi}_{ki}(P_{kij})$, so (5.54) proves property (5).

Note that $\widetilde{E}_{ki} \subset \mathring{P}_{k-1,i} \setminus \Omega_{ki}$ and $\Phi_{ki} = \widetilde{\Phi}_{ki}$ in a neighborhood of $\widetilde{E}_{ki}$, so the first condition in (5.61) together with (5.59) imply that

$$D\Phi_{ki} = T \text{ on } \hat{E}_{ki} := E'_{ki} \cup \widetilde{E}_{ki} \subset \mathring{P}_{k-1,i}.$$

Note that $\hat{E}_{ki}$ is compact as a finite sum of compact sets. This proves property (2). We calculate

$$|\hat{E}_{ki}| = |E'_{ki}| + |\widetilde{E}_{ki}| > \beta|P_{k-1,i} \setminus \widetilde{E}_{ki}| + |\widetilde{E}_{ki}| = \beta|P_{k-1,i}| + (1-\beta)|E_{k-1,i} \setminus L_{ki}|,$$

where the last equality follows from $\widetilde{E}_{ki} = E_{k-1,i} \setminus L_{ki}$. This proves (3).

Since $\Phi_{ki} = \widetilde{\Phi}_{ki}$ on $\widetilde{E}_{ki} = E_{k-1,i} \setminus L_{ki}$, recalling (5.60) and (5.57), we see that (4) also holds.

We verified conditions (1)-(5) and that completes the proof of the theorem. □

5.5 Corollaries

We begin this section with a corollary saying that given $T : \mathbb{Q} \to GL(n)$ as in Theorem 1.5 but without the volume constraint (1.4), it is still possible to construct an a. e. approximately differentiable homeomorphism with the prescribed approximate derivative equal T but, obviously, without any boundary condition that would imply that $\Phi(\mathbb{Q}) = \mathbb{Q}$.

Corollary 5.21. *Let $\mathbb{Q} = [0,1]^n$ and let $T : \mathbb{Q} \to GL(n)$ be measurable and such that*

$$\int_{\mathbb{Q}} |\det T(x)|\, dx < \infty.$$

Then there is an a. e. approximately differentiable homeomorphism $\Phi : \mathbb{Q} \to \mathbb{R}^n$ which satisfies the Lusin (N) condition and such that $D_{\mathrm{a}}\Phi(x) = T(x)$ for a. e. $x \in \mathbb{Q}$.

Proof. Let
$$M := \left(\int_{\mathcal{Q}} |\det T(x)|\, dx \right)^{1/n} \quad \text{and} \quad \lambda(x) := Mx.$$
Then $\lambda : \mathcal{Q} \to \mathbb{R}^n$ is a diffeomorphism which maps $\mathcal{Q}$ onto the cube $[0, M]^n$.

Set $\widetilde{T}(x) := M^{-n} T(x)$ for $x \in \mathcal{Q}$ and note that $\int_{\mathcal{Q}} |\det \widetilde{T}| = 1$. Then Theorem 1.5 yields an a. e. approximately differentiable homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ which satisfies the Lusin (N) condition and such that
$$D_{\mathrm{a}} \widetilde{\Phi} = \widetilde{T} = M^{-n} T \text{ a. e. on } \mathcal{Q}.$$
Then $\Phi := \lambda \circ \widetilde{\Phi}$ is the required mapping. Indeed, Φ is a homeomorphism which satisfies the Lusin (N) condition as a composition of such mappings. As λ is a bi-Lipschitz homeomorphism, it suffices to use the chain rule from Lemma 4.13 to see that Φ is approximately differentiable a. e. on $\mathcal{Q}$ and that
$$D_{\mathrm{a}}\Phi(x) = D\lambda(\widetilde{\Phi}(x))\, D_{\mathrm{a}}\widetilde{\Phi}(x) = M^n \cdot M^{-n} \cdot T(x) = T(x) \text{ for a. e. } x \in \mathcal{Q}.$$
This concludes the proof. □

Prescribing a Jacobian of a homeomorphism is easy once we are able to prescribe the whole derivative, as we can see in Corollary 5.3.

Proof of Corollary 5.3. For $x \in \mathcal{Q}$, set
$$T(x) := \begin{bmatrix} f(x) & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$
By Theorem 1.5, there is an a. e. approximately differentiable homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$, $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$, which satisfies the Lusin (N) condition and such that $D_{\mathrm{a}}\Phi = T$ a. e. on $\mathcal{Q}$. Then $\det D_{\mathrm{a}}\Phi(x) = f(x)$ a. e. on $\mathcal{Q}$. By change of variables (2.15) in Corollary 2.31, (5.3) holds. □

Lemma 5.22. *Let $\mathcal{Q} = [0,1]^n$ and let $\Phi_k : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ be a sequence of orientation preserving C^1-diffeomorphisms that converge to a homeomorphism $\Phi : \mathcal{Q} \twoheadrightarrow \mathcal{Q}$ in the Lusin metric d_L. Then Φ is approximately differentiable a. e. on $\mathcal{Q}$ and*
$$\det D_{\mathrm{a}}\Phi(x) > 0 \text{ a. e. on } Q \quad \text{and} \quad \int_Q \det D_{\mathrm{a}}\Phi(x)\, dx \le 1. \tag{5.62}$$

Proof. It follows directly from Whitney's theorem, i. e., Lemma 2.26, that Φ is approximately differentiable a. e. on $\mathcal{Q}$, see Remark 2.28. Let $E_k := \{x \in \mathcal{Q} : \Phi(x) = \Phi_k(x)\}$. Clearly, $|\bigcup_{k=1}^{\infty} E_k| = |\mathcal{Q}|$. By Remark 2.27, $D_{\mathrm{a}}\Phi = D\Phi_k$ a. e. on E_k and since $\det D\Phi_k > 0$ on $\mathcal{Q}$, we know that $\det D_{\mathrm{a}}\Phi > 0$ a. e. on E_k. Consequently, $\det D_{\mathrm{a}}\Phi > 0$ a. e. on $\mathcal{Q}$.

By (2.14) in Corollary 2.31 for $E = \mathring{\mathcal{Q}}$,
$$\int_{\mathcal{Q}} \det D_{\mathrm{a}}\Phi(x)\, dx = \int_{\mathring{\mathcal{Q}}} \det D_{\mathrm{a}}\Phi(x)\, dx \le |\Phi(\mathring{\mathcal{Q}})| = 1,$$
which shows that (5.62) holds. □

Proof of Theorem 5.5. For T as assumed here, we have constructed in Section 5.4.2 in the proof of Theorem 1.5 a sequence of orientation preserving C^1-diffeomorphisms Φ_k of Ω and Borel sets $C_k \subset \Omega$ with the following properties for $k \geq 1$:

(i) $\Phi_k = \mathrm{id}$ near $\partial\Omega$;
(ii) $\Phi_{k+1} = \Phi_k$ on C_k;
(iii) $D\Phi_k = T$ on C_k;
(iv) C_k is an increasing family of sets, $C_1 \subset C_2 \subset \cdots$, with $\lim_{k\to\infty} |C_k| = 1$;
(v) $d(\Phi_k, \Phi_{k+1}) < 2^{-(k-1)}$ for $k \geq 2$.

The limit map $\Phi := \lim_{k\to\infty} \Phi_k$ is the required homeomorphism. It is easy to see (and we have checked it in Section 5.4.2) that $\Phi = \mathrm{id}$ on $\partial\Omega$, that Φ_k converge in uniform metric to Φ and that $D_{\mathrm{a}}\Phi = T$ a. e. It follows from (ii) and (iv) that Φ_k converge to Φ in the Lusin metric d_L as well. This finishes the proof. $\square$

Appendix A

Appendix

A.1 Lusin metric

We present below Lemma 8.1 from [39], which shows that the space of measurable function is complete with respect to the Lusin metric. Let us recall its definition. Let $m, n \in \mathbb{N}$. The space of measurable mappings $f, g : E \to \mathbb{R}^n$ defined on a measurable set $E \subset \mathbb{R}^m$ is equipped with *Lusin metric* defined as

$$d_L(f, g) := |\{x \in \Omega :\ f(x) \neq g(x)\}|.$$

This is the only lemma in which we include also $n = 1$.

Lemma A.1. *Let $E \subset \mathbb{R}^n$, $n \geq 1$, be a measurable set of finite measure. Then the space of measurable functions $f : E \to \mathbb{R}$ is complete with respect to the Lusin metric d_L.*

Proof. Let $\{f_k\}_k$ be a Cauchy sequence. It suffices to show that it has a convergent subsequence in the metric d_L. To this end, we will show that a subsequence $\{f_{k_\ell}\}_\ell$ such that

$$d_L(f_{k_\ell}, f_{k_{\ell+1}}) = |\{f_{k_\ell} \neq f_{k_{\ell+1}}\}| < 2^{-(\ell+1)}$$

is convergent. Let

$$A_\ell := \{f_{k_\ell} = f_{k_{\ell+1}}\} \quad \text{and} \quad C := \bigcup_{\ell=1}^{\infty} \bigcap_{i=\ell}^{\infty} A_i.$$

Note that for any ℓ,

$$|E \setminus C| \leq \Big|E \setminus \bigcap_{i=\ell}^{\infty} A_i\Big| \leq \sum_{i=\ell}^{\infty} |E \setminus A_i| < 2^{-\ell}, \quad \text{so} \quad |E \setminus C| = 0.$$

If $x \in C$, then $x \in \bigcap_{i=\ell}^{\infty} A_i$ for some ℓ and hence

$$f_{k_\ell}(x) = f_{k_{\ell+1}}(x) = f_{k_{\ell+2}}(x) = \ldots \quad \text{so} \quad f(x) := \lim_{\ell\to\infty} f_{k_\ell}(x)$$

exists for all $x \in C$ and hence for almost all $x \in E$. In fact, $f(x) = f_{k_\ell}(x)$ for $x \in \bigcap_{i=\ell}^{\infty} A_i$ and hence

$$d_L(f, f_{k_\ell}) \leq \Big|E \setminus \bigcap_{i=\ell}^{\infty} A_i\Big| < 2^{-\ell},$$

proving that $f_{k_\ell} \to f$ in the metric d_L. □

A.2 1-parameter groups of diffeomorphisms

Throughout the thesis, we have used the very convenient method of constructing diffeomorphisms using 1-parameter groups of diffeomorphisms. The following theorem asserts its validity

Theorem A.2. *Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞-smooth vector field, which equals zero outside a compact set $K \subset \mathbb{R}^n$. Then there exists a family of mappings $\{\Phi_t\}_{t\in\mathbb{R}}$, $\Phi_t : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ which is a solution to*

$$\frac{d}{dt}\Phi_t(x) = X(\Phi_t(x)), \tag{A.1}$$

$$\Phi_0(x) = x. \tag{A.2}$$

Moreover, for each $t \in \mathbb{R}$, Φ_t is a C^∞-diffeomorphism of $\mathbb{R}^n$, Φ_t depends continuously on t and

$$\Phi_{-t} = (\Phi_t)^{-1}. \tag{A.3}$$

For the proof, see e.g. Theorem 1 in Chapter 5, Section 35 in [4]. The statement and proof there are written for C^r-manifolds, $r \geq 2$.

Remark A.3. Since the right hand-side of (A.1) is smooth w. r. t. $\Phi_t(x)$, it follows from standard uniqueness theorems (see e.g. [4, Chapter 2, Section 3]) that given $x_o \in \mathbb{R}^n$, the mapping $t \mapsto \Phi_t(x_o)$ is unique.

The family $\{\Phi_t\}_t$ is often called a 1-parameter group of diffeomorphisms generated by the vector field X (and we have called it so throughout the thesis). It is clear from the statement where 'diffeomorphisms' and '1-parameter' come from. The family $\{\Phi_t\}_{t\in\mathbb{R}}$ indeed forms a group, in which $\Phi_0 = \mathrm{id}$ acts as the neutral element. More precisely, for any $s, t \in \mathbb{R}$, $\Phi_s \circ \Phi_t = \Phi_{s+t}$. It is this property which implies that (A.3) holds.

Alternatively, the family $\{\Phi_t\}_t$ is also called the *flow* of the vector field X. Indeed, for each $x \in \mathbb{R}^n$, the mapping

$$t \mapsto \Phi_t(x)$$

defines the trajectory of the point x. In this manner, we can see that Φ_t maps the point x to the point $\Phi_t(x)$. Actually, this is the gist of the *flow method* used by Moser in his seminal paper [76], which we repetitively quoted when discussing diffeomorphisms with prescribed Jacobians. The precise statement and proof of the existence and properties of the flow of a vector field which also depends on t can be found for example in [20, Theorem 12.1].

Corollary A.4. *Let U be a bounded domain in $\mathbb{R}^n$ and p, q two distinct points in U which can be connected with a segment contained in U. Then there is a C^∞-diffeomorphism $F : U \twoheadrightarrow U$ such that*

$$F(p) = q \text{ and } F = \mathrm{id} \text{ near } \partial U.$$

Proof. Let V_1, V_2 denote open neighborhoods of the segment connecting p with q such that $V_1 \Subset V_2 \Subset U$. Then choose $X : \mathbb{R}^n \to \mathbb{R}^n$ to be a smooth vector field satisfying

$$X = \begin{cases} 0 & \text{in } \mathbb{R}^n \setminus V_2, \\ q - p & \text{in } V_1. \end{cases}$$

Let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the 1-parameter group of diffeomorphisms generated by the vector field X according to Theorem A.2. We claim that $F := \Phi_1|_U$ is the required mapping. Indeed,

Φ_1 is a C^∞-diffeomorphism. For $x_o \in \mathbb{R}^n \setminus V_2$, the mapping $\Phi_t(x_o) = x_o$ for all $t \in \mathbb{R}$ satisfies (A.1) and it follows from uniqueness (see Remark A.3) that $\Phi_1(x_o) = x_o$ for $x_o \in \mathbb{R}^n \setminus V_2$. Therefore, $\Phi_1 = \mathrm{id}$ near ∂U and hence Φ_1 maps U onto itself. Similary, the mapping $\Phi_t(p) = tq + (1-t)p$ for $t \in [0,1]$ satisfies (A.1), as for any $t \in [0,1]$, $\Phi_t(p) \in V_1$. Therefore, again by uniqueness, $\Phi_1(p) = q$. This finishes the proof. □

Remark A.5. Observe that F also maps by translation a small neighborhood of p onto a small neighborhood of q. Indeed, let γ denote the segment connecting p and q, then since V_1 is open, there is $\varepsilon > 0$ for which

$$\{x : \operatorname{dist}(x,\gamma) < \varepsilon\} \Subset V_1.$$

It can be checked that for any $x \in B(p,\varepsilon)$ and for $t \in [0,1]$, $\Phi_t(x) = t(p-q)+x$ satisfies (A.1) and hence Φ_1 acts as a translation on $B(p,\varepsilon)$.

A.3 Measure preserving version of Lemma 2.16

In Lemma A.7 below, we construct a diffeomorphism as in Lemma 2.16 with the additional property of being measure preserving. I believe it is an interesting and possibly useful construction. The result is from [81, Chapter 5] and the approach from [3, Chapter 2.2] but as both pairs of authors were only interested in homeomorphisms, they do not show that it is possible to construct a C^∞-diffeomorphism with such properties.

The heart of the proof is the following construction

Lemma A.6. *Let p, q be two points in $\mathbb{R}^n$. For any $\varepsilon \in (0, |p-q|/2)$, there is a measure preserving C^∞-diffeomorphism $H : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ such that*

$$H = \mathrm{id} \ \textit{outside}\ B\big((p+q)/2, |p-q|/2 + 2\varepsilon\big) \tag{A.4}$$

and

$$H(x) = x + (q-p) \ \textit{for}\ x \in B(p,\varepsilon). \tag{A.5}$$

Proof. Fix $\varepsilon \in (0, |p-q|/2)$.

Step 1. Suppose for now that $n = 2$. We shall construct the required mapping in the planar case and extend the result for $n > 2$ in Step 2.

Suppose that $p = -q$. Firstly, we shall construct a measure preserving C^∞-diffeomorphism $\tilde{g}_{p,q}$, which satisfies (A.4) and maps $B(p,\varepsilon)$ onto $B(q,\varepsilon)$ by rotation and then correct it so that it satisfies (A.5).

Let $\alpha : [0,\infty) \to \mathbb{R}$ be a C^∞-differentiable function which equals π on $[0, |p|+\varepsilon]$ and 0 on $[|p|+2\varepsilon, \infty)$. We define

$$\tilde{g}_{p,q}(x,y) = \begin{pmatrix} \cos\alpha(|(x,y)|) & -\sin\alpha(|(x,y)|) \\ \sin\alpha(|(x,y)|) & \cos\alpha(|(x,y)|) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

The mapping $\tilde{g}_{p,q} : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ acts like rotation by angle π on $B(0, |p|+\varepsilon)$, hence it maps the ball $B(p,\varepsilon)$ onto $B(q,\varepsilon)$, and it is equal identity outside $B(0, |p|+2\varepsilon)$. Moreover, it follows from smoothness of α that $\tilde{g}_{p,q}$ is C^∞-differentiable and it can be easily checked that $\tilde{g}_{p,q}$ is injective. Indeed, for any two distinct points $x_1, x_2 \in \mathbb{R}^2$, if $|x_1| \neq |x_2|$, then

$$|\tilde{g}_{p,q}(x_1) - \tilde{g}_{p,q}(x_2)| \geq ||\tilde{g}_{p,q}(x_1)| - |\tilde{g}_{p,q}(x_2)|| > 0.$$

On the other hand, if $|x_1| = |x_2|$, then $|\tilde{g}_{p,q}(x_1) - \tilde{g}_{p,q}(x_2)| = |x_1 - x_2| \neq 0$.

Next, we show by direct computation that the Jacobian of $\tilde{g}_{p,q}$ equals 1. By the inverse function theorem, this implies C^∞-differentiability of $\tilde{g}_{p,q}^{-1}$ and, by the classical change of variables formula, that $\tilde{g}_{p,q}$ is measure preserving. In the end, this shows that $\tilde{g}_{p,q} : \mathbb{R}^2 \twoheadrightarrow \mathbb{R}^2$ is a measure preserving diffeomorphism.

To simplify the notation, set $u := u(x,y) := \cos \alpha(|(x,y)|)$, $v := v(x,y) := \sin \alpha(|(x,y)|)$ and $\alpha := \alpha(|(x,y)|)$. Clearly, $u^2 + v^2 = 1$. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$, f_x and f_y denote the partial derivative of f w. r. t. the first and second coordinate, respectively. We have

$$D\tilde{g}_{p,q}(x,y) = \begin{pmatrix} u_x x + u - v_x y & u_y x - v_y y - v \\ v_x x + v + u_x y & v_y x + u_y y + u \end{pmatrix}. \tag{A.6}$$

It is easy to check the following identities

$$u_x = -v\alpha_x \quad \text{and} \quad u_y = -v\alpha_y, \tag{A.7}$$

$$v_x = u\alpha_x \quad \text{and} \quad v_y = u\alpha_y. \tag{A.8}$$

This implies that

$$\begin{aligned} u_x v_y - v_x u_y &= -uv\alpha_x\,\alpha_y + uv\alpha_x\,\alpha_y = 0 \\ uu_x + vv_x &= -uv\alpha_x + uv\alpha_x = 0 \\ uu_y + vv_y &= -uv\alpha_y + uv\alpha_y = 0 \\ uv_y - vu_y &= \alpha_y\,(u^2 + v^2) = \alpha_y \\ u_x v - v_x u &= -\alpha_x\,(u^2 + v^2) = -\alpha_x. \end{aligned} \tag{A.9}$$

We compute

$$\begin{aligned} \det D\tilde{g}_{p,q}((x,y)) &= (u_x x + u - v_x y)(v_y x + u_y y + u) - (v_x x + v + u_x y)(u_y x - v_y y - v) \\ &= u^2 + v^2 + (x^2 + y^2)\,(u_x v_y - v_x u_y) + x(uu_x + vv_x + uv_y - vu_y) \\ &\quad + y(uu_y + vv_y - uv_x + u_x v) \overset{(A.9)}{=} 1 + x\alpha_y - y\alpha_x = 1. \end{aligned}$$

The last equality follows from the fact that α is radially symmetric and hence $x\alpha_y = y\alpha_x$.

We have shown that if $p = -q$, then $\tilde{g}_{p,q}$ satisfies the claim of Lemma A.6 except for (A.5), as $\tilde{g}_{p,q}$ maps $B(p,\varepsilon)$ onto $B(q,\varepsilon)$ by rotation and not by translation. Now for any $p, q \in \mathbb{R}^2$, set

$$g_{p,q} = \tilde{g}_{(p-q)/2,\,(q-p)/2}\left(x - \frac{p+q}{2}\right) + \frac{p+q}{2},$$

which is a measure preserving C^∞-diffeomorphism which satisfies (A.4) and maps $B(p,\varepsilon)$ onto $B(q,\varepsilon)$ by rotation.

Then, for any $p, q \in \mathbb{R}^2$ set

$$\tilde{h}_{p,q} = g_{\frac{p+q}{2},q} \circ g_{p,\frac{p+q}{2}}.$$

The mapping $\tilde{h}_{p,q}$ is a measure preserving C^∞-diffeomorphism, which maps the ball $B(p,\varepsilon)$ by translation onto the ball $B(q,\varepsilon)$. Indeed, it firstly rotates the ball $B(p,\varepsilon)$ onto the ball $B((p+q)/2,\varepsilon)$ and then rotates the latter onto the ball $B(q,\varepsilon)$. Clearly, it satisfies (A.4) as well. we have constructed the mapping required in this lemma for $n = 2$.

Step 2. We will now describe how to construct such a mapping for $n > 2$. For any $a, b \in \mathbb{R}^n$ such that $a = (a_1, a_2, 0, \dots, 0)$ and $b = (b_1, b_2, 0, \dots, 0)$, we set

$$h_{a,b}(x_1, \dots, x_n) := (\tilde{h}_{(a_1,a_2),(b_1,b_2)}(x_1, x_2), x_3, \dots, x_n).$$

This is a measure preserving C^∞-diffeomorphism, which maps the n-dimensional ball $B^n(a,\varepsilon)$ by translation onto the ball $B^n(a,\varepsilon)$ and such that

$$h_{a,b} = \text{id} \text{ outside } B\big((a+b)/2, |a-b|/2 + 2\varepsilon\big). \tag{A.10}$$

We are now ready to discuss the general case. Let p,q by any distinct points in $\mathbb{R}^n$. We choose a 2-dimensional plane P containing p and q. Then we find $\varphi : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$, a composition of rotation and translation and thus a measure preserving C^∞-diffeomorphism, which maps P onto the $x_1\,x_2$-plane. Therefore,

$$\varphi(B(p,\varepsilon)) = B(\varphi(p),\varepsilon) \text{ and } \varphi(B(q,\varepsilon)) = B(\varphi(q),\varepsilon). \tag{A.11}$$

Then

$$H := \varphi^{-1} \circ h_{\varphi(p),\varphi(q)} \circ \varphi.$$

is the required mapping. Indeed, H is a measure preserving C^∞-diffeomorphism as a composition of such mappings. It follows from construction that H satisfies (A.5) and by (A.10) and (A.11), (A.4) holds as well. This finishes the proof. □

Lemma A.7. *Let $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$ be given points in U, a domain in $\mathbb{R}^n$, with $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$. Then there exists an $\varepsilon > 0$ and a* ***measure preserving*** *C^∞-diffeomorphism $H : U \to U$, identity near ∂U, such that*

$$H(x) = x + (q_i - p_i) \textit{ for } x \in B(p_i,\varepsilon),$$

i.e., H maps by translation each ball $B(p_i,\varepsilon)$ onto $B(q_i,\varepsilon)$ with $H(p_i) = q_i$.

Proof. To begin with, assume that $N = 1$ and that $p,q \in U$ satisfy

$$\overline{B}\left(\frac{p+q}{2}, \frac{|p-q|}{2}\right) \subset U.$$

Then we find an $\varepsilon \in (0, |p-q|/2)$ so that

$$\overline{B}\left(\frac{p+q}{2}, \frac{|p-q|}{2} + 2\varepsilon\right) \subset U. \tag{A.12}$$

By Lemma A.6, we find a measure preserving C^∞-diffeomorphism $H^\varepsilon_{p,q}$, which maps $B(p,\varepsilon)$ by translation onto $B(q,\varepsilon)$ and equals identity outside the ball (A.12) and hence near ∂U.

Next, consider p,q which can be connected by a segment in U. We find $\varepsilon > 0$ such that V, the 4ε-neighborhood of the segment connecting p and q, is compactly contained in U, and $\ell + 1$ points $p = a_o, \ldots, q = a_\ell$ such that

$$2\varepsilon < |a_i - a_{i-1}| < 4\varepsilon \text{ for } i = 1,\ldots,\ell.$$

Then the balls (A.12) with a_{i-1}, a_i instead of p and q, respectively, are contained in V. We then set

$$H^\varepsilon_{p,q} = H^\varepsilon_{a_\ell,a_{\ell-1}} \circ \ldots \circ H^\varepsilon_{a_o,a_1}.$$

This is a measure preserving C^∞-diffeomorphism, which maps $B(p,\varepsilon)$ onto $B(q,\varepsilon)$ by translation.

We then have to show that the statement is true for any $p,q \in U$ and we do this exactly like in Lemma 2.16. Next, the proof of the general case of N pairs of distinct points is the same as in Lemma 2.16 as well. □

A.4 Appendix to Chapter 3

A.4.1 Orientation

In order to keep this dissertation as self-contained as possible, we shortly address in this section the issue of what it means for a homeomorphism to be orientation preserving. Let us firstly discuss the Euclidean setting.

Let Ω be a bounded domain in $\mathbb{R}^n$. We say that a diffeomorphism $f : \Omega \to \mathbb{R}^n$ is orientation preserving if its Jacobian is positive on Ω. Indeed, it is clear that the Jacobian of a diffeomorphism cannot change sign, so this definition is well-posed. Similarly, by Rademacher's theorem a bi-Lipschitz homeomorphism is differentiable a. e. and hence its Jacobian also cannot change sign (for the proof see [47, Theorem 5.22][1]). We can therefore say that a bi-Lipschitz homeomorphism $f : U \to \mathbb{R}^n$ is orientation preserving if its Jacobian (defined a. e.) is non-negative a. e. However, given a homeomorphism $f : U \to \mathbb{R}^n$, we can use only topological notions. We shall provide a definition using the notion of *degree*. In this very short exposition, we follow [32, Section 1.2], see also [79, Section IV.2].

We firstly define the degree for C^1 mappings. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ and let $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$, i. e., we assume that f admits a C^1 extension to an open set containing $\overline{\Omega}$. With J_f we denote the Jacobian of f, i. e., $J_f(x) := \det Df(x)$. Set $Z_f := \{x \in \overline{\Omega} : J_f(x) = 0\}$ and suppose that $p \in \mathbb{R}^n \setminus (f(Z_f) \cup f(\partial\Omega))$. Then,

$$\deg(f, \Omega, p) := \sum_{x \in f^{-1}(p)} \operatorname{sgn}(J_f(x)), \tag{A.13}$$

where $\deg(f, \Omega, p)$ denotes the degree of f with respect to the domain Ω at a point p. Consistently with (A.13), we can set $\deg(f, \Omega, p) = 0$ whenever $p \notin f(\overline{\Omega})$, i. e., whenever $f^{-1}(p)$ is empty.

Using Sard's lemma, it is then possible to define the degree for C^1 mappings at points also from $f(Z_f) \setminus f(\partial\Omega)$ and so, more precisely, for all $p \in \mathbb{R}^n \setminus f(\partial\Omega)$. Eventually, degree for continuous mappings can be defined, see [32, Definition 1.18]. The transition between C^1 and continuous setting can be done by approximating continuous mappings with C^1-smooth ones and showing that the degree is stable under homotopy.

It is clear from (A.13) that for a diffeomorphism $f : \overline{\Omega} \to \mathbb{R}^n$, we have $\deg(f, \Omega, p) = 1$ for every $p \in f(\Omega)$ or $\deg(f, \Omega, p) = -1$ for every $p \in f(\Omega)$. It is more difficult to show an analogous statement for homeomorphisms but it can be proved that

Theorem A.8. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism. Then either*

$$\deg(f, \Omega, p) = 1 \textit{ for every point } p \in f(\Omega) \tag{A.14}$$

or

$$\deg(f, \Omega, p) = -1 \textit{ for every point } p \in f(\Omega). \tag{A.15}$$

This follows e.g. from Theorem 3.35 and Theorem 2.3 (c) in [32]. This theorem asserts that the following definition is well-posed.

Definition A.9. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be a homeomorphism. We say that f is orientation preserving if (A.14) holds. We say that f is orientation reversing if (A.15) holds.

[1] The statement of this theorem assumes f to be in $W^{1,1}$ but this assumption is not used in the proof.

A linear homeomorphism $f(x) = Ax$ is orientation preserving if and only if $\det A > 0$ and orientation reversing if $\det A < 0$.

Remark A.10. Given a bounded domain Ω in $\mathbb{R}^n$, any homeomorphism $f : \overline{\Omega} \twoheadrightarrow \overline{\Omega}$ with $f|_{\partial\Omega} = \mathrm{id}$ is orientation preserving. This follows from the fact that the degree depends only on the boundary values, see [32, Theorem 2.4]. This implies that the homeomorphism constructed in [37] (recalled in this thesis as Proposition 4.12) as well as its generalization from Theorem 5.2 are orientation preserving.

What is more, given any measurable $T : \mathcal{Q} \to GL(n)$ as in Theorem 1.5, the homeomorphism Φ we construct there also satisfies $\Phi|_{\partial\mathcal{Q}} = \mathrm{id}$ and hence it is orientation preserving as well.

Remark A.11. Suppose that Ω is a bounded domain and $f : \Omega \to \mathbb{R}^n$ is a homeomorphism which is differentiable at a point $x_o \in \Omega$ with $J_f(x_o) \neq 0$. Then $S(x) := Df(x_o)(x - x_o) + f(x_o)$ can be connected to $f(x)$ via a linear homotopy. Since the degree is stable under homotopy, see [32, Theorem 2.3 (2)], it can be shown that

$$\deg(f, \Omega, f(x_o)) = \deg(S, \Omega, f(x_o)) = \operatorname{sgn}(J_f(x_o)).$$

For details, see the already mentioned [47, Theorem 5.22].[2] Therefore, if f is an orientation preserving homeomorphism differentiable at a point x_o, then $J_f(x_o) \geq 0$.

Using the notion of orientation preserving homeomorphism between domains in $\mathbb{R}^n$, we shall now define orientation preserving homeomorphism between oriented manifolds. To this end, we need to recall the notion of an oriented atlas.

Definition A.12 (Oriented atlas)**.** Let $\mathcal{M}^n$ be an n-dimensional connected manifold. We say that an atlas $\{(U_i, \varphi_i)\}_{i\in I}$, where U_i is an open subset of $\mathcal{M}^n$ and $\varphi_i : U_i \to \mathbb{R}^n$, is oriented if for every $i, j \in I$, the transition map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \twoheadrightarrow \varphi_i(U_i \cap U_j)$$

is orientation preserving. More precisely, $\varphi_i \circ \varphi_j^{-1}$ needs to be orientation preserving on each connected component of $\varphi_j(U_i \cap U_j)$.

We call an oriented atlas *maximal* if it is maximal in the sense of inclusion, i. e., if it is not possible to add another chart (U, φ) so that the enlarged atlas stays oriented.

We are now in a position to define an oriented manifold.

Definition A.13 (Oriented manifold)**.** Let $\mathcal{M}^n$ be an n-dimensional connected manifold. We say that $\mathcal{M}^n$ is orientable if there exists a maximal oriented atlas on $\mathcal{M}^n$. An oriented manifold $\mathcal{M}^n$ is then the pair of $\mathcal{M}^n$ and the maximal oriented atlas.

Eventually, we can define an orientation preserving homeomorphism.

Definition A.14 (Orientation preserving homeomorphism)**.** Let $\mathcal{M}^n, \mathcal{N}^n$ be two oriented n-dimensional connected manifolds. Let $\mathcal{A}_{\mathcal{M}} := \{(U_i, \varphi_i)\}_{i\in I}$ and $\mathcal{A}_{\mathcal{N}} := \{(U'_j, \psi_j)\}_{j\in J}$ denote the maximal oriented atlases on $\mathcal{M}^n$ and $\mathcal{N}^n$, respecitvely. We say that a homeomorphism $f : \mathcal{M}^n \twoheadrightarrow \mathcal{N}^n$ is orientation preserving if for every chart $(U'_j, \psi_j) \in \mathcal{A}_{\mathcal{N}}$, the chart $(f^{-1}(U'_j), \psi_j \circ f) \in \mathcal{A}_{\mathcal{M}}$.

[2] We recall that the statement of this theorem assumes f to be in $W^{1,1}$ but this assumption is not used in the proof.

It is clear from the definition that a composition of orientation preserving homeomorphisms is an orientation preserving homeomorphism.

There is more than one way to define orientation on a manifold and to say what an orientation preserving homeomorphism is (even for homeomorphisms of $\mathbb{R}^n$). Of course, whatever the method, the resulting definitions coincide. Another standard approach is through homology groups, see e.g. [44, Chapter 3.3]. For a concise explanation of how the different notions agree, see the note [64].

Lemma A.15. *Let $\mathcal{M}^n$ be an n-dimensional connected manifold with a maximal oriented atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$. Suppose that D, $D \subset \mathcal{M}^n$, is a flat topological closed ball, i. e., there is $\varepsilon > 0$ and a homeomorphism $\widetilde{\Phi} : B(0, 1+\varepsilon) \to \mathcal{M}^n$ such that $\widetilde{\Phi}(\bar{\mathbb{B}}^n) = D$. Then there is $\Phi : B(0, 1+\varepsilon) \to \mathcal{M}^n$ with $\Phi(\bar{\mathbb{B}}^n) = D$ and such that*

$$\big(\Phi(B(0, 1+\varepsilon)), \Phi^{-1}\big) \in \mathcal{A}. \tag{A.16}$$

Proof. It follows from Corollary 2.3 that $U_\varepsilon := \widetilde{\Phi}(B(0, 1+\varepsilon))$ is open in the topology of $\mathcal{M}^n$. Therefore, the pair $(U_\varepsilon, \widetilde{\Phi}^{-1})$ is a chart and if it belongs to $\mathcal{A}$, then $\Phi = \widetilde{\Phi}$ and the proof is concluded.

If not, then given any chart $(U, \varphi) \in \mathcal{A}$, the homeomorphism

$$\varphi \circ \widetilde{\Phi} : \widetilde{\Phi}^{-1}(U \cap U_\varepsilon) \twoheadrightarrow \varphi(U \cap U_\varepsilon)$$

is orientation reversing. Let $R : B(0, 1+\varepsilon) \twoheadrightarrow B(0, 1+\varepsilon)$ denote reflection w. r. t. the n-th coordinate. We claim that then $\Phi = \widetilde{\Phi} \circ R$ is the required mapping. Indeed, the pair $(U_\varepsilon, \Phi^{-1})$ is also a chart and for any chart $(U, \varphi) \in \mathcal{A}$, the homeomorphism

$$(\varphi \circ \widetilde{\Phi}) \circ R : \Phi^{-1}(U \cap U_\varepsilon) \twoheadrightarrow \varphi(U \cap U_\varepsilon)$$

is a composition of two orientation reversing homeomorphisms, which yields an orientation preserving one. This can be made precise by computing the degree of a composite mapping using the multiplication theorem [32, Theorem 2.10]. ☐

Remark A.16. If $\Phi : \bar{\mathbb{B}}^n \to \mathcal{M}^n$ is a parametrization of the closed topological ball $D \subset \mathcal{M}^n$ which satisfies (A.16), we say that Φ is orientation preserving. It then follows directly from Definition A.14 that if $f : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ is orientation preserving, then

$$\Phi \circ f \circ \Phi^{-1} : \mathbb{B}^n \twoheadrightarrow \mathbb{B}^n$$

is orientation preserving.

Remark A.17. Recall another definition from Chapter 3, that of a flat bi-Lipschitz closed ball, where the $\widetilde{\Phi}$ in the statement of Lemma A.15 is assumed to be bi-Lipschitz. If $\mathcal{M}^n$ is an n-dimensional connected Lipschitz manifold with a maximal oriented atlas $\mathcal{A}$ and D, $D \subset \mathcal{M}^n$, is a flat bi-Lipschitz closed ball, then there is $\varepsilon > 0$ and a bi-Lipschitz homeomorphism $\Phi : B(0, 1+\varepsilon) \twoheadrightarrow U_\varepsilon$ such that $\Phi(\bar{\mathbb{B}}^n) = D$ and (A.16) holds.

Indeed, the proof is the same as that of Lemma A.15 as the definition of a flat bi-Lipschitz closed ball yields a bi-Lipschitz homoemorphism $\widetilde{\Phi}$ and the reflection homeomorphism R is also bi-Lipschitz.

A.4.2 Construction of H in the proof of Theorems 1.1, 1.2 and 1.3

Lemma A.18. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented manifold of class C^k, $k \in \mathbb{N} \cup \{\infty\}$. Suppose that $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of* *pairwise disjoint C^k-diffeomorphic closed balls. Moreover, assume that $K \subset \mathcal{M}^n$ is also a C^k-diffeomorphic closed ball which is disjoint from D_i, D_i' for every $i = 1, \dots, \ell$. Then there is an orientation preserving C^k-diffeomorphism $H : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that*

$$H(D_i) \subset \mathring{K} \quad \text{and} \quad H(D_i') \subset \mathring{K} \quad \text{for every } i = 1, \dots, \ell. \tag{A.17}$$

Lemma A.19. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented Lipschitz manifold. Suppose $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of pairwise disjoint flat bi-Lipschitz closed balls. Moreover, assume that $K \subset \mathcal{M}^n$ is also a flat bi-Lipschitz closed ball which is disjoint from D_i, D_i' for every $i = 1, \dots, \ell$. Then there is an orientation preserving bi-Lipschitz homeomorphism $H : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that (A.17) holds.*

Lemma A.20. *Let $\mathcal{M}^n$ be an n-dimensional connected and oriented topological manifold. Suppose $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$, $D_i, D_i' \subset \mathcal{M}^n$, are two families of pairwise disjoint flat topological closed balls. Moreover, assume that $K \subset \mathcal{M}^n$ is also a flat topological closed ball which is disjoint from D_i, D_i' for every $i = 1, \dots, \ell$. Then there is an orientation preserving homeomorphism $H : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that (A.17) holds.*

We prove only Lemma A.18, as the proofs of Lemma A.19 and A.20 are nearly the same. The only difference is, of course, the regularity of the manifold and the mapping in question. In the proof of Lemma A.18, we do not use the C^k-smoothness of the manifold for any other purpose than to guarantee that C^k-diffeomorphisms can be defined. Therefore, this proof can be easily adapted to the Lipschitz and purely topological setting.

The proof is standard and easy-to-believe but requires a few details. Let us therefore recall its sketch in the proof of Theorem 1.1. In principle, construction of H follows from Lemma 2.17. Let us for now assume that the family $\{D_i\}_{i=1}^{\ell}$ is disjoint from the family $\{D_i'\}_{i=1}^{\ell}$. For each ball D_i (and D_i') we create a finite chain of connected coordinate systems that connect D_i (and D_i') to K in a way that consecutive systems have nonempty overlapping. Moreover, we assume that the first coordinate system in the chain contains a neighborhood of D_i (D_i'). Then we construct a diffeomorphism H_i (and H_i') as a composition of diffeomorphisms defined in the local coordinate systems that move the ball D_i from one coordinate system to the next one. We can guarantee that on the set where H_i or H_i' differs from the identity, all other diffeomorphisms H_j and H_j' are equal identity. Finally, we define H as a composition of all diffeomorphisms H_i and H_i'. If $D_i \cap D_j'$ for some $i, j \in \{1, \dots, \ell\}$, it is not possible to guarantee that if $H_i \neq \mathrm{id}$, then all H_j' for $j = 1, \dots, \ell$ are equal identity. Hence, one has to slightly modify the construction of $H_i = H_j'$ so that the sum $D_i \cup D_j'$ is mapped into $\mathring{K}$ without spoiling the construction of other diffeomorphisms.

Proof of Lemma A.18. Step 1. We firstly explain the construction under the additional assumption that the family $\{D_i\}_{i=1}^{\ell}$ is disjoint from the family $\{D_i'\}_{i=1}^{\ell}$. It will be convenient for now to denote D_i' as D_i for $i = \ell+1, \dots, 2\ell$ since now there is no difference whatsoever between the family of $\{D_i\}_{i=1}^{\ell}$ and $\{D_i'\}_{i=1}^{\ell}$.

As D_i for every $i = 1, \dots, 2\ell$ is a C^k-diffeomorphic ball, we can find $\varepsilon > 0$ and C^k-diffeomorphisms $\Phi_i : B(0, 1+\varepsilon) \to \mathcal{M}^n$ so that $\Phi_i(\overline{\mathbb{B}}^n) = D_i$. Let $V_i := \Phi_i(B(0, 1+\varepsilon))$. By choosing smaller ε, if necessary, we can guarantee that V_i are pairwise disjoint (up to closures).

Since a connected manifold is path-connected, for every $i = 1, \dots, 2\ell$, we can find paths γ_i connecting a point from $\mathring{D}_i$ to a point in $\mathring{K}$. Since D_i are pairwise disjoint, we can choose the paths so that for every $i \neq j$,

$$\gamma_i \cap \gamma_j = \varnothing \quad \text{and} \quad \gamma_i \cap V_j = \varnothing.$$

Moreover, it follows from compactness that for each $i = 1, \dots, 2\ell$, we can find a finite family of charts $\{(U^i_m, \varphi^i_m)\}_{m\in J(i)}$ which covers γ_i so that for $i \neq j$, the family $\{U^i_m\}_m$ is disjoint from $\{U^j_m\}_m$. We assume that U^i_1 contains the endpoint of γ_i which belongs to $\mathring{D}_i$ and $U^i_{J(i)}$ contains the point which belongs to $\mathring{K}$. Moreover, we can assume that $U^i_{J(i)}$ is contained in $\mathring{K}$ and that for every $m \geq 2$, U^i_m intersects only U^i_{m-1} and U^i_{m+1}.

We now construct for each $i = 1, \dots, 2\ell$, an orientation preserving C^k-diffeomorphism $H_i : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ such that $H_i(D_i) \subset \mathring{K}'$ and

$$H_i = \mathrm{id} \ \text{ in } \mathcal{M}^n \setminus \big(\bigcup\nolimits_{m=1}^{J(i)} U^i_m \cup V_i\big).$$

Then $H := H_{2\ell} \circ \ldots \circ H_1$ is a well-defined, orientation preserving C^k-diffeomorphism of $\mathcal{M}^n$. Note that due to the choices we have made, H_i differs from identity only on the set on which all H_j for $j \neq i$ are equal identity. Consequently, if $j < i$, then H_j does not move D_i and if $j > i$, then H_j does not move $H_i(D_i)$. Hence H satisfies $H(D_i) \subset \mathring{K}$ for every $i = 1, \dots, 2\ell$.

Let us fix $i \in \{1, \dots, 2\ell\}$. It follows from Lemma A.15 that (V_i, Φ^{-1}) is a chart and that we can assume that it belongs to the maximal oriented atlas on $\mathcal{M}^n$. Therefore, it is possible to find an open ball $B(p_o, r_o) \Subset B(0, 1+\varepsilon)$ satisfying

$$\overline{\mathbb{B}}^n \cap \overline{B}(p_o, r_o) = \varnothing \quad \text{and} \quad \Phi_i(B(p_o, r_o)) \Subset U^i_1.$$

By Lemma 2.17, we find a C^∞-diffeomorphism $\Theta_o : B(0, 1+\varepsilon) \twoheadrightarrow B(0, 1+\varepsilon)$, $\Theta_o = \mathrm{id}$ near $\partial B(0, 1+\varepsilon)$, such that $\Theta_o(\overline{\mathbb{B}}^n) \subset B(p_o, r_o)$. We then set

$$\widetilde{\Psi}_o := \Phi_i \circ \Theta_o \circ \Phi_i^{-1} : V_i \twoheadrightarrow V_i.$$

It is an orientation preserving C^k-diffeomorphism, which equals identity near ∂V_i so it can be extended by identity to a diffeomorphism $\Psi_o : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$. By construction, $\Psi_o(D_i) \subset U^i_1$ and $\Psi_o = \mathrm{id}$ outside V_i.

Similarly, we construct for $m = 1, \dots, J(i)-1$ orientation preserving C^k-diffeomorphisms $\Psi_m : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$, $\Psi_m = \mathrm{id}$ in $\mathcal{M}^n \setminus U^i_m$, such that

$$\Psi_m \circ \ldots \circ \Psi_1 \circ \Psi_o(D_i) \subset U^i_{m+1}.$$

Set

$$H_i := \Psi_{J(i)-1} \circ \ldots \circ \Psi_1 \circ \Psi_o.$$

It is an orientation preserving C^k-diffeomorphism of $\mathcal{M}^n$. Since $U^i_{J(i)}$ is contained in $\mathring{K}$,

$$H_i(D_i) \subset \mathring{K}.$$

This finishes the construction of H_i and the construction of H, as explained earlier.

Step 2. Now, we address the general case, when the family $\{D_i\}_{i=1}^\ell$ is not disjoint from the family $\{D'_i\}_{i=1}^\ell$. We change back the notation and for $i = 1, \dots, \ell$ we will write V'_i, γ'_i and H'_i instead of $V_{i+\ell}$, $\gamma_{i+\ell}$ and H_{i+1}.

If for some $i, j \in \{1, \dots, \ell\}$ we have $D_i \subset D'_j$ (or $D'_i \subset D_j$), we construct H_i and H'_i as before with the exception that we simply set $H_i := H'_j$ (or $H'_j := H_i$).

The subtlety lies in the case when

$$D_i \cap D_j \neq \varnothing \quad \text{but} \quad D_i \subsetneq D_j \quad \text{nor} \quad D'_j \subsetneq D_i. \tag{A.18}$$

Since there are finitely many sets, it suffices to understand the necessary modification when (A.18) is true for just one i and one j. Without loss of generality, we assume that (A.18) holds for $i = j = 1$ and that the sets D_i, D_i' are pairwise disjoint for $i = 2, \ldots, \ell$.

We choose γ_1 by connecting a point from $\mathring{D}_1 \setminus D_1'$ with a point from $\mathring{K}$ so that γ_1 does not intersect V_1' nor V_i, V_i' for $i = 2, \ldots, \ell$. We do not choose γ_1' and we do not construct H_1' but slightly modify H_1 from Step 1 to take care of the sum $D_1 \cup D_1'$.

As in Step 1, we can find a chain of overlapping open sets $V_1, V_1', U_1^1, \ldots, U_{J(1)}^1$ which cover D_1, D_1' and γ_1 and are disjoint (up to closures) from V_i, V_i', γ_i, γ_i' for all $i = 2, \ldots, \ell$. We assume that $U_{J(1)}^1 \subset \mathring{K}$. More precisely, we have a chain of overlapping charts

$$(V_1, \Phi_1^{-1}),\ (V_1', (\Phi')_1^{-1}),\ (U_1^1, \varphi_1), \ldots, (U_{J(1)}^1, \varphi_{J(1)}).$$

This is in fact the same situation which we encountered in the construction of G_1 in Step 1, the only difference being a somewhat special character of the chart $(V_1', (\Phi')_1^{-1})$. The sum $D_1 \cup D_1'$ is covered with $V_1 \cup V_2$, a sum of two images of an open ball under C^k-diffeomorphisms. We have no parametrization from $\mathbb{B}^n$ onto $V_1 \cup V_2$, which makes it difficult to map $D_1 \cup D_1'$ into $\mathring{K}$ as we did in Step 1. To overcome this technical difficulty, we construct a C^k-diffeomorphism Ψ_o, which maps D_1 into V_1' so that $\Psi_o(D_1 \cup D_1') \subset V_1'$. Even though $\Psi_o(D_1 \cup D_1')$ might not be a C^k-diffeomorphic closed ball, it is contained in one. Therefore, we can perform a similar construction as in Step 1, which 'pushes' the C^k-diffeomorphic closed ball containing $\Psi_o(D_1 \cup D_1')$ into the domain of the next coordinate system until it 'reaches' $\mathring{K}$.

We construct $\Psi_o : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$ exactly like in Step 1 so that $\Psi_o = \mathrm{id}$ outside V_1 and $\Psi_o(D_1) \subset V_1'$. Therefore, $\Psi_o(D_1 \cup D_1') \subset V_1'$. As in Step 1, we chose charts so that U_1^1 does not intersect V_1 and thus $\Psi_o(U_1^1) = U_1^1$.

The set $(\Psi_o \circ \Phi_1')^{-1}(\Psi_o(D_1 \cup D_1'))$ is a compact subset of $B(0, 1+\varepsilon)$ and therefore it is contained in a ball $B(0, 1+\eta)$ for some $\eta \in (0, \varepsilon)$. We can find a ball $B(\zeta, r) \Subset B(0, 1+\varepsilon)$ such that

$$\overline{B}(\zeta, r) \cap \overline{B}(0, 1+\eta) = \varnothing \quad \text{and} \quad \Psi_o \circ \Phi_1'(B(\zeta, r)) \Subset \Psi_o(U_1^1) = U_1^1.$$

Then by Lemma 2.17, we construct $\Theta' : B(0, 1+\varepsilon) \twoheadrightarrow B(0, 1+\varepsilon)$ such that $\Theta' = \mathrm{id}$ near $\partial B(0, 1+\varepsilon)$ and such that $\Theta'(\overline{B}(0, 1+\eta)) \subset B(\zeta, r)$. We then set

$$\widetilde{\Psi}_o' := \Psi_o \circ \Phi_1' \circ \Theta' \circ (\Phi_1')^{-1} \circ \Psi_o^{-1} : \Psi_o(V_1') \twoheadrightarrow \Psi_o(V_1').$$

It is an orientation preserving C^k-diffeomorphism, which can be extended by identity to a diffeomorphism $\Psi_o' : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$. By construction, $\Psi_o' = \mathrm{id}$ in $\mathcal{M}^n \setminus \Psi_o(V_1')$ and

$$\Psi_o'(\Psi_o(D_1 \cup D_1')) \subset U_1^1.$$

In the same manner, we construct for $m = 1, \ldots, J(1)-1$ orientation preserving C^k-diffeomorphisms $\Psi_m : \mathcal{M}^n \twoheadrightarrow \mathcal{M}^n$, $\Psi_m = \mathrm{id}$ in $\mathcal{M}^n \setminus U_m^1$, such that

$$\Psi_m \circ \ldots \circ \Psi_1 \circ \Psi_o(D_1 \cup D_1') \subset U_{m+1}^1.$$

Set

$$H_1 := \Psi_{J(1)-1} \circ \ldots \circ \Psi_1 \circ \Psi_o.$$

It is an orientation preserving C^k-diffeomorphism of $\mathcal{M}^n$. Since $U_{J(1)}^1$ is contained in $\mathring{K}$,

$$H_1(D_i \cup D_i') \subset \mathring{K}.$$

This finishes the construction of H_1.

For $i = 2, \ldots, \ell$ we construct H_i, H_i' exactly like in Step 1. We then set

$$H := H_\ell' \circ H_\ell \circ \ldots \circ H_2' \circ H_2 \circ H_1,$$

which is the required mapping. $\square$

Afterword

While preparing this dissertation I stumbled across a note written by William Thurston about *proof and progress in mathematics* [92]. He writes there that the job of a mathematician is to *increase people's understanding of mathematics*. I often found myself wondering if I do this job well—if a proof that I have just meticulously written down does not do more to obscure than to elucidate the reasoning behind it. I guess that the rigidity of proofs takes away some of the fun and beauty of the ideas and theorems that I feel look so much better in my head than on paper.

To conclude, on a perhaps slightly philosophical note, I wanted to convey some of it—the fun (and the pain) of working on problems presented in this thesis.

It is a result of an assignment given by dr hab. Piotr Wasylczyk during his short course on presentations in which we were asked to write a short fairy tale about the topic of our dissertation.

* * *

Once upon a time in a land not that far away two sisters decided to make Christmas cookies. It was a cozy December afternoon when they took all their cookie cutters out of the kitchen drawer and prepared all the well known ingredients: flour, butter, honey and a bit of cream. When the dough was already prepared, Lucy, the younger one, quickly reached out for her favourite (though irksome to use) cutter, the reindeer. Mary, twelve years older and at least that many times lazier, took a square-shaped cutter and set to work as well.

- Is it your favourite cutter? - Lucy asked.
- Yes, why? - Mary replied.
- It seems a bit... boring.
- It is not boring at all! You see, I like squares best because of what I study.
- You don't study squares, you study analysis. - Lucy proudly opposed (it took her some time to memorize the name of her sister's favourite branch of mathematics).
- That's true! I don't study squares, I study mappings, that is, all the things that can happen to a square. Look - Mary said and smoothed the corners of one of the squares so that it looked more like a disk - I deformed this square into a disk!

But by that time Lucy paid little attention since she desperately tried to glue the hind legs, which fell off her unlucky reindeer, to the rest of its cookie body.

- It seems that your reindeer was attacked by a discontinuous mapping... They can be really dangerous, they tear squares apart so that pieces that used to belong together, do not any more.

Lucy, not yet recovered from her reindeer's misfortune, felt convinced that she wanted to have nothing to do with these discontinuous monstrosities.

- Do you have your favourite mmm… mmaaa -
- Mapping - Mary finished her sister's question - Yes, I do. My favourite ones are called homeomorphisms and I'm sure you'd like them. They are continuous, of course - she quickly added as she sensed a doubt in Lucy's bright eyes. - And they also have this nice property that whenever I deform a square, let's say into a disk, I can easily turn the disk back into the square! Such mappings are called invertible.
- That seems nice...

Lucy absent-mindedly agreed although she was still wondering why one would like to change something and then change it back..? Then she recalled the time when she secretly curled her hair and wanted to straighten it before her mum came home and so her sympathy for homeomorphisms grew stronger.

- But Mary? What if something goes wrong on the way back? Like the horrible thing you said earlier, that something will be torn apart?
- Good point, Lucy! That, fortunately, cannot happen in case of homeomorphisms. They are continuous, invertible and the 'way back', as you nicely called it, has to be continuous as well. I hope that now you will find homeomorphisms quite acceptable?

They smiled at each other and continued to cut the dough into all kinds of shapes. Mary kept thinking about all these difficulties she has run into while trying to construct a quite peculiar homeomorphism recently. She explained to Lucy how things work on the plane, that is in the two-dimensional world of the kitchen table. In higher dimensions (and we really can have more than just the three that we see) it does not get easier... Lucy, oblivious to her sister's troubles, kept herself happily busy with her beloved reindeer. Only later that night, when all the cookies had already been baked and their little house smelled of nothing but honey and cinnamon, Lucy approached Mary and said in a quiet voice

- Mary, I don't understand. Is your analysis like baking cookies but with weird names?

Mary held her closer and hesitated before giving an answer.

- You see, my dear, there is a difference. When you make cookies, you really should have your eyes wide open, shouldn't you? And when you deal with homeomorphisms, you sometimes have to close them. There is no other way to see how to cut a square into thousands or millions of little squares and then how to deform those tiny ones and shift them around inside the bigger one - with so much care and attention that nothing gets torn apart. There is no other way to see it unless you see it in your head.

Having said that, Mary kissed her little sister and bid her goodnight.

That night Lucy dreamt of becoming a baker and indeed she did. It was then Mary's turn to wonder at the fancy cake names in Lucy's bakery. She came by the store the more often, the more troubles she had with her homeomorphisms. And Lucy knew well enough that mathematically tired look in her sister's eyes—the one she got from closing them and imagining all sorts of homeomorphic horrors. On such occasions she gave Mary her favourite Portuguese pastry, *pasteis de nata*, and watched her eyes grow brighter and less weary.

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