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A Unified Approach to Opetopic Algebra

PhD dissertation

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I hereby declare that this dissertation is my own work.

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A Unified Approach to Opetopic Algebra

We develop an approach to opetopic sets based on algebra, that is monoids in monoidal categories. These categories naturally assemble into bifibrations, as do their monoidal structures. Consequently, they form monoidal bifibrations. To expedite our constructions, we adapt several notions from the theory of ordinary categories to the relative context, that is the 2-category $\mathbf{Cat}/\mathcal{S}$. These include universal properties, adjoint functors, the theory of monads (including Kleisli and Eilenberg-Moore objects, monadicity, and distributive laws), and exponential objects.

The specific structures we work with are signatures. These are sets of function symbols, with multiple (typed) inputs and a single output. Signatures have natural monoidal structures given by the formation of formal composites, matching outputs to inputs. Several different categories are at play, differing in morphisms and possible extra structure on the function symbols.

The conceptual core of our approach is the notion of a distributivity structure. It formalizes the idea that some structures, such as trees, can have two independent types of inputs, for example leaves and nodes. Following this intuition, we construct monoidal signatures, which have two different monoidal structures, and a distributivity structure between them.

Trees can be grafted into leaves of other trees, or substituted for a single node. These operations commute with each other. This observation forms the basis of the construction of the web monoid. This functor, mapping monoids to monoids, is the algebraic device which allows us to construct higher dimensional opetopic cells from lower dimensional ones, starting with points and arrows. As such it is instrumental in our definition of opetopic sets. It is abstractly characterized by a commutativity condition, as that for grafting and substitution, which is stated using a distributivity structure.

We prove our approach generalizes, or is equivalent to other algebraic approaches, such as those of Hermida, Makkai and Power, and Kock, Joyal, Batanin and Mascari. The original approach of Baez and Dolan is also of this form, and is shown to be incorrect: it is inequivalent to those mentioned, and inconsistent with its own pictorial intuition.

Finally, we explain the relationship between the structures we use in our work and equational logic.

2010 AMS Mathematical Subject Classification: 18D10, 18D30, 18D50, 18C05, 18C10, 18C15, 18C20, 03G30

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Jednolite Ujęcie Algebry Opetopowej

W pracy rozwijam algebraiczne podejście do zbiorów opetopowych – oparte na monoidach w kategoriach monoidalnych. Kategorie te pochodzą z naturalnych birozwłóknień, tak jak ich struktury monoidalne. Aby ułatwić nasze konstrukcje, dostosowuję kilka pojęć teorii zwykłych kategorii do kontekstu relatywnego, tzn. 2-kategorii $\mathbf{Cat}/\mathcal{S}$. Wśród nich znajdują się własności uniwersalne, funktory sprzężone, teoria monad (obiekty Kleisliego i Eilenberga-Moore'a, monadyczność i prawa dystrybutywności) i obiekty wykładnicze.

Konkretne struktury, których używam, to sygnatury. Są to zbiory symboli funkcyjnch z kilkoma (otypowanymi) wejściami, i jednym wyjściem. Sygnatury posiadają naturalne struktury monoidalne polegające na tworzeniu formalnych złożeń symboli, dopasowując wyjścia do wejść. Istnieje kilka kategorii sygnatur, różniących się morfizmami i dodatkową strukturą na symbolach.

Zasadniczą ideą tego podejścia jest pojęcie struktury dystrybutywności. Formalizuje ona intuicję, że niektóre struktury, takie jak drzewa, mogą mieć dwa niezależnie rodzaje wejść – na przykład liście i węzły. Podążając za tą intuicją konstruujemy sygnatury monoidalne, które mają dwie struktury monoidalne wraz ze strukturą dystrybutywności między nimi.

Drzewa można zszywać wzdłuż liści i korzeni, ale można również wstawić całe drzewo za jeden węzeł. Te operacje są przemienne. Ta obserwacja jest podstawą konstrukcji *web monoidu*. Ten funktor, z monoidów w monoidy, jest algebraicznym urządzeniem, które pozwala nam skonstruować wyżej wymiarowe opetopowe komórki z niżej wymiarowych, zaczynając od punktów i strzałek. Jest to podstawa naszej definicji zbiorów opetopowych. Jest on abstrakcyjnie zcharakteryzowany przez warunek przemienności, podobnym do tego dla zszywania i podstawiania dla drzew, który wyrażony jest za pomocą struktury dystrybutywności.

Dowodzimy, że nasze podejście uogólnia, lub jest równoważne z innymi podejściami, takimi jak podejście Hermidy, Makkaia i Powera, oraz Kocka, Joyala, Batanina i Mascariego. Oryginalne podejście Baeza i Dolana też jest tej postaci, i wykazujemy, że jest niepoprawne: nie jest równoważne z powyższymi i jest sprzeczne z własną rysunkową intuicją.

Na koniec wyjaśniam związek między strukturami użytymi w pracy i logiką równościową.

Klasyfikacja Tematyczna AMS 2010: 18D10, 18D30, 18D50, 18C05, 18C10, 18C15, 18C20, 03G30

Słowa kluczowe: Kategoria Monoidalna, Operad, Opetop, Rozwłóknienie, Zbiór Opetopowy

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Introduction

Motivation

Higher Categories In Category Theory

A deep understanding of category theory rests on a simple foundation. It is the fact that the category of categories[1], **Cat**, is in fact a 2-category. The mere fact that $\mathbf{Cat}(-,-)$ is a 2-functor encodes all of the algebraic properties of natural transformations that are usually painstakingly proven independently of each other in the first few lectures of a basic course in category theory.

Algebraic constructions in **Cat** paint the broad strokes of the landscape of modern mathematics. Adjunctions provide universal constructions – a fundamental concept in all of mathematics. Monoids[2] define monoidal categories, whose monoidal products play a central role in many parts of algebra and geometry. Finally, monads express an intrinsically categorical notion of equipping objects with extra structure.

Universal constructions in **Cat** (and related 2-categories) are even more sweeping. The assignment of categories of algebras to monads, presheaves to categories, the localization of categories, and the consequent Kan extensions are all examples of operations without which modern mathematics would make little sense.

In short, understanding category theory starts with a 2-category. This pattern repeats itself indefinitely: to understand the mentioned universal constructions in **Cat** properly, one needs to work with 3-categories. In general, the understanding of n-categories requires knowledge of $(n+1)$-categories. The internal consistency of category theory demands the development of higher categories.

[1] We have chosen to ignore the eternal struggle against smallness conditions in the introduction. Someone should finally propose a serious solution to the semantic paradoxes of naive set theory. It would not only be a profound philosophical result, but would also provide annoyance-free foundations of mathematics. In any case, I do not look forward to using ascending chains of Grothendieck universes.

[2] More correctly: pseudomonoids.

Higher Categories In Mathematics

Homological algebra and homotopy theory have left a dominating imprint on the mathematics of the past century. It is difficult to find a structure that is not amenable to homotopical methods. This is both surprising and deep. It suggests an unexpected insight into the very notion of a mathematical structure.

Nothing in the definition of rings, modules, and other classical algebraic structures suggests their susceptibility to homological algebra. And yet essentially every structure, not only in algebra, has a homotopical interpretation and classification. From smooth structures on manifolds, through extensions of Lie algebras and orientations of sphere bundles, to group representations, abelian extensions of number fields, and the Weil conjectures, nothing escapes the reach of homotopy theory. Even the Riemann hypothesis would be solved by the existence of a suitable cohomology theory over $\mathbb{F}_1$, the mystical field with one element [D05, Ma95]. These classifications are effective, allowing proofs of highly nontrivial theorems, which without these methods seem completely magical. A trivial application of Stokes' theorem (the coincidence of the homological boundary with the geometrical one) can be equivalent to an impossibly difficult direct calculation.

Higher categories hold a promise in explaining these phenomena in a uniform way. The unexpected insight mentioned above is this: mathematical structures naturally form n-categories, not just categories, and therefore so do all of their invariants. There are several hints in this direction which we will now discuss.

Natural Categorification

The most interesting spaces are the members of some naturally defined class, such as manifolds and CW-complexes, but *spaces of things*, that is moduli spaces. We will adopt a broad view of what a moduli space can be. The space of curves studied in the in the brachistochrone problem is the moduli space of smooth curves in a plane with an action of gravity included. This example shows that calculus on moduli spaces can be a highly potent tool. Similarly, enumerative problems in linear algebra can be studied by means of intersection theory on the Grassmannian $G_k(\mathbb{R}^n)$, the space of all linear subspaces in $\mathbb{R}^n$. This is again a form of (homological!) calculus, since $G_k(\mathbb{R}^n)$ is a smooth manifold. Further examples lead to the theory of characteristic classes and deformation theory (the study of formal neighborhoods in moduli spaces).

Most of these examples have a common theme: the points of a moduli space naturally form a category, and not a set. Thus their space should be a category as well, and indeed the moduli spaces studied in algebraic geometry are stacks, and not ordinary spaces. Stacks form a 2-category. In trying to construct an invariant of an object of a category (the space of similar such objects), we have been led to a

2-category of all possible such invariants. At this point it would be foolish to study individual stacks in isolation. A complete understanding of stacks includes their place in the 3-category of 2-categories of stack-like objects. In this perspective, the idea of studying families instead of individuals is unmistakably categorical.

The categorical demands of geometry are the same as those of category theory. Mathematics requires consistent application of its principles, and if geometric structures form categories, then their understanding necessarily involves higher categories. Categorification is not an option, but a necessity. This process is completely natural: most algebraic geometers have no idea of what a 2-category is, and yet the study of the moduli space of curves began with Riemann, well before the concept of a 1-category was even formulated.

We have described one example of natural categorification: the serious study of geometry leads to higher categorical geometrical structures. Another, simpler example is given in the next subsection. The concept of natural categorification should be contrasted with categorification of the ordinary sort. The natural numbers can be categorified to the category **FinSet**, of finite sets and functions. But this does not happen automatically, and the choice of morphisms in **FinSet** – whether they are all functions, or just the bijections – is left undetermined. Particular circumstances dictate differing solutions in this case.

Homotopy Theory as Higher Groupoids

The profusion of homology and cohomology theories in contemporary use arises from the fact that every natural invariant of essentially anything (denoted X) is a linear n-groupoid[3], and the homology groups $H_*(X)$ are simply the sets of equivalence classes of cells in such a groupoid. This stems from a combination of the globular and simplicial Dold-Kan correspondences [GJ99, BH81], [BHS11, 14.8.1]. The chain complexes that define most of these (co)homology theories are equivalent to linear higher groupoids. The simplicial objects used in nonlinear settings should be equivalent to ordinary higher groupoids.

To illustrate this point, and to provide another example of natural categorification, consider and abelian category $\mathcal{A}$. In analyzing it, short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

are of critical importance. According to the correspondences mentioned above, these are the 2-groupoids in $\mathcal{A}$ in which every 1-automorphism is uniquely isomorphic to the identity. That there are plenty different such 2-groupoids for a fixed $C \in \mathcal{A}$ (which represents the objects of the groupoid) should not be surprising. For

[3]More precisely: an n-groupoid internal to the category of abelian groups, or some other abelian category.

example, in the logic of $Ab(Sh(X))$, the categories of sheaves of abelian groups on some topological space X, existence and uniqueness mean something very different than in **Set**.

Trying to understand the 1-category $\mathcal{A}$ we were immediately led to a 3-category of certain 2-groupoids in $\mathcal{A}$. The higher cells correspond to chain homotopies, homotopies of homotopies, and so on. Natural categorification is again at work. Note that exact sequences, like moduli spaces, were discovered long before higher categories.

In this interpretation the derived categories $\mathbf{D}(X)$, and homotopy categories $\mathbf{Ho}(X)$ would correspond to the categories of linear higher groupoids, and plain higher groupoids in X, respectively. Their extra structure (triangulation, enhancement, etc.), which has proven so intractable, would be explained by the fact that they are themselves higher categories of higher groupoids, not just ordinary 1-categories.

In short, abstract homotopy theory should be the study of (internal) higher groupoids and (internal) higher categories associated to these groupoids. The most concrete expression of this suggestion is the homotopy hypothesis. It asserts that the homotopy groupoid functor

$$\Pi\colon \mathbf{Top} \to \omega\mathbf{Gpd},$$

from topological spaces and all higher homotopies to ω-groupoids, is an ω-equivalence, which restricts to an n-equivalence for every $n \in \mathbb{N}$.

The homotopy hypothesis, if true, is staggeringly powerful. Form it follow universal properties of the groupoids $\Pi_{n+k}(S^k)$ which allow almost trivial calculations of the homotopy groups such as $\pi_3(S^2)$ and $\pi_4(S^3)$, not to mention $\pi_i(S^n)$ for $i \leq n$.

Implicit in this picture is the fact that extracting the homotopy n-type of a topological space amounts to a simple truncation of the ω-category **Top**. There is no need to inductively add cells to kill higher homotopy groups. Everything is taken care of by the formalism. The cellular approximation theorem, which states that every map of CW-complexes is homotopic to a cellular map, follows from the mere fact that the truncations of Π to lower dimensions are well defined. Other constructions, such as Postnikov towers, are similarly simplified.

This (speculative) directness and applicability of higher categories cannot be matched by the old formalism, which in this light appears as a clumsy veil.

Space as Algebra in Higher Categories

Our last hint is the emergence of ordinary spaces from the algebra of higher categories. The last example did this for topological spaces: they are simply the higher groupoids. This inseparably included homotopy theory into our notion of spaces. There are other examples, which do not include homotopical data. One such example is the tangle hypothesis [BD95]. It asserts that the n-category of framed n-tangles in $n+k$ dimensions is $(n+k)$-equivalent to the free weak k-tuply monoidal n-category with duals on one object.

This means that the cells of an n-category, which is the solution to a certain specific universal problem, are naturally identified with framed smooth manifolds. Space appears as a natural consequence of algebraic structures (monoidal structures and duality) in higher categories.

This hypothesis has special applications in physics, and has the effect of bringing the so-called defects in quantum field theory [Ka] to center stage. The importance of defects has recently been recognized by the physicists themselves [Mo14]. The defect of a point should, according to the TQFT hypothesis [BD95], completely determine a quantum field theory with no local degrees of freedom.

The Necessity of Weakness and the Problem of Coherence Conditions

Given the transformative nature of the picture of mathematics sketched above, one may wonder why mathematicians have not rushed into this paradise, never to leave again. The flaming sword guarding the gates of heaven is called the problem of coherence conditions, and no good progress on it has been made since, essentially, forever.

The problem arises from the fact that isomorphism, when taken seriously, is a very different concept from equality. Equalities hold, while isomorphisms need to be specified. The details of such specifications, as it happens, matter a lot.

What is easy to construct is the theory of *strict* higher categories, where composition is associative strictly (i.e. we have an equality) at all levels. But even in the construction of the fundamental group $\pi_1(X)$ one faces the problem that the concatenation of parametrized paths is not strictly associative. In that instance, passing to equivalence classes solves the problem. But the general problem remains, and is real: the 4-category of strict 3-groupoids is not equivalent to the 4-category of homotopy 3-types. Thus the homotopy hypothesis is false for strict higher categories.

A parallel problem, but perhaps not as sharply stated, plays itself out in pure category theory. Universal properties define functors only with suitably strong choice principles, allowing us to choose solutions for each instance of the problem. The solutions to iterated universal properties, the sort of which arise in higher categories (e.g. freely adding finite limits to a 1-category), therefore behave analogously to concatenated paths – each choice of parametrization requires further choices, or passing to equivalence classes (the choice of a category, and then for each finite diagram, the choice of its limit). The latter option is not tenable, since it leads to strictness and, as we have seen above, to the loss of structural information.

The appropriate setting for higher categorical mathematics is that of *weak* higher categories. In these structures, composition is not strictly associative, but only up to equivalence. Thus, composition of 1-cells in a weak 2-category is associative up to isomorphism, but composition of 1-cells in a weak 3-category is associative only up to equivalence. These associativity isomorphisms and equivalences satisfy equations and isomorphisms of their own, respectively, and so on: one additional level of conditions appears for each dimension added.

Specifying these coherence conditions is problematic. Even the coherence conditions for a strong monoidal category – which is the same thing as a weak 2-category with a single 0-cell – are usually omitted in non-categorical literature. This is not an option for higher dimensions, since these conditions encode crucial homotopical data, which would otherwise be lost.

The definition of a 1-category can be crammed into a couple of lines. The definition of a weak 2-category takes a page, with functors and natural transformations taking several more. The complete definition of a weak 3-category takes 4 pages [G07], and a definition of a weak 4-category based on associahedra takes a comical and tragic 51 pages [T06], not including functors or natural transformations.

There is simply no way such explicit definitions can ever enter the practice of mainstream mathematics, barring an essential reliance on computers. These coherence conditions, even the unknown ones in 42 dimensions, must be packaged into a compact and practical definition. This is the challenge of higher category theory.

The Dialectic Solution: Opetopes and Opetopic Sets

There are many proposed solutions to the problem of coherence conditions in the literature, and as of today none of them are in working order. We will describe, in the author's opinion, the most elegant and promising solution to this problem. It is important to keep in mind that this solution is speculative at this point in time, and not guaranteed to work out. There are several serious problems with the published and unpublished proposals in this style.

The central idea is to keep the coherence conditions implicit, and define composition by a universal property. This property should be flexible enough to allow a dialectic approach to coherence conditions: every condition can be derived as needed, on the spot. Coherence questions are answered as they arise, and no a priori list of conditions needs to be maintained. Everything is done through the universal property. The process of checking coherence conditions has the form of a game, and the universal property states that we have a winning strategy. Hence we have called this solution "dialectic".

Thus we are led to wonder how to define the composite of $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$. It should be some 1-cell, which we will predictably call $g \circ f$. We will by no means require that this arrow is unique. We will only specify what universal property it should have. The proposals which have appeared in the literature usually have the following form: in every possible context in which $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ is present, we can exchange it for $\cdot \xrightarrow{g \circ f} \cdot$. Thus our game has the following form: the opposing player challenges us with contexts containing our composable arrows, and we must respond with contexts in which these arrows are replaced with our candidate for their composition. The universal property dictates our answers, and maintains their consistency.

We will not solve the problem of what a "possible context" is here. The available proposals: [BD98] and the unpublished work [M04], do not seem to have ironed out all the difficulties with this notion, as it relates to implementing universal properties describing composition. Intuitively speaking, a context in higher category theory is a diagram of cells – since that is the only thing categories are capable of describing. Below we will try to convey some intuition about how the proposal of [M04] is meant to function.

First, note that commutativity in higher dimensions is rather more complicated than in ordinary categories. The commutativity of

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \cdot \end{tikzcd}$$

means that $f = g$, while the commutativity of

$$\begin{tikzcd} \cdot \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \end{tikzcd}$$

means essentially nothing.

Despite this vagueness, let us press on. Our definition should assert the equivalence of this diagram

$$\begin{tikzcd} \cdot \arrow[r, bend left=50, "g \circ f"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \end{tikzcd}$$

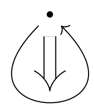
with some diagram of the form

$$
\begin{tikzcd}
& \bullet \arrow[dr, "g"] & \\
\bullet \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=5ex, "\Downarrow \alpha'"] & & \bullet
\end{tikzcd}
$$

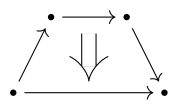
To continue our parallel with games, when presented with the latter diagram, we must find some answer with the shape of the former diagram. We are allowed to change α to α', but h must be kept fixed. The complete rules laid out in [M04] are complicated, and we will not elaborate on them here.

This diagram makes sense in the context of computads, and is the result of simply replacing $\cdot \xrightarrow{g\circ f} \cdot$ with $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$. This is a paradigmatic opetopic shape, and it immediately makes clear that globular cells are not enough to express composition through a universal property of the kind we have been describing.

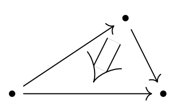
The existence of identities necessitates the inclusion of such curious looking shapes as

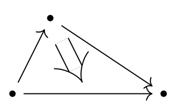


Associativity of such compositions would then correspond to our ability to manipulate the diagram



in the two obvious ways, iteratively composing pairs of cells, and obtaining shapes like





which can then be composed down to a single cell.

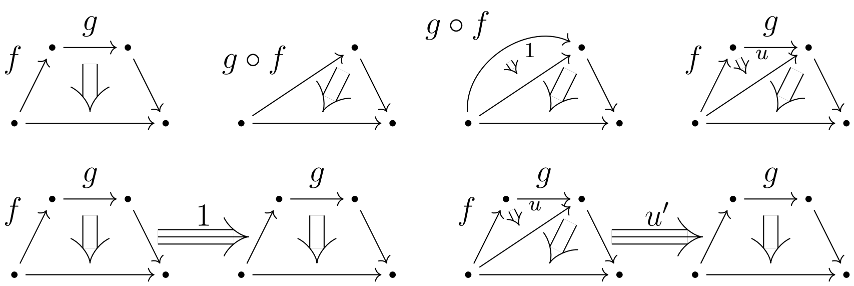
To discuss the combatibility of these two procedures, consider the identity cell

$$
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g \circ f"] \arrow[r, bend right=50, "g \circ f"'] \arrow[r, phantom, "\Downarrow 1"] & \bullet
\end{tikzcd}
$$

It should correspond to a cell

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & \arrow[u, phantom, "\Downarrow u"] & \bullet \end{tikzcd}$$

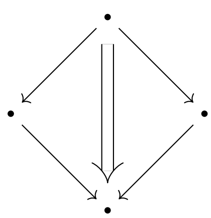
which would in some sense be an equivalence. Cells obtained this way are called universal cells. These are the cells out of which the coherence conditions would be built. For example, we could perform the following two manipulations:



the first of which shows that every cell factors through the universal composite, and the second one shows that the composition of the factorization is the original cell. Similar universal 3-cells (and their factorization properties) obtained from the different ways of composing three arrows would be the coherence conditions ensuring the associativity of composition.

As the reader can see, coherence conditions are derived objects in the opetopic approach, and as long as we can manipulate contexts, as above, they do not need to be mentioned.

Care must be taken with replacing single shapes with composites of cells. One may arrive at pictures of the following sort



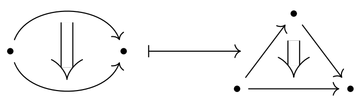
which do not correspond to any shapes at all. More formally, the computads including such cells do not form a presheaf category [MZ08]. The simplest way to stay in the realm of presheaf categories is to restrict our ability to replace cells only to the domains of other cells. We thus declare that opetopes must have as codomains only single cells, and not composites. Opetopes with the “obvious” face maps (there are no degeneracies) form a category, just like simplices. Presheaves on this category are called opetopic sets.

The required definition of weak n-category, would be that of an opetopic set in which every composable diagram has a composite in the sense of the equivalence of all their contexts. Some of these contexts for $g \circ f$ and $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ have been illustrated above. At the moment it is not clear if the proposals of [BD98] and [M04] are suitable for this purpose.

This Work

The Algebraic Approaches

In this thesis we focus our attention on the algebraic approaches to opetopic sets. These are the ones in which the replacement operation



is implemented by a monoid in a monoidal category. This includes the original approach of [BD98], as well as that of [KJBM10], and [HMP02]. Our own approach is also of this general form.

These types of definitions rely on the fact that free monoids in certain monoidal categories (such as signatures, polynomial functors, etc.) consist of tree-like structures, with composition being grafting families trees by their roots into the leaves of a single tree.

In such a situation there is another composition operation on the free monoid: replacing a node in a tree with a whole tree, whose leaves are matched to the children of the node. This is essentially what happens in the “operad of operads” defined by Baez and Dolan [BD98], and what is explicitly sought in the definition of the “multicategory of function replacement” of Hermida, Makkai, and Power [HMP02]. The same thing happens in [KJBM10] and our approach, the web monoid.

Defining this operation is usually very difficult. The complete proof of the central theorem 14 in [BD98], provided for the first time in this thesis (theorem 6.2.7), is rather complicated, and draws heavily on all the results we establish up to that point. The multicategory of function replacement likewise takes an entire part of the [HMP02] paper, and is filled with interesting ideas, many of which we adopt in our own arguments. The polynomial functor approach [KJBM10] uses the results of the book [Le04].

The web monoid is no exception. The proof of the abstract three tensors theorem 4.1.1, takes about 12 pages. Some more setup is necessary to see that it can be applied to our concrete situation. This is the main difference between our approach and all the others. We formalize the idea of an additional operation on the free monoid, and explicitly use an additional monoidal structure to turn it into a monoid. This allows us to abstractly characterize the common features of all the specific structures used in the algebraic approaches.

This forms the basis of our comparison theorems, presented in chapter 6. The abstract concepts needed to define the web monoid result in a monoidal property of the natural action of the category in which the web monoid is defined (theorem 3.4.1). Surprisingly, all three of our comparisons follow from this (and some details specific to each approach, of course). Our abstract approach bears fruit this way: without the concepts we introduce to define the web monoid, theorem 3.4.1 could not be proven, and the comparisons would remain opaque.

Definitions of Opetopic Sets and Their Uniqueness

The quickest (and least informative) definition of opetopic sets is that they are the many-to-one computads [HMZ08]. This means they are levelwise free strict n-categories, whose cells always have exactly one generator in their codomain, and an arbitrary cell (a formal composite of generators, including the empty composite) in their domain. It is a nontrivial fact that this category is a presheaf category. This exponent category is by definition the category of opetopes.

That this definition is correct and unique is part of an ongoing project, to which this thesis contributes. There are plenty of definitions of opetopic sets, and many are known to be equivalent. Adding to this knowledge, we prove the following things: the informal pictures in [BD98], which started the whole subject, and which can be formalized into mathematical structures (as is done in [KJBM10]), do not, unfortunately, correspond to the structures presented in that paper[4]. They do, however, correspond to our definition, which we prove to be equivalent to that of Hermida, Makkai, and Power [HMP02].

That definition, in turn, is known to be equivalent to the computad definition [HMZ08]. From there, many other equivalences are available, which we will not review.

We also provide a construction of the category of opetopes, different and less combinatorially involved, than the one presented in [HMP02]. Our construction uses Artin gluing and standard category theory. It is not completely explicit, but nonetheless affords us considerable leverage over the category of opetopes.

[4]Along the way we must also prove that the technical problems in [BD98] can be fixed, so that this statement has content.

Together these results suggest that the computad definition, and the ones equivalent to it, are the unique correct formalization of the fundamental ideas of [BD98].

Contents of This Thesis

Each chapter contains an introduction designed to guide the reader through its contents with minimal effort. They are not to be skipped. These individual introductions also tie the chapters into a single narrative from a technical perspective (as opposed to a conceptual one, which is the role of this introduction).

In chapter 1 we adapt category theory to the relative setting, which will be used troughout the text. Fibrations and opfibrations play an important role, but their morphisms are immaterial, and must be discarded in favor of fibered functors. Without this modification none of the following chapters could function.

Next we review (or rather adapt) the construction of free monoids in monoidal fibrations certain nice properties. This is fundamental to all subsequent results, since we will rely on the fact that this construction is the same in all the contexts in which we will work.

At the end of the chapter we define the notion of a distributivity structure. It is required to define the web monoid in chapter 4, and forms the conceptual basis for all our comparison theorems in chapter 6.

In chapter 2 we define the main concrete actors of the whole affair – the various species of signatures[5]. The ordinary signatures and their symmetric counterparts are well known. To this we add signatures with nonstandard amalgamation, a notion suggested by the structures used in [HMP02], which are essentially the symmetric signatures with free symmetric actions (although this is an extremely poor way to think of them), and monoidal signatures with amalgamation.

This latter category is the one which carries an obvious distributivity structure, and allows us to instantiate the abstract web monoid construction, for use in the construction of opetopic sets in chapter 5.

We provide all our signatures with monoidal structures and actions. This is the main technical content of this chapter. We spend the most effort constructing the second monoidal structure on monoidal signatures, which in turn allows the definition of the mentioned distributivity structure.

Monoids for these monoidal structures range from the well known to the obscure. Monoids in ordinary signatures are Lambek's multicategories. Monoids in symmetric signatures are symmetric multicategories, also known as typed symmetric operads. Monoids in signatures with nonstandard amalgamation are the

[5] According to theorem 2.6.2 this is a pun cf. [Jo86].

(one level version of) multicategories with nonstandard amalgamation introduced in [HMP02]. Monoids in monoidal signatures for the first monoidal structure (the vertical structure) are multicategories with nonstandard amalgamation whose objects are the morphisms of another multicategory with nonstandard amalgamation. Monoids for the second, horizontal, structure remain nameless.

The actions of signatures lead to their well known interpretation as polynomial and analytic endofunctors. This identification is monoidal, and identifies the various multicategories as analytic and polynomial monads. We review this equivalence, but refer for the details to [Z10].

At the end of the chapter we review the categorical properties of signatures needed in the other chapters.

In chapter 3 we study the non-categorical properties of signatures needed in the later chapters. We provide tools for working with signatures with nonstandard amalgamation without excessively comparing permutations. This saves a lot of work, since these permutations are often unmanageably complicated.

In the last two sections we discuss an alternative description of the category $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$. It is critical for our later work, specifically the comparison theorems of chapter 6. The constructions of these sections are based on the original construction of monoidal signatures, and provide some insight into their structure.

The end result of this work is theorem 3.4.1 and its corollary 3.4.2, which plays a central role in chapter 6.

In chapter 4 we define and construct the abstract web monoid. The proof of its existence is constructive, and split into a conceptual and technical part, for the reader's convenience. The instantiation of the web monoid in monoidal signatures is obvious at this point, since all the requisite structures and properties are already in place.

We provide an example showing that the complexity of signatures with nonstandard amalgamation is necessary if we are to capture the intuitions of [BD98]. This statement is made particularly sharp in the context of the results of chapter 6.

At the end of the chapter we prove some "combinatorial" theorems (they actually replace the combinatorics present in [HMP02]). They are used to prove that the opetopic sets, defined in the next chapter, form a presheaf category.

In chapter 5 we define and study the opetopic sets. The definition we give is intended to clarify the somewhat digressive style of [BD98].

Using this definition, the theory of Artin gluing, and the combinatorial theorems of chapter 4 we prove that the category of opetopic sets (and related categories of **O**-opetopic sets) form a presheaf category.

This argument, in turn, allows us enough insight into the limits and colimits in opetopic sets to explicitly construct the exponent category. We note some of its basic properties, for use in the comparison theorems, specifically in section 6.3.

In chapter 6 we prove our comparison theorems. This chapter uses essentially all of the material of the previous chapters, and relies on the notion of a distributivity structure, as used in theorem 3.4.1, to make progress.

The end results are as follows: the categories of opetopes and multitopes are isomorphic, as are the categories of opetopic and multitopic sets. The monads used in [KJBM10] to define opetopes can be reproduced by the web monoid in such a way that the pictorial formalism of [KJBM10] applies to our opetopes, and therefore the multitopes.

The original approach of Baez and Dolan is, unfortunately, not equivalent to these. We sketch this negative result after showing that the entire formalism of [BD98] can be rigorously established. This is not obvious, as we point out several technical problems with the original arguments. Our fixes are rather involved and rely, again, on almost all the previous material. The main obstruction standing in the way of equivalence is this: the sets of function symbols of a signature with nonstandard amalgamation, and the associated symmetric signature, are different, even though they define the same polynomial functor. Confusion between these sets led to the problems in the original paper.

In chapter 7 we discuss the relation to logic. We review the equivalence of (untyped) operads, finitary monads, Lawvere theories, and equational theories.

We then show how exactly these equivalences restrict to various subcategories. Many of these subcategories are most naturally defined in one of the mentioned approaches. Thus, operads come in flavors, such as symmetric, or freely symmetric, and we describe the corresponding categories of Lawvere and equational theories. Some of the more obscure comparisons, which require defining new variants of operads, are left to other sources [SZ].

Some of these equivalences are well known, but our uniform approach, especially the use of factorization systems in Lawvere theories, is new. Among the new results is a characterization of the equational theories corresponding to polynomial monads cf. [CJ04]. We also obtain a series of monadicity results, which are mediated by the existence of a certain monoidal monad. This gives a distributivity law, analogous to the combing distributive law used to construct the operad of operads (corollary 6.2.5). This is what gives monadicity. We provide a formula for one inductive step of this law, without proof.

We end the chapter by giving some examples. In particular, the existence of the operad of operads (and its relatives) implies that the associated (multisorted) equational theory has as its models other equational theories.

Some of the content of this thesis has already been published. Most of chapters 2, 3 and 4, as well as section 6.1, was first published in [SZ13]. Chapter 5 and the rest of chapter 6 are the subject of a forthcoming paper. The content of chapter 7 was first published as [SZ].

Notation

Categories, Functors and Fibers

- Concrete categories and 2-categories, such as **Set** or **Cat** will be denoted in boldface font.
- $\mathcal{C}, \mathcal{D}, \ldots$ are abstract categories
- $\mathcal{E}, \mathcal{F}, \ldots$ are abstract domain categories of functors (usually fibrations or bifibrations) $p \colon \mathcal{E} \to \mathcal{S}$
- $\mathcal{S}$ is an abstract base (codomain) category for objects of the slice fibrations and bifibrations (more generally for objects of $\mathbf{Cat}/\mathcal{S}$)
- $\mathcal{E}/O$, where $O \in \mathcal{S}$ is the fiber of the abstract fibration of bifibration (or just functor) $p \colon \mathcal{E} \to \mathcal{S}$.
- If $X \in \mathcal{C}$, then $\mathcal{C}/X$ is the usual slice, as defined in [CWM98]. Thus $(\mathcal{E}/O)/X$ is the slice category of a fiber, not an iterated slice.

Lists

- An abstract index i on some symbol, e.g. a_i should be expanded to its implicit range, which will always be determined by the context. Thus a_i is a shorthand for
$$a_1, a_2, \ldots, a_n,$$
for some $n \in \mathbb{N}$, determined by the object a.
- Double indices (i, j) are ordered lexicographically. Thus $a_{i,j}$ is a shorthand for
$$a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{2,1}, a_{2,2}, \ldots a_{2,n_2}, \ldots, a_{k,n_k},$$
where k and $n_i \in \mathbb{N}$ are determined by a. Note that for every specific i the second index j is allowed to have a different range.

- Longer indices are handled analogously.

Other Notation

- $[n] = \{0, \ldots, n\}$
- $(n] = \{1, \ldots, n\}$. This set carries a natural action of S_n, the n'th symmetric group. This action extends to $[n]$ by leaving 0 fixed.
- If $O \in \mathbf{Set}$, then $O^* = \bigcup_{n\in\mathbb{N}} O^{(n]}$ – finite lists with values in O.
- Similarly $O^\dagger = \bigcup_{n\in\mathbb{N}} O^{[n]}$ – finite lists with values in O, with a chosen first element. We have a canonical isomorphism $O^\dagger = O \times O^*$.

Chapter 1

The Setting

In this chapter we set the stage for all further constructions. Accordingly its contents are rather varied – we must adapt a sizeable part of standard category theory to the relative setting, i.e. to the slice 2-categories $\mathbf{Cat}/\mathcal{S}$. For this reason we assume mastery of (not just mere familiarity with) the material in [CWM98].

The adapted elements include universal properties (including limits/colimits and adjoint functors), the general theory of monads, including the monadicity theorems and distributive laws, as well as relative monoidal categories.

Throughout our work, it will become clear that the property of being a fibration or opfibration (the basics of which are briefly recalled below) is a regularity condition enabling our adaptations to take place, and not a specification of a new fundamental 2-category. The crucial point is to allow morphisms to *not* preserve the prone and supine arrows. This allows the necessary constructions mentioned above to function in $\mathbf{Cat}/\mathcal{S}$, where all our examples live. Very few of our later constructions would work correctly if they were carried out in 2-categories whose 1-cells preserve the prone or supine arrows.

Thus fibrations and opfibrations are nice objects in the categories $\mathbf{Cat}/\mathcal{S}$, but not things which belong to their own 2-category.

At the end of this chapter, after discussing exponentiation in $\mathbf{Cat}/\mathcal{S}$, we discuss a notion first introduced in [SZ13]: that of a *distributivity structure*. It formalizes the following intuition: trees can be considered as function symbols having two independent kinds of inputs. The first kind corresponds to leaves of the tree, with the natural composition operation using these inputs being grafting. This kind of composability is described by a monoidal structure (the substitution tensor product cf. e.g. [FGHW08]), and composition is described by a monoid structure on the set of trees. It leads to well known descriptions of free operads, multicategories or (certain) equational theories, all of which will be relevant in the later chapters.

The second kinds of inputs correspond to the nodes of the tree, with the natural operation being the replacement of a node with an entire tree. This is the basis for all of the algebraic constructs used in the construction of opetopic sets ([BD98, HMP02, SZ13, KJBM10]). This is also described by a monoidal structure, whose monoids (at least those which we will be considering) usually describe passage to the meta-level constructions[1], as was explicitly intended in [BD98].

The independence of these two kinds of inputs means that, when forming terms, or formal composites, of function symbols using both kinds of inputs, the order is immaterial. Thus if $\odot$ describes (formal) grafting, and $\otimes$ describes (formal) node replacement, then we expect an isomorphism

$$(A \otimes C) \odot (B \otimes C) \xrightarrow{\varphi} (A \odot B) \otimes C,$$

since we may first formally graft A and B terms, and then formally substitute terms from C (as is done on the right side of the isomorphism), or first formally substitute, and then formally graft (as is done on the left side).

This is the essence of a distributivity structure, and it is formalized by requiring the functor $(-) \otimes C$ to be $\odot$-monoidal. The isomorphism displayed above is just part of this monoidal structure. Further coherence conditions which we require are nicely packaged by a lifting condition to a naturally defined functor category.

We end this chapter by providing a theorem which characterizes distributivity structures in terms of extra data and coherence conditions, and allows us to construct distributivity structures in practice.

1.1 The Relevant 2-categories

1.1.1 Reminders on 2-categories

We will never work with abstract 2-categories, only specific ones, as our needs dictate. This subsection is therefore dedicated to fixing conventions.

Our notion of 2-category is the usual one, presented in [CWM98]. It coincides with the notion of **Cat**-enriched categories [Ke82]. Strictly speaking, this means there is an isomorphism of the associated 3-categories. In particular, **Cat**-enriched functors are 2-functors. For a brief overview of strict n-categories, including the mentioned isomorphism, see [Le04, chapter 1].

Universal properties. We will use limits in 2-categories, in particular products and pullbacks. It is easiest to define them as being **Cat**-enriched limits [Ke82, chapter 3]. This means that in the usual definition of a limiting cone for a functor $F\colon \mathcal{I} \to \mathcal{A}$ [CWM98, III.4] (a 2-functor between 2-categories in our case) the arrow

$$\mathcal{A}(x, \varprojlim F) \xrightarrow{\simeq} Cone(x, F) = Nat(\Delta x, F),$$

[1] See, for example, the first point in section 7.6.

is not just a natural bijection, but a natural isomorphism of categories. In 2-categorical language, such limits are called *strict 2-limits*. Note that the ordinary limit-style constructions in **Cat**, such as the product of categories, are in fact limits of this type. All the 2-categorical limits in this work are of this type, unless explicitly stated otherwise.

Set-theoretic assumptions. We postulate two Grothendieck universes U_1, U_2, such that $U_1 \in U_2$. We let **Set** be the category of sets in U_1 and **Cat** be the 2-category of categories in U_2. Therefore we may write **Set** $\in$ **Cat**. We shall suppress the U_i from notation. All required notions of smallness will be clear from their context. All the results about specific categories in this work may be proven without recourse to Grothendieck universes. Doing so, however, would result in a complete loss of conceptual clarity.

1.1.2 Recollections on Fibrations and Opfibrations

Definition 1.1.1. *A functor $p\colon \mathcal{E} \to \mathcal{S}$ is a fibration if the following condition is satisfied: for any $u\colon Q \to O \in \mathcal{S}$ and $Y \in \mathcal{E}$ there exists an arrow $\phi\colon X \to Y$ over u (i.e. $p(\phi) = u$), such that for any θ and v as in the diagram below there exists a unique ψ over v satisfying $\theta = \phi \circ \psi$.*

$$\begin{tikzcd}
Z \arrow[drr, "\theta"] \arrow[dr, dotted, "\exists!\psi"'] & & \\
& X \arrow[r, "\phi"'] \arrow[dd, dashed, mapsto, "p"] & Y \\
K \arrow[drr, "u \circ v"] \arrow[dr, "v"'] & & \\
& Q \arrow[r, "u"'] & O
\end{tikzcd}$$

Definition 1.1.2. *Any arrow ϕ satisfying the above definition for fixed u and Y is called a* prone *arrow over u with codomain Y.*

By their universality properties prone arrows are unique up to unique isomorphism (mapping to an identity under p). For this reason one sometimes writes $X = u^*Y$.

Remark 1.1.3. *Because of the universality properties of prone maps the operation $Y \mapsto u^*Y$ extends to a functor $\mathcal{E}/O \to \mathcal{E}/Q$, called the reindexing functor.*

Example. Let $\mathcal{C}$ be a category with pullbacks. Then the codomain functor $\mathcal{C}^{\cdot\to\cdot} \to \mathcal{C}$ is a fibration. The prone arrows are given by projections from the pullback $X = Y \times_I J \to Y$. This is another reason for the notation $X = u^*Y$ – frequently this means that X is the pullback of Y along u.

To appreciate how general this example is, consider the fact that bounded geometric morphisms of toposes have pullbacks along all other geometric morphisms (this is a paraphrase of B.3.3.6 of [J02]). Thus the restriction to bounded maps $\mathbf{BTop} \to \mathbf{Top}$ of the codomain functor $\mathbf{Top}^{\cdot\to\cdot} \to \mathbf{Top}$ is a fibration.

Thinking of toposes as generalized spaces, this generalizes (at least for sober spaces) the fact that the codomain functor for topological spaces $\mathbf{Spc}^{\cdot\to\cdot} \to \mathbf{Spc}$ is a fibration, since topological spaces have all pullbacks.

One may now add algebraic structures to these toposes (either to just the domain or also the codomain), such as rings, modules, groups and their actions. It is easy to check that this will still result in a fibration. Restricting it to suitable subcategories of sheaf toposes, one obtains all the usual structures in mathematics that allow "base change": (sober) fiber bundles, vector bundles, principal bundles, sheaves (possibly of modules or algebras), over all the common types of spaces (sober spaces, various kinds of manifolds[2], schemes, etc.).

Definition 1.1.4. *A functor $p\colon \mathcal{E} \to \mathcal{S}$ is called an opfibration if p^{op} is a fibration.*

The analogs of prone morphisms in opfibrations are called *supine* morphisms.

Remark 1.1.5. *The analogues of reindexing functors for opfibrations are denoted by u_*, and sometimes called coreindexing functors.*

Example. The codomain functor $\mathcal{C}^{\cdot\to\cdot} \to \mathcal{C}$ is always an opfibration. The coreindexing of $f\colon X \to O$ along $u\colon O \to Q$ is just $u \circ f\colon X \to Q$.

Definition 1.1.6. *A functor $p\colon \mathcal{E} \to \mathcal{S}$ is called a bifibration if it is both a fibration and an opfibration.*

We record the following well known and easy fact:

Proposition 1.1.7. *The pullback in* **Cat** *of a fibration or opfibration is a fibration or opfibration, respectively.*

1.1.3 Base Change

We will need to change the base category along a functor $\mathcal{S}' \to \mathcal{S}$. For this recall the following elementary facts:

[2] Described by an appropriate sheaf of rings. This includes supermanifolds and similar objects.

Proposition 1.1.8. *The 2-category* **Cat** *has small strict 2-limits and 2-colimits.*

Proof. Since the 2-category structure of **Cat** comes from the canonical enrichment given by its cartesian closedness as a 1-category, the claim follows from being small **Cat**-complete and cocomplete. This in turn follows from completeness and cocompleteness as a 1-category, and sections 2.2, 3.1 and 3.10 in [Ke80]. $\square$

Corollary 1.1.9. *The codomain 2-functor* $\mathbf{Cat}^{\cdot\to\cdot} \to \mathbf{Cat}$*, when considered as a functor of ordinary categories*[3]*, is a fibration.*

In the definition below, we similarly consider strict 2-functors as plain 1-functors.

Definition 1.1.10.

- *The fibration of fibrations,* $\mathbf{Fib} \to \mathbf{Cat}$ *is the subfibration of* $\mathbf{Cat}^{\cdot\to\cdot} \to \mathbf{Cat}$*, whose 0-cells consist of fibrations.*

- *The fibration of bifibrations,* $\mathbf{BF} \to \mathbf{Cat}$ *is the subfibration of* $\mathbf{Cat}^{\cdot\to\cdot} \to \mathbf{Cat}$*, whose 0-cells consist of bifibrations.*

- *The 2-category of fibrations over* $\mathcal{S} \in \mathbf{Cat}$ *is* $\mathbf{Fib}/\mathcal{S}$*, the fiber of* **Fib** *over* $\mathcal{S}$*.*

- *The 2-category of bifibrations over* $\mathcal{S} \in \mathbf{Cat}$ *is* $\mathbf{BF}/\mathcal{S}$*, the fiber of* **BF** *over* $\mathcal{S}$*.*

Since **Cat** has finite limits, and (op)fibrations are stable under pullback, we have:

Corollary 1.1.11. *The 2-categories* $\mathbf{Fib}/\mathcal{S}$ *and* $\mathbf{BF}/\mathcal{S}$ *have finite products.*

Since the fibration property of $\mathbf{Cat}^{\cdot\to\cdot} \to \mathbf{Cat}$ comes from pullbacks in **Cat**, and these same pullbacks give products in $\mathbf{Fib}/\mathcal{S}$ and $\mathbf{BF}/\mathcal{S}$, we obtain:

Corollary 1.1.12 (Pullbacks Preserve Algebra)**.** *Let* $F : \mathcal{S}' \to \mathcal{S}$ *be a functor. The operation* $\mathcal{E} \mapsto F^*\mathcal{E}$*, of pulling back along* F *is a 2-functor* $F^* \colon \mathbf{Fib}/\mathcal{S} \to \mathbf{Fib}/\mathcal{S}'$*, which preserves finite products. The same statement is true for the 2-categories* $\mathbf{BF}/\mathcal{S}$*.*

[3]This is possible due to strictness.

1.2 Universal Properties in Fibrations

The following theorem is the basis for extending colimit/left adjoint type constructions from the fibers of a fibration to the whole thing. Its dual applies to limit type constructions in opfibrations.

Theorem 1.2.1. *Consider $U : \mathcal{E} \to \mathcal{F}$, a morphism of fibrations over $\mathcal{S}$. Let $X \in \mathcal{F}/O$. Then a vertical arrow $X \to U(M)$ is universal from X to U if and only if it is universal from X to the restriction of U to $\mathcal{E}/O$.*

Proof. Using prone morphisms we can reduce morphisms between fibers to morphisms in the fiber over O, where we assumed universality. The other implication is trivial. □

To discuss fibered colimits, we must introduce the appropriate notion of a diagram. We will consider only the simplest form of diagrams, that is those whose shape remains constant from fiber to fiber. Such diagrams will suffice for all our needs. The reader may investigate more complicated diagrams using the exponential objects defined in section 1.6. The following definition was given in [St08].

Definition 1.2.2. *The* fibration of diagrams of type $\mathcal{D} \in \mathbf{Cat}$ *is the pullback of $\mathcal{E}^{\mathcal{D}} \to \mathcal{S}^{\mathcal{D}}$ along the constant diagram functor $\Delta_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}^{\mathcal{D}}$.*

Given this, $\Delta_{\mathcal{E}} \colon \mathcal{E} \to \mathcal{E}^{\mathcal{D}}$ factors into a morphism $\mathcal{E} \to \Delta_{\mathcal{S}}^{*}\mathcal{E}^{\mathcal{D}}$ over $\mathcal{S}$, which we will still call the constant diagram functor, followed by the canonical projection. A fibered colimit of an object F of such a fibration is a vertical universal arrow from F to the constant diagram functor $\Delta_{\mathcal{E}}$ (considered over $\mathcal{S}$), as usual.

Corollary 1.2.3.

1. *If the fibration $\mathcal{E} \to \mathcal{S}$ has a type of colimit (e.g. coproducts, pushouts, filtered colimits) fiberwise, then it has the fibered version of this type of colimit.*

2. *If a fibration has a type of colimit fiberwise, then taking the colimit extends to a fibered functor on the fibration of diagrams of the given type.*

Proof. The needed universal property follows immediately from theorem 1.2.1, since the constant diagram functor $\mathcal{E} \to \Delta_{\mathcal{S}}^{*}\mathcal{E}^{\mathcal{D}}$ (again, considered over $\mathcal{S}$) preserves prone morphisms. The second statement is a formal consequence of the first. □

Note that the condition in this corollary refers only to fibers. It is therefore stable under pullback. Thus existence of fibered colimits is stable under pullback.

Another immediate corollary of theorem 1.2.1 is a characterization of adjointness in fibrations.

Corollary 1.2.4. *If $U: \mathcal{E} \to \mathcal{F}$ is a morphism of fibrations which has an adjoint pointwise in every fiber, then U has a left adjoint in* $\mathbf{Cat}/\mathcal{S}$*.*

Proof. By theorem 1.2.1 the usual formula, using representability or universal properties, for extending the values $F(X)$ of a left adjoint to a functor can be used, and results in a fibered functor. □

1.3 Fibered Monads and Their Algebras

Monads can be defined in any 2-category [CWM98]. In particular they can be defined in $\mathbf{Cat}/\mathcal{S}$ and its various sub-2-categories. We are interested in the existence of Kleisli and Eilenberg-Moore objects for such monads, and the analog of the monadicity theorem.

Let $T: \mathcal{E} \to \mathcal{E}$ be a monad in $\mathbf{Cat}/\mathcal{S}$. Note that T restricts to a monad on every fiber $\mathcal{E}/O$, for $O \in \mathcal{S}$. This suggests the following guess at the definition of Eilenberg-Moore and Kleisli objects in $\mathbf{Cat}/\mathcal{S}$.

Definition 1.3.1.

1. *The* Eilenberg-Moore object *for T, $\mathcal{E}^T$ is defined as follows: objects of the category $\mathcal{E}^T$ are all the objects of the ordinary Eilenberg-Moore categories $(\mathcal{E}/O)^T$. Morphisms in $\mathcal{E}^T$ are the morphisms in $\mathcal{E}$, which make the diagrams for a homomorphism of algebras commute. Composition is inherited from $\mathcal{E}$. The projection $\mathcal{E}^T \to \mathcal{S}$ is likewise inherited from $\mathcal{E}$, ignoring the extra data of the algebra structures.*

2. *The* Kleisli object *for T, $\mathcal{E}_T$ is defined as follows: objects of the category $\mathcal{E}_T$ are all the objects of the ordinary Kleisli categories $(\mathcal{E}/O)_T$. Morphisms in $\mathcal{E}_T$ are defined as $\mathcal{E}_T(X,Y) = \mathcal{E}(X, T(Y))$, with composition defined as in ordinary Kleisli categories (again, using composition in $\mathcal{E}$). The projection map $\mathcal{E}_T \to \mathcal{S}$ is inherited from $\mathcal{E}$.*

Alternatively, one may define $\mathcal{E}_T$ as the essential image of the free algebra functor $\mathcal{E} \to \mathcal{E}^T$. This provides a more "concrete" construction of $\mathcal{E}_T$, but in practice the above definition is necessary to perform calculations. It will be used extensively in the following chapters.

Remark 1.3.2. *Beck's theory of distributive laws [Be69] immediately adapts to our setting. Due to our definitions, all the calculations required to establish this theory can be copied verbatim from the original work.*

The above definitions are justified only in light of fulfilling the defining properties of Eilenberg-Moore and Kleisli objects.

Proposition 1.3.3. *With the above definitions, $\mathcal{E}^T$ is an Eilenberg-Moore object for T in $\mathbf{Cat}/S$ and $\mathcal{E}_T$ is a Kleisli object for T in $\mathbf{Cat}/\mathcal{S}$.*

Proof. The required representability property for the Eilenberg-Moore object:

$$\mathbf{Cat}/\mathcal{S}(\mathcal{F}, \mathcal{E}^T) \simeq \mathbf{Cat}/\mathcal{S}(\mathcal{F}, \mathcal{E})^{\mathbf{Cat}/\mathcal{S}(\mathcal{F},T)},$$

is easily verified directly, just like in the case of ordinary categories. Here $\mathbf{Cat}/\mathcal{S}(\mathcal{F}, T)$ is the induced monad on the hom-categories. The same is true for the representability property of the Kleisli object:

$$\mathbf{Cat}/\mathcal{S}(\mathcal{E}_T, \mathcal{F}) \simeq \mathbf{Cat}/\mathcal{S}(\mathcal{E}, \mathcal{F})^{\mathbf{Cat}/\mathcal{S}(T,\mathcal{F})}$$

□

Due to our choices of 1- and 2-cells, $\mathcal{E}^T$ is also an Eilenberg-Moore object in $\mathbf{Fib}/\mathcal{S}$ and $\mathbf{BF}/\mathcal{S}$, since the same isomorphism restricts to those 2-categories. The same remark applies to $\mathcal{E}_T$.

Proposition 1.3.4.

1. *If $\mathcal{E}$ is a fibration, then so is $\mathcal{E}^T$.*

2. *If $\mathcal{E}$ is an opfibration, then so is $\mathcal{E}_T$.*

3. *If T is a morphism of fibrations, then $\mathcal{E}_T$ is a fibration.*

4. *If T is a morphism of opfibrations, then $\mathcal{E}^T$ is an opfibration.*

Proof. We construct the required maps, and leave checking their properties to the reader.

Ad 1. Consider the diagram:

$$\begin{tikzcd}
T(u^*X) \arrow[d, "\exists!"] \arrow[drr, "T(\phi_X)"] \arrow[dd, bend right, "\alpha'"'] & & \\
u^*T(X) \arrow[rr, "\phi_{T(X)}"] \arrow[d, "u^*\alpha"] & & T(X) \arrow[d, "\alpha"] \\
u^*X \arrow[rr, "\phi_X"] & & X
\end{tikzcd}$$

$$Q \xrightarrow{\quad u \quad} O$$

We start with an algebra $\alpha\colon T(X) \to X$ over $O \in \mathcal{S}$. Given u, as in the diagram, we form the reindexing functor u^* and the associated prone arrows $\phi_{(-)}$. To construct the required algebra structure on u^*X, we consider $T(\phi_X)$, and note that it factors through $\phi_{T(X)}$, by the latter's universal property and the fact that T is fibered. The composite $\alpha' = \exists! \circ u^*\alpha$ is easily seen to be a T-algebra structure on u^*X. The required prone map in $\mathcal{E}^T$ consists of ϕ_X and $T(\phi_X)$.

Ad 2. We start with $X \in \mathcal{E}_T$ over $Q \in \mathcal{S}$, and Y over O. We compute the morphisms in $\mathcal{E}_T$ over $u\colon Q \to O$:

$$\mathcal{E}_T/u(X,Y) = \mathcal{E}/u(X,T(Y)) = \mathcal{E}/O(u_*X,T(Y)) = \mathcal{E}_T/O(u_*X,Y).$$

This computation shows that the value of u_* on X can be taken to coincide for both $\mathcal{E}$ and $\mathcal{E}_T$. The resulting supine arrow in $\mathcal{E}_T$, as seen in $\mathcal{E}$, is

$$X \xrightarrow{\psi_X} u_*X \xrightarrow{\eta_{u_*X}} T(u_*X),$$

where ψ_X is the supine arrow over u in $\mathcal{E}$ for X.

Ad 3. Similarly to the above, we compute

$$\mathcal{E}_T/u(X,Y) = \mathcal{E}/u(X,T(Y)) = \mathcal{E}/Q(X,u^*(T(Y))) = \mathcal{E}/Q(X,T(u^*(Y))) = \mathcal{E}_T/Q(X,u^*Y),$$

where the natural isomorphism $Tu^* \simeq u^*T$ follows from the fact that T preserves prone maps. Again, we see that the value of u^* on X can be taken to coincide for both $\mathcal{E}$ and $\mathcal{E}_T$. The resulting prone map in $\mathcal{E}_T$, as seen in $\mathcal{E}$, is

$$u^*Y \xrightarrow{\eta_{u^*Y}} T(u^*Y) \xrightarrow{\simeq} u^*T(Y) \xrightarrow{\phi_{T(Y)}} T(Y),$$

where $\phi_{T(Y)}$ is the prone arrow over u for $T(Y)$ in $\mathcal{E}$.

Ad 4. Consider the diagram

$$\begin{tikzcd}
 & T(u_*X) \arrow[d, "\simeq"] \arrow[dd, bend left=60, "\alpha'"] \\
T(X) \arrow[ur, "T(\psi_X)"] \arrow[r, "\psi_{T(X)}"] \arrow[d, "\alpha"] & u_*T(X) \arrow[d, "u_*\alpha"] \\
X \arrow[r, "\psi_X"] & u_*X \\
Q \arrow[r, "u"] & O
\end{tikzcd}$$

As before, we construct the coreindexing functor u_*, and the associated supine maps $\psi_{(-)}$. Since T preserves supine arrows, we have a unique isomorphism $T(u_*X) \simeq u_*T(X)$, and α' is easily seen to be a T-algebra structure on u_*X. The required supine map in $\mathcal{E}^T$ consists of ψ_X and $\psi_{T(X)}$. □

We may now state the monadicity theorem for morphisms of fibrations.

Theorem 1.3.5. *Let $U \colon \mathcal{E} \to \mathcal{F}$ be a morphism of fibrations in* $\mathbf{Cat}/\mathcal{S}$*. Then U is monadic iff it is fiberwise monadic.*

Proof. By our construction of $\mathcal{F}^T$, if U is monadic then it is clearly fiberwise monadic.

For the converse implication note that if U has left adjoints fiberwise, then by corollary 1.2.4 it has a left adjoint F in $\mathbf{Cat}/\mathcal{S}$. Using this adjoint we obtain, by the universal property of $\mathcal{F}^T$, a comparison functor $\mathcal{E} \to \mathcal{F}^T$. In Beck's monadicity theorem the inverse of the comparison functor is explicitly defined in terms U-split coequalizers. By corollary 1.2.3 we may use the same formula here, and the result will be a fibered functor. The comparison natural transformations in Beck's monadicity theorem are again defined explicitly using the universal properties of U-split coequalizers. Thus we may also define them using the formula for ordinary categories. Since such data form an equivalence iff they form an equivalence fiberwise, and we assumed the fiberwise equivalence, the proof is complete. □

Corollary 1.3.6. *Any variant of Beck's monadicity theorem, such as strict, crude or vulgar monadicity, is also valid for morphisms of fibrations.*

Proof. By corollary 1.2.3 the criteria of these theorems may be checked fiberwise, and imply fiberwise monadicity. By the above theorem this suffices for monadicity in $\mathbf{Cat}/\mathcal{S}$. □

1.4 Relative Monoidal Categories

Here we recall the definitions associated to monoidal fibrations introduced in [Z10]. The reader is referred to there for a more complete discussion, including motivating comments and examples. We state the variants in which the coherenece morphisms are isomorphisms, since we will only use those. For this reason we omit "lax" from the name. Note however that this "laxness" also refers to the use of fibered functors – such functors can be considered "lax morphisms of fibrations", as opposed to the usual ones, which preserve prone arrows strongly.

The reader short on patience can recreate all the definitions below according to the following rule: take the diagrams in $\mathbf{Cat}$, including the 2-cells, defining the usual notion, and interpret them in $\mathbf{Cat}/\mathcal{S}$, as cartesian theories (i.e. substituting $\times_{\mathcal{S}}$ for $\times$).

1.4.1 Relative Monoidal Categories

Objects

A relative monoidal category over $\mathcal{S} \in \mathbf{Cat}$ consists of:

1. A category $\mathcal{E}$ over $\mathcal{S}$.
2. Equipped with two functors fibered over $\mathcal{S}$:

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{S}} \mathcal{E} \arrow[r, "\otimes"] \arrow[dr] & \mathcal{E} \arrow[d] & \mathcal{S} \arrow[l, "I"'] \arrow[dl, "1_{\mathcal{S}}"] \\
& \mathcal{S} &
\end{tikzcd}$$

3. Three fibered natural isomorphisms:

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{S}} \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \arrow[r, "\otimes \times_{\mathcal{S}} 1_{\mathcal{E}}"] \arrow[d, "1_{\mathcal{E}} \times_{\mathcal{S}} \otimes"'] & \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \arrow[d, "\otimes"] \\
\mathcal{E} \times_{\mathcal{S}} \mathcal{E} \arrow[r, "\otimes"'] \arrow[ur, phantom, "\Rightarrow\alpha"] & \mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S} \times_{\mathcal{S}} \mathcal{E} \arrow[r, "I \times_{\mathcal{S}} 1_{\mathcal{E}}"] \arrow[dr, "\pi_2"'] & \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \arrow[d, "\otimes"] \arrow[dl, phantom, "\lambda\Leftarrow"] \arrow[dr, phantom, "\Leftarrow\rho"] & \mathcal{E} \times_{\mathcal{S}} \mathcal{S} \arrow[l, "1_{\mathcal{E}} \times_{\mathcal{S}} I"'] \arrow[dl, "\pi_1"] \\
& \mathcal{E} &
\end{tikzcd}$$

Subject to the following coherence conditions:

$$\begin{tikzcd}
& A \otimes (B \otimes (C \otimes D)) \arrow[dl, "\alpha"'] \arrow[dr, "1 \otimes \alpha"] & \\
(A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha"] \\
((A \otimes B) \otimes C) \otimes D & & (A \otimes (B \otimes C)) \otimes D \arrow[ll, "\alpha \otimes 1"']
\end{tikzcd}$$

$$
\begin{tikzcd}
A \otimes (I \otimes B) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \lambda"'] & (A \otimes I) \otimes B \\
A \otimes B \arrow[r, "1"] & A \otimes B \arrow[u, "\rho \otimes 1"']
\end{tikzcd}
$$

and

$$\rho_I = \lambda_I^{-1}$$

Definition 1.4.1. *A monoidal fibration is a relative monoidal category which is a fibration.*

Remark 1.4.2.

1. *One should think of A, B, C and D above as generalized elements of $\mathcal{E}$ in $\mathbf{Cat}/\mathcal{S}$. This remark applies to all other coherence conditions in this section.*

2. *We will sometimes write $A \otimes_O B$ and $f \otimes_u g$ to indicate that A and B are over $O \in \mathcal{S}$, and f and g are over $u \in \mathcal{S}$. We will do this especially in contexts where several categories and monoidal structures are involved. The previous remark shows, however, that this special notation is redundant.*

3. *Note that ρ_I and λ_I in the last condition above are natural transformations, as opposed to ordinary arrows. Their components are $\lambda_{I(O)}$, for $O \in \mathcal{S}$.*

4. *By corollary 1.1.12 the pullback of a relative monoidal category is again a relative monoidal category. In particular, the fibers of a relative monoidal category are strong monoidal categories.*

5. *The definition 1.4.1 implies that the reindexing functors $u^* \colon \mathcal{E}/O \to \mathcal{E}/Q$ (see remark 1.1.3), for $u \colon Q \to O \in \mathcal{S}$, are naturally lax monoidal functors. More specifically, the pseudofunctor $\mathcal{S} \to \mathbf{Cat}$ associated to $\mathcal{E}$ takes values in strong monoidal categories and lax monoidal functors.*

6. *We will usually suppress the extra data from our notation. We will add the minimum amount of indexing to avoid ambiguity.*

Morphisms

A morphism of relative monoidal categories consists of:

1. Two relative monoidal categories $\mathcal{E} \to \mathcal{S}$ and $\mathcal{F} \to \mathcal{S}'$.

2. A 1-cell in $\mathbf{Cat}^{\cdot \to \cdot}$ between $\mathcal{E}$ and $\mathcal{F}$, consisting of functors $F \colon \mathcal{E} \to \mathcal{F}$ and $K \colon \mathcal{S} \to \mathcal{S}'$.

3. Fibered natural isomorphisms $\varphi_0\colon I \circ K \to F \circ I$, and $\varphi_2\colon \otimes \circ F \times_K F \to F \circ \otimes$.

Subject to the following coherence conditions:

$$\begin{tikzcd}
I(K) \otimes F(A) \arrow[r, "\varphi_0 \otimes 1"] \arrow[d, "\lambda"'] & F(I) \otimes F(A) \arrow[d, "\varphi_2"] \\
F(A) & F(I \otimes A) \arrow[l, "F(\lambda)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes I(K) \arrow[r, "1 \otimes \varphi_0"] & F(A) \otimes F(I) \arrow[d, "\varphi_2"] \\
F(A) \arrow[u, "\rho"] \arrow[r, "F(\rho)"] & F(A \otimes I)
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes (F(B) \otimes F(C)) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \varphi_2"'] & (F(A) \otimes F(B)) \otimes F(C) \arrow[d, "\varphi_2 \otimes 1"] \\
F(A) \otimes F(B \otimes C) \arrow[d, "\varphi_2"'] & F(A \otimes B) \otimes F(C) \arrow[d, "\varphi_2"] \\
F(A \otimes (B \otimes C)) \arrow[r, "F(\alpha)"] & F((A \otimes B) \otimes C)
\end{tikzcd}$$

Remark 1.4.3.

1. *We allow morphisms to change base categories.*

2. *One should think of K as a 1-cell in $\mathbf{Cat}^{\cdot \to \cdot}$ from $1_{\mathcal{S}}$ to $1_{\mathcal{S}'}$. This makes it clear how to treat it as a generalized element.*

3. *The distinctions between the monoidal structures for $\mathcal{E}$ and $\mathcal{F}$ follow uniquely from the notation.*

4. *It follows that the projection from the pullback of a relative monoidal category (with the structure given by corollary 1.1.12) to the original is a morphism of relative monoidal categories.*

Transformations

A transformation between two morphisms $F, F' \colon \mathcal{E} \to \mathcal{F}$ of relative monoidal categories is a 2-cell $F \to F'$ in $\mathbf{Cat}^{\cdot \to \cdot}$, consisting of natural transformations $\tau \colon F \to F'$ and $\sigma \colon K \to K'$, subject to the following conditions:

$$\begin{tikzcd}
I(K) \arrow[r, "I_\sigma"] \arrow[d, "\varphi_0"'] & I(K') \arrow[d, "\varphi_0"] \\
F(I) \arrow[r, "\tau_I"] & F'(I)
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "\tau_A \otimes \tau_B"] \arrow[d, "\varphi_2"'] & F'(A) \otimes F'(B) \arrow[d, "\varphi_2"] \\
F(A \otimes B) \arrow[r, "\tau_{A \otimes B}"] & F'(A \otimes B)
\end{tikzcd}$$

Again, the distinctions between the monoidal structures (also on F, F') follow uniquely from the notation.

Let us record the following obvious statement.

Proposition 1.4.4. *With the above definitions, monoidal fibrations, their morphisms and transformations form a* 2*-category, denoted* **MonFib**.

There are obvious forgetful functors $\mathbf{MonFib} \to \mathbf{Fib} \to \mathbf{Cat}$, which allow us to define, via fibers, monoidal fibrations over a fixed base $\mathcal{S} \in \mathbf{Cat}$, which we will denote $\mathbf{MonFib}/\mathcal{S}$. By construction it is a 2-category.

1.4.2 Monoids in a Relative Monoidal Category

A monoid M in a relative monoidal category $\mathcal{E} \to \mathcal{S}$ over an object $O \in \mathcal{S}$ is a monoid in the ordinary monoidal category $\mathcal{E}/O$, given by corollary 1.1.12. Thus, we are given a multiplication map $\mu \colon M \otimes M \to M$ and a unit map $\eta \colon I(O) \to M$, which are vertical in $\mathcal{E}$, subject to the usual associativity and unit conditions.

We allow morphisms of monoids to change fibers, so a morphism of monoids is a morphism $f \colon M \to N$ in $\mathcal{E}$, over some $u \colon O \to Q$ in $\mathcal{S}$, such that the following diagram commutes:

$$\begin{array}{ccc}
M \otimes M & \xrightarrow{f \otimes f} & N \otimes N \\
{\scriptstyle \mu^M}\big\downarrow & & \big\downarrow{\scriptstyle \mu^N} \\
M & \xrightarrow{\quad f \quad} & N \\
{\scriptstyle \eta^M}\big\uparrow & & \big\uparrow{\scriptstyle \eta^N} \\
I(O) & \xrightarrow{I(u)} & I(Q)
\end{array}$$

With this definition of homomorphisms, monoids in $\mathcal{E}$ form an object of $\mathbf{Cat}/\mathcal{S}$, denoted $Mon(\mathcal{E})$, with the projection $Mon(\mathcal{E}) \to \mathcal{S}$ given by $f \mapsto u$. There is also a fibered forgetful functor $Mon(\mathcal{E}) \to \mathcal{E}$, which forgets the multiplication and unit maps.

For the reader's convenience, we provide the basic properties of this construction below. Their proofs are elementary calculations.

Proposition 1.4.5. *If $\mathcal{E}$ is a fibration over $\mathcal{S}$, then so is $Mon(\mathcal{E})$, and the natural fibered forgetful functor $Mon(\mathcal{E}) \to \mathcal{E}$ is a morphism of fibrations.*

Proof. This follows easily from the fact that the reindexing functors u^* are naturally lax monoidal. □

Proposition 1.4.6. *The construction $\mathcal{E} \mapsto Mon(\mathcal{E})$ is a 2-functor* **MonFib** $\to$ **Fib***, and the natural forgetful functor is actually a 2-natural transformation $Mon \to U$, where $U\colon$* **MonFib** $\to$ **Fib** *is the natural forgetful functor.*

Since **MonFib** is essentially $Mon(\mathbf{Fib})$ considered as an object of $2-\mathbf{Cat}/\mathbf{Cat}$, we can see a glimpse of the microcosm in the above proposition, as defined by Baez and Dolan [BD98].

The definition of monoids we have given is actually predetermined by our notion of monoids in ordinary monoidal categories:

Proposition 1.4.7. *For any relative monoidal category $\mathcal{E}$ there is a natural isomorphism*

$$Mon(\mathbf{Cat}/\mathcal{S}(-,\mathcal{E})) \simeq \mathbf{Cat}/\mathcal{S}(-, Mon(\mathcal{E}))$$

1.4.3 Actions of Relative Monoidal Categories

Objects

An action of a relative monoidal category over $\mathcal{S} \in \mathbf{Cat}$ consists of:

1. A relative monoidal category $\mathcal{E} \to \mathcal{S}$ and a relative category $\mathcal{X} \to \mathcal{S}$.

2. Equipped with a functor fibered over $\mathcal{S}$:

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{S}} \mathcal{X} \arrow[rr, "\star"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}$$

3. Two fibered natural isomorphisms:

$$\begin{tikzcd}
\mathcal{E} \times_{\mathcal{S}} \mathcal{E} \times_{\mathcal{S}} \mathcal{X} \arrow[r, "\otimes \times_{\mathcal{S}} 1_{\mathcal{X}}"] \arrow[d, "1_{\mathcal{E}} \times_{\mathcal{S}} \star"'] & \mathcal{E} \times_{\mathcal{S}} \mathcal{X} \arrow[d, "\star"] \\
\mathcal{E} \times_{\mathcal{S}} \mathcal{X} \arrow[r, "\star"'] \arrow[ur, phantom, "\Uparrow \psi_2"] & \mathcal{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S} \times_{\mathcal{S}} \mathcal{X} \arrow[rr, "I \times_{\mathcal{S}} 1_{\mathcal{X}}"] \arrow[dr, "\pi_2"'] & \arrow[d, phantom, "\Rightarrow \psi_0"] & \mathcal{E} \times_{\mathcal{S}} \mathcal{X} \arrow[dl, "\star"] \\
& \mathcal{E} &
\end{tikzcd}$$

Subject to the following coherence conditions:

$$\begin{tikzcd}
& A \star (B \star (C \star X)) \arrow[dl, "\psi_2"'] \arrow[dr, "1 \star \psi_2"] & \\
(A \otimes B) \star (C \star X) \arrow[d, "\psi_2"'] & & A \star ((B \otimes C) \star X) \arrow[d, "\psi_2"] \\
((A \otimes B) \otimes C) \star X & & (A \otimes (B \otimes C)) \star X \arrow[ll, "\alpha \star 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \star X \arrow[r, "\psi_0"] \arrow[d, "1"'] & I \star (A \star X) \arrow[d, "\psi_2"] \\
A \star X & (I \otimes A) \star X \arrow[l, "\lambda \star 1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \star X \arrow[r, "1 \star \psi_0"] \arrow[d, "1"'] & A \star (I \star X) \arrow[d, "\psi_2"] \\
A \star X \arrow[r, "\rho \star 1"'] & (A \otimes I) \star X
\end{tikzcd}$$

Definition 1.4.8. *An action of a monoidal fibration is an action of relative monoidal categories in which $\mathcal{X}$ is also a fibration.*

Morphisms

A morphism of actions of relative monoidal categories consists of a (not strictly commutative) diagram:

$$
\begin{tikzcd}
\mathcal{E} \times_{\mathcal{S}} \mathcal{X} \arrow[rr, "\star"] \arrow[dr] \arrow[ddd, "F \times_K H"'] & & \mathcal{X} \arrow[dl] \arrow[ddd, "H"] \\
& \mathcal{S} \arrow[d, "K"] & \\
& \mathcal{S}' \arrow[dr, phantom, "\Uparrow \tau"] & \\
\mathcal{F} \times_{\mathcal{S}'} \mathcal{Y} \arrow[ur] \arrow[rr, "\star"] & & \mathcal{Y} \arrow[ul]
\end{tikzcd}
$$

where

1. K is a functor.
2. H is a fibered functor over K.
3. F is a morphism of relative monoidal categories over K.
4. $\star$ are actions of relative monoidal categories.
5. τ is a natural isomorphism $\star \circ (F \times_K H) \to H \circ \star$
6. The triangles and inner trapezoids commute, but the outer square is not required to commute.

Subject to the following coherence conditions:

$$
\begin{tikzcd}[column sep=small]
 & F(A) \star (F(B) \star H(X)) \arrow[dl, "1 \star \tau"'] \arrow[dr, "\psi_2"] & \\
F(A) \star H(B \star X) \arrow[d, "\tau"'] & & (F(A) \otimes F(B)) \star X \arrow[d, "\varphi_2 \star 1"] \\
H(A \star (B \star X)) \arrow[dr, "H(\psi_2)"'] & & F(A \otimes B) \star H(X) \arrow[dl, "\tau"] \\
 & H((A \otimes B) \star X) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
H(X) \arrow[r, "\psi_0"] \arrow[d, "H(\psi_0)"'] & I \star H(X) \arrow[d, "\varphi_0 \star 1"] \\
H(I \star X) & F(I) \star H(X) \arrow[l, "\tau"]
\end{tikzcd}
$$

Transformations

A transformation of morphisms of actions of relative monoidal categories (denoted briefly H and H') consists of:

1. A 2-cell $\zeta \colon H \to H'$ in $\mathbf{Cat}^{\cdot \to \cdot}$.
2. A transformation of morphisms of relative monoidal categories $\alpha \colon F \to F'$, with the same base (in $\mathbf{Cat}^{\cdot \to \cdot}$) as ζ.

Subject to the following condition:

$$
\begin{tikzcd}
F(A) \star H(X) \arrow[r, "\tau"] \arrow[d, "dom(\alpha) \star dom(\zeta)"'] & H(A \star X) \arrow[d, "dom(\zeta)"] \\
F' \star H' \arrow[r] & H'(A \star X)
\end{tikzcd}
$$

where $dom \colon \mathbf{Cat}^{\cdot \to \cdot} \to \mathbf{Cat}$ is the domain 2-functor.

Proposition 1.4.9. *With the above definitions of* $0, 1$- *and* 2-*cells, the actions of monoidal fibrations, their morphisms and transformations form a* 2-*category, denoted* **Act**.

Remark 1.4.10.

1. *Just like for relative monoidal categories, corollary 1.1.12 implies that the pullback of an action is again an action in a natural way.*

2. *The canonical projection of a pullback of an action is a morphism of actions over the functor used to construct the pullback, just as before.*

1.4.4 Actions of Monoids Along an Action

Consider a monoid M over $O \in \mathcal{S}$ in a relative monoidal category $\mathcal{E}$ over $\mathcal{S}$, acting by $\star$ on another relative category $\mathcal{X}$ over the same base. Then the monoid can act along $\star$ on objects in $\mathcal{X}$ over O.

An action of M on an object $X \in \mathcal{X}/O$ is given by a map $\nu\colon M \star X \to X$, such that

$$\begin{tikzcd} M \star (M \star X) \arrow[r, "\psi_2"] \arrow[d, "1 \star \nu"'] & (M \otimes M) \star X \arrow[r, "\mu^M \star 1"] & M \star X \arrow[d, "\nu"] \\ M \star X \arrow[rr, "\nu"] & & X \end{tikzcd}$$

$$\begin{tikzcd} I \star X \arrow[r, "\eta^M \star 1"] & M \star X \arrow[d, "\nu"] \\ X \arrow[u, "\psi_0"] \arrow[r, "1"] & X \end{tikzcd}$$

A homomorphism of actions is a homomorphism of monoids $h\colon M \to M'$ and a map $f\colon X \to X'$ in $\mathcal{X}$ of the objects being acted upon, over the same map in $\mathcal{S}$ as h, such that

$$\begin{tikzcd} M \star X \arrow[r, "h \star f"] \arrow[d, "\nu"'] & M' \star X' \arrow[d, "\nu'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

This defines a category $\mathbf{Act}(\mathcal{E}, \mathcal{X})$ of actions of monoids along an action of $\mathcal{E}$ on $\mathcal{X}$. The data for $\otimes$ and $\star$ are understood implicitly. This category has has a natural forgetful functor $\mathbf{Act}(\mathcal{E}, \mathcal{X}) \to Mon(\mathcal{E})$, which forgets the object being acted upon. This functor is a fibration if both $\mathcal{E}$ and $\mathcal{X}$ are fibrations (i.e. we are dealing with an action of a monoidal fibration). Its fibers are the categories of actions of a single monoid, and morphisms of actions which leave the monoid fixed.

A monoid M also determines a monad $T = M\star(-)$ on $\mathcal{X}/O$. The multiplication map $\mu^T : T^2 \to T$ is given by $\mu^T = (\mu^M \star (-)) \circ \psi_2$, where μ^M is the multiplication in M, and there is a similar obvious formula for the unit. This gives an alternate characterization of the category of actions of a single monoid.

Proposition 1.4.11. *The the category of actions of M along $\star$ is equivalent to that of T-algebras.*

Proof. This is a tedious calculation using the coherence conditions and the definitions of monoids and their actions. $\square$

1.4.5 The Fibered Slice

Recall that if $\mathcal{C}$ is a monoidal category and $M \in Mon(\mathcal{C})$ is a monoid, then the slice category $\mathcal{C}/M$ is also naturally a monoidal category, with monoidal product of $A \to M$ and $B \to M$ defined by the composite $A \otimes B \to M \otimes M \xrightarrow{\mu} M$, where $\otimes$ is the product in $\mathcal{C}$ and μ is multiplication in M. The unit is the unit of the monoid $e : I \to M$, and there are obvious correct choices for the coherence isomorphisms. If $\mathcal{C}$ has pullbacks, these categories are fibers of the monoidal fibration $\mathcal{C} \downarrow \mathcal{U}$ over $Mon(\mathcal{C})$, where $\mathcal{U} : Mon(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor.

This construction has a fibered analogue. Let $\mathcal{E}$ be a monoidal fibration over $\mathcal{S}$, and let $\mathcal{E}^{(\to)}$ be the fibration of diagrams of type $\to$ in $\mathcal{E}$ (recall definition 1.2.2). There is an obvious functor $(cod) : \mathcal{E}^{(\to)} \to \mathcal{E}$ sending each arrow to its codomain. If $\mathcal{E}$ has pullbacks then this functor is a fibration – the fibered analogue of a codomain fibration. As before let $\mathcal{U} : Mon(\mathcal{E}) \to \mathcal{E}$ be the forgetful functor. The structure we are looking for is the pullback of (cod) along $\mathcal{U}$,

$$\begin{tikzcd} \mathcal{E} \downdownarrows \mathcal{U} \arrow[r] \arrow[d, "\mathcal{U}^*(cod)"'] & \mathcal{E}^{(\cdot \to \cdot)} \arrow[d, "(cod)"] \\ Mon(\mathcal{E}) \arrow[r, "\mathcal{U}"] & \mathcal{E} \end{tikzcd}$$

which is a fibration over $Mon(\mathcal{E})$. Its fibers are precisely all the categories of the form $\mathcal{E}_O/M$, where $O \in \mathcal{B}$, $\mathcal{E}_O$ is the fiber of $\mathcal{E}$ over O, and M is a monoid in

$\mathcal{E}_O$. The above discussion gives us a monoidal structure on $\mathcal{E} \Downarrow \mathcal{U}$. The monoidal product of $A \to M$ and $B \to M$ is given by $A \otimes B \to M \otimes M \xrightarrow{\mu} M$ (as before) and the unit functor is $I(M) = I(O) \xrightarrow{e} M$, the unit of multiplication in M.

1.5 Free Monoids in Monoidal Fibrations

The three tensors theorem asserts the existence of a certain extra structure on a free monoid. To obtain this extra structure we will use an explicit construction of the free monoid, which we recall below. The basic ideas behind this construction seem to have been first stated explicitly in [A74]. A very general account of such constructions was given in [Ke80]. We will follow the very brief and readable appendix B of [BJT97], and refer the reader there for all the calculations omitted here. The context there is a single monoidal category, but the calculations adapt to monoidal fibrations verbatim.

Let $\mathcal{E}$ be a monoidal fibration over $\mathcal{S}$. We wish to construct a fibered left adjoint to the forgetful functor $\mathcal{U} : Mon(\mathcal{E}) \to \mathcal{E}$. Our assumptions are summarized in the following definition.

Definition 1.5.1. *We say that the monoidal fibration $\mathcal{E}$* admits the free monoid construction *for the monoidal structure $\otimes$ if the following conditions are satisfied:*

a) $\mathcal{E}$ has fiberwise finite coproducts and filtered colimits.

b) The monoidal product $\otimes$ preserves fibered filtered colimits in both variables, and fibered binary coproducts in the left variable.

The condition *a*) is stable under pullback of fibrations, since it refers only to fibers and the canonical projection is an isomorphism when restricted to any fiber. By corollary 1.2.3 we obtain from *a*) the existence of fibered filtered colimits in $\mathcal{E}$.

Let $X \in \mathcal{E}/O$. We define

$$\begin{aligned} X_0 &= I(O) \\ X_{n+1} &= I(O) \sqcup (X \otimes X_n), \end{aligned}$$

where $I(O)$ is the unit of $\otimes$ in the fiber over O and $\sqcup$ is the coproduct. We have arrows

$$\begin{aligned} &i_n : X_n \to X_{n+1} \\ &i_0 : I(O) \to I(O) \sqcup X \text{ is the coprojection} \\ &i_{n+1} = 1 \sqcup (1 \otimes i_n). \end{aligned}$$

We define X_∞, the universe of the free monoid on X, as the colimit of the X_i:

$$X_\infty = \varinjlim (X_0 \to X_1 \to X_2 \to X_3 \to \cdots)$$

To define multiplication we define the morphisms $\mu_{n,m} : X_n \otimes X_m \to X_{n+m}$:

$$\mu_{0,m} = \lambda_{X_m} : I(O) \otimes X_m \to X_m$$

and for $n \geq 1$ we have

$$X_n \otimes X_m \simeq (I(O) \sqcup (X \otimes X_{n-1})) \otimes X_m \simeq X_m \sqcup (X \otimes X_{n-1}) \otimes X_m,$$

and define

$$\mu_{n,m} = (i_{m,n+m}, j_{n+m}(1 \otimes \mu_{n-1,m})) : X_m \sqcup X \otimes X_{n-1} \otimes X_m \to X_{n+m},$$

where $i_{m,n+m} : X_m \to X_{n+m}$ is the inclusion (the composite of the appropriate i_k), and $j_k : X \otimes X_{k-1} \to X_k \simeq I \sqcup X \otimes X_{k-1}$ is the coprojection.

By the fact that $\otimes$ preserves filtered colimits, and the (easily checked) compatibility of the $\mu_{n,m}$ we may pass to the colimit $\mu : X_\infty \otimes X_\infty \to X_\infty$ of the maps $i_{n+m,\infty} \circ \mu_{n,m} : X_n \otimes X_m \to X_\infty$, where $i_{n+m,\infty} : X_{n+m} \to X_\infty$ is the canonical map to the colimit. We also have the unit of our monoid $\eta : I = X_0 \to X_\infty$, given again by the canonical map to the colimit.

This construction is functorial in X. Consider a morphism $f : X \to Y$ over $u : O \to Q$ in $\mathcal{S}$. We set

$$\begin{aligned} &f_0 = I(u) : X_0 = I(O) \to I(Q) = Y_0 \\ &f_{n+1} = I_u \sqcup f \otimes_u f_{n-1}. \end{aligned}$$

Again, the (obvious) compatibility implies the existence of a morphism $f_\infty : X_\infty \to Y_\infty$ (we define $\sqcup$ and f_∞ using corollary 1.2.3), and it can be checked that it is a monoid homomorphism over u, with respect to μ and η.

Theorem 1.5.2. *If $\mathcal{E}$ has fiberwise finite coproducts and $\otimes$ preserves fibered filtered colimits in both variables and binary coproducts in the left variable, then the free monoid functor is $X \mapsto \mathcal{F}(X) = (X_\infty, \mu, \eta)$ on objects, and $f \mapsto f_\infty$ on morphisms.*

Proof. All the calculations in appendix B of [BJT97] clearly apply in each fiber, and η is natural in the entire fibration. The universality of η in the entire fibration follows from theorem 1.2.1, since the forgetful functor from monoids is always a morphism of fibrations. $\square$

We will require some additional facts about the above construction. They were discovered in the course of the proof of the three tensors theorem, but a very similar phenomenon was used in part 2 of [HMP02] under the name "unique readability". If the structures under consideration are multicategories or operads, then the free monoids consist of trees[4] or terms. We will now see that we can "identify" the first vertex in these trees or the first function symbol in these terms.

Proposition 1.5.3. *Under the assumptions of theorem 1.5.2 the multiplication in the free monoid $\mu : X_\infty \otimes X_\infty \to X_\infty$ has a vertical section $\hat{s} : X_\infty \to X_\infty \otimes X_\infty$, which factors as $X_\infty \xrightarrow{s} X_1 \otimes X_\infty \xrightarrow{i \otimes 1} X_\infty \otimes X_\infty$, where $i : X_1 \to X_\infty$ is the canonical map.*

In fact the components of the map s (see the proof) will be more important than either s or $\hat{s}$, which are only necessary for the application of the bootstrap lemma 4.2.1.

Proof. We write $\mu_{1,\infty} : X_1 \otimes X_\infty \to X_\infty$ for the colimit of $\mu_{1,m} : X_1 \otimes X_m \to X_{m+1}$, from the construction above, with respect to m. Hence $\mu_{1,\infty} \circ 1 \otimes i_{m,\infty} = i_{m+1,\infty} \circ \mu_{1,m}$, where $i_{k,\infty}$ is the canonical map $X_k \to X_\infty$. Also $\mu_{1,\infty} = \mu \circ i_{1,\infty} \otimes 1$, as is easily seen by composing both sides on the right with $1 \otimes i_{m,\infty}$. We will construct $s : X_\infty \to X_1 \otimes X_\infty$ such that $\mu_{1,\infty} \circ s = 1_{X_\infty}$, and define $\hat{s}$ via the commutative diagram

$$\begin{tikzcd}
X_\infty \arrow[r, "s"] \arrow[dr, "\hat{s}"'] & X_1 \otimes X_\infty \arrow[r, "\mu_{1,\infty}"] \arrow[d, "i_{1,\infty} \otimes 1"] & X_\infty \\
& X_\infty \otimes X_\infty \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Since $\otimes$ preserves filtered colimits, it suffices to construct a compatible family of maps $s_m : X_m \to X_1 \otimes X_{m-1}$, for $m > 0$, such that $\mu_{1,m-1} \circ s_m = 1_{X_m}$. We have

$$X_n = I \sqcup X \otimes X_{n-1}$$
$$X_1 \otimes X_{n-1} = (I \sqcup X \otimes I) \otimes X_{n-1} \simeq X_{n-1} \sqcup X \otimes X_{n-1},$$

and define

$$s_n = I \sqcup X \otimes X_{n-1} \xrightarrow{i_{0,n-1} \sqcup 1_{X \otimes X_{n-1}}} X_{n-1} \sqcup X \otimes X_{n-1}.$$

In these terms $\mu_{1,n-1}$ is easily found to be

$$\mu_{1,n-1} = (i_{n-1,n}, j_n(1_X \otimes 1_{X_{n-1}})) = (i_{n-1,n}, j_n).$$

[4]With additional structure of course. Note also, that vertices of these trees represent operations and leaves represent inputs, and these are different parts of the structure – we are *not* dealing with ordinary graphs!

We can now calculate $\mu_{1,n-1} \circ s_n$:

$$(i_{n-1,n}, j_n) \circ (i_{0,n-1} \sqcup 1_{X \otimes X_{n-1}}) = (i_{0,n}, j_n),$$

which is the identity $I \sqcup X \otimes X_{n-1} \to X_n$. The compatibility condition for s_n is implied by the stronger condition

$$1 \otimes i_{n-1} \circ s_n = s_{n+1} \circ i_n.$$

Expanding the definitions, it asserts the commutativity of the square

$$\begin{tikzcd}
I \sqcup X \otimes X_n \arrow[r, "i_{0,n} \sqcup 1"] & X_n \sqcup X \otimes X_n \\
I \sqcup X \otimes X_{n-1} \arrow[u, "1 \sqcup 1 \otimes i_{n-1}"] \arrow[r, "i_{0,n-1} \sqcup 1"] & X_{n-1} \sqcup X \otimes X_{n-1} \arrow[u, "i_{n-1} \sqcup 1 \otimes i_{n-1}"']
\end{tikzcd}$$

which is obvious. We may therefore pass to the colimit, and conclude the proof. □

We will call the maps $s, \hat{s}$, constructed above, the *canonical sections* of μ, or *unique readability morphisms*. The following technical lemma is needed in the proof of the three tensors theorem. In the context of operads it asserts the obvious fact that the "first" vertex stays first if we multiply its tree with something on the right (i.e. attach other trees to the leaves).

Lemma 1.5.4 (Coherence lemma)**.** *The following diagram commutes (for $n > 0$)*

$$\begin{tikzcd}
X_n \otimes X_m \arrow[r, "\mu_{n,m}"] \arrow[d, "s_n \otimes 1"'] & X_{n+m} \arrow[r, "s_{n+m}"] & X_1 \otimes X_{n+m-1} \\
(X_1 \otimes X_{n-1}) \otimes X_m \arrow[rr, "\alpha^{-1}"] & & X_1 \otimes (X_{n-1} \otimes X_m) \arrow[u, "1 \otimes \mu_{n-1,m}"']
\end{tikzcd}$$

Proof. We use the coherence theorem to ignore the coherence isomorphisms. Since $X_n \otimes X_m$ is the coproduct $I \otimes X_m \sqcup X \otimes X_{n-1} \otimes X_m$ it suffices to check the commutativity on each factor. Since every s_k is the identity on the second factor it is easy to see that both second factors are $j \circ (1 \otimes \mu_{n-1,m})$, where j is the coprojection as in the construction of μ.

The second factor can be calculated as follows. The lower way is straightforward. It is $\mu_{n-1,m} \circ i_{0,n-1} \otimes 1$, which is (by the unit laws for μ) $i_{m,n+m-1} : W_m \to W_{n+m-1}$. The upper way unfortunately mixes the components, so we must unwind one more level of definition. The relevant component of $\mu_{n,m}$ is $i_{m,n+m}$, which is $1 \sqcup 1 \otimes i_{m-1,n+m-1}$. Composing it with $s_{n+m} = (i_{0,n+m-1}, 1_{X \otimes X_{n+m-1}})$ yields $i_{0,n+m-1} \sqcup 1 \otimes i_{m-1,n+m-1} = i_{m,n+m-1}$, as required. □

Remark 1.5.5. *From now on we will occasionally abuse notation and write i_k for any of the maps $i_{k,l}$. The codomain will always be clear form context.*

1.6 Some Exponentiable Objects in Cat/$\mathcal{S}$

It is well known that **Cat** is not locally cartesian closed. Fortunately there are enough exponentiable objects in $\mathbf{Cat}/\mathcal{S}$ for our purposes.

Theorem 1.6.1. *The bifibrations in the* 2*-category* $\mathbf{Cat}/\mathcal{S}$ *are exponentiable.*

This is well known [G64, J99]. From this theorem we can read off a description of the exponential objects. A more detailed exposition is given in [Z10].

Consider a map $1 \to \mathcal{S}$ picking out an object O. Then, just by the definitions of the various objects involved we obtain a sequence of bijections of morphisms over $\mathcal{S}$:

$$\begin{array}{c} 1 \longrightarrow \mathcal{F}^{\mathcal{E}} \\ \hline 1 \times_S \mathcal{E} \longrightarrow \mathcal{F} \\ \hline \mathcal{E}/O \longrightarrow \mathcal{F} \\ \hline \mathcal{E}/O \longrightarrow \mathcal{F}/O \end{array}$$

Thus, the objects of the category $\mathcal{F}^{\mathcal{E}}$ are functors $\mathcal{E}/O \to \mathcal{F}/O$, with the projection to $\mathcal{S}$ sending them to O. To see what the morphisms are, consider a functor $(\cdot \to \cdot) \longrightarrow \mathcal{S}$ picking out an arrow $u\colon O \to Q \in \mathcal{S}$. Repeating the previous procedure we obtain:

$$\begin{array}{c} (\cdot \to \cdot) \longrightarrow \mathcal{F}^{\mathcal{E}} \\ \hline (\cdot \to \cdot) \times_S \mathcal{E} \longrightarrow \mathcal{F} \\ \hline \end{array}$$

$$\frac{\mathcal{E}/u \longrightarrow \mathcal{F}}{\mathcal{E}/u \longrightarrow \mathcal{F}/u}$$

The interpretation here is not quite as clear as in the previous case. These morphisms correspond to natural transformations fibered over u. The arrows in $\mathcal{E}/u$ mapping to u should be thought of as components of the natural transformation. This strange packaging of the components results from the various type restrictions imposed by working in a fibered context. For a precise statement of what this means, see lemma 4.1 in [Z10]. For a hint of what this might mean (looking at transformations fibered over an identity arrow), see propositions 1.7.3 and 1.7.5 below.

1.7 Distributivity Structures

For any category $\mathcal{C}$ the category of endofunctors $End(\mathcal{C})$ is strict monoidal under composition of functors. The monoidal structure is composition in diagrammatic order (that is $(x)f \circ g$ means "first apply f to x, then g to (x) f", but only if f and g are objects of $End(\mathcal{C})$). If in addition $\mathcal{C}$ is itself monoidal, we obtain functors $\mathcal{C} \to End(\mathcal{C})$ which send each $X \in \mathcal{C}$ to either $X \otimes (-)$ or $(-) \otimes X$. We will always be interested in the latter functor, which we will denote by R. Interestingly these functors are always monoidal in a natural way. Namely we have

$$\begin{aligned} (A)(-) \otimes (X \otimes Y) &= A \otimes (X \otimes Y) \\ (A)(-) \otimes (X) \circ (-) \otimes Y &= (A \otimes X) \otimes Y, \end{aligned}$$

and a natural isomorphism between these two is given by $\alpha^{-1}_{A,X,Y}$. The unit isomorphism is given by the appropriate components of ρ. The coherence diagrams for this monoidal functor are the defining coherence diagrams of a monoidal category (as in [CWM98]), with some morphisms replaced by their inverses.

If $\odot$ is yet another monoidal structure on $\mathcal{C}$, then we can also define a monoidal category $End^{\odot}(\mathcal{C})$ of strong $\odot$-monoidal endomorphisms of $\mathcal{C}$, as follows. The identity functor $1_{\mathcal{C}}$ has an obvious monoidal structure, and will serve as a unit. The tensor is composition of monoidal functors. It is easy to see that the horizontal composite of monoidal transformations is again monoidal, and so we can take as arrows the monoidal natural transformations. This category is of course still strict monoidal.

There is an obvious strict monoidal functor $U : End^{\odot}(\mathcal{C}) \to End(\mathcal{C})$, which forgets the additional data.

Definition 1.7.1. *Let $\mathcal{C}$ be a category, and suppose we are given two monoidal structures on $\mathcal{C}$, denoted $(\odot, I_\odot, \alpha^\odot, \lambda^\odot, \rho^\odot)$ and $(\otimes, I_\otimes, \alpha^\otimes, \lambda^\otimes, \rho^\otimes)$. As above, let R denote the functor $X \mapsto (-) \otimes X$. A* distributivity structure *of $\otimes$ over $\odot$ is given by a lift of R to $End^\odot(\mathcal{C})$ along U, as a monoidal functor:*

$$\begin{tikzcd} & End^\odot(\mathcal{C}) \arrow[d, "U"] \\ \mathcal{C} \arrow[ur, dashed, "\tilde{R}"] \arrow[r, "R"'] & End(\mathcal{C}) \end{tikzcd}$$

which means that we require $R = U \circ \tilde{R}$ as monoidal functors.

We can unravel this definition and state it explicitly as extra data and properties for $\mathcal{C}$. First, every functor $R(X) = (-) \otimes X$ becomes $\odot$-monoidal. This gives us isomorphisms

$$\varphi_{A,B,X} : (A \otimes X) \odot (B \otimes X) \to (A \odot B) \otimes X$$
$$\psi_X : I_\odot \to I_\odot \otimes X$$

which make $R(X)$ into a $\odot$-monoidal functor (which is $\tilde{R}(X)$). Of course for every morphism f in $\mathcal{C}$, the natural transformation $R(f) = (-) \otimes f$ is required to be $\odot$-monoidal (giving $\tilde{R}(f)$). Second, since we require equality of R and $U \circ \tilde{R}$ as monoidal functors, we see that the isomorphisms $(\alpha^\otimes)^{-1} : \tilde{R}(X) \circ \tilde{R}(Y) \to \tilde{R}(X \otimes Y)$ and $\rho^\otimes : 1_\mathcal{C} \to \tilde{R}(I_\otimes)$ giving the monoidal structure of $\tilde{R}$ become $\odot$-monoidal natural transformations. These properties are expressed as commuting diagrams in theorem 1.7.8. Conversely, natural transformations[5] $\varphi_{A,B,X}$ and ψ_X subject to the coherence diagrams in theorem 1.7.8 determine a unique distributivity structure.

Remark 1.7.2. *For this reason we will denote distributivity structures, also in the fibered case, by (φ, ψ).*

Example. Let $(\mathcal{C}, \otimes)$ be a monoidal category with finite coproducts, and suppose that $\otimes$ preserves them in the left variable. Then the natural maps

$$\begin{aligned} A \otimes X \sqcup B \otimes X &\to (A \sqcup B) \otimes X \\ 0 &\to 0 \otimes X \end{aligned}$$

are isomorphisms which define a distributivity structure of $\otimes$ over $\sqcup$. All the conditions are satisfied because of universality.

The preservation of products also defines a distributivity structure, but the directions of the natural arrows are opmonoidal rather than monoidal.

[5]Note that at this point it is not clear that φ and ψ are natural in X. This is demonstrated in next subsection.

1.7.1 Fibered Distributivity

We will now explain how to adapt this definition to the fibered context, and give a theorem characterizing fibered distributivity structures as concrete extra data subject to certain coherence conditions. This theorem will allow us to construct the most important example, the distributivity structure on monoidal signatures with amalgamation 2.4.1, and allow us to apply the three tensors theorem there. The coherence conditions are also used in the proof of the three tensors theorem 4.1.1, and are stated in a form conducive to that proof.

To motivate the passage from monoidal categories to monoidal fibrations, consider a pullback square

$$\begin{tikzcd}
(\cdot \to \cdot) \times \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\cdot \to \cdot \arrow[r] & \cdot
\end{tikzcd}$$

where $\mathcal{C}$ is a monoidal category. The result is a monoidal fibration over $\cdot \to \cdot$, and we have the following proposition.

Proposition 1.7.3. *Let $\mathcal{C}, \mathcal{D}$ be monoidal categories, with $\mathcal{D}$ considered as a monoidal fibration over $\cdot$. Then morphisms of monoidal fibrations*

$$\begin{tikzcd}
(\cdot \to \cdot) \times \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\cdot \to \cdot \arrow[r] & \cdot
\end{tikzcd}$$

are exactly pairs of monoidal functors $F, G \colon \mathcal{C} \to \mathcal{D}$ with a monoidal natural transformation between them.

Proof. We use the usual correspondence between natural transformations and functors from $\mathcal{C} \times (\cdot \to \cdot)$. Writing out all the monoidal definitions explicitly, one sees the equivalence of the extra data immediately. $\square$

Now consider two monoidal bifibrations $\mathcal{E}$ and $\mathcal{F}$ over $\mathcal{S}$. We denote their monoidal structures by $\otimes$ and $\odot$, respectively.

Definition 1.7.4. *We define $\underline{Hom}_{\mathcal{S}}^{\otimes,\odot}(\mathcal{E}, \mathcal{F})$, an object of $\mathbf{Cat}/\mathcal{S}$, as follows:*

1. *Objects are objects of the exponential $\mathcal{F}^{\mathcal{E}}$, together with a monoidal structure on the corresponding functors $\mathcal{E}/O \to \mathcal{F}/O$.*

2. *Morphisms are the morphisms in the exponential $\mathcal{F}^{\mathcal{E}}$, together with a monoidal structure on the corresponding functors $\mathcal{E}/u \to \mathcal{F}/u$ (where $u\colon O \to Q \in \mathcal{S}$), which restricts to the given ones upon restriction to the domain and codomain of u.*

3. *Composition is inherited from $\mathcal{F}^{\mathcal{E}}$, along with the obvious composition of monoidal structures on functors.*

4. *The projection map $\underline{Hom}_{\mathcal{S}}^{\otimes,\odot}(\mathcal{E},\mathcal{F}) \to \mathcal{S}$ is also inherited from $\mathcal{F}^{\mathcal{E}}$.*

By the above proposition, this generalizes the construction of the category of monoidal functors between monoidal categories. Essentially by construction we have the following proposition.

Proposition 1.7.5. *Fibered morphisms $\mathcal{G} \to \underline{Hom}_{\mathcal{S}}^{\otimes,\odot}(\mathcal{E},\mathcal{F})$ over $\mathcal{S}$ correspond to morphisms of monoidal fibrations $\mathcal{G} \times_{\mathcal{S}} \mathcal{E} \to \mathcal{F}$ over $\mathcal{G} \to \mathcal{S}$.*

Proof. Given the existence of the exponential $\mathcal{F}^{\mathcal{E}}$, this is a trivial generalization of the previous proposition. □

Remark 1.7.6.

1. *Note that the distinction between functors and natural transformations is blurred more strongly in the fibered case than it is for ordinary categories.*

2. *Unfortunately, unless the structures $\otimes$ and $\odot$ preserve the prone and supine arrows, $\underline{Hom}_{\mathcal{S}}^{\otimes,\odot}(\mathcal{E},\mathcal{F})$ is not a fibration or opfibration. Since preservation of prone and supine arrows never happens in our examples, we will have to treat it as an ordinary object of* $\mathbf{Cat}/\mathcal{S}$.

3. *There is a natural forgetful fibered functor $\underline{Hom}_{\mathcal{S}}^{\otimes,\odot}(\mathcal{E},\mathcal{F}) \to \mathcal{F}^{\mathcal{E}}$, which forgets the extra data.*

We are now ready to define fibered distributivity structures. Let $\mathcal{E}$ be a monoidal bifibration over $\mathcal{S}$ with two monoidal structures $\otimes$ and $\odot$. As in any category with a cartesian closed subcategory of exponentiable objects, the exponential $\mathcal{E}^{\mathcal{E}}$ is a monoid in $\mathbf{Cat}/\mathcal{S}$, and so in particular a monoidal fibration (even a bifibration). The operation is composition of endomorphisms. The above construction gives us a relative monoidal category

$$\underline{End}_{\mathcal{S}}^{\odot}(\mathcal{E}) = \underline{Hom}_{\mathcal{S}}^{\odot,\odot}(\mathcal{E},\mathcal{E})$$

of $\odot$-monoidal endofunctors of $\mathcal{E}$ and $\odot$-monoidal natural transformations between them. Like before, the monoidal structure is composition – monoidal structures on functors and natural transformations compose strictly. This makes the natural forgetful functor $\underline{End}^{\odot}_{\mathcal{S}}(\mathcal{E}) \to \mathcal{E}^{\mathcal{E}}$ a monoidal functor (in fact a homomorphism of monoids). This allows us to state the definition of a fibered distributivity structure in the same form as for ordinary categories.

Definition 1.7.7. *Let $\mathcal{E}$ be a bifibration over $\mathcal{S}$ with two monoidal structures $\otimes$ and $\odot$. A* fibered distributivity structure *of $\otimes$ over $\odot$ is given by a lift of $R\colon \mathcal{E} \to \mathcal{E}^{\mathcal{E}}$, the exponential adjoint of $\otimes\colon \mathcal{E} \times_{\mathcal{S}} \mathcal{E} \to \mathcal{E}$ for the right variable, as a monoidal functor, to $\underline{End}^{\odot}_{\mathcal{S}}(\mathcal{E})$ along the obvious forgetful functor $U\colon \underline{End}^{\odot}_{\mathcal{S}}(\mathcal{E}) \to \mathcal{E}^{\mathcal{E}}$:*

$$\begin{tikzcd}
 & \underline{End}^{\odot}_{\mathcal{S}}(\mathcal{E}) \arrow[d, "U"] \\
\mathcal{E} \arrow[ur, dashed, "\tilde{R}"] \arrow[r, "R"'] & \mathcal{E}^{\mathcal{E}}
\end{tikzcd}$$

The map R is of course monoidal for the same reason as in the case of ordinary categories.

Note that this is a generalization of the previous definition – a fibered distributivity structure give an ordinary distributivity structure on every fiber, together with some compatibility conditions between the fibers.

We may unwind this definition in the same way as for ordinary categories, and arrive at the following simple theorem, which states the practical criteria for the construction of fibered distributivity structures.

Theorem 1.7.8. *A fibered distributivity structure determines and is determined by fibered natural transformations $\varphi_{A,B,X}$ and ψ_X, as above, subject to coherence conditions I - VII below.*

- *Condition I*

$$\begin{tikzcd}
(A \otimes X) \odot ((B \otimes X) \odot (C \otimes X)) \arrow[r, "\alpha^{\odot}"] \arrow[d, "1 \odot \varphi_{B,C,X}"'] & ((A \otimes X) \odot (B \otimes X)) \odot (C \otimes X) \arrow[d, "\varphi_{A,B,X} \odot 1"] \\
(A \otimes X) \odot (B \odot C) \otimes X \arrow[d, "\varphi_{A,B\odot C,X}"'] & ((A \odot B) \otimes X) \odot (C \otimes X) \arrow[d, "\varphi_{A\odot B,C,X}"] \\
(A \odot (B \odot C)) \otimes X \arrow[r, "\alpha^{\odot} \otimes 1"] & ((A \odot B) \odot C) \otimes X
\end{tikzcd}$$

- *Condition II*

$$\begin{tikzcd}
(A \otimes (X \otimes Y)) \odot (B \otimes (X \otimes Y)) \arrow[r, "\alpha^{\otimes} \odot \alpha^{\otimes}"] \arrow[dd, "\varphi_{A,B,X\otimes Y}"'] & ((A \otimes X) \otimes Y) \odot ((B \otimes X) \otimes Y) \arrow[d, "\varphi_{A\otimes X, B\otimes X, Y}"] \\
 & ((A \otimes X) \odot (B \otimes X)) \otimes Y \arrow[d, "\varphi_{A,B,X} \otimes 1"] \\
(A \odot B) \otimes (X \otimes Y) \arrow[r, "\alpha^{\otimes}"] & ((A \odot B) \otimes X) \otimes Y
\end{tikzcd}$$

- *Condition III*

$$\begin{tikzcd}
 & A \odot B \arrow[dl, "\rho^{\otimes} \odot \rho^{\otimes}"'] \arrow[dr, "\rho^{\otimes}"] & \\
(A \otimes I_{\otimes}) \odot (B \otimes I_{\otimes}) \arrow[rr, "\varphi_{A,B,I_{\otimes}}"] & & (A \odot B) \otimes I_{\otimes}
\end{tikzcd}$$

- *Condition IV, for every morphism* $f : A \otimes X \to Y$

$$\begin{tikzcd}
(I_{\odot} \otimes X) \odot (A \otimes X) \arrow[r, "\psi_X^{-1} \odot f"] \arrow[d, "\varphi"] & I_{\odot} \odot Y \arrow[dd, "\lambda^{\odot}"] \\
(I_{\odot} \odot A) \otimes X \arrow[d, "\lambda^{\odot} \otimes 1"'] & \\
A \otimes X \arrow[r, "f"] & Y
\end{tikzcd}$$

- *Condition V*

$$\begin{tikzcd}
I_{\odot} \otimes (X \otimes Y) \arrow[r, "\alpha^{\otimes}"] \arrow[dr, "\psi_{X\otimes Y}^{-1}"'] & (I_{\odot} \otimes X) \otimes Y \arrow[r, "\psi_X^{-1} \otimes 1"] & I_{\odot} \otimes Y \arrow[dl, "\psi_Y^{-1}"] \\
 & I_{\odot} &
\end{tikzcd}$$

- *Condition VI*

$$\psi_{I_{\otimes}} = \rho^{\otimes}_{I_{\odot}}$$

- *Condition VII*

$$\begin{tikzcd}
(A \otimes X) \odot I_\odot \arrow[d, "1 \odot \psi_X"'] & & A \otimes X \arrow[ll, "\rho^\odot"'] \arrow[d, "\rho^\odot \otimes X"] \\
(A \otimes X) \odot (I_\odot \otimes X) \arrow[rr, "\varphi_{A,I_\odot,X}"] & & (A \otimes I_\odot) \otimes X
\end{tikzcd}$$

Proof. First, the fact that $\tilde{R}(f)$ is $\odot$-monoidal for all morphisms f is equivalent to the fact that ϕ and ψ giving the monoidal structure for $\tilde{R}(X)$ are natural in X. Conditions I, IV and VII say that $\tilde{R}(X)$ is $\odot$-monoidal. Condition IV is the left unit condition combined with the naturality of $\lambda^\odot$. Conditions III and VI are the requirement that $\rho : 1_{\mathcal{C}} \to \tilde{R}(I_\otimes)$ is an $\odot$-monoidal natural transformation. Conditions II and V say the same for $(\alpha^\otimes)^{-1} : \tilde{R}(X) \circ \tilde{R}(Y) \to \tilde{R}(X \otimes Y)$. $\square$

Remark 1.7.9. *The conditions are listed in the form in which they are used in the proof of the three tensors theorem. Condition VII will not be used in the proof. Thus the three tensors theorem is true if we lift R to $\underline{End}_{\mathcal{S}}^{\odot,L}(\mathcal{E})$, the category of left-unital $\odot$-monoidal functors.*

Chapter 2

Bifibrations of Signatures and Their Actions

In this chapter we construct the three basic bifibrations of signatures that will be used throughout our work. These are the symmetric signatures, signatures with nonstandard amalgamation, and monoidal signatures with nonstandard amalgamation.

All three types of signatures are constructed from ordinary multisorted (or typed) signatures, as used in logic and universal algebra, the category of which is denoted by **Sig**. These are sets of function symbols, with arbitrary arity (arity 0 indicating a constant – a nullary function). Every input, as well as the output, of every function symbol has a specified type, and types form an additional set, which is considered part of the structure of the signature. These types control which outputs can be matched to which inputs, that is, which function symbols can be composed with each other. The bifibration projection maps every signature to its set of types, and the prone and supine arrows correspond to maps which universally retype or refactor, respectively, the function symbols of a signature, according to a function which changes the types.

Here is an example: the signature of "rings and modules" would have two types – one for the elements of a ring, and another for the elements of a module, as well as function symbols that define ring and module operations (all those derivable from the basic ones listed in the axioms), one of which would allow the ring to act on the module.

The requirement of matching output and input types leads directly to the monoidal structure on ordinary signatures. If A and B are signatures with the same set of types, then

$$A \otimes_O B$$

is a new signature with the same set of types, and consists of formal, correctly typed, composites of function symbols from A with B. Monoids for this monoidal structure correspond to certain very specific equational theories. Returning to our example of rings and modules, this signature has a composition map

$$M \otimes M \to M$$

which maps formal composites to actual composites. This encodes the axioms of the theory of rings and modules – the fact that addition and multiplication in rings is associative is expressed by the formal composites

$$\mu(\mu(x_1, x_2), x_3),$$

where the x_i are formal variables, and

$$\mu(x_1, \mu(x_2, x_3)),$$

where μ is the appropriate multiplication (or addition) map, are sent by composition $M \otimes M \to M$ to the same function symbol in M. Associativity of the composition map $M \otimes M \to M$, and the other monoid axioms, express the inference rules for equational logic. We will investigate equational theories in much more detail in chapter 7.

From this basic bifibration we construct all the others. This is done with the aid of the symmetrization monad $\mathcal{S}$. This monad arises from the ability to permute the inputs of a function symbol (while simultaneously permuting the typing). Since permutations can be composed, we obtain a monad, which acts on the n-ary function symbols as

$$\mathcal{S}(X)_n = X_n \times S_n,$$

where S_n is the n-th symmetric group.

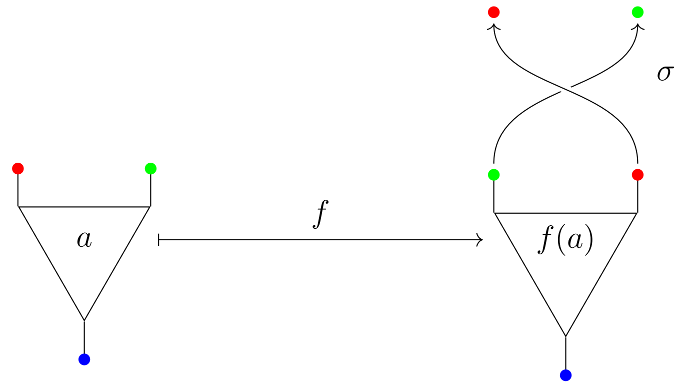
The symmetric signatures and signatures with nonstandard amalgamation are defined, respectively, as the Eilenberg-Moore and Kleisli algebras for this monad. The interpretation of symmetric signatures should be clear – these are just signatues eqiupped with right S_n-actions on the n-ary function symbols, and equivariant morphisms. The traditional interpretation of Kleisli algebras as the free algebras is, however, likely to confuse the reader. We will systematically use the intrinsic construction of Kleisli algebras (as in [CWM98]), without reference to the Eilenberg-Moore category. This is essential to understanding our use of these signatures.

To understand the Kleisli algebras on their own, consider their morphisms:

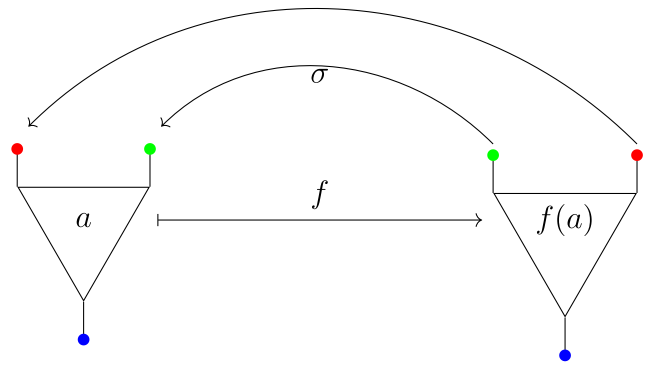
$$X \to \mathcal{S}(Y).$$

Thus, a morphism of Kleisli algebras $f\colon X \to Y$ is a morphism of ordinary signatures $X \to \mathcal{S}(Y)$. For us this means that every morphism of signatures with nonstandard amalgamation has, in some sense, two components: a function assigning function symbols to function symbols (but *not* a morphism of signatures, since the typings do not match), and a parallel assignment of permutations to function symbols, which together form a legitimate morphism of ordinary signatures.

These permutations, called amalgamation permutations, should be thought of as follows: if a is mapped by f to (b, σ), then σ maps the inputs of b to those of a (note the order) in a way that respects the typing. This shift in thinking may be presented pictorially as follows. This morphism of Kleisli algebras:



should be thought of as this morphism of signatures with nonstandard amalgamation



Both symmetric signatures and signatures with nonstandard amalgamation are monoidal bifibrations. This follows from abstract considerations, which are, however, too lengthy to reproduce here. We will simply write down explicit definitions for these structures. This makes their relationship to the original structure on **Sig** clear enough.

Monoids with respect to these structures range from the well known, to the exotic. Monoids in **Sig** are simply Lambek's multicategories. Accordingly, monoids in symmetric signatures are the symmetric multicategories, which are also known as typed symmetric operads.

The exotic part comes from signatures with nonstandard amalgamation. These were originally introduced (in a 2-leveled version) in part 2 of [HMP02] as multicategories with nonstandard amalgamation". This is how the signatures $\mathbf{Sig}_{\mathcal{S}}$ got their name. A monoid in signatures with nonstandard amalgamation is a multicategory, in which after each composition operation the inputs of the formal composite (i.e. the symbols we are composing) are permuted into the actual composite (compatibly with types). These permutations must, of course, also be associative in an appropriate way. All this is built into the monoidal structure on signatures with nonstandard amalgamation.

One of the central goals of this chapter is the construction of monoidal signatures with amalgamation, denoted $\mathbf{Sig}_{ma}$. These are constructed from signatures with nonstandard amalgamation by base change, which makes the types of each monoidal signature remember that they form (the set of function symbols of) a monoid in $\mathbf{Sig}_a$. By the ever-useful "pullbacks preserve algebra" corollary 1.1.12, they inherit a monoidal structure $\otimes$ from $\mathbf{Sig}_{\mathcal{S}}$, as well as an action, called the pullback action (we will discuss actions of signatures in a moment).

But the most important structures on $\mathbf{Sig}_{ma}$ must be constructed by hand. They are the horizontal monoidal structure $\odot$ (the obvious one, $\otimes$, is called vertical), and a distributivity structure of $\otimes$ over $\odot$. These structures formalize the intuitions presented in the introduction to chapter 1 – that trees can be considered to have two independent kinds of inputs. The horizontal structure $\odot$ corresponds to leaves, and $\otimes$ corresponds to the nodes.

They also explain why we bother with nonstandard amalgamation (and, as the reader will see, it is a huge bother). The inputs are defined easily enough, but for the outputs, we must have a way to reduce a tree to a single node, so that a tree can be matched to a node. This is exactly what the structure of a monoid (i.e. multicategory) on the types allows us to accomplish. Unfortunately, this monoid must have nonstandard amalgamation – if we want to replace nodes with trees, then the lists of nodes in the incoming trees and the final result must be permuted nontrivially. This is proven in the example in section 4.4, and stems from the fact that nodes can have many neighbors (children and a parent), and the elements of a list can have only two. Thus, if we want to iterate this construction, as we must in the construction of opetopic sets, the multicategory structure on the types must be allowed to have nonstandard amalgamation.

This is how the requirements of opetopic sets determine the bifibration $\mathbf{Sig}_{ma}$

in an essentially unique way. The results of chapter 6 reinforce this view.

Monads have algebras, as do operads, multicategories, and even ordinary categories (presheaves). These are all described as actions of a monoid along an action of a monoidal category. Our monoids are no different: all our bifibrations come equipped with actions, which will play a huge role in our work. For symmetric signatures and signatures with nonstandard amalgamation, these are the so-called "tautological actions". They are all uniformly defined, and correspond to the usual notion of an algebra of a symmetric multicategory (operad), or an action of a multicategory, in the case of ordinary signatures. For nonstandard amalgamation, we simply take the permutations into account (but do not add any new ones, beyond those coming from the monoid structure).

For monoidal signatures, the action is defined by pullback, and so is called the pullback action. It is fiberwise isomorphic to the action of signatures with nonstandard amalgamation.

These actions provide an important link with monads, which is contained in theorem 2.6.2. It states that symmetric signatures are monoidally equivalent to analytic functors (of many variables) and analytic natural transformations, and that signatures with nonstandard amalgamation are monoidally equivalent to polynomial functors. Especially the second kind of functor seems to be much better known and liked than the rather exotic signatures with nonstandard amalgamation.

Theorem 2.6.2 implies that symmetric multicategories and multicategories with nonstandard amalgamation are equivalent to analytic and polynomial monads, respectively. This fact will play a prominent role in our comparisons in chapter 6.

We end this chapter by collecting the various categorical properties of signatures, such as the existence and preservation of filtered colimits. This is a technical section, and the reader may safely skip it, only to refer to its results as needed.

Notation Reminder

Recall that $[n] = \{0, \ldots, n\}$, and $(n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. In particular $[n] = [0] \cup (n]$ and $(0] = \emptyset$. For a set O we define $O^\dagger_n = O^{[n]}$, $O^*_n = O^{(n]}$ and $O^\dagger = \bigcup_{n\in\mathbb{N}} O^{[n]}$, $O^* = \bigcup_{n\in\mathbb{N}} O^{(n]}$. Thus O^* is the set of finite lists of elements of O, and $O^\dagger$ is the set of lists with a chosen first element.

S_n, the n-th symmetric group, acts on $O^\dagger_n$ i O^*_n on the right by precomposition (the lists are functions on $[n]$), leaving 0 fixed.

If $d\colon [n] \to O$ is a list, then we denote its restrictions of positive numbers by $d^+\colon (n] \to O$, and its restriction to $[0]$ by $d^-\colon [0] \to O$. This establishes a bijection $\langle (-)^-; (-)^+ \rangle : O^\dagger \to O \times O^*$. We have an obvious functor $(-)^\dagger\colon \mathbf{Set} \to \mathbf{Set}$.

2.1 Basic Definitions

2.1.1 Ordinary Signatures

Let $(-)^\dagger\colon \mathbf{Set} \to \mathbf{Set}$ denote the functor of finite lists with a chosen first element. That is

$$X^\dagger = X \times X^*,$$

where $(-)^*$ denotes the functor of finite (possibly empty) lists.

Definition 2.1.1. *The bifibration of* ordinary signatures $\mathbf{Sig} \to \mathbf{Set}$ *is defined by the pullback square*

$$\begin{tikzcd}
\mathbf{Sig} \arrow[r] \arrow[d] & \mathbf{Set}^{\cdot\to\cdot} \arrow[d, "cod"] \\
\mathbf{Set} \arrow[r, "(-)^\dagger"] & \mathbf{Set}
\end{tikzcd}$$

Since **Set** has all pullbacks, this is indeed a bifibration, owing to our introductory examples. The objects of **Sig** can be identified with functions

$$\partial\colon X \to O^\dagger$$

which should be thought of as assigning typings to functions symbols from X. The typing of $f \in X$ is $\partial(f) \in O^\dagger$, and consists of an element of O – the output type, and a list of elements of O – the input types. Thus the objects of **Sig** are indeed multisorted signatures, as used in logic and universal algebra, but without relation symbols. The morphisms are exactly what one would expect – functions transforming function symbols along compatible type changes.

We will refer to signatures (of all the various kinds defined below) by their codomain, and denote by X_n the set of function symbols in X which have input lists of length exactly n – the n-ary function symbols in X.

Typing functions for signatures will be decorated by indices, to indicate which signature they apply to, and to use currying. Thus for example

$$\partial^X_f(n)$$

means the evaluation at $n \in \mathbb{N}$ of the function

$$\partial(f)\colon |f| \to O,$$

in $O^\dagger$ (where $|f|$ is the arity of f) for the signature X, i.e. $\partial\colon X \to O^\dagger$.

2.1.2 The Symmetrization Monad

We can define a fibered monad $\mathcal{S}\colon \mathbf{Sig} \to \mathbf{Sig}$ as follows:

$$\mathcal{S}(X)_n = X_n \times S_n,$$

where S_n is the n-th symmetric group. The typing of $(f, \sigma) \in S(X)_n$ is defined by

$$\partial(f, \sigma)^+ = \partial(f)^+ \circ \sigma,$$

with the output types unchanged. This means that the permutation σ attached to the function symbol f permutes its inputs.

The composition map $\mathcal{S}^2 \to \mathcal{S}$ and unit $1_{\mathbf{Sig}} \to \mathcal{S}$ are defined by composition in the symmetric groups and their units. It is clear that this defines a monad. Since taking products commutes with taking pullbacks, the following proposition is obvious.

Proposition 2.1.2. *$\mathcal{S}$ is a morphism of bifibrations.*

2.1.3 Signatures with Nonstandard Amalgamation, Symmetric Signatures

Definition 2.1.3.

1. *The bifibration of* symmetric signatures *is the Eilenberg-Moore bifibration $\mathbf{Sig}^{\mathcal{S}}$. It will be denoted by $\mathbf{Sig}_s$.*

2. *The bifibration of* signatures with nonstandard amalgamation *is the Kleisli bifibration $\mathbf{Sig}_{\mathcal{S}}$. It will be denoted by $\mathbf{Sig}_a$.*

By propositions 1.3.4 and 2.1.2, these are indeed bifibrations over **Set**. The connection with previous work is contained in the following proposition.

Proposition 2.1.4. *The bifibrations of symmetric signatures and signatures with nonstandard amalgamation are isomorphic to the objects with the same names defined in sections 6 and 7 of [Z10].*

Proof. A morphism of signatures with nonstandard amalgamation $(f, \sigma)\colon A \to B$ is equivalent to a morphism of Kleisli algebras $\tilde{f}\colon A \to \mathcal{S}(B)$, with

$$\tilde{f}(a) = (f(a), \sigma_a) \in B_n \times S_n = \mathcal{S}(B)_n.$$

Composition is easily seen to coincide in both pictures. This establishes one isomorphism.

The other isomorphism is even simpler: since $\mathcal{S}$ is just multiplication by the symmetric groups, its algebras are families of right S_n-sets, indexed by the arity n. This is exactly what the symmetric signatures are. □

2.2 Monoidal Structures

2.2.1 Monoidal Structure on Ordinary Signatures

If A and B are signatures over O, then we set

$$A \otimes_O B = \{\langle a, b_i \rangle_i : a \in A, b_i \in B, \partial_a(i) = \partial_{b_i}(0), \text{for } i = 1 \dots |a|\},$$

which is to be thought of as the signature of formal composites of symbols from A and B. Note that we allow $|a| = 0$, which means a nullary formal composite (no b_i)[1]. For the typing we set

$$\begin{aligned} \partial^{A \otimes_O B, -}_{\langle a, b_i \rangle} &= \partial^-_a \\ \partial^{A \otimes_O B, +}_{\langle a, b_i \rangle} &= \coprod_i \partial^+_{b_i}, \end{aligned}$$

which means that the output type of $\langle a, b_i \rangle$ is the output type of a, and the input types are those of the b_i placed side by side, in order of increasing i.

For morphisms f, g over $u \colon O \to Q$ we set

$$f \otimes_u g(\langle a, b_i \rangle) = \langle f(a), g(b_i) \rangle,$$

For the unit we set $I(O) = \partial^{I_O} \colon O \to O^\dagger$, which assigns to every $o \in O$ the unary typing with constant value o. This defines a fibered functor in an obvious way.

The coherence isomorphisms α, λ, and ρ are given by

$$\begin{aligned} \alpha_{A,B,C}(\langle a, \langle b_i, c_{i,j} \rangle \rangle) =& \langle \langle a, b_i \rangle, c_{i,j} \rangle \\ \lambda_A(\langle 1_{\partial^A_a(0)}, a \rangle) =& a \\ \rho_A(a) =& \langle a, 1_{\partial^A_a(1)}, \dots, 1_{\partial^A_a(n)} \rangle, \end{aligned}$$

with the double index (i, j) is ordered lexicographically.

Theorem 2.2.1. *The structure given above defines a monoidal structure on the bifibration* **Sig***.*

Proof. This is a trivial computation consisting of rearranging brackets. ☐

[1] Without such composites associativity fails, among many other important things.

2.2.2 Monoidal Structure on The Symmetrization Monad

The Operad of Symmetries

This ordinary operad is necessary to describe the monoidal structures on the monad S and consequently on $\mathbf{Sig}_s$ and $\mathbf{Sig}_a$. Using it systematically will allow us to minimize our computational effort. Let S_n be the n-th symmetric group. The operad of symmetries $\mathbf{S}$ has the S_n as the sets of n-ary operations, and composition is defined by ([Le04]):

$$\sigma * (\rho_1, \ldots, \rho_n)(k_1 + \ldots + k_{i-1} + j) = k_{\sigma^{-1}(1)} + \ldots + k_{\sigma^{-1}(\sigma(i)-1)} + \rho_i(j),$$

where $\sigma \in S_n$, $\rho_i \in S_{k_i}$, and $1 \leq i \leq n$, $1 \leq j \leq k_i$

This just means that we permute n disjoint blocks according to σ, and apply ρ_i in the block that was i-th in the beginning.

It will be convenient to adopt a notation which allows us to compute compositions of permutations of the form $\sigma * (\rho_1, \ldots, \rho_n)$ in certain cases. First we will assume the number of blocks is fixed and equal to n. We will then write (i, j) for the j-th entry in the i-th block. For each block the number of entries is arbitrary. Thus, by definition, we have

$$\sigma * (\rho_1, \ldots, \rho_n)(i, j) = (\sigma(i), \rho_i(j))$$

This will allow us to compose such permutations easily, but only if the block lengths match. This will never be a problem, since we will always know that this happens beforehand.

With this notation in hand we can compute compositions and inverses of permutations arising from the operad of symmetries.

Lemma 2.2.2. *If $\sigma, \sigma' \in S_n$ and ρ_i, ρ'_i are permutations, such that the domains of ρ_i and $\rho'_{\sigma(i)}$ are equal, then*

$$\sigma' * (\rho'_1, \ldots, \rho'_n) \circ \sigma * (\rho_1, \ldots, \rho_n) = (\sigma' \circ \sigma) * (\rho'_{\sigma(1)} \circ \rho_1, \ldots, \rho'_{\sigma(n)} \circ \rho_n) \quad (2.1)$$

$$\sigma * (\rho_1, \ldots, \rho_n)^{-1}(i, j) = (\sigma^{-1}(i), \rho^{-1}_{\sigma^{-1}(i)}(j)) \quad (2.2)$$

Proof. The assumptions allow us to use the double index notation. We have

$$\sigma * (\rho_1, \ldots, \rho_n)(i, j) = (\sigma(i), \rho_i(j))$$

applying $\sigma' * (\rho'_1, \ldots, \rho'_n)$ to this (this is where we use the assumptions on the domains) yields

$$(\sigma'(\sigma(i)), \rho'_{\sigma(i)}(\rho_i(j)))$$

proving the first formula. The second one follows from the first by applying $\sigma * (\rho_1, \ldots, \rho_n)$ to both sides. $\square$

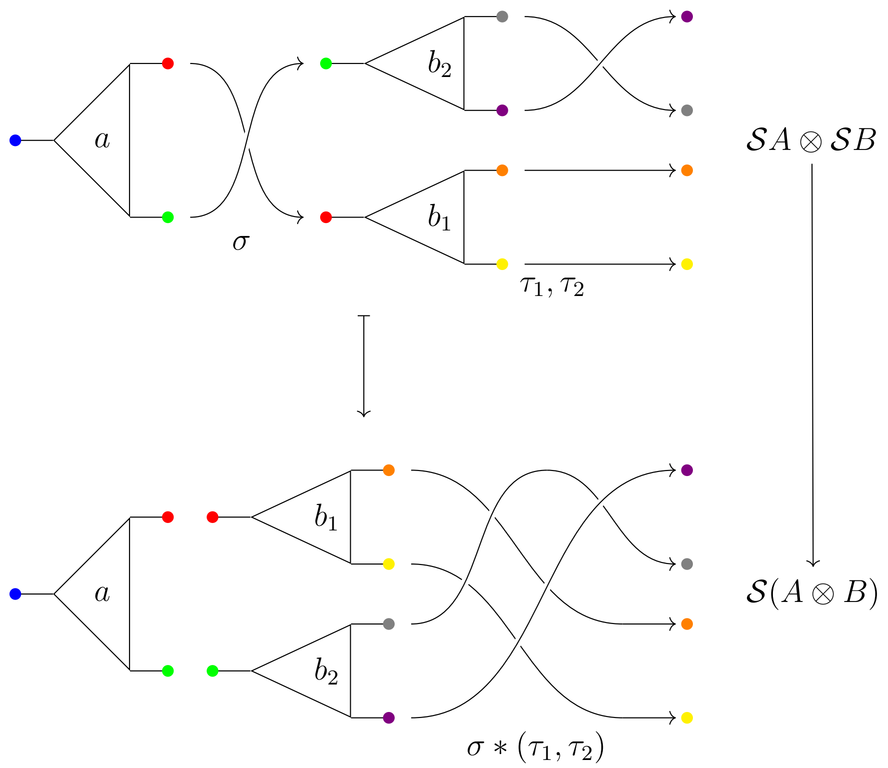
We will occasionaly use longer indices, for example (i, j, k), but we will not calculate with them. However such calculations would be justified, since $*$ is associative.

The Monoidal Structure Proper

We are now ready to define the lax monoidal structure on the monad $\mathcal{S}$. The structure morphisms are defined by

$$\begin{gathered} \mathcal{S}(X) \otimes \mathcal{S}(Y) \to \mathcal{S}(X \otimes Y) \\ \langle (x, \sigma), (y_i, \tau_i) \rangle \mapsto (\langle x, y_{\sigma^{-1}(i)} \rangle, \sigma * (\tau_i)) \\ I \to \mathcal{S}(I) \\ o \mapsto (o, id), \end{gathered}$$

where $*$ refers to composition in the operad of symmetries $\mathbf{S}$ (not to be confused with the symmetrization monad $\mathcal{S}$). Note that these are not isomorphisms. This monoidal structure can be illustrated by the picture below, following the pictorial conventions set in the introduction.



Theorem 2.2.3. *With the above definitions, the monad $\mathcal{S}$ is a monad on* **Sig***, considered as an object of the* 2*-category* **MonFib***.*

Proof. The coherence conditions are satisfied because multiplication in the operad of symmetries is associative and unital, and one can use the formulas in lemma 2.2.2 to explicitly calculate everything[2]. $\square$

2.2.3 Monoidal Structures on $\mathbf{Sig}_a$ and $\mathbf{Sig}_s$

Given the monoidal structure on the symmetrization monad, the algebra objects $\mathbf{Sig}_{\mathcal{S}}$ and $\mathbf{Sig}^{\mathcal{S}}$ are also monoidal[3], for general reasons. These reasons are, unfortunately, somewhat complicated in the Eilenberg-Moore case, so we will simply write down the monoidal structures explicitly.

The Monoidal Structure on $\mathbf{Sig}_a$

The formulas in this case are essentially the same as for **Sig**, but taking into account the amalgamation permutations.

If A and B are signatures over O, then we set

$$A \otimes_O B = \{\langle a, b_i\rangle : a \in A, b_i \in B, \partial_a(i) = \partial_{b_i}(0), \text{for } i = 1 \ldots |a|\},$$

This formula is the same as before, and we define the typing map in exactly the same way. For morphisms, we must know how to treat the permutations. For morphisms f, g over $u \colon O \to Q$, with permutations given by σ and τ, respectively, we set

$$f \otimes_u g(\langle a, b_i\rangle) = \langle f(a), g(b_{\sigma_a^{-1}(i)})\rangle,$$

and set the permutations of the morphism $f \otimes_u g$ to be

$$(\sigma \otimes_u \tau)_{\langle a, b_i\rangle} = \sigma_a * (\tau_{b_1}, \ldots, \tau_{b_{|a|}}).$$

The unit object and structure maps are given by the same formulas as for **Sig**.

The Monoidal Structure on $\mathbf{Sig}_s$

We set

$$A \otimes_O B = \{\langle a, b_i, \sigma\rangle : a \in A, b_i \in B, \partial_a(i) = \partial_{b_i}(0), i = 1 \ldots |a|, \sigma \in S_n, n = \Sigma_i |b_i|\}/\sim,$$

[2] A significantly more complicated example of such calculations is presented in full detail in the next section, when discussing the horizontal monoidal structure $\odot$. See also theorem 3.3 in [SZ13].

[3] They have the requisite universal properties in **MonFib**.

where the equivalence relation $\sim$ is given by

$$\langle a \cdot \tau, b_{\tau(i)} \cdot \sigma_{\tau(i)}, \sigma \rangle \sim \langle a, b_i, \tau * (\sigma_{\tau(1)}, \ldots, \sigma_{\tau(|a|)}) \circ \sigma \rangle,$$

where, as usual, $*$ denotes composition in the operad of symmetries.

The typing map is defined by

$$\partial^{A \otimes_O B, -}_{[\langle a, b_i, \sigma \rangle]_\sim} = \partial^{A,-}_a$$

and the requirement that the square

$$\begin{tikzcd}
\Sigma_i |b_i| \arrow[r, "\sigma^{-1}"] & \Sigma_i |b_i| \arrow[d, "\partial^{A \otimes_O B, +}"] \\
|b_i| \arrow[u] \arrow[r, "\partial^B_{b_i}"] & O
\end{tikzcd}$$

commutes, where the unnamed arrow is the canonical coprojection into the coproduct. This is essentially the definition we gave above, with the inputs permuted by σ. The symmetric action is given by

$$[\langle a, b_i, \sigma \rangle]_\sim \cdot \tau = [\langle a, b_i, \sigma \circ \tau \rangle]_\sim$$

With these definitions we obtain a well defined object $A \otimes_O B$ in $\mathbf{Sig}_s/O$.

The unit object is defined as for ordinary signatures. The associativity and unit isomorphisms are defined analogously to the case of ordinary signatures, at the level of representatives of equivalence classes.

We also define the action of morphisms via representatives of equivalence classes:

$$f \otimes_u g([\langle a, b_i, \sigma \rangle]_\sim) = [\langle f(a), g(b_i), \sigma \rangle]_\sim$$

Proposition 2.2.4. *The isomorphisms of proposition 2.1.4 are strict monoidal.*

Proof. In both cases all the formulas are the same on both sides of the isomorphism. □

Proposition 2.2.5. *The canonical functor* $\mathbf{Sig}_{\mathcal{S}} \to \mathbf{Sig}^{\mathcal{S}}$ *is strong monoidal.*

Proof. Direct computation. □

This proposition will become obvious later on, once we define the tautological actions.

2.3 Monoidal Signatures with Amalgamation

Definition

The bifibration of monoidal signatures with amalgamation, denoted $\mathbf{Sig}_{ma}$, is defined by the following pullback square

$$\begin{tikzcd} \mathbf{Sig}_{ma} \arrow[r] \arrow[d] & \mathbf{Sig}_a \arrow[d] \\ Mon(\mathbf{Sig}_a) \arrow[r, "\mathcal{U}"] & \mathbf{Set} \end{tikzcd}$$

where $\mathcal{U}\colon Mon(\mathbf{Sig}_a) \to \mathbf{Set}$ is the functor that maps a monoid to its set of function symbols.

Remark 2.3.1. *Note that the canonical functor $\mathbf{Sig}_a \to \mathbf{Sig}_s$ does not preserve this set of function symbols, because it maps a signature X to $\mathcal{S}(X)$ – a significantly larger set. This is the ultimate reason for the failure of the original Baez-Dolan approach to be equivalent to all the others. We will come back to this point throughout section 6.2.*

Thus a monoidal signature is a map $X \to M^\dagger$, where M is the set of function symbols of a monoid (and this monoid structure is retained). The morphisms are maps of signatures with nonstandard amalgamation over morphisms of monoids in $\mathbf{Sig}_a$. It is important to keep in mind that these morphisms of monoids have their own amalgamation permutations, which are retained. This will be important in the construction of the second, horizontal, monoidal structure $\odot$, below.

Since pullbacks preserve algebra (corollary 1.1.12), the monoidal signatures carry a monoidal structure, denoted by $\otimes$, and a strict monoidal forgetful functor $\mathbf{Sig}_{ma} \to \mathbf{Sig}_a$, fibered over $\mathcal{U}$. This means that all the formulas we have given for signatures with nonstandard amalgamation are also valid for $\mathbf{Sig}_{ma}$, and we will not repeat them here.

In particular, every $\otimes$-monoid in $\mathbf{Sig}_{ma}$ determines a monoid in $\mathbf{Sig}_a$. This applies especially to all instances of the web monoid (defined later) – if we are dealing with a single fiber, we will not distinguish between $\mathbf{Sig}_{ma}$ and $\mathbf{Sig}_a$, since the forgetful functor $\mathbf{Sig}_{ma} \to \mathbf{Sig}_a$ is a fiberwise isomorphism.

2.3.1 The Second Monoidal Structure $\odot$ on $\mathbf{Sig}_{ma}$

Consider an object $A \in \mathbf{Sig}_{ma}$ over a monoid M. It has a typing $A \to M^\dagger$, and we can consider the output type $A \to M^\dagger \to M$. Since M is a monoid over some set O, it has its own typing $M \to O^\dagger$. The composite $A \to M^\dagger \to M \to O^\dagger$ gives us a typing of A over O. Thus every $a \in A$ has *two* kinds of inputs and outputs. The ones just defined will be called horizontal, the old ones will be called vertical. As a set we define

$$A \odot_M B = A \otimes_O B,$$

which means

$$\{\dot{\langle} a, b_i \dot{\rangle} \colon \partial^M_{\partial^A_a(0)}(i) = \partial^M_{\partial^B_{b_i}(0)}(0)\},$$

where i ranges over $(0,k]$ with $k = |\partial^A_a(0)|$ (arity computed in M). We will use the notation $\dot{\langle} \dots \dot{\rangle}$ and $\langle \dots \rangle$ to distinguish between elements of $A \odot_M B$ and $A \otimes_M B$. To ease notation we will write $\check{a}$ for $\partial^A_a(0)$. We must define the typing of this set, that is a function $A \odot_M B \to M^\dagger$. The output type is the composite

$$\partial^{\odot,-}_{\dot{\langle} a,b_i \dot{\rangle}} = A \otimes_O B \to M \otimes_O M \xrightarrow{\mu} M,$$

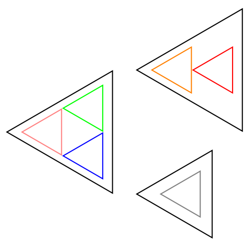
where the arrows from A and B to M are the output types (considered as morphisms in $\mathbf{Sig}_a$ with trivial amalgamation), used to define the horizontal typing, and μ is multiplication in M. Using our notation we can write

$$\partial^{\odot,-}_{\dot{\langle} a,b_i \dot{\rangle}} = \mu(\check{a}, \check{b}_i).$$

The inputs are just the inputs of a and b_i concatenated in order

$$\partial^{\odot,+}_{\dot{\langle} a,b_i \dot{\rangle}} = [\partial^{A,+}_a, \partial^{B,+}_{b_1}, \dots, \partial^{B,+}_{b_k}] \colon (n_0 + n_1 + \dots n_k] \to M$$

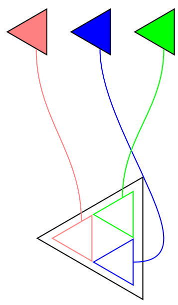
The reader should imagine that a and b_i are arranged on a level surface ("horizontally") and all the vertical inputs (including those of a) are visible "from above", and are available in forming $(A \odot_M B) \otimes_M C$. We may illustrate this by the following picture:



In the above picture, the big black triangles represent function symbols. The interior triangles are the vertical inputs of different types (colors). The whole picture is a single element of $A \odot B$, and all the interior triangles are its vertical inputs.

The horizontal inputs, used to construct this formal composite, are implicit, and should be imagined as dangling off the black triangles, like in the introduction. Similarly, the outputs of either kind are not shown, since this picture will be part of more complicated wholes, above theorem 2.4.1 and below theorem 4.1.1.

In this convention, the $\otimes$ monoidal structure on $\mathbf{Sig}_{ma}$ looks as follows:



Here we have illustrated a single element of $A \otimes B$. The solid triangles are from B, and attach to appropriately typed horizontal inputs in an element of A (the large black triangle).

We need to describe the values of $\odot$ on morphisms. It is easy to see that a morphism in $\mathbf{Sig}_{ma}$ is completely described by a quintuple (f, τ, u, σ, v), where (u, σ, v) is a homomorphism of monoids $M \to N$ in $\mathbf{Sig}_a$ over some function v in $\mathbf{Set}$, and (f, τ) is a morphism in $\mathbf{Sig}_a$ from $A \to M^\dagger$ to $B \to N^\dagger$ over u. Let f and f' be two such homomorphisms. We define

$$f \odot f'(\dot{\langle} a, b_1, \ldots, b_k \dot{\rangle}) = \dot{\langle} f(a), f'(b_{\sigma_{\breve{a}}^{-1}(1)}), \ldots, f'(b_{\sigma_{\breve{a}}^{-1}(k)}) \dot{\rangle}$$

as a function. The permutation $\tau \odot \tau'$ permutes the *vertical* inputs of the formal composites according to τ for the inputs from a and τ' for inputs from the b_i, and places the blocks in which these inputs are arranged on the block belonging to its image. Formally we define $\tau \odot \tau'$ as follows

$$\tau \odot \tau'_{\dot{\langle} a, b_i \dot{\rangle}} = (1, \sigma_{\breve{a}}) * (\tau_a, \tau'_{b_1}, \ldots, \tau'_{b_k}).$$

where $(1, \sigma)$ means the coproduct of the identity on the singleton and σ (conjugated by a translation to act on $[2, k+1]$). Using the same notation for more general permutations we could have written

$$\tau \odot \tau'_{\dot{\langle} a, b_i \dot{\rangle}} = (\tau_a, \sigma_{\breve{a}} * (\tau'_{b_1}, \ldots, \tau'_{b_k}))$$

The entire morphism $f \odot f'$ is now the quintuple $(f \odot f', \tau \odot \tau', u, \sigma, v)$.

The unit $I_\odot$ takes values $I_\odot(M)$, which are defined to be $\partial_{I_\odot(M)} \colon O \to M^\dagger \simeq M \times M^*$. The first factor is the output type, and $\partial_{I_\odot(M)}(o)$ takes the value $e(o)$

on this factor, where $e\colon O \to M$ is the unit of multiplication in M. The inputs are defined to be empty for all $o \in O$.

We must define the coherence isomorphisms. Both $\rho^{\odot}$ and $\lambda^{\odot}$ are defined analogously to the previous case, replacing $\langle \dots \rangle$ with $\dot{\langle} \dots \dot{\rangle}$. They are given by

$$\begin{array}{rcl} \lambda_A(\dot{\langle} 1_{\partial^M_{\check{a}}(0)}, a \dot{\rangle}) & = & a \\ \rho_A(a) & = & \dot{\langle} a, 1_{\partial^M_{\check{a}}(1)}, \dots, 1_{\partial^M_{\check{a}}(|\check{a}|)} \dot{\rangle} \end{array}$$

We take the amalgamation permutations to be the identity, since the unit $I_{\odot}$ has no vertical inputs. If it did, there would be no bijection between the inputs of both sides.

The only problem is the definition of $\alpha^{\odot}$. The problem is that M may have nonstandard amalgamation. That is the multiplication $M \otimes_O M \to M$ need not be strict – it can shuffle the inputs according to some nontrivial permutation. We have used it to define the horizontal typing. Because of this the naive associativity isomorphism $A \odot (B \odot C) \to (A \odot B) \odot C$ is not even well defined, since the values it "should" have are not necessarily among the elements of $(A \odot B) \odot C$.

The correct solution, for geometrical and other reasons[4], is the following. Denote by γ (more precisely γ^M) the amalgamation permutations of the multiplication map of M (that is of $\mu\colon M \otimes_O M \to M$, which lives in $\mathbf{Sig}_a/O$). Define

$$\alpha^{\odot}_{A,B,C}(\dot{\langle} a, \dot{\langle} b_i, c_{i,j} \dot{\rangle} \dot{\rangle}) = \dot{\langle} \dot{\langle} a, b_i \dot{\rangle}, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i \rangle}(i,j)} \dot{\rangle}$$

as a function. We leave it to the reader to see that the term on the left is well defined. Indeed, this is the only formula which works when the horizontal inputs of $\dot{\langle} a, b_i \dot{\rangle}$ are all distinct.

We must define the vertical amalgamation permutations. For this consider a, b_i and $c_{i,j}$ as formal variables. Let $\kappa_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j} \dot{\rangle} \dot{\rangle}}$ be the permutation which sends each formal variable on the list $\dot{\langle} a, \dot{\langle} b_i, c_{i,j} \dot{\rangle} \dot{\rangle}$ to itself on the list $\dot{\langle} \dot{\langle} a, b_i \dot{\rangle}, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i \rangle}(i,j)} \dot{\rangle}$. More precisely these lists are

$$\begin{array}{c} \dot{\langle} a, \dot{\langle} b_1, c_{1,1}, \dots, c_{1,l_1} \dot{\rangle} \dots \dot{\langle} b_k, c_{k,1}, \dots, c_{k,l_k} \dot{\rangle} \dot{\rangle} \\ \dot{\langle} \dot{\langle} a, b_1, \dots, b_k \dot{\rangle}, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i \rangle}(1,1)}, \dots, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i \rangle}(k,l_k)} \dot{\rangle} \end{array}$$

[4]For example, the constructions in chapter 6 depend critically on this definition

To define the amalgamation permutations of $\alpha^{\odot}$, which we will denote by π, we make κ act on blocks of the appropriate length (the length of the inputs of each function symbol)

$$\pi_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j} \dot{\rangle}\dot{\rangle}} = \kappa_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j} \dot{\rangle}\dot{\rangle}} * (1_{(|a|]}, \dots 1_{(|c_{k,l_k}|]})$$

thus, each block of inputs "tracks" the position of its corresponding function symbol. Again, this permutation is the only well defined one when all the inputs of a, b_i and $c_{i,j}$ are distinct. Therefore by the (proof of the) separation principle it is the only formula that can be natural, given what we want to do with the function symbols.

We will now prove that $\alpha^{\odot}$ is natural and satisfies the pentagon identity. The rest of the proof that $(\odot, \alpha, \lambda, \rho)$ defines a monoidal fibration structure is very easy and formally identical to the corresponding part of the proof for $\otimes$ in $\mathbf{Sig}_a$.

Naturality of $\alpha^{\odot}$. We consider the diagram

$$\begin{tikzcd}
A \odot (B \odot C) \arrow[r, "\alpha^{\odot}_{A,B,C}"] \arrow[d, "f \odot (g \odot h)"'] & (A \odot B) \odot C \arrow[d, "(f \odot g) \odot h"] & M \arrow[d, "u"] \\
A' \odot (B' \odot C') \arrow[r, "\alpha^{\odot}_{A',B',C'}"] & (A' \odot B') \odot C' & N
\end{tikzcd}$$

All three morphisms are over the homomorphism of monoids u. We will check that they are equal as functions first, and then consider the amalgamation permutations. Consider the term

$$\dot{\langle} a, \dot{\langle} b_i, c_{i,j} \dot{\rangle}\dot{\rangle}$$

Applying both ways to go around the diagram we obtain

$$\dot{\langle}\dot{\langle} f(a), g(b_{\sigma_{\breve{a}}^{-1}(i)})\dot{\rangle}, h(c_{\xi_1^{-1}(i,j)})\dot{\rangle}$$

and

$$\dot{\langle}\dot{\langle} f(a), g(b_{\sigma_{\breve{a}}^{-1}(i)})\dot{\rangle}, h(c_{\xi_2^{-1}(i,j)})\dot{\rangle}$$

Where σ are the amalgamation permutations of u, and ξ_1 and ξ_2 are given as follows[5]:

[5] The "check" symbol over the lowermost index a was replaced by ∂ due to TeX-nical issues.

$$\begin{aligned}
\xi_1 &= \gamma^N_{\langle f(\check{a}), g(b_{\sigma^{-1}_{\check{\partial a}}(i)})\rangle} \circ \sigma \otimes \sigma_{\langle \check{a}, \check{b}_i\rangle} \\
\xi_2 &= \sigma_{\mu(\check{a}, \check{b}_i)} \circ \gamma^M_{\langle \check{a}, \check{b}_i\rangle}
\end{aligned}$$

Their equality follows from the fact that u is a homomorphism of monoids – this is the equality required from the amalgamation permutations of a homomorphism.

We are left with proving that the amalgamation permutations are equal, thus we must prove that

$$\pi_{\dot{\langle} f(a), \dot{\langle} g(b_{\sigma_{\check{a}}^{-1}(i)}), h(c_{\sigma\otimes\sigma^{-1}_{\langle \check{a}, \check{b}_i\rangle}(i,j)})\dot{\rangle}\dot{\rangle}} \circ \tau \odot (\delta \odot \zeta)_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j}\dot{\rangle}\dot{\rangle}} =$$
$$(\tau \odot \delta) \odot \zeta_{\dot{\langle}\dot{\langle} a, b_i\dot{\rangle}, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i\rangle}(i,j)}\dot{\rangle}} \circ \pi_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j}\dot{\rangle}\dot{\rangle}}$$

Both these permutations permute the input blocks of the function symbols a, b_i and $c_{i,j}$, and apply some permutation inside each block. This follows from our definitions of π and $\beta \odot \chi$. We will prove their equality in two (concurrent) steps: we will show that the block permutations are equal, and then that the same permutation is applied inside each block.

To see the equality of block permutations we argue for each function symbol. The argument for a is trivial. The arguments for b_i are similar to (and simpler than) the arguments for $c_{i,j}$. We will therefore only consider those last symbols. Each c_{i_0,j_0} is part of a larger symbol $\dot{\langle} b_{i_0}, c_{i_0,j}\dot{\rangle}$. Let us analyze what both sides do to this block and its elements.

The left permutation applies $\delta \odot \zeta_{\dot{\langle} b_{i_0}, c_{i_0,j}\dot{\rangle}}$ to the input block of this larger symbol, and moves it to its position in $\dot{\langle} f(a), \dot{\langle} g(b_{\sigma_{\check{a}}^{-1}(i)}), h(c_{\sigma\otimes\sigma^{-1}_{\langle \check{a}, \check{b}_i, \rangle}(i,j)})\dot{\rangle}\dot{\rangle}$ (i.e. applies $(1, \sigma_{\check{a}})$ to the input blocks). In particular $\zeta_{c_{i_0,j_0}}$ is applied to our input block, and it is placed on the block of $h(c_{i_0,j_0})$ in $\dot{\langle} f(a), \dot{\langle} g(b_{\sigma_{\check{a}}^{-1}(i)}), h(c_{\sigma\otimes\sigma^{-1}_{\langle \check{a}, \check{b}_i, \rangle}(i,j)})\dot{\rangle}\dot{\rangle}$. The final π moves the block to its position in $\dot{\langle}\dot{\langle} f(a), g(b_{\sigma_{\check{a}}^{-1}(i)})\dot{\rangle}, h(c_{\xi_2^{-1}(i,j)})\dot{\rangle}$, with ξ_2 given above.

The right permutation breaks up this bigger block, since π is applied first. By definition of π, the input block of c_{i_0,j_0} is moved to its position in $\dot{\langle}\dot{\langle} a, b_i\dot{\rangle}, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i\rangle}(i,j)}\dot{\rangle}$. Then we must apply $(\tau \odot \delta) \odot \zeta_{\dot{\langle}\dot{\langle} a, b_i\dot{\rangle}, c_{\gamma^{-1}_{\langle \check{a}, \check{b}_i\rangle}(i,j)}\dot{\rangle}}$. Looking at the definition, we see that $\zeta_{c_{i_0,j_0}}$ is applied to our block, and then all the blocks (including those of a and b_i, which are permuted by $(1, \sigma_{\check{a}})$ before this) are permuted by $(1, \sigma_{\mu(\check{a}, \check{b}_i)})$. This means that the input blocks of $c_{i,j}$ are permuted by $\sigma_{\mu(\check{a},\check{b}_i)}$. This means that our block lands on the block of $h(c_{i_0,j_0})$ in $\dot{\langle}\dot{\langle} f(a), g(b_{\sigma_{\check{a}}^{-1}(i)})\dot{\rangle}, h(c_{\xi_1^{-1}(i,j)})\dot{\rangle}$.

But we know that $\xi_1 = \xi_2$. Therefore the block permutations are equal. We have also seen that $\zeta_{c_{i_0,j_0}}$ is applied to our block in both cases. Thus both permutations are equal.

The pentagon identity. Since we have established naturality, we can apply the separation principle 3.1.5. All the functors in the pentagon diagram are jointly agreeable and separated – their values on prone morphisms (which are constructed by pullback, as in $\mathbf{Sig}_a$) are strict. To see separability consider a term

$$\dot{\langle} a, \dot{\langle} b_i, \dot{\langle} c_{i,j}, d_{i,j,k}\dot{\rangle}\dot{\rangle}\dot{\rangle}$$

label the inputs of a by consecutive natural numbers, starting with 0, then label the inputs of b_1 with consecutive numbers after the ones used for a. Repeat this process until the last $d_{i,j,k}$ is reached. Label the outputs so as to maintain composability. This defines the needed lift. By the separation principle we are reduced to checking the pentagon identity on function symbols.

After some amount of calculation we find that we must compare

$$\dot{\langle}\dot{\langle}\dot{\langle} a, b_i\dot{\rangle}, c_{\gamma^{-1}_{\langle\check{a},\check{b}_i\rangle}(i,j)}\dot{\rangle}, d_{(1\otimes\gamma)^{-1}_{\langle\check{a},\langle\check{b}_i,\check{c}_{i,j}\rangle\rangle}\circ\gamma^{-1}_{\langle\check{a},\mu(\check{b}_i,\check{c}_{i,j})\rangle}(i,j,k)}\dot{\rangle}$$

and[6]

$$\dot{\langle}\dot{\langle}\dot{\langle} a, b_i\dot{\rangle}, c_{\gamma^{-1}_{\langle\check{a},\check{b}_i\rangle}(i,j)}\dot{\rangle}, d_{(\gamma\otimes 1)^{-1}_{\langle\dot{\langle}\check{a},\check{b}_i\dot{\rangle},\check{c}_{i,j}\rangle}\circ\gamma^{-1}_{\langle\mu(\check{a},\check{b}_i),\check{c}_{\gamma^{-1}_{\langle\check{a},\check{b}_i\rangle}(i,j)}\rangle}(i,j,k)}\dot{\rangle}$$

which comes down to the equality

$$(1\otimes\gamma)^{-1}_{\langle\check{a},\langle\check{b}_i,\check{c}_{i,j}\rangle\rangle}\circ\gamma^{-1}_{\langle\check{a},\mu(\check{b}_i,\check{c}_{i,j})\rangle} = (\gamma\otimes 1)^{-1}_{\langle\langle\check{a},\check{b}_i\rangle,\check{c}_{i,j}\rangle}\circ\gamma^{-1}_{\langle\mu(\check{a},\check{b}_i),\check{c}_{\gamma^{-1}_{\langle\check{a},\check{b}_i\rangle}(i,j)}\rangle}$$

This equality is satisfied by the amalgamation permutations of μ by virtue of associativity.

2.4 Distributivity For Monoidal Signatures

We can now define the distributivity structure on $\mathbf{Sig}_{ma}$ which will give us the web monoids used in the definition of opetopic sets. By theorem 1.7.8, to define a distributivity structure we need only specify $\varphi_{A,B,X}$ and ψ_X, which satisfy certain coherence conditions.

[6]Note the five levels of indexing.

By definition, $\varphi_{A,B,X} \colon (A \otimes X) \odot (B \otimes X) \to (A \odot B) \otimes X$ maps the term

$$\dot{\langle}\langle a, x_{0,1}, \ldots, x_{0,l_0}\rangle, \langle b_1, x_{1,1}, \ldots, x_{1,l_1}\rangle, \ldots, \langle b_k, x_{k,1}, \ldots, x_{k,l_k}\rangle\dot{\rangle}$$

to

$$\langle\dot{\langle} a, b_1, \ldots, b_k\dot{\rangle}, x_{0,1}, \ldots, x_{0,l_0}, x_{1,1}, \ldots, x_{1,l_1}, \ldots, x_{k,1}, \ldots, x_{k,l_k}\rangle$$

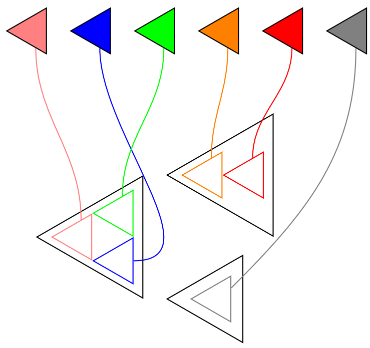
with trivial amalgamation permutations.

$\psi_X \colon I_\odot \to I_\odot \otimes X$ is defined by

$$1_o \mapsto \langle 1_o, -\rangle$$

for $o \in O$ (the set of types of M). The $(-)$ represents an empty list, since the vertical inputs of elements of $I_\odot$ are empty.

The morphism φ expresses the fact that the following picture, following the pictorial conventions of subsection 2.3.1, is unambiguous. It illustrates both the function symbols in $(A \odot B) \otimes X$ and the symbols in $(A \otimes X) \odot (B \otimes X)$ – the order of operations leading to this picture is irrelevant.



Theorem 2.4.1. *The above definitions give* $\mathbf{Sig}_{ma}$ *a distributivity structure of* $\otimes$ *over* $\odot$.

The long, but trivial, proof is given in the next subsection.

2.4.1 Proof of Theorem 2.4.1

We must check whether $\varphi_{A,B,X}$ and ψ_X define a distributivity structure for $\mathbf{Sig}_a$. To do this we will show that they are natural and prove that they satisfy all the coherence diagrams listed in theorem 1.7.8.

Before we dive into the calculations two remarks are in order. First, conditions $III - VII$ are very easy. To make our point we will leave the last diagram (which is unnecessary anyway) as a rather trivial exercise. The only real problem is the complexity of the terms in conditions I and II. Our calculations for these conditions will look somewhat like a physicist's version of tensor calculus – crawling with indices. To ease our problems we will use the separation principle – all the functors involved in these conditions are agreeable and separated (as we will see).

Proposition 2.4.2. *$\varphi_{A,B,X}$ and ψ_X are isomorphisms, fibered natural in all variables.*

Proof. The statement for ψ is obvious, as is the isomorphism part. We consider naturality for φ:

$$\begin{tikzcd}
(A \otimes X) \odot (B \otimes X) \arrow[r, "(f \otimes h) \odot (g \otimes h)"] \arrow[d, "\varphi_{A,B,X}"'] & (A' \otimes X') \odot (B' \otimes X') \arrow[d, "\varphi_{A',B',X'}"] \\
(A \odot B) \otimes X \arrow[r, "(f \odot g) \otimes h"] & (A' \odot B') \otimes X'
\end{tikzcd}$$

$$M \xrightarrow{\quad\quad u \quad\quad} N$$

where f, g and h are over the homomorphism u. The amalgamation permutations of these morphisms are denoted σ, τ, δ and θ respectively. We consider a term

$$\dot{\langle}\langle a, x_{0,k}\rangle, \langle b_i, x_{i,k}\rangle\dot{\rangle}$$

in $(A \otimes X) \odot (B \otimes X)$ and apply $(f \odot g) \otimes h \circ \varphi_{A,B,X}$ to it, obtaining

$$\langle \dot{\langle} f(a), g(b_{\theta_{\breve{a}}^{-1}(i)})\dot{\rangle}, h(x_{\sigma \odot \tau^{-1}_{\dot{\langle} a, b_i \dot{\rangle}}(i,k)})\rangle \tag{2.3}$$

On the other hand, we can apply $(f \otimes h) \odot (g \otimes h)$ and obtain

$$\dot{\langle}\langle f(a), h(x_{0,\sigma_a^{-1}(k)}\rangle, \langle g(b_{\theta_{\breve{a}}^{-1}(i)}), h(x_{\theta_{\breve{a}}^{-1}(i), \tau^{-1}_{b_{\theta_{\breve{a}}^{-1}(i)}}(k)})\rangle\dot{\rangle},$$

which $\varphi_{A',B',X'}$ maps to

$$\begin{aligned}
&\langle \dot{\langle} f(a), g(b_{\theta_{\breve{a}}^{-1}(1)}), \ldots, g(b_{\theta_{\breve{a}}^{-1}(k)})\dot{\rangle}, \\
h(x_{0,\sigma_a^{-1}(1)}), \ldots, h(x_{0,\sigma_a^{-1}(l_0)}), &h(x_{\theta_{\breve{a}}^{-1}(1), \tau^{-1}_{b_{\theta_{\breve{a}}^{-1}(1)}}(1)}), \ldots h(x_{\theta_{\breve{a}}^{-1}(k), \tau^{-1}_{b_{\theta_{\breve{a}}^{-1}(k)}}(l_k)})\rangle,
\end{aligned}$$

which is equal to term 2.3 by our definition of $\tau \odot \delta$ (and the formula for inverses in the operad of symmetries).

We must still prove that the amalgamation permutations are equal. This means that

$$(\sigma \otimes \delta) \odot (\tau \otimes \delta)_{\dot{\langle}\langle a,x_{0,k}\rangle,\langle b_i,x_{i,k}\rangle\dot{\rangle}} = (\sigma \odot \tau) \otimes \delta_{\langle\dot{\langle} a,b_i\dot{\rangle},x_{i,k}\rangle}$$

Fortunately, we can write out both sides in this case, using just our definitions. The left side is

$$(1, \theta_{\check{a}}) * (\sigma_a * (\delta_{x_{0,k}}), \tau_{b_i} * (\delta_{x_{i,k}}))$$

and the right side is

$$[(1, \theta_{\check{a}}) * (\sigma_a, \tau_{b_i})] * (\delta_{x_{i,k}})$$

They are equal by the associativity of the operad of symmetries. □

Let us see why all our functors are jointly agreeable. This is a simple consequence of our formulas. Prone morphisms in $\mathbf{Sig}_{ma}$ are defined using **Set**-pullback (just like in $\mathbf{Sig}_a$), and are therefore strict, and the projection $\pi\colon M_{\mathbb{N}} \to M$ is strict. Therefore all possible combinations of $\otimes$ and $\odot$ on these morphisms will have standard amalgamation – the formulas for amalgamation give identities if they are supplied only with identities. Nonstandard amalgamation does not appear out of thin air, so to speak.

Separation can be seen, in some sense, in the same way. All our functors are combinations of $\otimes$ and $\odot$, and the typing of their values is defined from the typings of their arguments. Thus the arguments contain the simplest building blocks of the terms we will consider. If our term is, for example $\langle f, g_1, \ldots, g_n\rangle \in A \otimes B$, then the simplest building blocks are f and the g_i. We can attach consecutive natural numbers to the inputs of g_1, bigger numbers to the inputs of g_2, and so on up to g_n. We can attach numbers to outputs of g_i to maintain composability with f – in this case they can be arbitrary. This will define a term in $\pi^*A \otimes \pi^*B$ with injective typing mapping to the original one when we forget the added numbers. This proves separability of the functor $\otimes$ for $\mathbf{Sig}_{ma}$ (and $\mathbf{Sig}_a$ also). This procedure works for terms of arbitrary complexity, in particular for those which are elements of our diagrams[7].

We start with the easy diagrams (the last one is an exercise).

Condition III. An element of $A \odot B$ is of the form

$$\dot{\langle} a, b_i \dot{\rangle},$$

[7] Formally we should use induction on complexity of the terms, but this only obscures the idea. An argument essentially equivalent to the separability of $\mathcal{F}_\odot \odot \mathcal{F}_\odot$ can be found in lemma 7 of [HMP02], part 2.

where i ranges over the horizontal inputs of a. The map $\rho^{\otimes} \odot \rho^{\otimes}$ maps this to

$$\dot{\langle}\langle a, 1_{\partial^A_a(j)}\rangle, \langle b_i, 1_{\partial^B_{b_i}(j')}\rangle\dot{\rangle},$$

where j and j' range over the vertical inputs. The map φ maps this to

$$\langle\dot{\langle} a, b_i\dot{\rangle}, 1_{\partial^A_a(j)}, 1_{\partial^B_{b_i}(j')}\rangle,$$

which is exactly what $\rho^{\otimes}$ does to the original term. By separability we are finished.

Condition IV. We start with

$$\dot{\langle}\langle 1_{\partial^M_{\check{a}}(0)}, -\rangle, \langle a, x_i\rangle\dot{\rangle},$$

where $(-)$ represents the empty list. There is only one a since $1_{\partial^M_{\check{a}}(0)}$ is unary. φ maps this to

$$\langle\dot{\langle} 1_{\partial^M_{\check{a}}(0)}, a\dot{\rangle}, x_i\rangle,$$

which $\lambda^{\odot} \otimes 1$ maps to

$$\langle a, x_i\rangle.$$

The final result of this way is thus f applied to the above term. The other way around the diagram goes like this. Starting with the original term we obtain in the first step

$$\dot{\langle} 1_{\partial^M_{\check{a}}(0)}, f(\langle a, x_i\rangle)\dot{\rangle},$$

and then, applying $\lambda^{\odot}$,

$$f(\langle a, x_i\rangle)$$

in the second step. Thus both ways agree.

Condition V. The condition says very little in our case. We start with

$$\langle 1_o, -\rangle,$$

which is mapped by ψ^{-1} to 1_o.
Alternately it is mapped by $\alpha^{\otimes}$ to (surprise!)

$$\langle\langle 1_o, -\rangle, -\rangle,$$

and then by $\psi^{-1} \otimes 1$ to

$$\langle 1_o, -\rangle,$$

which the final ψ^{-1} maps to 1_o, thus agreeing with the first way.

Condition VI. This is entirely trivial – there is only one way to add an empty list to a unary term. We start with $1_o \in I_\odot$ and both ψ and $\rho^\otimes$ map it to

$$\langle 1_o, -\rangle,$$

by definition for ψ, and for $\rho^\otimes$ because $I_\odot$ has no vertical inputs.

Now we turn to the more complicated cases

Condition II. We start with a rather unwieldy

$$\dot{\langle}\langle a, \langle x_i, y_{i,j}\rangle\rangle, \langle b_k, \langle x'_{i'}, y'_{i',j'}\rangle\rangle\dot{\rangle},$$

which φ maps to

$$\langle\dot{\langle} a, b_k\dot{\rangle}, \langle x_i, y_{i,j}\rangle, \langle x'_{i'}, y'_{i',j'}\rangle\rangle,$$

which after $\alpha^\otimes$ becomes

$$\langle\langle\dot{\langle} a, b_k\dot{\rangle}, x_i, x'_{i'}\rangle, y_{i,j}, y'_{i',j'}\rangle.$$

The other way maps the original term by $\alpha^\otimes \odot \alpha^\otimes$ to

$$\dot{\langle}\langle\langle a, x_i\rangle, y_{i,j}\rangle\rangle, \langle\langle b_k, x'_{i'}\rangle, y'_{i',j'}\rangle\rangle\dot{\rangle},$$

and then by φ to

$$\langle\dot{\langle}\langle a, x_i\rangle, \langle b_k, x'_{i'}\rangle\dot{\rangle}, y_{i,j}, y'_{i',j'}\rangle.$$

Applying the final $\varphi \otimes 1$ yields

$$\langle\langle\dot{\langle} a, b_k\dot{\rangle}, x_i, x'_{i'}\rangle, y_{i,j}, y'_{i',j'}\rangle,$$

in agreement with our previous calculation.

Condition I. Up to now we did not have to deal with any permutations acting on terms (the other ones were taken care of by the separation principle). The first condition is the hardest one because this is not true in this case. Fortunately, the permutations are manageable. We begin with

$$\dot{\langle}\langle a, x_i\rangle, \dot{\langle}\langle b_k, x_{k,j}\rangle, \langle c_{k,l}, x_{k,l,m}\rangle\dot{\rangle}\dot{\rangle}.$$

The last index of each instance of x ranges over the vertical inputs of a, b_k or $c_{k,l}$. The indices k and (k,l) range over horizontal inputs of a and b_k.

After applying $1 \odot \varphi$ to this term we obtain

$$\dot{\langle}\langle a, x_i\rangle, \langle\dot{\langle} b_k, c_{k,l}\dot{\rangle}, x_{k,j}, x_{k,l,m}\rangle\dot{\rangle},$$

which φ maps to

$$\langle\dot{\langle} a, \dot{\langle} b_k, c_{k,l}\dot{\rangle}\dot{\rangle}, x_i, x_{1,j}, x_{1,l,m}, \dots, x_{2,j}, x_{2,l,m}, \dots\rangle.$$

We must now determine what $\alpha^{\odot} \otimes 1$ does to this term. This means looking at the definition of the amalgamation permutations of $\alpha^{\odot}$, which we have denoted by π:

$$\pi_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j}\dot{\rangle}\dot{\rangle}} = \kappa_{\dot{\langle} a, \dot{\langle} b_i, c_{i,j}\dot{\rangle}\dot{\rangle}} * (1_{(|a|]}, \dots 1_{(|c_{k,l_k}|]}),$$

where κ is the permutation that implements the movements of the function symbols between $\dot{\langle} a, \dot{\langle} b_i, c_{i,j}\dot{\rangle}\dot{\rangle}$ and $\dot{\langle}\dot{\langle} a, b_i\dot{\rangle}, c_{\gamma^{-1}_{\langle\breve{a},\breve{b}_i\rangle}(i,j)}\dot{\rangle}$. Therefore $\alpha^{\odot} \otimes 1$ acts on our term as follows

$$\langle\dot{\langle}\dot{\langle} a, b_k\dot{\rangle}, c_{\gamma^{-1}_{\langle\breve{a},\breve{b}_k\rangle}(k,l)}\dot{\rangle}, x_i, x_{k,j}, x_{\gamma^{-1}_{\langle\breve{a},\breve{b}_k\rangle}(k,l),m}\rangle,$$

where γ are the amalgamation permutations of multiplication in M.

We must determine what happens when we take the diagram the other way around. By our typing conventions $\alpha^{\odot}$ maps our original term to

$$\dot{\langle}\dot{\langle}\langle a, x_i\rangle, \langle b_k, x_{k,j}\rangle\dot{\rangle}, \langle c_{\gamma^{-1}_{\langle\breve{a},\breve{b}_i\rangle}(k,l)}, x_{\gamma^{-1}_{\langle\breve{a},\breve{b}_k\rangle}(k,l),m}\rangle\dot{\rangle},$$

which $\varphi \odot 1$ makes into

$$\dot{\langle}\langle\dot{\langle} a, b_k\dot{\rangle}, x_i, x_{k,j}\rangle, \langle c_{\gamma^{-1}_{\langle\breve{a},\breve{b}_i\rangle}(k,l)}, x_{\gamma^{-1}_{\langle\breve{a},\breve{b}_k\rangle}(k,l),m}\rangle\dot{\rangle}.$$

Applying the final φ we obtain

$$\langle\dot{\langle}\dot{\langle} a, b_k\dot{\rangle}, c_{\gamma^{-1}_{\langle\breve{a},\breve{b}_i\rangle}(k,l)}\dot{\rangle}, x_i, x_{k,j}, x_{\gamma^{-1}_{\langle\breve{a},\breve{b}_k\rangle}(k,l),m}\rangle,$$

as we should. This concludes the proof of theorem 2.4.1.

2.5 Tautological Actions

If we forget that an object of **Sig** has inputs (but not the output!) we obtain an object of $\mathbf{Set}^{\cdot\to\cdot}$, since $O^\dagger \simeq O \times O^*$ gives a decomposition of the typing into (output, inputs) – forgetting the second factor leaves us with a map $A \to O$, which is an object of $\mathbf{Set}^{\cdot\to\cdot}$. This defines a fibered forgetful functor $U : \mathbf{Sig} \to \mathbf{Set}^{\cdot\to\cdot}$

We can also construct a fibered functor $\overline{-} : \mathbf{Set}^{\cdot\to\cdot} \to \mathbf{Sig}$, which defines the inputs to be empty. This is neither a left nor right adjoint to U. We will call $\overline{X}$ the *sterile signature* associated to X.

Definition 2.5.1 (Tautological Action of Signatures on $\mathbf{Set}^{\cdot\to\cdot}$)**.** *For $A \in \mathbf{Sig}/O$ and $d\colon X \to O \in \mathbf{Set}/O = \mathbf{Set}^{\cdot\to\cdot}/O$, we define:*

$$A \star X = U(A \otimes \overline{X}) = \{(a, x_i) : a \in A, x_i \in X, \partial_a^A(i) = d(x_i), i = 1, \ldots, |a|\},$$

with the map to O given by $d^{A\star X}(a, x_i) = \partial_a^A(0)$.

This formula is universal for all our fibrations of signatures over **Set** – the same formula (using their own respective monoidal structures) works for $\mathbf{Sig}, \mathbf{Sig}_a$ and $\mathbf{Sig}_s$. We will distinguish these actions by a subscript: $\star, \star_a, \star_s$.

The verification that this defines an action in all three cases is essentially the same as checking that the monoidal structures satisfy the coherence conditions. The details of this simple computation are left to the reader.

By our construction of colimits in $\mathbf{Sig}_a$ (see 2.7.1), and their preservation by the monoidal structure by theorem 2.7.2, we immediately obtain the following corollary.

Corollary 2.5.2. *The actions $\star$ and $\star_a$ preserve coproducts in the left variable. They also perserve filtered colimits and reflexive coequalizers in both variables.*

Proof. We need to see that all the functors in 2.5.1 preserve the listed colimits (the construction of these colimits for $\mathbf{Sig}_a$ will be given in 2.7.1). This is simple for $\overline{-}$: it is cocontinuous since it has a fibered right adjoint $G\colon \mathbf{Sig} \to \mathbf{Set}^{\cdot\to\cdot}$, which is defined as follows:

$$G(A) = U(A \setminus \{\text{symbols in } A \text{ with non-empty inputs}\}).$$

U trivially preserves coproducts. It preserves filtered colimits and reflexive coequalizers by their construction, which is given in 2.7.1. □

2.5.1 The Pullback Action

Since we defined $\mathbf{Sig}_{ma}$ as the pullback of the monoidal fibration $\mathbf{Sig}_a$, all of its actions pull back as well to give actions of $\mathbf{Sig}_{ma}$. The tautologous action pulls back to the pullback action, defined by the diagram

$$\begin{tikzcd}
 & \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot} \arrow[rr] \arrow[dd, "\mathcal{U}^*cod"] & & \mathbf{Set}^{\cdot\to\cdot} \arrow[dd, "cod"] \\
\mathbf{Sig}_{ma} \times_{Mon(\mathbf{Sig}_a)} \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot} \arrow[ur, "\mathcal{U}^*\star_a"] \arrow[dr] \arrow[rr] & & \mathbf{Sig}_a \times_{\mathbf{Set}} \mathbf{Set}^{\cdot\to\cdot} \arrow[ur, "\star_a"] \arrow[dr] & \\
 & Mon(\mathbf{Sig}_a) \arrow[rr, "\mathcal{U}"] & & \mathbf{Set}
\end{tikzcd}$$

Again, the formula for $\mathcal{U}^*\star_a$ is the same as the one for $\star_a$, but the set of types (or the codomain of $d\colon X \to M$) forms a monoid in $\mathbf{Sig}_a$. We will denote the pullback action by $\star$ in the sequel, and also denote $\mathcal{U}^*U$ (the functor $\mathbf{Sig}_{ma} \to \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$, which forgets the vertical inputs, the pullback of the input forgetting functor for $\mathbf{Sig}_a$) as U. This will not cause any confusion, since the pullback action will always be called such. Note that the formula of definition 2.5.1 is still true for the pullback action, as is its corollary 2.5.2.

2.6 Interpretation as Endofunctors

2.6.1 Polynomial and Analytic Functors

Definition 2.6.1. *Let O be a set. An* analytic functor *over O is a finitary functor*

$$\mathbf{Set}/O \longrightarrow \mathbf{Set}/O$$

which weakly perserves wide pullbacks (weak means the uniqueness clause in the relevant universal property is dropped), or equivalently preserves weak wide pullbacks (for the definition of weak limits see [CWM98, p. 235]). A wide pullback in a category $\mathcal{C}$ is simply a product (of any number of factors) in $\mathcal{C}/X$ for some object $X \in \mathcal{C}$.

A polynomial functor *is an analytic functor, but which strongly preserves wide pullbacks.*

To define a categories of analytic and polynomial functors, we must specify which natural transformations are analytic and polynomial. An analytic morphism of analytic functors F, G over O is a natural transformation $\tau\colon F \to G$, which is weakly cartesian. This means that the naturality squares

$$\begin{tikzcd}
F(X) \arrow[r, "\tau_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\tau_Y"] & G(Y)
\end{tikzcd}$$

are weak pullbacks. A polynomial morphism between polynomial functors is a cartesian natural transformation – the naturality squares are required to be ordinary pullbacks. For polynomial functors this is not stronger that the requirement of being weakly cartesian – polynomial functors are a full subcategory of analytic functors [Z10, 7.19].

These functors naturally assemble into bifibrations $\mathbf{Poly} \subset \mathbf{An} \to \mathbf{Set}$ over $\mathbf{Set}$, such that $\mathbf{Poly}/O$ and $\mathbf{An}/O$ are the categories of polynomial and analytic functors over O, respectively. These are subbifibrations of the exponential $Exp(\mathbf{Set}^{\cdot\to\cdot}) = (\mathbf{Set}^{\cdot\to\cdot})^{\mathbf{Set}^{\cdot\to\cdot}}$.

Since analytic and polynomial functors are closed under composition, both $\mathbf{Poly}$ and $\mathbf{An}$ are strict monoidal, just like the bifibration of endofunctors $Exp(\mathbf{Set}^{\cdot\to\cdot})$

2.6.2 Relation to Signatures

The tautologous actions

$$\star_? \colon \mathbf{Sig}_? \times_{\mathbf{Set}} \mathbf{Set}^{\cdot\to\cdot} \to \mathbf{Set}^{\cdot\to\cdot},$$

where $? = a, s$ or is blank, have exponential adjoints, with codomain the exponential monoidal fibration $Exp(\mathbf{Set}^{\cdot\to\cdot})$,

$$rep_{a,s} \colon \mathbf{Sig}_{a,s} \to Exp(\mathbf{Set}^{\cdot\to\cdot})$$

are characterized by the following theorem:

Theorem 2.6.2 (6.12 and 7.5 of [Z10])**.**

1. *The morphism rep_a is full on isomorphisms, and is a monoidal equivalence onto its essential image, which consists of polynomial functors and cartesian natural transformations.*

2. *The morphism rep_s is full on isomorphisms, and is a monoidal equivalence onto its essential image, which consists of analytic functors and weakly cartesian natural transformations.*

The morphism *rep* (for ordinary signatures) is not full on isomorphisms.

2.7 Categorical Properties of Signatures

In this section we collect the categorical properties of signatures necessary for later proofs. They are elementary, and do not seem to be of intrinsic interest. The reader may skip this section, referring back to it as necessary.

2.7.1 Colimits in Categories of Signatures

Signatures without amalgamation have very nice (co)completeness properties – they are complete and cocomplete (both globally and fiberwise). Unfortunately the addition of amalgamation permutations spoils some of these properties, as the following example shows.

Let A be a signature with one binary function symbol, over a singleton set $O = \{*\}$. Then we have two obvious morphisms $A \to A$ – the identity and a morphism which permutes the inputs of our function symbol. Since permutations are invertible, these two morphisms are not coequalized by any other morphism. Therefore $\mathbf{Sig}_a$ does not have all coequalizers (fibered or not). A similar argument shows that signatures with amalgamation do not have a terminal object (fiberwise, or globally).

An analogous example shows that $Mon(\mathbf{Sig}_a)$ has no terminal object, fiberwise or globally. We now turn to the positive results:

Proposition 2.7.1. *The fibration* $\mathbf{Sig}_a$ *has the following cocompleteness properties:*

1. *Small coproducts (fibered or not).*
2. *All (small) fibered filtered colimits.*
3. *Fibered reflexive coequalizers.*

Proof. We leave the first item as a warm-up exercise. The other constructions unfortunately require some work.

Filtered colimits. By corollary 1.2.3 we may restrict our attention to individual fibers. We will consider the fiber $\mathbf{Sig}_a/O$. There is a forgetful functor $U\colon \mathbf{Sig}_a/O \to \mathbf{Set}/O$, which forgets the input types, but not the output (it corresponds to the projection $O^\dagger \simeq O \times O^* \to O$). The category $\mathbf{Set}/O$ is obviously cocomplete. We will use it to build our colimits.

Consider a filtered diagram $F\colon \mathcal{D} \to \mathbf{Sig}_a/O$. Each signature is the coproduct of countably many signatures consisting of all the n-ary function symbols of the original signature, for $n \in \mathbb{N}$. Morphisms preserve arity, so this is also true for the entire diagram F. We may therefore assume that all the values of F consist of signatures with function symbols of a fixed arity $n \in \mathbb{N}$.

By cocompleteness $U \circ F$ has a colimiting cone $\tau\colon U \circ F \stackrel{\cdot}{\to} X$ in $\mathbf{Set}/O$. We will show that it can be lifted to a cone in $\mathbf{Sig}_a/O$. The fact that any such lift is colimiting is trivial, since permutations are invertible.

If all the values of F are empty, the colimit is empty, and we are done. We may assume that F has nonempty values. Since all the function symbols in the values of F are n-ary, we declare that each element of X is also an n-ary symbol. We must define the typing of each symbol and the amalgamation permutations of the components of the colimiting cone. For each $x \in X$ (which exist, since F has nonempty values) consider its inverse image in the diagram $U \circ F$ – those function symbols which map to x under the components of the colimiting cone. These inverse images are disjoint, and therefore we can consider them separately.

Choose an $f \in F(d)$ which maps to x under τ_d. We declare that the amalgamation permutations of τ_d are the identity for f. This gives us a typing of x. This also determines the amalgamation permutations of all other symbols which map to x – the diagram is filtered, and permutations are invertible, so considering only the permutations we can get anywhere in the inverse image of x starting from f. Such a procedure may result in a contradiction – and it does in the example we gave above for nonexistence of coequalizers. But in our case the diagram is filtered, so any two parallel morphisms are equalized by a third one, and no contradiction can arise. Any two potentially different ways for getting from f to another symbol have equal amalgamation permutations.

Reflexive coequalizers. Like in the previous case, we are lucky – the amalgamation permutations of the two maps f and g in the reflexive coequalizer diagram

$$\begin{tikzcd} \mathrm{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathrm{B} \arrow[l, "s" description] \end{tikzcd}$$

must be equal, since they have a common inverse (the permutations of s). Consider the coequalizer $e\colon U(B) \to E$ of $U(f)$ and $U(g)$ (i.e. the coequalizer in $\mathbf{Set}$). We can turn E into a signature, by declaring the inputs of $[b] \in E$ to be the inputs of any of its representatives $b \in B$. Like before, no contradictions can arise from such choices, since the permutation amalgamations for f and g are equal. We can then make e into a morphism of signatures by declaring the amalgamation permutations to be identities.

The verification that this is indeed a coequalizer in $\mathbf{Sig}_a$ proceeds as above – we first check in **Set**, at the level of function symbols, and then argue for the amalgamation permutations, which in this case is trivial. $\square$

Theorem 2.7.2. *The all the functors in the monoidal structures on* $\mathbf{Sig}_a$ *and* $\mathbf{Sig}_{ma}$ *introduced above preserve fibered filtered colimits and fibered reflexive coequalizers. They also preserve coproducts in the left variable.*

Proof. The second statement is obvious: the symbol a in the formal composite $\langle a, b_i\rangle$, where $a \in \coprod_i A_i$ must come from exactly one A_i, and this means that the natural map

$$\coprod_i (A_i \otimes B) \to (\coprod_i A_i) \otimes B$$

is an isomorphism. This is also true for $\odot$, since it is defined in terms of $\otimes$.

The first statement is also easy: as we have seen, fibered filtered colimits and reflexive coequalizers in $\mathbf{Sig}_a$ are constructed, essentially, in **Set**. By corollary 1.2.3 this is also true for $\mathbf{Sig}_{ma}$, since the projection $\mathbf{Sig}_{ma} \to \mathbf{Sig}_a$ is a fiberwise isomorphism.

It is well known that in **Set** filtered colimits and reflexive coequalizers commute with finite products. Now note that $A \otimes B$, with A and B being signatures over O, is nothing but a coproduct of finite products: lists of the form

$$\langle a, b_i\rangle,$$

with the typings of a and b_i held fixed, are simply elements of the products

$$A(\partial^A(a)) \times B(\partial^B(b_1)) \times \ldots \times B(\partial^B(b_n)),$$

where a has arity n, and the notation C(some typing) is the set of function symbols in the signature C with that typing.

The whole $A \otimes B$ is just the coproduct of these products, with the sum being over $O^\dagger \otimes O^\dagger$, the set of all possible compatible typings (the product above is the summand corresponding to $\langle \partial^A(a), \partial^B(b_i)\rangle$). Since we know that filtered colimits and reflexive coequalizers commute with other colimits, and with finite products in **Set**, the whole operation $A \otimes B$ also preserves filtered colimits and reflexive coequalizers, by their construction in $\mathbf{Sig}_a$.

The same argument applies to $\odot$, since again, it is based on matching outputs to inputs. $\square$

2.7.2 Wide Pullbacks in Categories of Signatures

Theorem 2.7.3. *The categories* $\mathbf{Sig}_a$ *and* $\mathbf{Sig}_{ma}$ *have wide pullbacks, which are preserved by all their monoidal structures and their associated free monoid constructions.*

The proof of this theorem takes the rest of this subsection. We start with a lemma that does a lot of the work for us. Its proof is elementary category theory, and is omitted.

Lemma 2.7.4. *Let $\mathcal{B}, \mathcal{E}, \mathcal{F}, \mathcal{G}$ be categories with wide pullbacks, and let the functors $p_1 \colon \mathcal{E} \to \mathcal{B}$, $p_2 \colon \mathcal{F} \to \mathcal{B}$, $p_3 \colon \mathcal{G} \to \mathcal{B}$ and $F \colon \mathcal{E} \to \mathcal{G}$ preserve them. Then:*

1. *The category $\mathcal{E} \times_{\mathcal{B}} \mathcal{F}$ has wide pullbacks, and the projections to $\mathcal{E}$ and $\mathcal{F}$ preserve them.*
2. *If $\mathcal{E}$ is a relative monoidal category over $\mathcal{B}$, and the monoidal structure and unit preserve wide pullbacks (which exist by the above point), then the relative category of monoids $Mon(\mathcal{E})$ has wide pullbacks, and the forgetful functor $Mon(\mathcal{E}) \to \mathcal{E}$ preserves them.*
3. *If F is over $\mathcal{B}$ then its pullback $p_2^* F \colon \mathcal{E} \times_{\mathcal{B}} \mathcal{F} \to \mathcal{G} \times_{\mathcal{B}} \mathcal{F}$ also preserves wide pullbacks.*

Now suppose we can prove that $\mathbf{Sig}_a$ has wide pullbacks preserved by the projection to $\mathbf{Set}$, and that the monoidal structure preserves them. Then $Mon(\mathbf{Sig}_a)$ has wide pullbacks by the above proposition. $\mathbf{Sig}_{ma}$ is defined as the pullback

$$\begin{tikzcd}
\mathbf{Sig}_{ma} \arrow[r] \arrow[d, "p_{ma}"'] & \mathbf{Sig}_a \arrow[d, "p_a"] \\
Mon(\mathbf{Sig}_a) \arrow[r, "\mathcal{U}"] & \mathbf{Set}
\end{tikzcd}$$

Since $\mathcal{U}$ preserves pullbacks, we obtain from the above lemma that $\mathbf{Sig}_{ma}$ has wide pullbacks, and its projection preserves them. By the last part of the lemma, the vertical monoidal structure $\otimes, I_\otimes$ on $\mathbf{Sig}_{ma}$ also preserves wide pullbacks, since it is a pullback of the structure on $\mathbf{Sig}_a$.

To prove theorem 2.7.3 we need to show that $\mathbf{Sig}_a$ has wide pullbacks preserved by the projection to $\mathbf{Set}$, that its monoidal structure preserves them, and that the horizontal structure $\odot, I_\odot$ on $\mathbf{Sig}_{ma}$ preserves wide pullbacks, as well as the free monoid functor $\mathcal{F}_\odot$.

Consider a family $f_i \colon X_i \to Y$ in $\mathbf{Sig}_a$ for which we wish to compute a pullback. To simplify the situation, we use the functorial factorization constructed in lemma 3.2.3. Thus we write

$$f_i = X_i \xrightarrow{a_i} X[i] \xrightarrow{\bar{f}_i} Y,$$

where the first map is an isomorphism over O_i with nonstandard amalgamation, and the second map is strict. The signature $X[i]$ is simply X with function symbols retyped according to the amalgamation permutations of f_i, so that $\bar{f}_i\colon X[i] \to Y$ is strict, and is identical to f_i at the level of function symbols.

To define $\prod_Y X_i$, consider the pullback of the $X[i]$ over Y, considered as ordinary signatures (it is elementary to see that **Sig** is complete, since it has small products and equalizers). To see that this is a pullback in $\mathbf{Sig_a}$, consider the strict projections $p_i\colon \prod_Y X[i] \to X[i]$, and their obvious compositions

$$\pi_i\colon \prod_Y X[i] \xrightarrow{p_i} X[i] \xrightarrow{a_i^{-1}} X_i.$$

Note that these are maps with nonstandard amalgamation. We claim that this defines the desired pullback in $\mathbf{Sig}_a$. It is obvious that this is a cone, so we only need to see that is a limiting cone. Consider any other cone $g_i\colon Z \to X_i$. We can obtain a cone $a_i g_i\colon Z \to X[i]$. These maps can have nonstandard amalgamation, but since they form a cone over Y, all the permutations are determined by the constant $f = f_i \circ g_i\colon Z \to Y$. Thus, the obvious and unique set map $Z \to \prod_Y X[i]$ can be given a unique structure of a morphism in $\mathbf{Sig}_a$, which completes our proof. Note that the set of types for $\prod_Y X[i]$ is $\prod_O O_i$, where O is the set of types for Y and O_i are the types for X_i. Thus the projection $\mathbf{Sig}_a \to \mathbf{Set}$ preserves wide pullbacks.

Wide Pullbacks and Monoidal Structures

Now we must show that the functors $I\colon \mathbf{Set} \to \mathbf{Sig}_a$ and $\otimes\colon \mathbf{Sig}_a \times_{\mathbf{Set}} \mathbf{Sig}_a \to \mathbf{Sig}_a$ preserve wide pullbacks. This fact is obvious for I, since $I(O)$ is just the set O with a unary typing of each of its elements with itself. The functor $\otimes$ is not much more difficult. The natural map

$$\prod_Y X_i \otimes \prod_Z T_i \to \prod_{Y \otimes Z} X_i \otimes T_i$$

is given by

$$\langle (x_i)_i, ((t_i)_i)_j \rangle \mapsto (\langle x_i, t_{i,j} \rangle)_i,$$

where $(x_i)_i$ is an element of $\prod_Y X_i$, and $(t_i)_i$ are elements of $\prod_Z T_i$, as j ranges over the inputs (any one) of the x_i. The map above transforms a formal composite of compatible families into a single compatible family of formal composites. It is clearly a bijection and hence an isomorphism (whatever the amalgamation permutations are on the types, we can always invert them). It is obvious that the associativity and unit isomorphisms are compatible with the above one.

Thus it follows from point 3 of lemma 2.7.4, and the 2-functoriality of pullbacks in **Cat**, that the vertical monoidal structure $(\otimes, I_\otimes)$ on $\mathbf{Sig}_{ma}$ preserves wide pullbacks. This leaves the horizontal structure $(\odot, I_\odot)$. Again, the unit is easy – $I_\odot(M)$ is just $I_\otimes(M)$ with the vertical inputs removed. It is thus obvious that it preserves wide pullbacks. The computation for $\odot$ is formally the same as for $\otimes$ with $\langle \ldots \rangle$ replaced by $\dot{\langle} \ldots \dot{\rangle}$. Indeed $A \odot_M B$ as a set is defined as $A \otimes_O B$, where O is the set of types for M, and the typing of A and B over O is defined as

$$A \xrightarrow{\text{typing}} M^\dagger \xrightarrow{\text{output}} M \xrightarrow{\text{typing}} O^\dagger$$

Thus, since we know how to compute pullbacks in $\mathbf{Sig}_{ma}$ in terms of those in $\mathbf{Sig}_a$, the natural map

$$\prod_X A_i \odot \prod_Y B_i \to \prod_{X \odot Y} A_i \odot B_i$$

will be the same as the natural map for $\otimes$ at the level of function symbols. But we already know it is a bijection in that case, and so it is an isomorphism.

Compatibility with the associativity and unit isomorphisms is again obvious.

The Free Monoid Construction and Wide Pullbacks

To finish the proof, we must show that $\mathcal{F}_\odot$ preserves wide pullbacks. The computation for $\mathcal{F}_\odot$ is the same as for $\mathcal{F}_\otimes$ in both $\mathbf{Sig}_{ma}$ and $\mathbf{Sig}_a$. Thus we will show that the free monoid construction in $\mathbf{Sig}_a$ preserves wide pullbacks.

Proposition 2.7.5. *The fibrations $\mathcal{E} = \mathbf{Sig}_a, \mathbf{Sig}_{ma}$ have the following exactness property: if J is a diagram category describing a wide pullback, then the wide pullback functor $\mathcal{E}^J \to \mathcal{E}$ preserves coproducts.*

Proof. Since the proposition asserts that a certain naturally defined map is an isomorphism, we can forget about the input typing of the function symbols, and check for a bijection. This means applying the forgetful functor $\mathbf{Sig} \to \mathbf{Set}^{\cdot\to\cdot}$, to the codomain fibration. But all of the limits and colimits involved are computed in $\mathbf{Set}^{\cdot\to\cdot}$ at the level of function symbols. For wide pullbacks we proved this above. For coproduts it is obvious.

Thus it suffices to show this property for the codomain fibration, where it is clear. $\square$

The free monoid construction is defined in the same way for all our monoidal structures by theorems 2.7.2, 2.7.1 and 1.5.2. By the free monoid formula, the elements of $\mathcal{F}(X)$, where X is over O, are therefore either (but not both) of the following

i_o a unit, where $o \in O$
$\langle x, t_1, \ldots, t_n \rangle$

where $x \in X$, and the t_i are terms in $\mathcal{F}(X)$, and n is the arity of x. Thus $\mathcal{F}(X)$ consists of trees colored by function symbols in X, with arity corresponding to the number of children of the node.

We wish to see that the natural map

$$\mathcal{F}(\prod_Y X_i) \to \prod_{\mathcal{F}(Y)} \mathcal{F}(X_i)$$

is an isomorphism. Like always, it suffices to check that it is a bijection. Had proposition 2.7.5 stated that colimits along $\mathbb{N}$ preserve wide pullbacks, we would be done, since every part of the formula for the free monoid would preserve them. Unfortunately this is not true. The map above is obviously injective, as it is for any such colimit of wide pullbacks. The problem lies in the surjectivity. A typical element of

$$\prod_{\varinjlim A_j} \varinjlim_j X_{i,j}$$

consists of compatible equivalence classes $([x_i])_{i \in I}$, and the representatives x_i need not all be definable over the same A_j. Thus there is no element $[(x_i)]$ in $\varinjlim_j \prod_{A_j} X_{i,j}$ that would map to it. This situation can only happen, however, if the fibers of the $X_{i,j}$ over a compatible family of elements of A_j (i.e. an element in $\varprojlim_j A_j$) grow as j increases, so that the x_i can be selected in such a way that they cannot all exist at once in the fibers of $X_{i,j}$ over A_j for a single j. Otherwise the element $([x_i]_{i \in I})$ is already in $\varinjlim_j \prod_{A_j} X_{i,j}$.

In our case the fibers do not grow. To see this, consider that trees have a height (the longest distance from the root to a leaf), and the X_n in the free monoid construction consists precisely of trees of height at most n. Thus the elements of $\prod_{\mathcal{F}(Y)} \mathcal{F}(X_i)$ consist of families of trees of at most some specific height, since they map to a single tree in $\mathcal{F}(Y)$. Once trees this height are constructed, no new trees of this size can appear in the later stages of the construction of $\mathcal{F}(X_i)$ – only higher trees are added later on. Thus the unfortunate situation we have described above cannot happen, and our natural map is a surjection.

Chapter 3

Fundamental Properties of Signatures

This chapter contains the results and constructions for signatures which are specific to them, and are not related to general category theory. They are of critical importance in the later chapters.

The first of these properties is the separation principle. It was abstracted from the ideas in the proofs in part 2 of [HMP02]. It allows us to ignore the amalgamation permutations in $\mathbf{Sig}_a$ when comparing natural transformations between two sufficiently nice fibered functors into that category. This simplifies later arguments greatly, since the amalgamation permutations are often defined using iterated recursion, and manageable formulas for them simply cannot be written down. Thus we save a great deal of time avoiding iterated inductive comparisons of permutations, of which there are more than enough already.

The next two sections contain an alternative view of $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$, where $\mathcal{U}\colon Mon(\mathbf{Sig}_a) \to \mathbf{Set}$ is the functor which assigns to monoids their sets of function symbols. These sections are highly technical, but unfortunately of fundamental importance. In essence, they detail the original construction of monoidal signatures with amalgamation. The construction given in the previous chapter is in fact a clever and convenient shortcut – using it risks missing the foundational intuitions behind $\mathbf{Sig}_{ma}$. These two sections make them clearer: monoidal signatures with amalgamation are the result of assembling the slices $(\mathbf{Sig}_{ma}/O)/M$, where M is a monoid in $\mathbf{Sig}_a$ over O, into a monoidal fibration, getting rid of the amalgamation permutations (i.e. strictifying the structure maps $A \to M$), and adding vertical inputs.

This is why the natural monoidal structure on the fibered slice $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ is related to the horizontal structure $\odot$ on $\mathbf{Sig}_{ma}$ – the second is obtained from the first by a strictification procedure, which is why the formulas associated with its construction are so complicated.

In the tradeoff between an easy and efficient construction of $\mathbf{Sig}_{ma}$ and a slightly less complicated construction of $\odot$ we have chosen the former[1].

We finish this chapter by discussing the monoidal property of the pullback action. This builds in the previous two sections, and provides a reflection of the distributivity structure on monoidal signatures in its pullback action. This may sound rather esoteric, but in fact this property underlies all of the comparisons in chapter 6. The main theorem of the last section, 3.4.1, states that the pullback action is monoidal, in the sense that all of the functors $(-) \star X$ (where $d\colon X \to M \in \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$) are coherently monoidal:

$$(-) \star X\colon (\mathbf{Sig}_{ma}/M, \odot) \to (\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}/M, \otimes).$$

From this one may easily deduce the relation between free monoids in both fibrations, as seen in corollary 3.4.2 – a result which will be repeatedly used in all three of our comparisons in chapter 6.

3.1 The Separation Principle

We will often need to verify equality of certain natural transformations. The problem can be split into two parts – check equality on function symbols, and on amalgamation permutations. The first part is usually easy, but the second part is often intractable – the formulas are just too complicated.

The separation principle is inspired by the construction of the multicategory of function replacement in [HMP02], where the permutations are completely avoided (at a cost of definiteness of the construction). We will settle for a little less, exploiting naturality to get rid of the second part. Establishing naturality will be difficult enough. We note the following trivial lemma.

Lemma 3.1.1. *If in the following diagram in $\mathbf{Sig}_{ma}$ (or $\mathbf{Sig}_a$) the morphisms h and k are strict, then for $a \in A$ we have $\sigma_a = \theta_{h(a)}$, where σ are the amalgamation permutations of f and θ are the permutations of g.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

Now consider the following construction. Let $M \in Mon(\mathbf{Sig}_a)$ be a monoid. We can construct a new monoid $M_{\mathbb{N}}$ over the same set of types as follows. The

[1] The most difficult part of the argument – verifying the naturality of $\alpha^{\odot}$ for permutations arising from the new inputs, would be unaffected.

universe of $M_{\mathbb{N}}$ is $M \times \mathbb{N}$, the typing is defined by projecting onto M: $M \times \mathbb{N} \to M \to O^{\dagger}$. The unit is the composite $I \to M \simeq M \times \{0\} \hookrightarrow M_{\mathbb{N}}$, and multiplication is defined by

$$\mu^{M_{\mathbb{N}}}\langle (f, n_0), (g_1, n_1), \ldots, (g_k, n_k)\rangle = (\mu^M(f, g_1, \ldots, g_k), \sum_{i=0}^{k} n_i)$$

The fact that these formulas define a monoid follows from the fact that $\mathbb{N}$ is a monoid in **Set**. Obviously the projection map $\pi\colon M_{\mathbb{N}} \to M$ is a homomorphism of monoids. It will be essential in applications that this homomorphism is strict. In fact we have defined a functor $M \mapsto M_{\mathbb{N}}$, and $\pi\colon (-)_{\mathbb{N}} \to 1_{Mon(\mathbf{Sig}_a)}$ is a natural transformation.

Let $\mathcal{E}$ be a fibration over $Mon(\mathbf{Sig}_a)$ and let $F, G\colon \mathcal{E} \to \mathbf{Sig}_{ma}$ be two fibered functors. Consider the following two properties of F and G (which are to hold for any M):

Definition 3.1.2. *The functors F and G are called* agreeable *if for every $X \in \mathcal{E}$ over M there is a prone morphism $\theta\colon Y \to X$ over π_M such that both $F(\theta)$ and $G(\theta)$ are strict morphisms.*

The notion of agreeability can be extended to any set of functors (we require all of them to be strict on a single prone morphism). We will then say that functors in this set are *jointly agreeable*.

Definition 3.1.3. *A functor F is called* separated *if for some (and hence every) prone morphism $\theta\colon Y \to X$ over π the following holds: for every $f \in F(X)$ there is an $\tilde{f} \in F(Y)$ in the fiber over f (i.e. mapping to f under $F(\theta)$) whose typing is injective.*

The notion of separability can be extended to finite sets of function symbols (we require that any finite set of function symbols can be lifted to ones with injective typing, such that the typings of the lifted symbols are pairwise disjoint). We will call such functors *strongly separated*.

Remark 3.1.4. *All pairs of functors we will deal with will be agreeable and separated. In fact our functors will preserve strict morphisms, making agreeability trivial to check.*

Theorem 3.1.5 (The Separation Principle)**.** *Let $\mathcal{E}$ be a fibration over $Mon(\mathbf{Sig}_a)$, let $F, G\colon \mathcal{E} \to \mathbf{Sig}_{ma}$ be two fibered functors, and let $\phi^1, \phi^2\colon F \to G$ be two fibered natural transformations whose components are equal on function symbols. If F and G are agreeable and F is separated, then $\phi^1 = \phi^2$.*

Proof. We must prove equality of all components. Since they are equal on function symbols, we must check the equality of amalgamation permutations. Consider an $f \in F(X)$ and choose a prone θ over π for which both $F(\theta)$ and $G(\theta)$ are strict. We have the following situation:

$$\begin{tikzcd}
\tilde{f} \arrow[d, mapsto] & F(Y) \arrow[r, shift left, "\phi^1_Y"] \arrow[r, shift right, "\phi^2_Y"'] \arrow[d, "F(\theta)"'] & G(Y) \arrow[d, "G(\theta)"] \\
f & F(X) \arrow[r, shift left, "\phi^1_X"] \arrow[r, shift right, "\phi^2_X"'] & G(X)
\end{tikzcd}$$

By separability there is an $\tilde{f} \in F(Y)$ in the fiber over f whose typing is injective. We know that ϕ^1_Y and ϕ^2_Y are equal on $\tilde{f}$. Their amalgamation permutations on $\tilde{f}$ are uniquely determined, since its typing is injective. Thus they are equal. Now lemma 3.1.1 implies that the amalgamation permutations are equal for f also. $\square$

The separation principle is also true for $\mathbf{Sig}_a$. Instead of considering $M_{\mathbb{N}}$ we consider $O \times \mathbb{N}$ for $O \in \mathbf{Set}$.

There is considerable room in the above argument – one need not consider $M_{\mathbb{N}}$, but some other monoid with infinite fibers over M. In our applications is also important that the projection π has standard amalgamation. The choices we have made work in general and make the statement of the separation principle short enough to be applicable.

A similar argument can be used to *define* natural transformations between agreeable functors, when we know what to do on function symbols. This is what is done in part 2 of [HMP02] to construct multiplication in the multicategory of function replacement.

3.2 An Alternative Description of $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$

Our first result is that $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ (the fibered slice) is equivalent to $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$. The proof requires some preliminary constructions.

Lemma 3.2.1. *For any $M \in \mathbf{Sig}_a/O$ there is a bijection* $\{\text{set maps } X \to M\} \simeq \{\text{strict morphisms } X \to M \text{ over } O\}$.

Proof. To a function $X \to M$ we assign a strict morphism, with X typed by the composition $X \to M \xrightarrow{\partial} O^\dagger$. Conversely, if the morphism is strict, then the typing is defined by that formula, so we can forget it. $\square$

The full subfibration of $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ of strict objects is defined as follows. Recall that the objects of $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ are morphisms $A \to \mathcal{U}(M)$ in $\mathbf{Sig}_a$ over some $O \in \mathbf{Set}$, where M is a monoid in $\mathbf{Sig}_a/O$. An object is called strict if the morphism $A \to \mathcal{U}(M)$ is strict. This fibration will be denoted by $\mathbf{Sig}_a \downdownarrows \mathcal{U}_{str}$.

Corollary 3.2.2. *The subfibration of* $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ *of strict objects is isomorphic to* $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$.

Proof. The above lemma defines a bijection on objects. A morphism between strict objects has, by lemma 3.1.1, the same amalgamation permutations as the morphism in the base, and can therefore be regarded as a function. Conversely any function between strict objects can be made into a morphism by setting the amalgamation permutations to what lemma 3.1.1 says they should be. These constructions are clearly inverse to each other. □

We will now show that the subfibration of $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ of strict objects is in fact fibered equivalent over $Mon(\mathbf{Sig}_a)$ to $\mathbf{Sig}_a \downdownarrows \mathcal{U}$. We will use a functorial factorization for this purpose. This construction was first used in [HMP02] for monoids.

Consider a morphism $f\colon A \to B$ in $\mathbf{Sig}_a$. We will factor it into two morphisms $A \xrightarrow{\zeta_f} A[f] \to B$, with the first morphism an isomorphism and the second morphism strict. The construction is simple: $A[f]$ is the same set as A, but with typing defined by $\partial^A \circ \sigma^{-1}$, which means $\partial^{A[f]}(a) = \partial^A(a) \circ \sigma_a^{-1}$, where σ are the permutations of f, and ∂^A is the original typing.

We now set the morphism $\zeta_f\colon A \to A[f]$ to be the identity on function symbols and have permutations given by σ. Obviously it is an isomorphism. The second morphism acts as f on the function symbols, but is strict.

Recall that a functorial factorization is a section of the composition functor $\mathcal{C}^{\cdot\to\cdot\to\cdot} \longrightarrow \mathcal{C}^{\cdot\to\cdot}$.

Lemma 3.2.3. *The above construction uniquely defines a functorial factorization on* $\mathbf{Sig}_a$ *and on* $Mon(\mathbf{Sig}_a)$.

Proof. Since the morphisms are factorized into an isomorphism followed by some other morphism, the middle of the factorization is uniquely defined by the commutativity conditions. Functoriality is then trivial. It is also easy to check that if we require the morphism $\zeta_f\colon A \to A[f]$ to be an isomorphism of monoids, then the middle factorization will also be a homomorphism if f was one. □

This functorial factorization is not fibered in any good sense – the first morphism is always over the identity, and the second is over whatever the original morphism was over.

Now let $f\colon A \to \mathcal{U}(M)$ be an object of $\mathbf{Sig}_a \downdownarrows \mathcal{U}$. Then $A[f] \to \mathcal{U}(M)$ is a strict object. Since the factorization was functorial, this defines a fibered functor $fct\colon \mathbf{Sig}_a \downdownarrows \mathcal{U} \to \mathbf{Sig}_a \downdownarrows \mathcal{U}_{str}$. There is also the obvious inclusion $i\colon \mathbf{Sig}_a \downdownarrows \mathcal{U}_{str} \hookrightarrow \mathbf{Sig}_a \downdownarrows \mathcal{U}$

Theorem 3.2.4. *The above functors form an adjoint equivalence over* $Mon(\mathbf{Sig}_a)$.

Proof. Save the adjoint part, this is a purely formal consequence of having a functorial factorization which factors a morphism into an isomorphism followed by another morphism. Inclusion followed by factorization is the identity on $\mathbf{Sig}_a \downdownarrows \mathcal{U}_{str}$. A factorization followed by inclusion is isomorphic to the identity functor on $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\zeta_h"'] \arrow[dd, bend right=60, "h"'] & B \arrow[d, "\zeta_k"] \arrow[dd, bend left=60, "k"] \\
A[h] \arrow[r, "fct(f)"] \arrow[d] & B[k] \arrow[d] \\
M \arrow[r, "u"] & N
\end{tikzcd}$$

The components ζ_h of the functorial factorization form an isomorphism from the identity functor to the composite of factorization and inclusion.

For the adjunction we take the components ζ_h^{-1} to be the counit – it is the identity on function symbols, so we only need to worry about its amalgamation permutations. The unit is the identity. The triangular identities then state that the following two composites are identities

$$i(X) \xrightarrow{1} i \circ fct \circ i(X) \xrightarrow{\zeta_{i(X)}^{-1}} i(X)$$
$$fct(X) \xrightarrow{1} fct \circ i \circ fct(X) \xrightarrow{fct(\zeta_X^{-1})} fct(X)$$

They are true, since $fct(\zeta_h^{-1})$ is the identity by lemma 3.1.1 (or direct calculation), and $\zeta_{i(X)}^{-1}$ is the identity for strict objects X. $\square$

Combining this theorem with corollary 3.2.2 we have

Theorem 3.2.5. *There is an adjoint equivalence* $\mathbf{Sig}_a \downdownarrows \mathcal{U} \to \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$.

3.3 A Monoidal Structure on $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$

The fibration $\mathbf{Sig}_a \downdownarrows \mathcal{U}$ is monoidal, and we have shown that it is equivalent to $\mathbf{Sig}_a \downdownarrows \mathcal{U}_{str} \simeq \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$. We can get a monoidal structure on the latter fibration by the following general construction

Let $\mathcal{C}, \mathcal{D}$ be categories (or fibrations over some base $\mathcal{S}$), and let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ be an adjoint equivalence of categories (fibrations) with counit and unit isomorphisms $\varepsilon\colon GF \to 1_{\mathcal{C}}$ and $\eta\colon 1_{\mathcal{D}} \to FG$. If $\mathcal{C}$ is monoidal, then we can make F and G into a monoidal equivalence using the following obvious formulas:

$$\begin{aligned} I_{\mathcal{D}} &= F(I_{\mathcal{C}}) \\ A \otimes_{\mathcal{D}} B &= F(G(A) \otimes_{\mathcal{C}} G(B)) \end{aligned}$$

as for the rest of the needed data (α, λ, ρ and monoidal structures on F and G), the obvious choices are the correct ones.

Theorem 3.3.1. *The above construction defines a monoidal structure on $\mathcal{D}$, F and G, for which ϵ and η are monoidal transformations.*

Proof. Exercise. Everything follows from naturality of various transformations except (all) diagrams involving η, where the triangular identities are also needed. $\square$

We can calculate what this structure looks like in our case. For example the units are unchanged, since they have only unary function symbols. The associativity isomorphism is the following

$$\alpha_{A,B,C}(\langle a, \langle b_i, c_{i,j}\rangle\rangle) = \langle\langle a, b_i\rangle, c_{\gamma^{-1}_{\langle \partial a, \partial b_i\rangle}(i,j)}\rangle$$

for A, B, C in the fiber over M. ∂ denotes the structure morphisms to M, and γ the amalgamation permutations of the multiplication map in M. This structure will be denoted by $\otimes$.

Corollary 3.3.2. *The functor $U\colon (\mathbf{Sig}_{ma}, \odot) \to (\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}, \otimes)$ is strict monoidal.*

Proof. All the formulas for parts of both monoidal structures coincide. $\square$

This gives an alternative construction of $\mathbf{Sig}_{ma}$ – take $\mathbf{Sig}_a \downdownarrows \mathcal{U}$, strictify and add vertical inputs. The construction by pullback is significantly more efficient.

3.4 Monoidal Property of the Pullback Action

Consider the exponential adjoint $\hat{\star}\colon \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot} \to \underline{Hom}_{Mon(\mathbf{Sig}_a)}(\mathbf{Sig}_{ma}, \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot})$ of the pullback action $\star\colon \mathbf{Sig}_{ma} \times \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot} \to \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$. We have the following theorem, which reflects the existence of a distributivity structure on $\mathbf{Sig}_{ma}$ in the pullback action.

Theorem 3.4.1. *This adjoint lifts to* $(\odot, \otimes)$*-monoidal functors:*

$$\begin{tikzcd} & \underline{Hom}^{\odot,\otimes}_{Mon(\mathbf{Sig}_a)}(\mathbf{Sig}_{ma}, \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}) \arrow[d] \\ \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot} \arrow[ur, dashed, "\tilde{\star}"] \arrow[r, "\hat{\star}"'] & \underline{Hom}_{Mon(\mathbf{Sig}_a)}(\mathbf{Sig}_{ma}, \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}) \end{tikzcd}$$

Proof. By definition 2.5.1 (which still holds true for $\mathbf{Sig}_{ma}$, by corollary 1.1.12) we have

$$\hat{\star} = U_* \circ R \circ \overline{-},$$

where R is the functor $X \mapsto (-) \otimes X$ for $\mathbf{Sig}_{ma}$, $U_* = \underline{Hom}_{Mon(\mathbf{Sig}_a)}(1, U)$ is the action of U by postcomposition, and $\overline{-}$ is the sterile signature functor.

Since we have a lift $\tilde{R}$ of R to $\underline{End}^{\odot}_{Mon(\mathbf{Sig}_a)}(\mathbf{Sig}_{ma})$, and U is strict $(\odot, \otimes)$-monoidal we can define $\tilde{\star}$ by

$$\tilde{\star} = U_* \circ \tilde{R} \circ \overline{-}$$

$\square$

Concretely, this gives us the following natural isomorphisms, where $A, B \in \mathbf{Sig}_{ma}$, and $X \in \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$:

$$\begin{array}{rcl} (A \star X) \otimes (B \star X) & \xrightarrow{\phi_{A,B,X}} & (A \odot B) \star X \\ \langle (a, x_{i,j}), (b_1, x_{i',j'}), \ldots, (b_k, x_{i'',j''}) \rangle & \mapsto & (\dot{\langle} a, b_1, \ldots, b_k \dot{\rangle}, x_{m,n}) \end{array}$$

$$\begin{array}{rcl} I_\odot \star X & \to & I_\otimes \\ (1_o, -) & \mapsto & 1_o, \end{array}$$

which are given by formulas formally identical to those for distributivity in $\mathbf{Sig}_{ma}$. They give each functor $(-)\star X$ the structure of a monoidal functor $(\mathbf{Sig}_{ma}/M, \odot) \to (\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}/M, \otimes)$ where X is over M.

Corollary 3.4.2. *The pullback action has the following properties:*

1. *Every functor* $(-) \star X$ *maps* $\odot$*-monoids in* $\mathbf{Sig}_{ma}/M$ *to* $\otimes$*-monoids in* $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}/M$*, where* X *is over* M*.*

2. $\mathcal{F}_\odot(I_\otimes) \star X \simeq \mathcal{F}_\otimes(X)$*. In particular, this isomorphism maps multiplication to multiplication* $\mu^{\mathcal{F}_\odot}_{I_\otimes} \star X \simeq \mu^{\mathcal{F}_\otimes}_X$*, and the units and counits:* $\eta^{\mathcal{F}_\odot}_{I_\otimes} \star X \simeq \eta^{\mathcal{F}_\otimes}_X$ *and* $\varepsilon_{I_\otimes} \star X \simeq \varepsilon_X$*.*

Proof. The first point is trivial. The second one follows from the formula for free monoids in theorem 1.5.2, the fact that $\star$ preserves filtered colimits and coproducts in the left variable (by corollaries 2.5.2, 1.1.12 and 1.2.3), and the fact that $\star$ is an action, which gives $I_\otimes \star X \simeq X$. Thus $(-) \star X$ maps the free $\odot$-monoid construction in $\mathbf{Sig}_{ma}$ to the free $\otimes$-monoid construction in $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$. Combining these facts gives $\mathcal{F}_\odot(I_\otimes) \star X \simeq \mathcal{F}_\otimes(I_\otimes \star X) \simeq \mathcal{F}_\otimes(X)$ along with all the associated structure. $\square$

Chapter 4

The Web Monoid

The web monoid construction lies at the heart of our construction of opetopic sets, the study of their structure, and all the comparison theorems. It is an algebraic construction, in the sense that the result is a monoid in a monoidal category. Accordingly, we only compare our work with other approaches of the same (or similar) kind.

The specific monoids we use are instances of an abstract construction, characterized by the three tensors theorem. It states that in any fibration with two sufficiently nice, and independent monoidal structures (in the sense of distributivity structures, as explained in the introduction to chapter 1), the free monoid $\mathcal{F}_{\odot}(I_{\otimes})$ has a unique $\otimes$-monoid structure, which commutes with the $\odot$-free monoid structure. This commutativity condition is called the main diagram, and is made possible by the distributivity structure. In our case this commutativity is quite literal, at least if we interpret our monoids as monads – this is what theorem 6.1.7 amounts to.

The proof of the three tensors theorem is long and arduous. For this reason we have split it into two parts. The first part is an outline, which contains all the ideas and constructions needed to complete the proof. The second part consists of the tedious computations necessary to check that the constructed structures satisfy all the conditions of the proof, i.e. that the web monoid is indeed a monoid.

The proof is made possible by the explicit inductive construction of free monoids we have given (theorem 1.5.2). This allows us to make an educated guess about what the multiplication in the web monoid should be, and complete its definition using a recursive formula, which is forced upon us by the main diagram.

In the section following the proof we give an example, showing that in the web monoid $\mathcal{W}(M)$, considered as a monoid in $\mathbf{Sig}_a$, cannot always have standard amalgamation. It is not (usually) isomorphic to a monoid with standard amalgamation, even if M is. The example is purely pictorial[1], and the reason for its existence is simple – in a tree each node can have many neighbors, and in a list of inputs, each input can only have two. Thus, if nodes of a tree are to be treated as inputs of function symbols, some shuffling upon replacing a node by a tree is necessary, no matter the order in which we listed the nodes. The complexity of nonstandard amalgamation is an unfortunate requirement, and there is no way around it.

From this example it is easy to see that the frame monoids $\mathbf{S}_n(X)$ for opetopic sets X (defined in the next chapter) usually have nonstandard amalgamation for $n \geq 2$.

In the last section of this chapter we prove two combinatorial theorems. They state the conditions under which the web monoid functor $\mathcal{W}$ preserves wide pullbacks, and assert that these conditions are satisfied for $\mathbf{Sig}_{ma}$. These theorems are used in the next chapter to prove that opetopic sets form a presheaf category. They are called combinatorial, since they do, in fact, require some combinatorial arguments with trees (or equivalently terms), but more importantly, they completely replace the extremely complicated combinatorial constructions used in part 3 of [HMP02] to build the multitopes and prove that multitopic sets form a presheaf category. This result can be deduced from the equivalence of opetopic and multitopic sets from chapter 6 and these combinatorial theorems.

The name "web monoid" comes from 2-dimensional pasting diagrams, which look like spider webs.

4.1 The Three Tensors Theorem

Theorem 4.1.1 (The Fibered Three Tensors Theorem)**.** *If the fibration $\mathcal{E} \to \mathcal{B}$ admits the free monoid construction (definition 1.5.1) for monoidal structures $\odot$ and $\otimes$, and $\otimes$ distributes over $\odot$ by a distributivity structure (φ, ψ), then there is a unique lift of the functor $\mathcal{F}_\odot(I_\otimes(-))$, the free $\odot$-monoid on the $\otimes$-unit functor, to the category of $\otimes$-monoids, such that the unit of the adjunction $\mathcal{F}_\odot \dashv \mathcal{U}_\odot$, $\eta_{I_\otimes(-)} \colon I_\otimes(-) \to \mathcal{F}_\odot(I_\otimes(-))$ is the unit of the multiplication $\nu \colon \mathcal{F}_\odot(I_\otimes(-)) \otimes \mathcal{F}_\odot(I_\otimes(-)) \to \mathcal{F}_\odot(I_\otimes(-))$, which in turn makes the following* main diagram *commute (we abbreviate $\mathcal{F}_\odot(I_\otimes(-))$ to $\mathcal{W}$):*

[1] It therefore applies to all other algebraic approaches equivalent to ours.

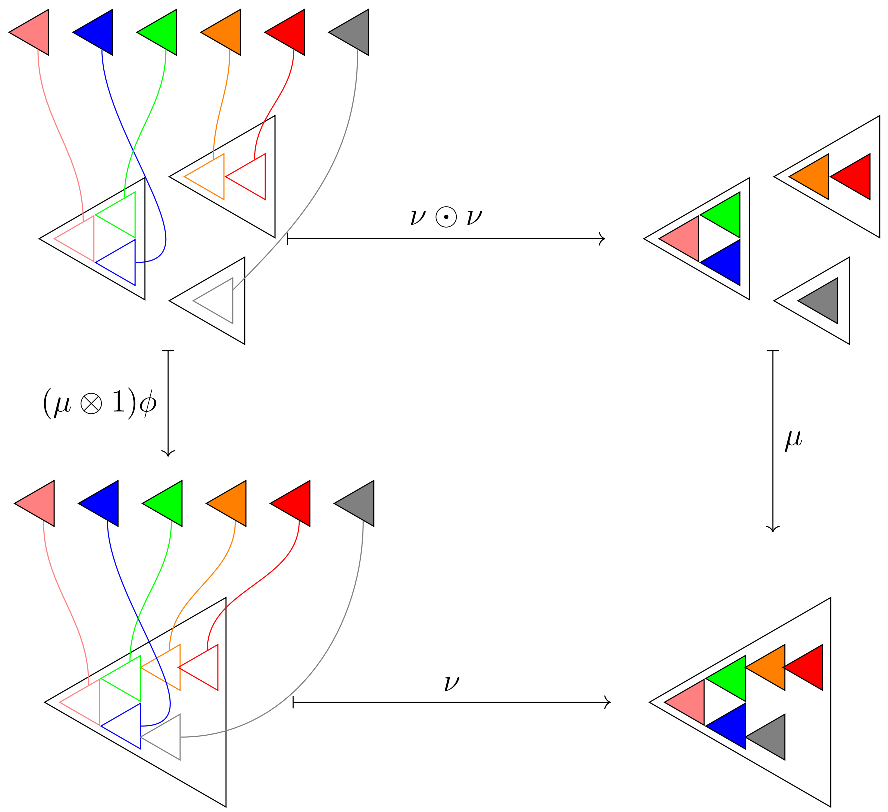
$$\begin{tikzcd}
(\mathcal{W} \otimes \mathcal{W}) \odot (\mathcal{W} \otimes \mathcal{W}) \arrow[r, "\nu \odot \nu"] \arrow[d, "\varphi_{\mathcal{W},\mathcal{W},\mathcal{W}}"'] & \mathcal{W} \odot \mathcal{W} \arrow[dd, "\mu"] \\
(\mathcal{W} \odot \mathcal{W}) \otimes \mathcal{W} \arrow[d, "\mu \otimes 1_{\mathcal{W}}"'] & \\
\mathcal{W} \otimes \mathcal{W} \arrow[r, "\nu"] & \mathcal{W}
\end{tikzcd}$$

In the above diagram μ is the free multiplication in $\mathcal{F}_{\odot}(I_{\otimes}(-))$.

The third, implicit, monoidal structure in the theorem is the coproduct, which is assumed to exist through the free monoid construction.

Note that all the structures mentioned in this theorem are stable under pullback – the conclusion of holds in each fiber separately.

Using the pictorial conventions of subsection 2.3.1, we can illustrate this theorem, at least for signatures $\mathbf{Sig}_{ma}$, as follows:



4.2 Outline of The Proof

We must check that ν is unique if it exists, construct it, and verify the conditions of the theorem. We will do the first two of these steps here, and carry out the remaining technical calculations in the next section. These first steps contain the key idea of the proof, while the rest consists of tedious inductive computations.

As $\otimes$ preserves filtered colimits we need only determine compatible components $\nu_n \colon \mathcal{W}_n \otimes \mathcal{W} \to \mathcal{W}$, where $\mathcal{W}_n$ is the n-th stage of the construction of $\mathcal{W} = \mathcal{F}_\odot(I_\otimes)$ from theorem 1.5.2. We will prove that these components are uniquely defined by the conditions of the theorem and define ν using these components. We begin with a trivial lemma.

Lemma 4.2.1 (Bootstrap lemma)**.** *If the following diagram commutes,*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[dd, "g"] \\
C \arrow[d, "k"'] & \\
D \arrow[r, "l"] & E
\end{tikzcd}$$

where h is an isomorphism and k has a section s, then

$$l = g \circ f \circ h^{-1} \circ s$$

In addition, if k is an isomorphism, then the diagram commutes if and only if the above equation holds.

Applying this lemma to the main diagram and the canonical section $\hat{s}$ of the free multiplication $\mu \colon \mathcal{W} \odot \mathcal{W} \to \mathcal{W}$ we obtain the equation:

$$\nu = \mu \circ (\nu \odot \nu) \circ \varphi^{-1} \circ \hat{s}$$

which must be satisfied by ν, but is unhelpful until we precompose it with $i_{n,\infty} \otimes 1 \colon \mathcal{W}_n \otimes \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$, and obtain the following diagram:

$$
\begin{tikzcd}
(\mathcal{W}_1 \otimes \mathcal{W}) \odot (\mathcal{W}_{n-1} \otimes \mathcal{W}) \arrow[r] \arrow[rr, bend left, "\nu_1 \odot \nu_{n-1}"] & (\mathcal{W} \otimes \mathcal{W}) \odot (\mathcal{W} \otimes \mathcal{W}) \arrow[r, "\nu \odot \nu"'] & \mathcal{W} \odot \mathcal{W} \arrow[dd, "\mu"] \\
(\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes \mathcal{W} \arrow[u, "\varphi^{-1}"] \arrow[r, "(i_1 \odot i_{n-1}) \otimes 1"] & (\mathcal{W} \odot \mathcal{W}) \otimes \mathcal{W} \arrow[u, "\varphi^{-1}"'] & \\
\mathcal{W}_n \otimes \mathcal{W} \arrow[u, "s_n \otimes 1"] \arrow[r, "i_n \otimes 1"] \arrow[rr, bend right, "\nu_n"'] & \mathcal{W} \otimes \mathcal{W} \arrow[u, "\hat{s} \otimes 1"'] \arrow[r, "\nu"] & \mathcal{W}
\end{tikzcd}
$$

The boundary of this diagram provides an inductive definition of ν_n, starting from ν_1. The unnamed arrow is $(i_1 \otimes 1) \odot (i_{n-1} \otimes 1)$, and all the i maps should have an additional ∞ subscript (omitted for readability). This diagram is commutative *if* ν exists. The top and bottom "bigons" or "biangles" are commutative, since by definition $\nu_n = \nu \circ i_{n,\infty} \otimes 1$, and $\odot$ is a functor. The upper small rectangle is commutative by naturality of φ. To prove commutativity of the lower small rectangle note the diagram

$$
\begin{tikzcd}
\mathcal{W}_1 \odot \mathcal{W}_{n-1} \arrow[r, "1 \odot i_{n-1}"] & \mathcal{W}_1 \odot \mathcal{W} \arrow[r, "i_1 \odot 1"] & \mathcal{W} \odot \mathcal{W} \\
\mathcal{W}_n \arrow[u, "s_n"] \arrow[r, "i_n"] & \mathcal{W} \arrow[u, "s"] \arrow[r, "1"] & \mathcal{W} \arrow[u, "\hat{s}"]
\end{tikzcd}
$$

which is the lower small rectangle with s added in the middle. The left square commutes since s is by definition the limit of $1 \odot i_{n-1,\infty} \circ s_n$. The right square commutes by proposition 1.5.3 (this is how we defined $\hat{s}$).

So far we have obtained that if ν exists, then the ν_n must satisfy

$$\nu_n = \mu \circ (\nu_1 \odot \nu_{n-1})\varphi^{-1}(s_n \otimes 1), \tag{4.1}$$

which means that any candidate for ν is uniquely determined by ν_1. The equation immediately gives the compatibility condition $\nu_n \circ i_n \otimes 1 = \nu_{n-1}$ – just add $\mathcal{W}_{n-1} \otimes \mathcal{W}$ in the lower left corner of the diagram above, two analogous small rectangles above it, and use induction. We define $\nu_0 = \nu_1 \circ i_0$.

We will show in the next section that ν_1 is uniquely determined by the unit conditions. We note that

$$\mathcal{W}_1 \otimes \mathcal{W} \simeq (I_\odot \sqcup I_\otimes) \otimes \mathcal{W} \simeq I_\odot \otimes \mathcal{W} \sqcup I_\otimes \otimes \mathcal{W}$$

Thus the map ν_1 is determined by what happens on both of these components. The calculations in the next section give these components as

$$\nu_1 = (i_0 \psi_{\mathcal{W}}^{-1}, \lambda_{\mathcal{W}}^{\otimes})$$

Where $i_0 \colon I_\odot \to \mathcal{W}$.

Proposition 4.2.2 (Uniqueness of ν)**.** *If ν exists, then it is the colimit of the arrows $\nu_n \colon \mathcal{W}_n \otimes \mathcal{W} \to \mathcal{W}$, with ν_0, ν_1 defined above, and ν_k defined by induction using equation 4.1, for $k > 1$.*

Proof. ν is determined by the family $\nu_n = \nu \circ i_{n,\infty} \otimes 1$. The calculations above (or in the next section, in the case $n = 1$) determine these components uniquely. □

Definition 4.2.3 (The definition of ν)**.** *We define $\nu \colon \mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$ as the colimit of the arrows $\nu_n \colon \mathcal{W}_n \otimes \mathcal{W} \to \mathcal{W}$.*

We are now left with checking that this definition works. We do so in the next section.

4.3 Details of The Proof

4.3.1 Determination of ν_1 and ν_0

In this subsection we still assume that ν exists. We must prove that $\nu_1 = (i_0 \psi_{\mathcal{W}}^{-1}, \lambda_{\mathcal{W}}^{\otimes})$ as claimed above. This follows from the unit conditions. They are

$$\begin{tikzcd}
I_\otimes \otimes \mathcal{W} \arrow[r, "\eta \otimes 1"] \arrow[dr, "\lambda^\otimes"'] & \mathcal{W} \otimes \mathcal{W} \arrow[d, "\nu"] & \mathcal{W} \otimes I_\otimes \arrow[l, "1 \otimes \eta"'] \arrow[dl, "(\rho^\otimes)^{-1}"] \\
 & \mathcal{W} &
\end{tikzcd}$$

We can expand them to the following commutative diagram

$$
\begin{tikzcd}
\mathcal{W}_1 \otimes \mathcal{W} \arrow[dr, "i_1 \otimes 1"] \arrow[ddr, dotted, "\nu_1"'] & & & \mathcal{W}_1 \otimes \mathcal{W}_1 \arrow[lll, "1 \otimes i_1"'] \\
I_\otimes \otimes \mathcal{W} \arrow[u, "j \otimes 1"] \arrow[dr, "\lambda"'] & \mathcal{W} \otimes \mathcal{W} \arrow[d, "\nu"] & \mathcal{W} \otimes I_\otimes \arrow[l, "1 \otimes \eta"'] & \mathcal{W}_1 \otimes I_\otimes \arrow[l, "i_1 \otimes 1"'] \arrow[u, "1 \otimes j"'] \arrow[dll, "(\rho^\otimes)^{-1}(i_1 \otimes 1)"] \\
& \mathcal{W} & &
\end{tikzcd}
$$

where some of the original maps was omitted for readability. Maps labeled j are coprojections of coproducts. They are factorizations of η (hence the commutativity). We wish to determine the dotted arrow ν_1. Note that

$$\mathcal{W}_1 \otimes \mathcal{W} \simeq (I_\odot \sqcup I_\otimes) \otimes \mathcal{W} \simeq I_\odot \otimes \mathcal{W} \sqcup I_\otimes \otimes \mathcal{W}$$

Thus the map ν_1 is determined by what happens on both of these components. The left unit condition immediately implies that the right component is mapped to $\mathcal{W}$ by $\lambda^\otimes_{\mathcal{W}}$. To see what happens to $I_\odot \otimes \mathcal{W}$ consider the top map composed with $1 \otimes j$ and the inclusion of $I_\odot \otimes I_\otimes$ into $\mathcal{W}_1 \otimes I_\otimes$. An easy calculation gives that this is $1 \otimes \eta \colon I_\odot \otimes I_\otimes \to I_\odot \otimes \mathcal{W}$ followed by the inclusion $I_\odot \otimes \mathcal{W} \to \mathcal{W}_1 \otimes \mathcal{W}$. Consider now the right unit condition. We obtain the diagram

$$
\begin{tikzcd}
I_\odot \arrow[d, "\psi_{\mathcal{W}}"'] & I_\odot \arrow[l, "1"'] \arrow[d, "\psi_{I_\otimes}"] \\
I_\odot \otimes \mathcal{W} \arrow[d] & I_\odot \otimes I_\otimes \arrow[l, "1 \otimes \eta"'] \arrow[d] \\
I_\odot \otimes \mathcal{W} \sqcup I_\otimes \otimes \mathcal{W} \arrow[d, "{(?, \lambda^\otimes_{\mathcal{W}})}"'] & I_\odot \otimes I_\otimes \sqcup I_\otimes \otimes I_\otimes \arrow[l, "1 \otimes \eta \sqcup 1 \otimes \eta"'] \arrow[dl, "i_0(\rho^\otimes)^{-1} \sqcup (\rho^\otimes)^{-1}\eta"] \\
\mathcal{W} &
\end{tikzcd}
$$

We want to determine the map "?". The unnamed maps are coprojections. On the right components of the coproducts this diagram commutes by naturality of $\lambda^\otimes$ and the condition $\lambda_{I_\otimes} = \rho^{-1}_{I_\otimes}$. The second component of the diagonal map is determined by the naturality of $\rho^\otimes$ applied to the inclusion $I_\odot = \mathcal{W}_0 \to \mathcal{W}_1 \to \mathcal{W}$. The top square commutes by naturality of ψ. From this we obtain the equation

$$? \circ \psi_{\mathcal{W}} \psi^{-1}_{I_\otimes} = i_0(\rho^\otimes_{I_\odot})^{-1}$$

from which follows, using coherence condition VI, that the map "?" is

$$i_0 \circ \psi_{\mathcal{W}}^{-1} \colon I_\odot \otimes \mathcal{W} = \mathcal{W}_0 \otimes \mathcal{W} \to \mathcal{W}.$$

Note that these calculations also determine that $\nu_0 = i_{0,\infty}\psi_{\mathcal{W}}^{-1}$, since this map is ν_1 precomposed with the inclusion $I_\odot \otimes \mathcal{W} \to \mathcal{W}_1 \otimes \mathcal{W}$, and we have just determined exactly this composite.

From now on we use the definition 4.2.3 for ν since we have already showed that it is the only possible choice. We still have to check that ν defines a monoid and makes the main diagram commute. We will intensely use induction – the first component will satisfy an appropriate equality, usually because of the coherence conditions, and then equality for all the other components will follow by applying the inductive definition 4.1. The original condition will be recovered by applying the colimit functor.

4.3.2 The Unit Conditions

The left unit condition holds as part of our definition of ν, since it factors through ν_1, which was defined in part by this condition. This leaves the right unit condition. We will prove it using induction on n starting with $n = 1$, which consists of the calculations above. For the inductive step we need to check that

$$\nu_n \circ 1 \otimes \eta = i_n(\rho^{\otimes})^{-1}.$$

Consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{W}_1 \odot \mathcal{W}_{n-1} \arrow[dr, bend left, "i_1 \odot i_{n-1}"] & \\
(\mathcal{W}_1 \otimes I_\otimes) \odot (\mathcal{W}_{n-1} \otimes I_\otimes) \arrow[ur, bend left, "\rho^{-1} \odot \rho^{-1}"] \arrow[r, "(1 \otimes \eta) \odot (1 \otimes \eta)"] & (\mathcal{W}_1 \otimes \mathcal{W}) \odot (\mathcal{W}_{n-1} \otimes \mathcal{W}) \arrow[r, "\nu_1 \odot \nu_{n-1}"] & \mathcal{W} \odot \mathcal{W} \arrow[dd, "\mu"] \\
 & (\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes \mathcal{W} \arrow[u, "\varphi_{1,n-1}^{-1}"'] & \\
 & \mathcal{W}_n \otimes \mathcal{W} \arrow[u, "s_n \otimes 1"'] \arrow[r, "\nu_n"] & \mathcal{W} \\
(\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes I_\otimes \arrow[uuu, "\varphi_{1,n-1}^{-1}"] \arrow[uur, "1 \otimes \eta"] & \mathcal{W}_n \otimes I_\otimes \arrow[l, "s_n \otimes 1"] \arrow[u, "1 \otimes \eta"] \arrow[ur, "i_n \rho^{-1}"'] &
\end{tikzcd}$$

All the regions in it commute except possibly the small triangle below ν_n, which we are investigating. This follows from the naturality of φ (note the abbreviation we have introduced here), the definition of ν_n, and the inductive hypothesis (for the top region). From this we obtain that

$$\nu_n 1 \otimes \eta = \mu(i_1 \odot i_{n-1})(\rho^{-1} \odot \rho^{-1})\varphi^{-1}_{1,n-1}(s_n \otimes 1)$$

using the explicit definition of μ given in theorem 1.5.2. Thus if we can check that

$$\mu(i_1 \odot i_{n-1})(\rho^{-1} \odot \rho^{-1})\varphi^{-1}_{1,n-1}(s_n \otimes 1) = i_n \rho^{-1},$$

we would be done. But this comes down to the commutativity of

$$\begin{tikzcd}
(\mathcal{W}_1 \otimes I_\otimes) \odot (\mathcal{W}_{n-1} \otimes I_\otimes) \arrow[r, "\rho^{-1} \odot \rho^{-1}"] & \mathcal{W}_1 \odot \mathcal{W}_{n-1} \arrow[r, "i_1 \odot i_{n-1}"] \arrow[dr, "\mu_{1,n-1}"] & \mathcal{W} \odot \mathcal{W} \arrow[r, "\mu"] & \mathcal{W} \\
(\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes I_\otimes \arrow[u, "\varphi^{-1}_{1,n-1}"] \arrow[r, "\rho^{-1}"] & \mathcal{W}_1 \odot \mathcal{W}_{n-1} \arrow[u, "1"] & \mathcal{W}_n \arrow[ur, "i_n"] & \\
\mathcal{W}_n \otimes I_\otimes \arrow[u, "s_n \otimes 1"] \arrow[r, "\rho^{-1}"] & \mathcal{W}_n \arrow[u, "s_n"] \arrow[ur, "1"] & &
\end{tikzcd}$$

which follows from the definition of μ (top triangle), naturality of $\rho^\otimes$ (bottom rectangle), coherence condition III (top rectangle), and proposition 1.5.3 – the defining property of s_n (bottom triangle).

4.3.3 Commutativity of the Main Diagram

We will consider the diagram

$$\begin{tikzcd}
(\mathcal{W}_n \otimes \mathcal{W}) \odot (\mathcal{W}_m \otimes \mathcal{W}) \arrow[r, "\nu_n \odot \nu_m"] \arrow[d, "\varphi_{n,m}"] & \mathcal{W} \odot \mathcal{W} \arrow[dd, "\mu"] \\
(\mathcal{W}_n \odot \mathcal{W}_m) \otimes \mathcal{W} \arrow[d, "\mu_{n,m} \otimes 1_{\mathcal{W}}"] & \\
\mathcal{W}_{n+m} \otimes \mathcal{W} \arrow[r, "\nu_{n+m}"] & \mathcal{W}
\end{tikzcd}$$

and prove its commutativity by induction on n (for arbitrary m), starting with $n = 0$. In this case $\mu_{0,m} = \lambda^\odot$ is an isomorphism, and the second part of the bootstrap lemma 4.2.1 tells us that we must prove

$$\nu_m = \mu((i_0 \circ \psi^{-1}_{\mathcal{W}}) \odot \nu_m)\varphi^{-1}_{0,m}(\lambda^{-1} \otimes 1),$$

since $\nu_0 = i_0 \circ \psi_{\mathcal{W}}^{-1}$. After applying the unit condition for μ to the right side of the above equation we find that it is

$$\lambda \circ (\psi_{\mathcal{W}}^{-1} \odot \nu_m)\varphi_{0,m}^{-1}(\lambda^{-1} \otimes 1).$$

But by coherence condition IV for ψ and naturality of $\lambda^{\odot}$ this is exactly ν_m, and we are done.

The inductive hypothesis is

$$\mu(\nu_{n-1} \odot \nu_m) = \nu_{n+m-1}(\mu_{n-1,m} \otimes 1)\varphi_{n-1,m}$$

and we must show that

$$\mu(\nu_n \odot \nu_m) = \nu_{n+m}(\mu_{n,m} \otimes 1)\varphi_{n,m}.$$

Expanding the left side, we can calculate [2]

$$\begin{aligned}
&\mu(\nu_n \odot \nu_m) = && \text{(definition of } \nu_n)\\
&\mu([\mu(\nu_1 \odot \nu_{n-1})\varphi_{1,n-1}^{-1}(s_n \otimes 1)] \odot \nu_m) = && \text{(functoriality of } \odot)\\
&\mu((\mu \odot 1) \circ [(\nu_1 \odot \nu_{n-1})\varphi_{1,n-1}^{-1}(s_n \otimes 1)] \odot \nu_m = && \text{(associvativity of } \mu)\\
&\mu((1 \odot \mu) \circ (\alpha^{\odot})^{-1} \circ [(\nu_1 \odot \nu_{n-1})\varphi_{1,n-1}^{-1}(s_n \otimes 1)] \odot \nu_m = && \text{(functoriality of } \odot)\\
&\mu((1 \odot \mu) \circ (\nu_1 \odot (\nu_{n-1} \odot \nu_m)) \circ (\alpha^{\odot})^{-1} \circ [\varphi_{1,n-1}^{-1}(s_n \otimes 1)] \odot 1) = && \text{(inductive hypothesis)}\\
&\mu(\nu_1 \odot \nu_{n+m-1}(\mu_{n-1,m} \otimes 1)\varphi_{n-1,m} \circ (\alpha^{\odot})^{-1} \circ [\varphi_{1,n-1}^{-1}(s_n \otimes 1)] \odot 1), &&
\end{aligned}$$

similarly for the right side

$$\begin{gathered}
\nu_{n+m}(\mu_{n,m} \otimes 1)\varphi_{n,m}\\
=\\
\mu((\nu_1 \odot \nu_{n+m-1})\varphi_{1,n+m-1}^{-1}(s_{n+m} \otimes 1))(\mu_{n,m} \otimes 1)\varphi_{n,m}.
\end{gathered}$$

We will show that

$$\begin{gathered}
\varphi_{1,n+m-1}^{-1}(s_{n+m} \otimes 1))(\mu_{n,m} \otimes 1)\varphi_{n,m}\\
=\\
(1 \odot (\mu_{n-1,m} \otimes 1))\varphi_{n-1,m} \circ (\alpha^{\odot})^{-1} \circ [\varphi_{1,n-1}^{-1}(s_n \otimes 1)] \odot 1.
\end{gathered}$$

This follows from the commutativity of the following diagram (specifically the commutativity of the boundary)

[2] Unfortunately the diagrams involved are simply too big to include here.

$$
\begin{tikzcd}
(\mathcal{W}_n \otimes \mathcal{W}) \odot (\mathcal{W}_m \otimes \mathcal{W}) \arrow[r, "\overbrace{(s_n \otimes 1) \odot 1}"] \arrow[d, "\varphi_{n,m}"] \arrow[dr, phantom, "\mathrm{I}"] & ((\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes \mathcal{W}) \odot (\mathcal{W}_m \otimes \mathcal{W}) \arrow[r, "\overbrace{\varphi^{-1}_{1,n-1} \odot 1}"] \arrow[d, "\varphi_{1\odot(n-1),m}"] \arrow[dr, phantom, "\mathrm{II}"] & ((\mathcal{W}_1 \otimes \mathcal{W}) \odot (\mathcal{W}_{n-1} \otimes \mathcal{W})) \odot (\mathcal{W}_m \otimes \mathcal{W}) \arrow[d, "(\alpha^{\odot})^{-1}"] \\
(\mathcal{W}_n \odot \mathcal{W}_m) \otimes \mathcal{W} \arrow[r, "(s_n \odot 1) \otimes 1"] \arrow[d, "\mu_{n,m} \otimes 1"] \arrow[dr, phantom, "\mathrm{III}"] & ((\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \odot \mathcal{W}_m) \otimes \mathcal{W} \arrow[d, "(\alpha^{\odot})^{-1} \otimes 1"] & (\mathcal{W}_1 \otimes \mathcal{W}) \odot ((\mathcal{W}_{n-1} \otimes \mathcal{W}) \odot (\mathcal{W}_m \otimes \mathcal{W})) \arrow[d, "1 \odot \varphi_{n-1,m}"] \\
\mathcal{W}_{n+m} \otimes \mathcal{W} \arrow[dr, "s_{n+m} \otimes 1"'] & (\mathcal{W}_1 \odot (\mathcal{W}_{n-1} \odot \mathcal{W}_m)) \otimes \mathcal{W} \arrow[r, "\varphi^{-1}_{1,(n-1)\odot m}"] \arrow[d, "(1 \odot \mu_{n-1,m}) \otimes 1"] \arrow[dr, phantom, "\mathrm{IV}"] & (\mathcal{W}_1 \otimes \mathcal{W}) \odot ((\mathcal{W}_{n-1} \odot \mathcal{W}_m) \otimes \mathcal{W}) \arrow[d, "1 \odot (\mu_{n-1,m} \otimes 1)"] \\
 & (\mathcal{W}_1 \odot \mathcal{W}_{n+m-1}) \otimes \mathcal{W} \arrow[r, "\varphi^{-1}_{1,n+m-1}"] & (\mathcal{W}_1 \otimes \mathcal{W}) \odot (\mathcal{W}_{n+m-1} \otimes \mathcal{W})
\end{tikzcd}
$$

This diagram commutes, since all the indicated regions commute. I and IV commute by naturality of φ, II commutes by coherence condition I for φ, and III commutes by the coherence lemma 1.5.4.

4.3.4 Associativity of ν

We will now check that ν is associative. Note that this condition has not been used to define ν, so as we have said earlier, it is a consequence of the main diagram and the unit conditions.

Lemma 4.3.1. *The composite $\nu_n \circ (1 \otimes i_m)$ factors through $i_{n\cdot m}$, as in the diagram*

$$
\begin{tikzcd}
\mathcal{W}_n \otimes \mathcal{W}_m \arrow[r, "1 \otimes i_m"] \arrow[dr, "\nu_{n,m}"'] & \mathcal{W}_n \otimes \mathcal{W} \arrow[r, "\nu_n"] & \mathcal{W} \\
 & \mathcal{W}_{n\cdot m} \arrow[ur, "i_{n\cdot m}"'] &
\end{tikzcd}
$$

Proof. By induction on n. For $n = 1$ we have $\nu_1 = (i_0\psi^{-1}_{\mathcal{W}}, \lambda_{\mathcal{W}})$, and the claim follows from the naturality of $\lambda^{\otimes}$, ψ and the fact that $i_0 = i_m \circ i_0$ (recall our abuse of notation). We obtain $\nu_{1,m} = (i_0\psi^{-1}_{\mathcal{W}_m}, \lambda_{\mathcal{W}_m})$.

The inductive step immediately follows from this commutative diagram:

$$
\begin{tikzcd}
(\mathcal{W}_1 \otimes \mathcal{W}_m) \odot (\mathcal{W}_{n-1} \otimes \mathcal{W}_m) \arrow[r] \arrow[rrr, bend left, "\nu_{1,m} \odot \nu_{n-1,m}"] & (\mathcal{W}_1 \otimes \mathcal{W}) \odot (\mathcal{W}_{n-1} \otimes \mathcal{W}) \arrow[r, "\nu_n \odot \nu_{n-1}"'] & \mathcal{W} \odot \mathcal{W} \arrow[dd, "\mu"] & \mathcal{W}_m \odot \mathcal{W}_{(n-1)m} \arrow[l] \arrow[dd, "\mu_{m,(n-1)m}"'] \\
(\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes \mathcal{W}_m \arrow[u, "\varphi^{-1}"] \arrow[r, "1 \otimes i_m"] & (\mathcal{W} \odot \mathcal{W}) \otimes \mathcal{W} \arrow[u, "\varphi^{-1}"'] & & \\
\mathcal{W}_n \otimes \mathcal{W}_m \arrow[u, "s_n \otimes 1"] \arrow[r, "1 \otimes i_m"] & \mathcal{W}_n \otimes \mathcal{W} \arrow[u, "s_n \otimes 1"'] \arrow[r, "\nu_n"] & \mathcal{W} & \mathcal{W}_{nm} \arrow[l, "i_{nm}"]
\end{tikzcd}
$$

The top arrow implements the inductive hypothesis, and the unnamed arrows are the obvious ones. □

We will show, by induction on n, for all m, the commutativity of

$$
\begin{tikzcd}
\mathcal{W}_n \otimes (\mathcal{W}_m \otimes \mathcal{W}) \arrow[r, "\alpha^{\otimes}"] \arrow[d, "1 \otimes \nu_m"] & (\mathcal{W}_n \otimes \mathcal{W}_m) \otimes \mathcal{W} \arrow[r, "\nu_{n,m} \otimes 1"] & \mathcal{W}_{nm} \otimes \mathcal{W} \arrow[d, "\nu_{nm}"] \\
\mathcal{W}_n \otimes \mathcal{W} \arrow[rr, "\nu_n"] & & \mathcal{W}
\end{tikzcd}
$$

Passing to the limit gives the desired associativity law. For $n = 0$ we need to show that

$$i_0 \psi^{-1}_{\mathcal{W}_m}(1 \otimes \nu_m) = \nu_m((i_0 \psi^{-1}_{\mathcal{W}_m}) \otimes 1)\alpha^{\otimes},$$

which is an easy calculation following from the fact that ψ is natural (anything on the right of ψ^{-1} can be canceled), and coherence condition V. The inductive hypothesis to be used in passing from $n-1$ to n is

$$\nu_{(n-1)m}(\nu_{n-1,m} \otimes 1)\alpha^{\otimes}_{n-1,m} = \nu_{n-1}(1 \otimes \nu_m)$$

Again we calculate:

$$
\begin{aligned}
&\nu_n(1 \otimes \nu_m) = && \text{(definition of } \nu_n\text{)} \\
&\mu((\nu_1 \odot \nu_{n-1}))\varphi^{-1}_{1,n-1}(s_n \otimes 1)(1 \otimes \nu_m) = && \text{(functoriality of } \odot\text{)} \\
&\mu((\nu_1 \odot \nu_{n-1}))\varphi^{-1}_{1,n-1}(1 \otimes \nu_m)(s_n \otimes 1) = && \text{(naturality of } \varphi\text{)} \\
&\mu((\nu_1(1 \otimes \nu_m) \odot \nu_{n-1}(1 \otimes \nu_m)))\varphi^{-1}_{1,n-1,m\otimes w}(s_n \otimes 1) = && \text{(inductive hypothesis)} \\
&\mu((\nu_m(\nu_{1,m} \otimes 1)\alpha^{\otimes}_{1,m}) \odot (\nu_{(n-1)m}(\nu_{n-1,m} \otimes 1)\alpha^{\otimes}_{n-1,m})) && \\
&\varphi^{-1}_{1,n-1,m\otimes w}(s_n \otimes 1) = && \text{(diagram below)} \\
&\mu(\nu_m \odot \nu_{(n-1)m})\varphi^{-1}_{m,(n-1)m}[(\nu_{1,m} \odot \nu_{n-1,m}) \otimes 1] && \\
&(\varphi_{1,n-1,m} \otimes 1)\alpha^{\otimes}_{1\odot(n-1),m\otimes w}(s_n \otimes 1) &&
\end{aligned}
$$

The relevant diagram is

$$
\begin{tikzcd}
(\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes (\mathcal{W}_m \otimes \mathcal{W}) \arrow[r, "\alpha^{\otimes}_{1\odot(n-1),m}"] \arrow[d, "\varphi^{-1}_{1,n-1,m\otimes w}"'] & ((\mathcal{W}_1 \odot \mathcal{W}_{n-1}) \otimes \mathcal{W}_m) \otimes \mathcal{W} \arrow[dd, "\varphi^{-1}_{1,n-1,m} \otimes 1"] \\
(\mathcal{W}_1 \otimes (\mathcal{W}_m \otimes \mathcal{W})) \odot (\mathcal{W}_{n-1} \otimes (\mathcal{W}_m \otimes \mathcal{W})) \arrow[d, "\alpha^{\otimes}_{1,m} \odot \alpha^{\otimes}_{n-1,m}"'] & \\
((\mathcal{W}_1 \otimes \mathcal{W}_m) \otimes \mathcal{W}) \odot ((\mathcal{W}_{n-1} \otimes \mathcal{W}_m) \otimes \mathcal{W}) \arrow[d, "(\nu_{1,m} \otimes 1) \odot (\nu_{n-1,m} \otimes 1)"'] & ((\mathcal{W}_1 \otimes \mathcal{W}_m) \odot (\mathcal{W}_{n-1} \otimes \mathcal{W}_m)) \otimes \mathcal{W} \arrow[l, "\varphi^{-1}_{1\otimes m,(n-1)\otimes m}"] \arrow[d, "(\nu_{1,m} \odot \nu_{n-1,m}) \otimes 1"] \\
(\mathcal{W}_m \otimes \mathcal{W}) \odot (\mathcal{W}_{(n-1)m} \otimes \mathcal{W}) & (\mathcal{W}_m \odot \mathcal{W}_{(n-1)m}) \otimes \mathcal{W} \arrow[l, "\varphi^{-1}_{m,(n-1)m}"]
\end{tikzcd}
$$

The top rectangle commutes by coherence condition II for φ. The bottom one commutes by naturality of φ.

We will now calculate the left side of the associativity condition. To do this we first calculate $\nu_{n,m}$, using the diagram from the proof of lemma 4.3.1:

$$\nu_{n,m} = \mu_{m,(n-1)m}(\nu_{1,m} \odot \nu_{n-1,m})\varphi^{-1}_{1,n-1,m}(s_n \otimes 1).$$

Putting this into

$$\nu_{nm}(\nu_{n,m} \otimes 1)\alpha^{\otimes}_{n,m},$$

we obtain

$$\nu_{nm}((\mu_{m,(n-1)m}(\nu_{1,m} \odot \nu_{n-1,m})\varphi^{-1}_{1,n-1,m}(s_n \otimes 1)) \otimes 1)\alpha^{\otimes}_{n,m}.$$

Now consider the diagram:

$$
\begin{tikzcd}
 & (\mathcal{W}_m \otimes \mathcal{W}) \odot (\mathcal{W}_{(n-1)m} \otimes \mathcal{W}) \arrow[r, "\nu_m \odot \nu_{(n-1)m}"] & \mathcal{W} \odot \mathcal{W} \arrow[dd, "\mu"] \\
 & (\mathcal{W}_m \odot \mathcal{W}_{(n-1)m}) \otimes \mathcal{W} \arrow[u, "\varphi^{-1}_{m,(n-1)m}"'] \arrow[d, "\mu_{m,(n-1)m} \otimes 1"] & \\
(\mathcal{W}_n \otimes \mathcal{W}_m) \otimes \mathcal{W} \arrow[ur] \arrow[r, "\nu_{n,m} \otimes 1"] & \mathcal{W}_{nm} \otimes \mathcal{W} \arrow[r, "\nu_{nm}"] & \mathcal{W}
\end{tikzcd}
$$

which is commutative by the commutativity of the main diagram. The unnamed arrow is

$$[(\nu_{1,m} \odot \nu_{n-1,m})\varphi^{-1}_{1,n-1,m}(s_n \otimes 1)] \otimes 1.$$

From this, we obtain that the left side of the associativity condition is

$$\mu(\nu_m \odot \nu_{(n-1)m})\varphi^{-1}_{m,(n-1)m}[(\nu_{1,m} \odot \nu_{n-1,m})\varphi^{-1}_{1,n-1,m}(s_n \otimes 1)] \otimes 1 \circ \alpha^{\otimes}_{n,m},$$

which is the same as the right side of the associativity condition.

4.3.5 $\mathcal{W}$ as a Functor

The only thing left to check is that the $\mathcal{W}(u)$ are homomorphisms with respect to ν. To see this, note that by theorem 1.5.2 the map η is fibered and natural. This means that the units of $\mathcal{W}$ are preserved across fibers. That theorem also tells us that the unit of μ and μ itself are preserved, since $\mathcal{F}_\odot$ is fibered, and takes values in the category of $\odot$-monoids. The structure maps λ and ψ are preserved by assumption.

From this it follows that ν_0 is preserved, by naturality of ψ and the fact that $i_0 \colon I_\odot \to \mathcal{W}$ is preserved:

$$\begin{tikzcd}
I_\odot(Q) \arrow[r, "\psi_{\mathcal{W}(Q)}"] \arrow[rr, bend left, "i_0(Q)"] & \mathcal{W}_0(Q) \otimes_Q \mathcal{W}(Q) \arrow[r, "\nu_0(Q)"] & \mathcal{W}(Q) \\
I_\odot(O) \arrow[u, "I_\odot(u)"] \arrow[r, "\psi_{\mathcal{W}(O)}"'] \arrow[rr, bend right, "i_0(O)"'] & \mathcal{W}_0(O) \otimes_O \mathcal{W}(O) \arrow[u, "\mathcal{W}_0(u) \otimes_u \mathcal{W}(u)"] \arrow[r, "\nu_0(O)"'] & \mathcal{W}(O) \arrow[u, "\mathcal{W}(u)"']
\end{tikzcd}$$

The triangles (top and bottom) and the left square commute, as does the boundary of the diagram (the outermost arrows). Since ψ are isomorphisms, the right square also commutes. Similarly ν_1 is preserved, since it is defined using $\lambda^{\otimes}, \psi$ and i_0, all of which are preserved by assumption, and coproducts, for which the appropriate equalities are easy to check (using corollary 1.2.3). The inductive step is taken care of by the following lemma.

Lemma 4.3.2 (Functoriality lemma)**.** *The following diagram commutes, for* $n > 0$

$$\begin{tikzcd}
\mathcal{W}_1(O) \odot_O \mathcal{W}_{n-1}(O) \arrow[r, "\mathcal{W}_1(u) \odot_u \mathcal{W}_{n-1}(u)"] & \mathcal{W}_1(Q) \odot_Q \mathcal{W}_{n-1}(Q) \\
\mathcal{W}_n(O) \arrow[u, "s_n(O)"] \arrow[r, "\mathcal{W}_n(u)"] & \mathcal{W}_{n-1}(Q) \arrow[u, "s_n(Q)"']
\end{tikzcd}$$

Proof. Expanding the definitions we have

$$\begin{tikzcd}
\mathcal{W}_{n-1}(O) \sqcup I_{\otimes,O} \odot \mathcal{W}_{n-1}(O) \arrow[r, "\underbrace{\mathcal{W}_{n-1}(u) \sqcup I_{\otimes,u} \odot_u \mathcal{W}_{n-1}(u)}"] & \mathcal{W}_{n-1}(Q) \sqcup I_{\otimes,Q} \odot \mathcal{W}_{n-1}(Q) \\
I_{\odot,O} \sqcup I_{\otimes,O} \odot \mathcal{W}_{n-1}(O) \arrow[u, "i_{0,n-1}(O) \sqcup 1"] \arrow[r, "I_{\odot,u} \sqcup I_{\otimes,u} \odot \mathcal{W}_{n-1}(u)"] & I_{\odot,Q} \sqcup I_{\otimes,Q} \odot \mathcal{W}_{n-1}(Q) \arrow[u, "i_{0,n-1}(Q) \sqcup 1"']
\end{tikzcd}$$

which commutes, since $i_{0,n-1}$ is preserved, and the other vertical components are identities. $\square$

The preservation of ν_n now follows by induction from the following diagram

$$
\begin{tikzcd}
 & (\mathcal{W}_1(O) \otimes \mathcal{W}(O)) \odot (\mathcal{W}_{n-1}(O) \otimes \mathcal{W}(O)) \arrow[drr, "\nu_1(O) \odot \nu_{n-1}(O)"] \arrow[ddddd, dotted] & & \\
(\mathcal{W}_1(O) \odot \mathcal{W}_{n-1}(O)) \otimes \mathcal{W}(O) \arrow[ur, "\varphi^{-1}_{1,n-1,O}"] \arrow[ddd, bend right=60, "(\mathcal{W}_1(u) \odot \mathcal{W}_{n-1}(u)) \otimes \mathcal{W}(u)"'] \arrow[ddd, phantom, bend right=30, "\mathrm{I}"] \arrow[drr, phantom, "\mathrm{II}_O"] & & & \mathcal{W}(O) \odot \mathcal{W}(O) \arrow[dl, "\mu(O)"'] \arrow[ddd, bend left=60, "\mathcal{W}(u) \odot \mathcal{W}(u)"] \arrow[dd, phantom, "\mathrm{III}"'] \\
\mathcal{W}_n(O) \otimes \mathcal{W}(O) \arrow[u, "s_n(O) \otimes 1"] \arrow[rr, "\nu_n(O)"] \arrow[d, "\mathcal{W}_n(u) \otimes \mathcal{W}(u)"'] & & \mathcal{W}(O) \arrow[d, "\mathcal{W}(u)"] & \\
\mathcal{W}_n(Q) \otimes \mathcal{W}(Q) \arrow[rr, "\nu_n(Q)"] \arrow[d, "s_n(Q) \otimes 1"] \arrow[drr, phantom, "\mathrm{II}_Q"] & & \mathcal{W}(Q) & \\
(\mathcal{W}_1(Q) \odot \mathcal{W}_{n-1}(Q)) \otimes \mathcal{W}(Q) \arrow[dr, "\varphi^{-1}_{1,n-1,Q}"'] & & & \mathcal{W}(Q) \odot \mathcal{W}(Q) \arrow[ul, "\mu(Q)"] \\
 & (\mathcal{W}_1(O) \otimes \mathcal{W}(O)) \odot (\mathcal{W}_{n-1}(O) \otimes \mathcal{W}(O)) \arrow[urr, "\nu_1(Q) \odot \nu_{n-1}(Q)"'] & &
\end{tikzcd}
$$

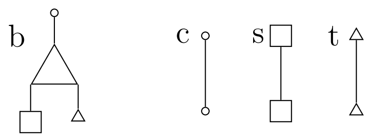
The dotted arrow is $(\mathcal{W}_1(u) \otimes \mathcal{W}(u)) \odot (\mathcal{W}_{n-1}(u) \otimes \mathcal{W}(u))$. The reader should imagine two copies of the main diagram defining $\nu(O)$ and $\nu(Q)$, side by side, connected by the various $\mathcal{W}_k(u)$. The picture above is a flattening of that situation. We need to check the commutativity of the central square. To do this we check that every other region commutes. I commutes by the functoriality lemma 4.3.2, applied to the left variable. II_O and II_Q commute by the definition of ν_n, and III commutes by the fact that $\mathcal{W}(u)$ are $\odot$-monoid homomorphisms, which they are by definition. We now turn to unnamed regions. The region defined by the dotted arrow, the leftmost arrow in region I, and φ commutes by naturality of φ. The region defined by the dotted arrow and the solid boundary of the diagram to the right of it commutes by the inductive hypothesis. Hence the central square commutes.

The preservation of ν now follows from the second part of corollary 1.2.3. This completes the proof of theorem 4.1.1.

4.4 Nonstandard Amalgamation is Necessary

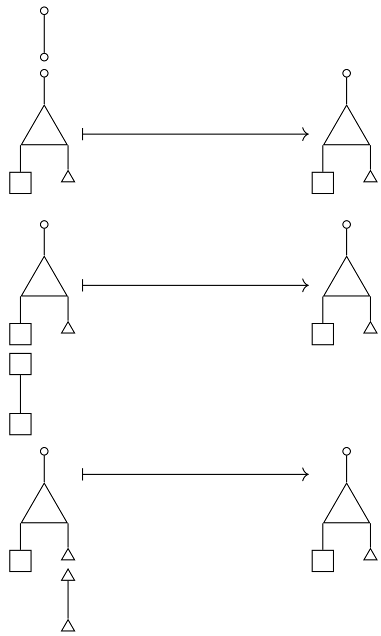
The following simple example shows that the web monoid $\mathcal{W}(M)$ need not be isomorphic to any monoid with standard amalgamation, even if M is standard. Such an isomorphism amounts to being able to retype the elements of the web monoid in such a way as to get standard amalgamation for multiplication. The example consists purely of pictures, and hence applies to most other constructions in the literature (for example the multicategory of function replacement).

Consider a set O of three distinct types { circle, square, triangle } and a signature M consisting of the following function symbols: $\{b, c, s, t, 1_c, 1_s, 1_t\}$. The symbol b (like binary) is binary, and its typing is arbitrary, but injective. We have fixed one such typing in the pictures below. The symbols c, s, t are unary of input and output type circle, square and triangle, respectively. The rest of the symbols are to be considered as identities on their respective types (we will want to consider M as a monoid). We will draw the nonidentity symbols like this



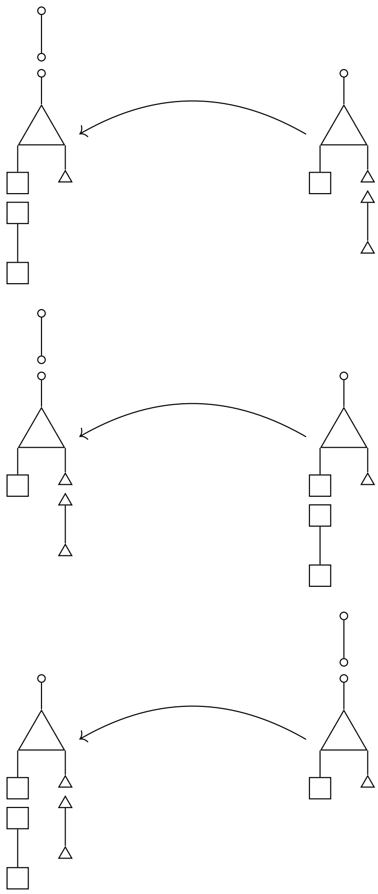
The shapes indicate input/output types of the symbols. We will never draw the identity symbols.

Note that M has, because of our choice of typing, a unique structure of a monoid in $\mathbf{Sig}_a$ (up to a choice of identity arrows, one of which we have indicated), and this monoid has standard amalgamation. Below we draw part of the multiplication table for M. It shows the result of computing $\mu(\langle c, b\rangle)$, $\mu(\langle b, 1_t, s\rangle)$ and $\mu(\langle b, t, 1_s\rangle)$, where μ is the multiplication map.

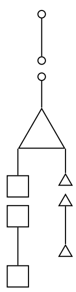


The web monoid $\mathcal{W}(M)$ consists of formal composites of these symbols (its universe is $\mathcal{F}_{\odot}(I_{\otimes}(M))$). The list of input types of a formal composite is the list of the symbols used to build it (in "tree order", but this is irrelevant since we will consider all the orderings), and its output type is its composite in M (image under the counit).

Looking at the multiplication table for M, we see that in $\mathcal{W}(M) \otimes \mathcal{W}(M)$ the following elements are well-defined. Each formal composite on the right is input to the central binary symbol in the formal composite on the left (again, other inputs get identities):



and all of them compose to



(this ultimately follows from the definition of ν_n given in equation 4.1).

Consider the amalgamation permutations of multiplication in $\mathcal{W}(M)$. The above composite has four distinct types – the list of function symbols used to build it – as do the above elements of $\mathcal{W}(M) \otimes \mathcal{W}(M)$ which compose to it. Thus the amalgamation permutations are determined uniquely once we determine the order in which these types are listed for all the four elements we are considering. We must list them in such a way that all the amalgamation permutations arising from the above compositions can be taken to be the identity. But in the above elements of $\mathcal{W}(M) \otimes \mathcal{W}(M)$ the binary symbol was listed next to each nonidentity unary symbol (because of our convention for typing tensor products). The identity permutation preserves the "was listed next to" relation. Thus, if we want standard amalgamation, the binary function symbol in the above formal composite must have as neighbors all three unary symbols. Three neighbors in a list is one too many – a contradiction.

It is easy to see that the situation above actually arises in our opetopic sets. Therefore some pasting diagram monoids must have nonstandard amalgamation.

This example is in some sense minimal – we need at least one non-unary symbol to obtain a contradiction, and here we use exactly one.

4.5 Combinatorial Theorems

Theorem 4.5.1 (Abstract Combinatorial Theorem)**.** *If the categories $\mathcal{E}, \mathcal{B}$ have wide pullbacks, and the projection $\mathcal{E} \to \mathcal{B}$ as well as all the structures $\odot, \otimes, (\varphi, \psi)$ and $I_\otimes, I_\odot \colon \mathcal{B} \to \mathcal{E}, \mathcal{F}_\odot \colon \mathcal{E} \to Mon_\odot(\mathcal{E})$ preserve them, then $\mathcal{W} \colon \mathcal{B} \to Mon_\otimes(\mathcal{E})$ preserves wide pullbacks as well.*

A few remarks before the proof: when we say that a monoidal structure preserves wide pullbacks, we also mean that the associativity and unit isomorphisms are compatible with this preservation in the obvious way (similar to the one defined below for (φ, ψ)). It follows from our assumptions that the categories of monoids in $\mathcal{E}$ have wide pullbacks for both structures $\odot$ and $\otimes$, and that the forgetful functors create them. Note that we need $I_\odot$ and $I_\otimes$ to preserve wide pullbacks to state this, as well as to state what it means for $\mathcal{F}_\odot$ to preserve wide pullbacks, since we are working in the fibered context.

It is clear what preserving wide pullbacks means for functors. For a distributivity structure (φ, ψ) it means that the following diagrams are commutative:

$$\begin{tikzcd}
(\prod_X A_i \otimes \prod_Z C_i) \odot (\prod_Y B_i \otimes \prod_Z C_i) \arrow[r] \arrow[d, "\varphi"'] & \prod\limits_{(X\otimes Z)\odot(Y\otimes Z)} (A_i \otimes C_i) \odot (B_i \otimes C_i) \arrow[d, "\prod \varphi_{A_i,B_i,C_i}"] \\
(\prod_X A_i \odot \prod_Y B_i) \otimes \prod_Z C_i \arrow[r] & \prod\limits_{(X\odot Y)\otimes Z} (A_i \odot B_i) \otimes C_i
\end{tikzcd}$$

$$\begin{tikzcd}
I_{\odot}(\prod_Y X_i) \arrow[r] \arrow[d, "\psi_A"'] & \prod_{I_{\odot}(Y)} I_{\odot}(X_i) \arrow[d, "\prod \psi_A"] \\
I_{\odot}(\prod_Y X_i) \otimes A \arrow[r] & \prod_{I_{\odot}(Y)} I_{\odot}(X_i) \otimes A
\end{tikzcd}$$

where the horizontal maps are the natural ones, defined by wide pullbacks.

Proof of theorem 4.5.1. By our assumptions the natural map $\mathcal{W}(\prod_Y X_i) \to \prod_{\mathcal{W}(Y)} \mathcal{W}(X_i)$ is an isomorphism in $\mathcal{E}$ and preserves the $\otimes$-units of both monoids. We only need to check that it is a homomorphism with respect to multiplication.

There are two ways to do this. One way is a direct computation. In the proof of the three tensors theorem an explicit recursive formula for multiplication in $\mathcal{W}$ is given 4.1, and it is obvious by induction that it is stable under wide pullbacks, since it is constructed out of $\mu, \varphi, \psi, \odot, \otimes$ and the unique readability map s, all of which are stable. The details of this approach are left to the interested reader.

Another way is to appeal to the uniqueness clause in the three tensors theorem. Consider the diagram

$$\begin{tikzcd}
(\mathcal{W} \otimes \mathcal{W}) \odot (\mathcal{W} \otimes \mathcal{W}) \arrow[rr, dashed] \arrow[d, "\varphi_{\mathcal{W},\mathcal{W},\mathcal{W}}"'] \arrow[ddr] & & \mathcal{W} \odot \mathcal{W} \arrow[ddd, "\mu"] \arrow[ddr] & \\
(\mathcal{W} \odot \mathcal{W}) \otimes \mathcal{W} \arrow[dd, "\mu \otimes 1_{\mathcal{W}}"'] \arrow[dddr] & & & \\
& (\Pi\mathcal{W} \otimes \Pi\mathcal{W}) \odot (\Pi\mathcal{W} \otimes \Pi\mathcal{W}) \arrow[rr, "\Pi\nu \odot \Pi\nu"] \arrow[dd] & & \Pi\mathcal{W} \odot \Pi\mathcal{W} \arrow[ddd, "\mu"] \\
\mathcal{W} \otimes \mathcal{W} \arrow[rr, dashed] \arrow[ddr] & & \mathcal{W} \arrow[ddr] & \\
& (\Pi\mathcal{W} \odot \Pi\mathcal{W}) \otimes \Pi\mathcal{W} \arrow[d] & & \\
& \Pi\mathcal{W} \otimes \Pi\mathcal{W} \arrow[rr, "\Pi\nu"] & & \Pi\mathcal{W}
\end{tikzcd}$$

Where we have abbreviated $\mathcal{W}(\prod_Y X_i)$ by $\mathcal{W}$ and $\prod_{\mathcal{W}(Y)} \mathcal{W}(X_i)$ by $\Pi\mathcal{W}$. Without the dotted lines, this diagram is commutative, since we assumed enough preservation of wide pullbacks. Since the diagonal arrows are isomorphisms in $\mathcal{E}$ we can, by transport of structure, fill in the dotted arrows by a $\otimes$-monoid structure on $\mathcal{W}(\prod_Y X_i)$ whose unit is η. Thus we can make the main diagram for $\mathcal{W}(\prod_Y X_i)$ commute by using this structure. But there is only one such structure (recall that this structure is determined by the fiber only, not the functor structure of $\mathcal{W}$), so transport of structure along the isomorphism $\mathcal{W}(\prod_Y X_i) \to \prod_{\mathcal{W}(Y)} \mathcal{W}(X_i)$ must result in the unique correct multiplication on $\mathcal{W}(\prod_Y X_i)$, and thus this isomorphism is in fact a homomorphism with respect to the correct multiplication. $\square$

Theorem 4.5.2 (Concrete Combinatorial Theorem)**.** *The fibration* $\mathbf{Sig}_{ma} \to Mon(\mathbf{Sig}_a)$ *satisfies the assumptions of the abstract combinatorial theorem.*

Proof. Most of the work has already been done in the proof of theorem 2.7.3. We are only left with checking that the distributivity structure preserves wide pullbacks. But this is obvious from the defining formulas: the natural transformation $\varphi_{A,B,X}$ is given by mapping

$$\dot{\langle}\langle a, x_{0,1}, \dots, x_{0,l_0}\rangle, \langle b_1, x_{1,1}, \dots, x_{1,l_1}\rangle, \dots, \langle b_k, x_{k,1}, \dots, x_{k,l_k}\rangle\dot{\rangle}$$

to

$$\langle\dot{\langle} a, b_1, \dots, b_k\dot{\rangle}, x_{0,1}, \dots, x_{0,l_0}, x_{1,1}, \dots, x_{1,l_1}, \dots, x_{k,1}, \dots, x_{k,l_k}\rangle$$

with trivial amalgamation permutations.

It is clear that if we replace a, b_i and $x_{i,j}$ by compatible families of symbols in a wide pullback, the formula will remain the same. The case of ψ_X is similarly trivial. □

Chapter 5

Opetopic Sets

In this chapter we construct and analyze the category of opetopic sets. We adapt the full formalism of [BD98], defining $\mathbf{O}$-opetopic sets, where $\mathbf{O}$ is a monoid in $\mathbf{Sig}_a$. We prove that the $\mathbf{O}$-opetopic sets assemble into a fibration over $Mon(\mathbf{Sig}_a)$, a fact which was implicitly used in certain constructions in [BD98]. We call this fibration the anytopic fibration, since $\mathbf{O}$ is any monoid.

The opetopic sets are a special case of $\mathbf{O}$-opetopic sets, corresponding to the choice $\mathbf{O} = I$, the monoid with a single type, and a single unary operation (the identity on the single type). Since all of our arguments in this chapter work for arbitrary $\mathbf{O}$, without any simplification for ordinary opetopic sets, we retain this generality throughout.

The $\mathbf{O}$-opetopic sets for $\mathbf{O}$ consisting of two types and a single unary non-identity operation between them were deemed appropriate by Baez and Dolan to describe weak n-functors in their definition of weak higher categories. The resulting shapes look like opetopes multiplied by an arrow $\cdot \to \cdot$. Due to the fibration property, the ends of this arrow correspond to opetopic sets (interpreted by Baez and Dolan as the domain and codomain of a weak n-functor). The middle describes (at least in low dimensions) double category-style arrows between them. It would be interesting to see if natural transformations could be definied in a similar way. It seems to me that operads, the basic ones we have been using, are only appropriate for this first dimension, and new ideas are required to go higher up.

With the definition of opetopic sets accounted for, we proceed with the analysis of their category. The critical insight is that opetopic sets are constructed in stages, dimension by dimension. Having constructed an opetopic set up to dimension n, to proceed further, the only thing that is needed is a *set* X_{n+1} of $(n+1)$-dimensional cells, and a specification of their boundaries, called frames (these are ultimately parallel domains and codomains, can be seen from the comparison to multitopic sets in section 6.3). This is encoded in a function $\vartheta_n \colon X_{n+1} \to S_n(X)$, which assigns each cell its frame. This is enough to determine the $(n+1)$-dimensional frames, and repeat the procedure. It is at this point that the web monoid plays a critical role, and it is the most complicated and subtle aspect of opetopic sets.

Despite this complication of intermediate steps, it should be clear, and in any case it will be proven, that $(n+1)$-dimensional opetopic sets arise from the n-dimensional ones by Artin gluing:

$$\mathbf{O} - \mathbf{OSet}_{n+1} = \mathbf{Set}/S_n.$$

This is the content of lemma 5.2.3. The complexity of opetopic sets only arises in the construction of $\mathbf{S}_{n+1}$ from this data. Armed with this insight, and the combinatorial theorem 4.5.2 for the web monoid, the structure of opetopic sets is open to analysis.

The standard theory of Artin gluing, and some simple arguments, quickly lead to the conclusion that all the categories of finite dimensional opetopic sets $\mathbf{O} - \mathbf{OSet}_n$ are presheaf toposes. This is enough to show that the whole category is a presheaf topos, using an ad hoc argument inspired by [Ch03].

Lemma 5.2.3 also allows us to easily analyze the limits and colimits in $\mathbf{O}-\mathbf{OSet}$, somewhat independently of the abstract arguments leading to its identification as a presheaf category. This allows us to establish representability for a naturally defined family of functors, whose representing objects are easily seen to be the $\mathbf{O}$-opetopes.

Since we construct the opetopes as opetopic sets first, the rigid structure of opetopic sets is reflected in the category of opetopes. We record several results to this end, for later use in the comparisons of chapter 6. The extent of these results may be succinctly summarized by saying that the $\mathbf{O}$-opetopes form a FOLDS signature [M95].

5.1 The Anytopic Fibration

The following definition is an adaptation of the definition given in [BD98] to the language used here, with more details spelled out explicitly. A comparison of the resulting categories is sketched in section 6.2. In the definition below monoids are denoted using boldface, and their underlying sets of signatures are denoted normally. Thus $\mathbf{M}$ is a monoid, considered in the category of monoids, and M is the underlying object.

An $\mathbf{O}$-opetopic set X is given by:

1. A monoid $\mathbf{O} \in Mon(\mathbf{Sig}_a)$ with a set of types $M \in \mathbf{Set}$.

2. A sequence of objects $X_n \in \mathbf{Sig}_a/X_{n-1}$, each in the fiber over the previous one (considered as a set), for $n > 0$. By definition X_0 is a set.

3. A sequence of monoids $\mathbf{S}_n \in Mon(\mathbf{Sig}_a)/X_n$, for $n \in \mathbb{N}$.

4. A sequence of functions $X_{n+1} \xrightarrow{\vartheta_n^X} S_n$ for $n \in \mathbb{N}$, and a function $\vartheta_{-1}^X \colon X_0 \to M$, equipped with strict prone arrows ξ_n^X (in $Mon(\mathbf{Sig}_a)$):

$$
\begin{tikzcd}
\mathbf{O} \arrow[d] & \mathbf{S}_0 \arrow[l, "\xi_{-1}^X"'] \arrow[d, "\partial^{\mathbf{S}_0}"] \\
M^\dagger & X_0^\dagger \arrow[l, "(\vartheta_{-1}^X)^\dagger"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{W}(\mathbf{S}_n) \arrow[d, "\partial^{\mathcal{W}(\mathbf{S}_n)}"'] & \mathbf{S}_{n+1} \arrow[l, "\xi_n^X"'] \arrow[d, "\partial^{\mathbf{S}_{n+1}}"] \\
S_n^\dagger & X_{n+1}^\dagger \arrow[l, "(\vartheta_n^X)^\dagger"']
\end{tikzcd}
$$

The functions ϑ_n^X can therefore be regarded as strict morphisms of signatures $X_n \to \mathbf{S}_n$.

A morphism $f \colon X \to Y$ of opetopic sets, where X is $\mathbf{O}$-opetopic, and Y is $\mathbf{O}'$-opetopic, is:

1. A homomorphism of monoids $h \colon \mathbf{O} \to \mathbf{O}'$ in $\mathbf{Sig}_a$ over a function $f_{-1} \colon M \to N$ in $\mathbf{Set}$.

2. A sequence of functions $f_n \colon X_n \to Y_n$, for $n \in \mathbb{N}$.

These data are subject to the following conditions:

1. The morphism $f_{n+1} \colon X_{n+1} \to Y_{n+1}$ is well defined as a strict morphism in $\mathbf{Sig}_a$ over f_n, for $n > 0$.

2. The induced homomorphisms $\bar{f}_n \colon \mathbf{S}_n \to \mathbf{T}_n$ make the following diagrams commute:

$$
\begin{tikzcd}
 & \mathbf{O} \arrow[dl, "h"'] \arrow[dd] & & \mathbf{S}_0 \arrow[ll, "\xi_{-1}^X"'] \arrow[dl, "\bar{f}_0"] \arrow[dd, "\partial^{\mathbf{S}_0}"] \\
\mathbf{O}' \arrow[dd] & & \mathbf{T}_0 \arrow[ll, "\xi_{-1}^Y"' near start] \arrow[dd, "\partial^{\mathbf{T}_0}" near start] & \\
 & M^\dagger \arrow[dl, "f_{-1}^\dagger"'] & & X_0^\dagger \arrow[ll, "(\vartheta_{-1}^X)^\dagger"' near end] \arrow[dl, "f_0^\dagger"] \\
N^\dagger & & Y_0^\dagger \arrow[ll, "(\vartheta_{-1}^Y)^\dagger"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathcal{W}(\mathbf{S}_n) \arrow[dl, "\mathcal{W}(\bar{f}_n)"'] \arrow[dd, "\partial^{\mathcal{W}(\mathbf{S}_n)}"] & & \mathbf{S}_{n+1} \arrow[ll, "\xi_n^X"'] \arrow[dl, "\bar{f}_{n+1}"] \arrow[dd, "\partial^{\mathbf{S}_{n+1}}"] \\
\mathcal{W}(\mathbf{T}_n) \arrow[dd, "\partial^{\mathcal{W}(\mathbf{T}_n)}"'] & & \mathbf{T}_{n+1} \arrow[ll, "\xi_n^Y"'] \arrow[dd, "\partial^{\mathbf{T}_{n+1}}"] & \\
 & S_n^\dagger \arrow[dl, "\bar{f}_n^\dagger"'] & & X_{n+1}^\dagger \arrow[ll, "(\vartheta_n^X)^\dagger"'] \arrow[dl, "f_{n+1}^\dagger"] \\
T_n^\dagger & & Y_{n+1}^\dagger \arrow[ll, "(\vartheta_n^Y)^\dagger"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
S_n \arrow[d, "\bar{f}_n"'] & X_{n+1} \arrow[l, "\vartheta_n^X"'] \arrow[d, "f_{n+1}"] \\
T_n & Y_{n+1} \arrow[l, "\vartheta_n^Y"']
\end{tikzcd}
$$

considered in $\mathbf{Sig}_a$, for $n \in \mathbb{N}$.

Definition 5.1.1. *The category defined above will be denoted* $\mathcal{A} - \mathbf{OSet}$*, and will be called the category of* anytopic sets*, since the monoid* $\mathbf{O}$ *is arbitrary.*

There is an obvious functor $\mathcal{A} - \mathbf{OSet} \to Mon(\mathbf{Sig}_a)$, which maps the opetopic set X to its underlying monoid $\mathbf{O}$, and maps the map $f \colon X \to Y$ to its underlying homomorphism of monoids $h \colon \mathbf{O} \to \mathbf{O}'$.

Proposition 5.1.2. *The functor* $p_{Opt} \colon \mathcal{A} - \mathbf{OSet} \to Mon(\mathbf{Sig}_a)$ *defined above is a fibration.*

Proof. We construct the prone arrows. Suppose we are given a morphism $h \colon \mathbf{O} \to \mathbf{O}'$, and an $\mathbf{O}'$-opetopic set X. We construct h^*X inductively, as follows:

$$
\begin{aligned}
(h^*X)_0 &= X_0 \times_M N \\
(h^*X)_{n+1} &= X_{n+1} \times_{S_n(X)} S_n(h^*X),
\end{aligned}
$$

where $\mathbf{S}_n(h^*X)$ is the monoid determined from h^*X_k for $k < n$. It is important to note that this definition defines not only the sets of n-cells of h^*X, but also the structure maps $(h^*X)_{n+1} \to S_n(h^*X)$, by induction, as follows:

$$\vartheta_{-1}^{h^*X} = X_0 \times_M N \to N \text{ is the pullback projection}$$
$$\vartheta_{n+1}^{h^*X} = X_{n+1} \times_{S_n(X)} S_n(h^*X) \to S_n(h^*X) \text{ is the pullback projection.}$$

Note that the well definedness of the next term follows from that of the previous term. The elementary verification that this is indeed a prone arrow for p_{Opt} over h proceeds again by induction, and is left to the reader. $\square$

In fact the inductive step consists of showing that the truncated **O**-opetopic sets, defined below 5.2.1, form a fibration, and that lemma 5.2.3 allows us to establish this in the next dimension.

Definition 5.1.3. *The functor p_{Opt} is called the anytopic fibration. Its fiber over $\mathbf{O} \in Mon(\mathbf{Sig}_a)$ is called the category of* **O***-opetopic sets. For the special case $\mathbf{O} = I$, the monoid with one type and one unary operation, we reserve the name* opetopic sets *and the notation* **OSet**.

5.2 The Presheaf Property

In this section we use the concrete combinatorial theorem 4.5.2 to prove that **O**-opetopic sets form a presheaf category.

Definition 5.2.1 (Truncated opetopic sets)**.** *The category $\mathbf{O} - \mathbf{OSet}_k$ is defined, for $k \in \mathbb{N}$, as the category resulting from restricting n in the definition of $\mathcal{A} - \mathbf{OSet}$ to be less than or equal to k.*

For example $\mathbf{O} - \mathbf{OSet}_0$ is isomorphic to $\mathbf{Set}/M$, where M is the set of types of **O**.

Lemma 5.2.2. *The monoids $\mathbf{S}_n$ in the definition of $\mathbf{O} - \mathbf{OSet}$ determine functors $S_n \colon \mathbf{O} - \mathbf{OSet} \to \mathbf{Set}$. This functor is also defined on $\mathbf{O} - \mathbf{OSet}_k$, for $k \geq n$.*

Proof. $S_n(X)$ is the set of function symbols in $\mathbf{S}_n$, the sequence of monoids which determines the **O**-opetopic set X. The first claim follows from the definition of a morphism in $\mathbf{O} - \mathbf{OSet}$, the second claim follows from the fact that we can define $\mathbf{S}_n$ from data of dimension n and less, as is clear from the definition. $\square$

Lemma 5.2.3. $\mathbf{O} - \mathbf{OSet}_{n+1} \simeq \mathbf{Set}/S_n$*, for $n \geq 0$.*

Proof. This isomorphism is obvious: to specify an object of $\mathbf{O}-\mathbf{OSet}_{n+1}$ from an object $X \in \mathbf{O}-\mathbf{OSet}_n$, it suffices to give a set map $\vartheta_{n+1}\colon X_{n+1} \to S_n(X)$. The same goes for morphisms. □

We have obvious functors $i_n\colon \mathbf{O}-\mathbf{OSet}_n \to \mathbf{O}-\mathbf{OSet}_{n+1}$, and $tr_{n+1}\colon \mathbf{O}-\mathbf{OSet}_{n+1} \to \mathbf{O}-\mathbf{OSet}_n$, which, respectively, define $X_{n+1} = \emptyset$ and forget X_{n+1}. Under the above isomorphism the functor tr_{n+1} corresponds to the projection $\mathbf{Set}/S_n \to \mathbf{O}-\mathbf{OSet}_n$.

Proposition 5.2.4. $i_n \dashv tr_{n+1}$

Proof. Since i_n defines $X_{n+1} = \emptyset$ we always have a unique set map $X_{n+1} \to Y_{n+1}$, which is easily checked to extend any given morphism of truncated opetopic sets $X \to tr_{n+1}(Y)$. Thus, to give a morphism $i_n(X) \to Y$ is to give a morphism $X \to tr_{n+1}(Y)$. □

Theorem 5.2.5. *The category $\mathbf{O}-\mathbf{OSet}_n$ has wide pullbacks, and the functor $S_n\colon \mathbf{O}-\mathbf{OSet}_n \to \mathbf{Set}$ preserves them.*

Proof. By induction. We argue that the functor $\mathbf{S}_n\colon \mathbf{O}-\mathbf{OSet}_n \to Mon(\mathbf{Sig}_a)$, with values in monoids, preserves wide pullbacks. The weaker claim needed for the theorem then follows from the construction of wide pullbacks for monoids in subsection 2.7.2, which stated that, at the level of function symbols, all the relevant pullbacks are constructed in $\mathbf{Set}$.

For $n = 0$ we have $\mathbf{O}-\mathbf{OSet}_0 = \mathbf{Set}/M$, and $\mathbf{S}_0(X) = \mathbf{O} \times_{M^\dagger} X^\dagger$, so the claim follows from the fact that wide pullbacks commute with other limits (other pullbacks in this case) and the free monoid construction (since $(-)^\dagger$ is isomorphic to the functor of finite sequences, which is a special case of the free monoid construction).

The inductive step: we know that if $T\colon \mathcal{C} \to \mathcal{D}$ preserves wide pullbacks, and $\mathcal{D}$ has wide pullbacks, then the projection $\mathcal{D}/T \to \mathcal{D}$ creates them. Thus we know exactly how to compute wide pullbacks in $\mathbf{Set}/S_n$. The wide pullbacks of the sets of cells X_n are computed in $\mathbf{Set}$, and by the inductive assumption, so is $\mathbf{S}_n$. Thus the functor $\mathcal{W}(\mathbf{S}_n(-))$, by the concrete combinatorial theorem 4.5.2, preserves wide pullbacks. $\mathbf{S}_{n+1}$ is by definition a pullback (computed in $\mathbf{Set}$) of $\mathcal{W}(\mathbf{S}_n)$, and so $\mathbf{S}_{n+1}(-)$ also preserves wide pullbacks. □

Corollary 5.2.6. *$\mathbf{O}-\mathbf{OSet}_k$ is a presheaf topos.*

Proof. We have $\mathbf{O}-\mathbf{OSet}_0 \simeq \mathbf{Set}/M \simeq \mathbf{Set}^M$ – a presheaf topos – so we can, by lemma 5.2.3, inductively invoke Artin gluing [CJ95, 4.1 (v)], [W74]. □

Corollary 5.2.7. *The functor tr_{n+1} has a right adjoint, and therefore is continuous and cocontinuous.*

Proof. Since this functor corresponds to the projection $\mathbf{Set}/S_n \to \mathbf{O}-\mathbf{OSet}_n$, this is just part of the general theory of Artin gluing [CJ95, 4.7]. $\square$

Corollary 5.2.8. *The functor* $i_n\colon \mathbf{O}-\mathbf{OSet}_n \to \mathbf{O}-\mathbf{OSet}_{n+1}$ *preserves projective indecomposables.*

Proof. We have $\mathbf{O}-\mathbf{OSet}_{n+1}(i_n(X), -) \simeq \mathbf{O}-\mathbf{OSet}_n(X, tr_{n+1}(-))$, and tr_{n+1} preserves colimits. $\square$

Consider the diagram

$$\mathbf{O}-\mathbf{OSet}_0 \xleftarrow{tr_1} \mathbf{O}-\mathbf{OSet}_1 \xleftarrow{tr_2} \cdots \xleftarrow{tr_n} \mathbf{O}-\mathbf{OSet}_n \xleftarrow{tr_{n+1}} \cdots$$

Proposition 5.2.9. $\mathbf{O}-\mathbf{OSet} \simeq \varprojlim \mathbf{O}-\mathbf{OSet}_k$ *in* $\mathbf{Cat}$.

Proof. Since limits can be constructed in $\mathbf{Cat}$ by products and equalizers, it is enough to see that an opetopic set is the same as a compatible sequence of truncated opetopic sets, which is obvious. The same argument works for morphisms. $\square$

Theorem 5.2.10 (Existence of limits and colimits in $\mathbf{OSet}$)**.** *The category* $\mathbf{O}-\mathbf{OSet}$ *is small complete and cocomplete.*

Proof. Let $F\colon \mathcal{C} \to \mathbf{O}-\mathbf{OSet}$ be a small diagram and let $X \in \mathbf{O}-\mathbf{OSet}$. For colimits we have, by the above proposition and cocompleteness of $\mathbf{O}-\mathbf{OSet}_k$:

$$Cone(F, X) = \varprojlim_n Cone(tr_n F, tr_n(X)) =$$

$$\varprojlim_n \mathbf{O}-\mathbf{OSet}_n(\varinjlim tr_n F, tr_n(X)) = \mathbf{O}-\mathbf{OSet}(\varprojlim_n \varinjlim tr_n F, X)$$

The family $\varinjlim tr_n F$ is compatible because tr_n preserves colimits. An analogous argument works for limits, due to the continuity of tr_n. $\square$

Corollary 5.2.11. *The inclusion* $\mathbf{O}-\mathbf{OSet}_n \to \mathbf{O}-\mathbf{OSet}$ *preserves projective indecomposables.*

Proof. By the construction of colimits in $\mathbf{O}-\mathbf{OSet}$ the truncation functor $tr^n\colon \mathbf{O}-\mathbf{OSet} \to \mathbf{O}-\mathbf{OSet}_n$ preserves colimits. It is also easy to see that it is right adjoint to the inclusion, so the same argument as in corollary 5.2.8 can be repeated. $\square$

Theorem 5.2.12. *The category* $\mathbf{O}-\mathbf{OSet}$ *is a presheaf category.*

Proof. Choose an exponent category $\mathcal{C}_k$ for each $\mathbf{O}-\mathbf{OSet}_k$, and let $I_k \subset \mathbf{O}-\mathbf{OSet}_k$ be the resulting set of representables. By corollary 5.2.11 the set $I = \cup_k I_k \subset \mathbf{O}-\mathbf{OSet}$ is a set of projective indecomposables. Since the I_k are strongly generating for each $\mathbf{O}-\mathbf{OSet}_k$, it is easily seen using 5.2.9 that I is strongly generating for $\mathbf{O}-\mathbf{OSet}$. Therefore by theorem 5.26 of [Ke80] $\mathbf{O}-\mathbf{OSet}$ is a presheaf category. $\square$

5.3 The Category of Opetopes

The above proof is abstract, and gives no indication of how the category of **O**-opetopes looks. To investigate it we need further arguments. We first need to see what the terminal opetopic set T looks like. Like all opetopic sets, it is constructed inductively. Starting from the terminal object in $\mathbf{O} - \mathbf{OSet}_0 = \mathbf{Set}/M$ at each step we add exactly one of each possible cell type (i.e. element of $S_n(T)$). This construction can be read off of lemma 5.2.3. Thus, directly from the definition of an opetopic set, we have

$$T_n = \underbrace{\mathcal{W} \circ \cdots \circ \mathcal{W}}_{n \text{ times}}(\mathbf{O}).$$

Any cell $\tau \in T_n$ will be called a cell type. Since there is a unique morphism $X \to T$ from any **O**-opetopic set to T we will say that cells from X mapping to τ *are of type* τ.

Definition 5.3.1. *For any cell type $\tau \in T$ we define $X(\tau)$, for $X \in \mathbf{O} - \mathbf{OSet}$, to be the set of cells of type τ in X.*

Since any map $X \to Y$ commutes over T i.e. preserves cell types, this defines a functor $X \mapsto X(\tau)$ from $\mathbf{O} - \mathbf{OSet}$ to **Set**. Similarly we have types of frames $\sigma \in S_n(T)$, which also define functors $\mathbf{O} - \mathbf{OSet} \to \mathbf{Set}$.

With this in hand we can give a concrete description of limits and colimits in $\mathbf{O} - \mathbf{OSet}$:

Theorem 5.3.2 (Construction of limits and colimits in $\mathbf{O} - \mathbf{OSet}$). *Limits and colimits in $\mathbf{O} - \mathbf{OSet}$ can be computed as follows:*

1. *Colimits can be computed at the level of sets of cells:* $(\varinjlim F)_n = \varinjlim F_n$
2. *Equalizers can be computed at the level of sets of cells*
3. *Products are computed at the level of types of cells:* $(\prod_i X_i)(\tau) = \prod_i X_i(\tau)$

Proof. Since we already know that $\mathbf{O} - \mathbf{OSet}_k$ is cocomplete, and $\mathbf{O} - \mathbf{OSet}_{k+1} = \mathbf{Set}/S_k$, part 1 follows from general category theory and the proof of theorem 5.2.10. Part 2 follows from part 3 and the construction of equalizers from products and pullbacks. Part 3 follows from the construction of the terminal object and the construction of wide pullbacks in $\mathbf{Set}/S_n$, along with theorem 5.2.10. □

Corollary 5.3.3. *The functor $X \mapsto X(\tau)$ is continuous and cocontinuous.*

Proof. This follows from the fact that $X_n = \coprod_{\tau \in T_n} X(\tau)$, the fact that maps of opetopic sets preserve this decomposition, and the above description of limits and colimits. □

Corollary 5.3.4. *The functor $X \mapsto X(\tau)$ is representable*

Proof. The opetopic sets such that $\coprod_n X_n \subset \mathbb{N}$ is a finite set (the finite opetopic sets with cells chosen from $\mathbb{N}$) provide a solution set. To see this note that for each cell $x \in X$, there is a smallest opetopic subset[1] $\langle x \rangle \subset X$ that contains x, and this set is finite. This can be seen by induction, since $\langle x \rangle$ is just x and the union of the subsets generated by the typing of its frame $\vartheta^X(x)$ – there are only finitely many of those. □

We denote the representing object by $\underline{\tau}$. This set represents $X \mapsto X(\tau)$, and hence has a universal cell of type τ, which we denote by $\tau^u \in \underline{\tau}$.

Corollary 5.3.5. *$\underline{\tau}$ is a projective indecomposable.*

Proof. Follows immediately from the previous two corollaries. □

Corollary 5.3.6. *The objects $\{\underline{\tau}\}_{\tau \in T}$ constitute a strong generator for* $\mathbf{O}-\mathbf{OSet}$.

Proof. Two maps $X \rightrightarrows Y$ differ iff they differ on some cell $x \in X(\tau)$ for some $\tau \in T$. Likewise a map $X \to Y$ is an isomorphism iff the maps $X_n \to Y_n$ are, and hence iff the maps $X(\tau) \to Y(\tau)$ are for all $\tau \in T$. □

Corollary 5.3.7. *In the proof of theorem 5.2.12 we may assume that $I = \{\underline{\tau}\}_{\tau \in T}$.*

The full subcategory of $\mathbf{O} - \mathbf{OSet}$ on the objects $\underline{\tau}$ is therefore the exponent identifying it as a presheaf category. What does it look like? We can describe its generators as follows. Let $\tau \in T_n$. Now consider the following composition applied to τ:

$$T_n \xrightarrow{\vartheta_n^T} S_{n-1}(T) \xrightarrow{\partial} T_{n-1}^{\dagger} \xrightarrow{p_i} T_{n-1},$$

where ∂ is the typing map, and $p_i \colon T_{n-1}^{\dagger} = T_{n-1} \times T_{n-1}^{*} \to T_{n-1}$ is a projection from the list of input/output types to some specific place, which we assume nonempty (at the very least the output type is always available). This defines a cell $\sigma = p_i(\partial(\vartheta_n^T(\tau)))$ in T_{n-1} depending on τ. This operation can be performed on any cell of type τ giving a cell of type σ. We thus obtain a natural transformation $X(\tau) \to X(\sigma)$ and hence a morphism $\underline{\sigma} \to \underline{\tau}$.

Theorem 5.3.8. *The subcategory of* $\mathbf{O}-\mathbf{OSet}$ *generated by the morphisms $\underline{\sigma} \to \underline{\tau}$ defined above is full.*

[1]in the naive sense of inclusions $\langle x \rangle_n \subset X_n$. It will later become obvious that these are the subobjects.

Proof. We have $\underline{\tau} = \langle \tau^u \rangle - \underline{\tau}$ is generated by its single universal cell. To see this consider the inclusion $i\colon \langle \tau^u \rangle \to \underline{\tau}$. By the universal property of τ^u the map $\mathbf{O}-\mathbf{OSet}(i,-)$ is an isomorphism, and hence i is an isomorphism. By the construction of $\langle \tau^u \rangle$, this tells us that every cell in $\underline{\tau}$ is in the image of the typing of some typing ...of the typing of τ^u. But the maps $\underline{\sigma} \to \underline{\tau}$ are just the cells $\underline{\tau}(\sigma)$ – which we have just seen to come from typing maps. □

Definition 5.3.9. *The category $Opt(\mathbf{O})$ of $\mathbf{O}$-opetopes is the full subcategory of $\mathbf{O}-\mathbf{OSet}$ on the objects $\{\underline{\tau}\}_{\tau \in T}$.*

By the above discussion we have an equivalence $\mathbf{O}-\mathbf{OSet} \simeq \mathbf{Set}^{(Opt(\mathbf{O}))^{op}}$ given by $X \mapsto \mathbf{O}-\mathbf{OSet}(-,X)|_{Opt(\mathbf{O})}$

Corollary 5.3.10. *In the category of $\mathbf{O}$-opetopes no object has a nontrivial endomorphism. In particular there are no nontrivial idempotents.*

Proof. Define the dimension of an opetope $\underline{\tau}$ to be the unique $n \in \mathbb{N}$ such that $\tau \in T_n$ (equivalently, for a general opetopic set X, the least n such that $X_{n+1} = \emptyset$[2]). Then by theorem 5.3.8, nonidentity maps $\underline{\sigma} \to \underline{\tau}$ can exist only if $dim(\underline{\sigma}) < dim(\underline{\tau})$. Thus there are no nontrivial endomorphisms $\underline{\tau} \to \underline{\tau}$. □

Corollary 5.3.11. *The category of $\mathbf{O}$-opetopes is skeletal*

Proof. Since morphisms in $Opt(\mathbf{O})$ cannot decrease dimension, we must show that if $\underline{\tau}$ and $\underline{\sigma}$ have the same dimension and $\underline{\tau} \neq \underline{\sigma}$, then they are not isomorphic. But this is obvious, since the universal cells $\tau^u \in \underline{\tau}$ and $\sigma^u \in \underline{\sigma}$, the only cells in their dimension, have different types, and hence cannot be mapped to each other. So we have in fact $Opt(\mathbf{O})(\underline{\tau}, \underline{\sigma}) = \emptyset$ for any two different τ and σ of the same dimension. □

[2]It is then true that $X_k = \emptyset$ for $k > n$.

Chapter 6

Comparison Theorems

In this chapter we establish the relationships between our work, and all the other algebraic approaches to opetopes and opetopic sets. To achieve this we will draw heavily on the results of chapter 3.

The first comparison is already almost finished. Corollary 3.4.2 establishes the necessary isomorphisms at the level of objects (i.e. functors in this case). After recalling the setting of [KJBM10], we fill in the remaining detail, which is that the multiplication maps in both approaches coincide. This means that the monoids in both approaches are isomorphic.

It is important to emphasize that our comparison necessarily exists at the level of individual fibers, since that is how the formalism in [KJBM10] is set up. At this level – of monads on slices of **Set** – both approaches are equivalent. However, our gathering of these fibers into the bifibration $\mathbf{Sig}_{ma} \to Mon(\mathbf{Sig}_a)$ provides a nontrivial distinction between both approaches. The fibers of $\mathbf{Sig}_{ma}$ are by themselves isomorphic to the appropriate fibers of $\mathbf{Sig}_a$. This isomorphism includes the monoidal structure $\otimes$ (the vertical one in the case of $\mathbf{Sig}_{ma}$), and the associated actions, and thus also the equivalence between signatures and polynomial endofunctors. Once we take into account the entire category $Mon(\mathbf{Sig}_a)$ as the base for $\mathbf{Sig}_{ma}$, this equivalence no longer holds.

As we have seen in the previous chapter, the fact that $\mathcal{W}\colon Mon(\mathbf{Sig}_a) \to Mon_{\otimes}(\mathbf{Sig}_{ma})$ is a functor (of the base of the bifibration $\mathbf{Sig}_{ma}$) is essential in the construction of the category of opetopic sets. It is also critical in its analysis, including the construction of the *category* of opetopes, something which was not achieved in [KJBM10].

The second comparison concerns the original approach of Baez and Dolan, presented in [BD98]. We begin the section by providing the first (to the author's knowledge) complete and rigorous proofs that the formalism of [BD98] can actually be made to work. As the reader will see, this is at least somewhat nontrivial, and we rely heavily on our previous results – both on the general theory of monads established in chapter 1, and on the fundamental properties of signatures from chapter 3.

The central technical problem in [BD98] is that the "operad of operads" construction is almost correct: Baez and Dolan correctly identify its function symbols[1], and have a good idea about what its composition map should be. But this is unfortunately done at the level of signatures with nonstandard amalgamation – where the symmetric actions are implicit (hidden in the morphisms). In their setting of symmetric operads and analytic functors, this construction is invalid.

Next, we consider the relationship of this construction with our web monoid. This is frustrated by the fact that the diagram

$$\begin{tikzcd} \mathbf{Sig}_a \arrow[rr, "K"] \arrow[dr] & & \mathbf{Sig}_s \arrow[dl] \\ & \mathbf{Set} & \end{tikzcd}$$

where the functors to **Set** maps signatures to their sets of function symbols, and the functor K is the canonical comparison between Kleisli and Eilenberg-Moore algebras, *does not commute*, not even up to isomorphism.

Because of this, and the fact that sets of function symbols of a monoid become types for both the associated web monoid and the (sliced) operad of operads, both monoids can be considered in $\mathbf{Sig}_s$, or $\mathbf{Sig}_a$, but with different types. They cannot be isomorphic, since their sets of types are not. Despite this, the categories of actions of both monoids are equivalent as abstract categories. Even in the (rare) cases where we can trick the types to coincide, there is only a canonical homomorphism, which corresponds to a forgetful functor between the categories of actions.

We finish this section with a sketch of proof of something many have conjectured (e.g. [Ch04]): the categories of Baez-Dolan opetopic sets and "the usual" opetopic sets are not equivalent. Given the results of the last section in this chapter – the equivalence of our approach to that of Hermida, Makkai, and Power – and the huge web of equivalences including the multitopic sets, we consider our approach to be one possible definition of the *standard* category of opetopic sets, to which all others should be compared (and found equivalent, or discarded). For this reason, we consider these results to be the end of the technical life of [BD98], with its conceptual core being extolled by a vast body of work, including this thesis.

[1]But then incorrectly identify it with the set of symbols of another operad, see remark 6.2.8.

The last section studies the relationship between opetopic sets and the multitopic sets constructed in [HMP02]. Its result is simple enough: the categories are isomorphic. Establishing it, however, is rather problematic.

Multitopic sets, like opetopic sets, are constructed by iterated Arting gluing. This is very easy to see from the arguments in part 3 of [HMP02]. We are thus reduced to comparing the gluing procedure, which in turn rests on the following problem: given an n-truncated opetopic set, and the corresponding multitopic set, establish that opetopic and multitopic parallelism of $(n+1)$-dimensional cells is the same thing. An affirmative answer immediately finishes the proof of the comparison theorem.

The equivalence of parallelism follows from a comparison of frames in both approaches, and is contained in theorem 6.3.5. Proving this theorem is the main technical problem. After wrestling with the specific 2-level multicategories used in [HMP02], we must finally compare the web monoid construction and the multicategory of function replacement from part 2 of [HMP02].

In general, there is no equivalence between these constructions, since their domain categories are not equivalent. We are only able to establish a relationship in the narrow case of applying both to the construction of opetopic and multitopic sets. This requires, as usual, the results of chapter 3, specifically corollary 3.4.2, a detailed analysis of the proof of the main theorem in part 2 of [HMP02] and, quite curiously, the computation used to finish our first comparison – the proof of theorem 6.1.7.

The end result of our work in this chapter is rather pleasing. The informal pictures drawn in [BD98], and formalized in [KJBM10], correspond to the objects of the category of opetopes, presheaves on which give opetopic sets. This category is, in turn, equivalent to the multitopic sets, and therefore to a vast network of different definitions.

We see these results as establishing a canonical version of the category of opetopic sets and opetopes, which expresses the original intuitions of [BD98]. It seems that all other approaches not already known to be equivalent to multitopic sets (or some other connected approach) should be compared to this category, and rejected if they are not equivalent. It is sad that this happens to the technical content of the original proposal.

6.1 Comparison to Kock, Joyal, Batanin, and Mascari

6.1.1 The Slice Construction from [KJBM10]

There are no fibrations in [KJBM10], and thus we will be forced to restrict our discussion to individual fibers. We will fix this in a moment, but first we will give the original construction. Our notation for polynomial functors is as in 2.6.1.

Recall that the slice of a monoidal category by a monoid is naturally a monoidal category. Let $M \in Mon(\mathbf{Poly}/O)$ be a polynomial monad over O. We obtain a natural monoidal structure on $(\mathbf{Poly}/O)/M$. Then monoids over M are the same as monoids in $(\mathbf{Poly}/O)/M$.

Note that free polynomial monads (free monoids) on a polynomial functor exist: by theorems 2.6.2, 2.7.2 and proposition 2.7.1 we may apply the construction of theorem 1.5.2. An explicit construction can be found in [KJBM10]. The same is true for monoids $(\mathbf{Poly}/O)/M$.

Remark 6.1.1. *In fact $Mon((\mathbf{Poly}/O)/M)$ is monadic over $(\mathbf{Poly}/O)/M$.*

Lemma 6.1.2. *$(\mathbf{Poly}/O)/M$ is equivalent to $\mathbf{Set}/M$*

Here we treat M as the corresponding set of function symbols given by theorem 2.6.2.

Proof. By theorem 2.6.2 $(\mathbf{Poly}/O)/M$ is equivalent to $(\mathbf{Sig}_a/O)/M$. We have seen in theorem 3.2.5 (upon restriction to fibers) that this category is equivalent to $\mathbf{Set}/M$. □

Of course in the above proof the fact that M is a monoid plays no role. It is only needed to construct the monoidal structure on $\mathbf{Poly}(O)/M$.

We can now give the slice construction. Let $M \in Mon(\mathbf{Poly}/O)$ be a polynomial monad. The category $(\mathbf{Poly}/O)/M$ is monoidal and has a free monoid functor. This gives rise to the free monoid monad $T_M : (\mathbf{Poly}/O)/M \to (\mathbf{Poly}/O)/M$. By lemma 6.1.2 this is equivalent to a monad $M^+ : \mathbf{Set}/M \to \mathbf{Set}/M$. This monad is polynomial (one can check this directly, but it will also follow from our results).

Definition 6.1.3. *The [KJBM10] Baez-Dolan slice construction is the assignment $M \mapsto M^+$ (cf. [KJBM10, 3.18]).*

By remark 6.1.1 M^+ is the "operad for operads over M", as it should be.

We can now state the comparison theorem.

Theorem 6.1.4. *For any $M \in Mon(\mathbf{Sig}_a)$ we have an isomorphism*

$$rep_a(\mathcal{W}(M)) \simeq rep_a(M)^+,$$

where $\mathcal{W}(M)$ is considered as a monoid in $\mathbf{Sig}_a/M$ with the ν multiplication.

Using theorem 2.6.2 to identify monoids in $\mathbf{Sig}_a$ with polynomial monads we can write more clearly

$$\mathcal{W}(M) \simeq M^+$$

Formation of the web monoid is therefore very close to the Baez-Dolan construction. However, the construction of the web monoid is completely different.

We will need to reformulate theorem 6.1.4 in order to prove it. The equivalence between **Poly** and $\mathbf{Sig}_a$ is monoidal, and hence induces an equivalence between polynomial monads and monoids in signatures. This in turn gives an equivalence of fibered slices of these fibrations:

$$\mathbf{Sig}_a \downdownarrows \mathcal{U}_{sig} \simeq \mathbf{Poly} \downdownarrows \mathcal{U}_{poly}$$

over the equivalence of monoid fibrations $Mon(\mathbf{Sig}_a) \simeq Mon(\mathbf{Poly})$. The fibers of the fibered slice of **Poly** are, by construction, exactly all the categories of the form $\mathbf{Poly}(O)/M$, where M is a monoid in $\mathbf{Poly}(O)$. Thus we have assembled into a fibration all the categories used in the Baez-Dolan construction. This fibration has a free monoid monad, which will be denoted $(-)^+$. The $(-)$ stands for an argument from the base: $M^+(X)$ is the free monoid on X in the fiber over M. This is the natural way to extend the Baez-Dolan construction into a functor by working with fibrations. It is also the same (up to equivalence) as the extension given by hand in [KJBM10]. At this point we will forget about polynomial functors and work exclusively with signatures.

Theorem 3.2.5 gives us an equivalence of fibration which on fibers is exactly the equivalence asserted in lemma 6.1.2. This allows us to carry out the Baez-Dolan construction in all fibers at once.

It should be clear that the fiber of $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$ over $M \in Mon(\mathbf{Sig}_a)$ is isomorphic to $\mathbf{Set}/M$, the slice of **Set** over the set of function symbols of M. Theorem 3.2.5 gives an adjoint equivalence $\mathbf{Sig}_a \downdownarrows \mathcal{U} \simeq \mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$, and hence we can view the monad $(-)^+$ as acting on the latter fibration. This allows us to state the fibered version of the comparison theorem.

Theorem 6.1.5 (Comparison Theorem – Fibered Version)**.** *There is an isomorphism, fibered over $Mon(\mathbf{Sig}_a)$, of monad valued functors*

$$M \mapsto \mathcal{W}(M) \star (-) \colon Mon(\mathbf{Sig}_a) \to End(\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot})$$

and

$$M \mapsto M^{+}(-)\colon Mon(\mathbf{Sig}_a) \to End(\mathcal{U}^{*}\mathbf{Set}^{\cdot\to\cdot}),$$

where $\mathcal{W}$ is the web monoid functor, and $End(\mathcal{U}^{*}\mathbf{Set}^{\cdot\to\cdot})$ is the endomorphism object in $\mathbf{Cat}/Mon(\mathbf{Sig}_a)$.

The original comparison theorem 6.1.4 follows immediately from this one when we apply the forgetful functor in each fiber to the pullback action and return to the action of $\mathbf{Sig}_a$ on the codomain fibration. This theorem makes it clear that the proper base category for these constructions is not **Set** but rather $Mon(\mathbf{Sig}_a)$, and this cannot be easily seen without using fibrations.

This theorem also neatly summarizes the differences between our approach and that of [KJBM10]. Their construction takes place in two different categories (or fibrations). In each only one type of inputs is visible (vertical or horizontal in our terminology). We have found a third fibration which sees both kinds of inputs, and all the relevant structure in the two original fibrations. It is a somewhat amusing fact that $\mathbf{Sig}_{ma}$ knows what free monoids look like in those other fibrations.

6.1.2 Proof of Theorem 6.1.5

We begin with a necessary lemma.

Lemma 6.1.6. *Let φ be the distributivity isomorphism in $\mathbf{Sig}_{ma}$, ϕ the isomorphism defined in theorem 3.4.1 and written out above corollary 3.4.2, and let a be the associativity isomorphism for the pullback action. Then the following diagram commutes:*

$$\begin{tikzcd}
{[A\star(Y\star X)]\otimes[B\star(Y\star X)]} \arrow[r, "a\otimes a"] \arrow[dd, "\phi_{A,B,Y\star X}"] & {[(A\otimes Y)\star X]\otimes[(B\otimes Y)\star X]} \arrow[d, "\phi_{A\otimes Y,B\otimes Y,X}"] \\
 & {[(A\otimes Y)\odot(B\otimes Y)]\star X} \arrow[d, "\varphi_{A,B,Y}\star X"] \\
(A\odot B)\star(Y\star X) \arrow[r, "a"] & {[(A\odot B)\otimes Y]\star X}
\end{tikzcd}$$

where $A, B, Y \in \mathbf{Sig}_{ma}$ and $X \in \mathcal{U}^{*}\mathbf{Set}^{\cdot\to\cdot}$.

Proof. Direct calculation. $\square$

The above diagram is analogous to diagram II in theorem 1.7.8. Indeed ϕ is in some sense φ and a is in some sense $\alpha^{\otimes}$ for $\mathbf{Sig}_{ma}$, just like $\star$ is in some sense $\otimes$ by definition 2.5.1. The analogy has to be left imprecise for the present moment, because in forming a we need to know that the middle variable is in $\mathbf{Sig}_{ma}$, and the formula of definition 2.5.1 cannot possibly remember this.

Theorem 6.1.7. *$M^+(-) \simeq \mathcal{W}(M) \star (-)$ as monoids, naturally in M.*

Proof. $(-)^+$ acts as the free monoid monad on $\mathbf{Sig}_a \downdownarrows \mathcal{U}$, which is monoidally equivalent to $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$ with the $\otimes$-structure, by theorem 3.2.5 and construction of $\otimes$. Thus $(-)^+$ is isomorphic to the free monoid monad on $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$. By the second point of the above corollary the universe of the web monoid acts as the free $\otimes$-monoid monad on $\mathcal{U}^*\mathbf{Set}^{\cdot\to\cdot}$, and a natural isomorphism drops out.

We must check whether it is an isomorphism of monads. The units are mapped to each other by definition – in both cases they are the unit of the same adjunction (for $\mathcal{W}$ this follows from corollary 3.4.2). This leaves multiplication. We must check if $\nu \star X \circ a$ is $\varepsilon_{\mathcal{F}_{\otimes}(X)} \simeq \varepsilon_{\mathcal{F}_{\odot}(I_{\otimes})} \star X$. Consider the following diagram (we abbreviate $\mathcal{W} = \mathcal{W}(M)$):

$$\begin{tikzcd}
\mathcal{F}_{\otimes}^2(X) \otimes \mathcal{F}_{\otimes}^2(X) \arrow[r] \arrow[d, "\simeq\otimes\simeq"'] \arrow[dddd, bend right=60, "\mu^{\mathcal{F}_{\otimes}}_{\mathcal{F}_{\otimes}(X)}"'] & {[(\mathcal{W}\otimes\mathcal{W})\star X]\otimes[(\mathcal{W}\otimes\mathcal{W})\star X]} \arrow[r, "(\nu\star X)\otimes(\nu\star X)"] \arrow[d, "\phi"'] & (\mathcal{W}\star X)\otimes(\mathcal{W}\star X) \arrow[r, "\simeq\otimes\simeq"] \arrow[d, "\phi"] & \mathcal{F}_{\otimes}(X)\otimes\mathcal{F}_{\otimes}(X) \arrow[ddl, "\mu^{\mathcal{F}_{\otimes}}_{X}"] \\
{[\mathcal{W}\star(\mathcal{W}\star X)]\otimes[\mathcal{W}\star(\mathcal{W}\star X)]} \arrow[ur, "a\otimes a"] \arrow[d, "\phi"] & {[(\mathcal{W}\otimes\mathcal{W})\odot(\mathcal{W}\otimes\mathcal{W})]\star X} \arrow[r, "(\nu\odot\nu)\star X"] \arrow[d, "\varphi\star X"'] & (\mathcal{W}\odot\mathcal{W})\star X \arrow[dd, "\mu\star X"] & \\
(\mathcal{W}\odot\mathcal{W})\star(\mathcal{W}\star X) \arrow[r, "a"] \arrow[d, "\mu\star 1"] & {[(\mathcal{W}\odot\mathcal{W})\otimes\mathcal{W}]\star X} \arrow[d, "(\mu\otimes 1)\star X"'] & & \\
\mathcal{W}\star(\mathcal{W}\star X) \arrow[r, "a"] \arrow[d, "\simeq"] & (\mathcal{W}\otimes\mathcal{W})\star X \arrow[r, "\nu\star X"] & \mathcal{W}\star X & \\
\mathcal{F}_{\otimes}^2(X) \arrow[ur] & & &
\end{tikzcd}$$

The central square is just the main diagram starred with X. It commutes by the definition of ν. The square above it commutes by naturality of ϕ. The rightmost triangle commutes by corollary 3.4.2, as does the leftmost "bigon" or "biangle". The trapezoid commutes by lemma 6.1.6. The square below it commutes by naturality. Since $\mathcal{F}_{\otimes}$ is determined only up to natural isomorphism, the isomorphisms marked $\simeq$ are irrelevant, and can be taken to be the identity.

Thus we see that $\nu \star X \circ a$ gives a natural homomorphism of monoids $\mathcal{F}_{\otimes}^2(X) \to \mathcal{F}_{\otimes}(X)$. Since $\mathcal{F}_{\otimes}$ is the free monoid monad, all such homomorphisms are determined by what they do to the unit $\eta_{\mathcal{F}_{\otimes}(X)}$. But by the unit conditions for ν we see that $\nu \star X \circ a \circ \eta_{\mathcal{F}_{\otimes}(X)}$ is the identity, as the following diagram shows

$$\begin{tikzcd}
\mathcal{F}_{\otimes}(X) \arrow[r, "\simeq"] \arrow[d, "\eta^{\mathcal{F}_{\otimes}}_{\mathcal{F}_{\otimes}(X)}"] & \mathcal{W} \star X \arrow[r] & I_{\otimes} \star (\mathcal{W} \star X) \arrow[r, "a"] \arrow[d, "\eta^{\mathcal{F}_{\odot}}_{I_{\otimes}} \star 1"'] & (I_{\otimes} \otimes \mathcal{W}) \star X \arrow[d, "(\eta^{\mathcal{F}_{\odot}}_{I_{\otimes}} \otimes 1) \star X"'] \arrow[dd, bend left, "\lambda^{\odot}_{\mathcal{W}} \star X"] \\
\mathcal{F}_{\otimes}^2(X) \arrow[rr, "\simeq"] \arrow[drr, dashed] & & \mathcal{W} \star (\mathcal{W} \star X) \arrow[r, "a"] & (\mathcal{W} \otimes \mathcal{W}) \star X \arrow[d, "\nu \star X"'] \\
& & \mathcal{F}_{\otimes}(X) \arrow[r, "\simeq"] & \mathcal{W} \star X
\end{tikzcd}$$

The top left rectangle commutes by 3.4.2. The right bigon commutes by the unit conditions for ν. The top right square commutes by naturality of a. The isomorphisms $\simeq$ are again irrelevant, and can be taken to be identities. The unnamed isomorphism is the canonical one, given by the action $\star$. The dashed arrow is determined by the other composites. By the left unit condition for $\star$ and the commutativity of the above diagram, the map $\nu \star X \circ a \circ \eta_{\mathcal{F}_{\otimes}(X)}$ is, in fact, the identity.

By the universality of $\eta_{\mathcal{F}_{\otimes}(X)}$ only the counit can satisfy this equation, and thus $\nu \star X \circ a = \varepsilon^{\mathcal{F}_{\otimes}}_{\mathcal{F}_{\otimes}(X)}$ concluding the proof. $\square$

It should be clear from this argument that distributivity really does tell us that ν commutes with μ. This is made literal by the pullback action, as we saw above. We can also read off an endearing identity from the preceding proof: trees of trees are the same as trees indexed by a tree.

Corollary 6.1.8. $\mathcal{F}_{\odot}^2(I_{\otimes}) \simeq \mathcal{F}_{\odot}(I_{\otimes}) \otimes \mathcal{F}_{\odot}(I_{\otimes})$

Proof. Since both objects mentioned in this isomorphism live in the same fiber of $\mathbf{Sig}_{ma}$, we may as well restrict our attention to it. This fiber is isomorphic to the corresponding fiber of $\mathbf{Sig}_{a}$. Thus theorem 2.6.2 applies, and all isomorphisms of the induced monads originate from the corresponding signatures. But we have seen above that $\mathcal{F}_{\otimes}^2(X) \simeq \mathcal{F}_{\otimes}(X) \otimes \mathcal{F}_{\otimes}(X)$, naturally in X, in the bifibration $\mathcal{U}^*\mathbf{Set}^{\cdot \to \cdot}$. Therefore this isomorphism comes from an isomorphism of signatures $\mathcal{F}_{\odot}^2(I_{\otimes}) \simeq \mathcal{F}_{\odot}(I_{\otimes}) \otimes \mathcal{F}_{\odot}(I_{\otimes})$. $\square$

6.2 Comparison to Baez and Dolan

6.2.1 Prerequisites for the Comparison

First, we must establish, that the structures used by Baez and Dolan are the same as ours. This is essentially a tautology, and is well known [FGHW08]. We summarize the result in the following proposition.

Proposition 6.2.1. *For every set O, the category $sig(O)$ introduced by Baez and Dolan is monoidally equivalent to $\mathbf{An}/O$.*

This result may also be deduced from the constructions in the next chapter. Since analytic functors are equivalent to symmetric signatures, we will henceforth work exclusively with signatures.

Remark 6.2.2. *An analytic functor is or is not polynomial. In stark contrast to this, the set of function symbols associated to such a functor may be a symmetric signature or a signature with nonstandard amalgamation. These sets are not naturally isomorphic, and often not isomorphic at all. This distinction is not heeded in [BD98], and is the source of several mistakes, the most critical of which is pointed out below in remark 6.2.8.*

Theorem 6.2.3. *The forgetful functor $Mon(\mathbf{Sig}) \to \mathbf{Sig}$ is monadic.*

Proof. This is obvious by Beck's monadicity theorem: we have already seen that the left adjoint exists, and U-split pairs are automatically absolute coequalizers, and so we can construct multiplication for the coequalizer by taking tensor powers of the relevant pair. This shows that for any monoidal category $\mathcal{C}$ with free monoids, the forgetful functor $Mon(\mathcal{C}) \to \mathcal{C}$ is monadic (even strictly monadic). $\square$

Corollary 6.2.4. *The forgetful functor $Mon(\mathbf{Sig}_s) \to \mathbf{Sig}_s \to \mathbf{Sig}$ is monadic.*

Note that both displayed forgetful functors are obviously monadic.

Proof. By the previous theorem, the algebras for the monad $\mathcal{UF}$ (which we will still denote by $\mathcal{F}$) of ordinary multicategories are exactly the ordinary multicategories. Since the symmetrization monad $\mathcal{S}$ is monoidal, it lifts to the fibration of monoids – i.e. the ordinary multicategories. We have just seen that this is the fibration of algebras $\mathbf{Sig}^{\mathcal{F}}$. Let us denote the lifted monad by $\tilde{\mathcal{S}}\colon Mon(\mathbf{Sig}) \to Mon(\mathbf{Sig})$.

Examining the definition of an $\tilde{\mathcal{S}}$-algebra, we immediately see that it is a typed symmetric operad. Thus $Mon(\mathbf{Sig}_s) \simeq Mon(\mathbf{Sig})^{\tilde{\mathcal{S}}} \simeq (\mathbf{Sig}^{\mathcal{F}})^{\tilde{\mathcal{S}}}$. This establishes monadicity directly from the definition. $\square$

Corollary 6.2.5 (The Combing Distributive Law)**.** *There is a distributive law* $\lambda\colon \mathcal{FS} \to \mathcal{SF}$*, whose resulting composite monad* $\mathcal{SF}$ *has symmetric operads as its algebras.*

This is immediate from the proof of corollary 6.2.4, and our basic theory of monads 1.3.2. We can describe this distributive law explicitly. An element $T \in \mathcal{F}(X)$, where $X \in \mathbf{Sig}/O$, can be uniquely represented by either an identity 1_o on some type $o \in O$, or a composite $\langle x, t_i \rangle$, where $x \in X$, the $t_i \in \mathcal{F}(X)$, and i runs through the arity of x. This property is called unique readability, and it was established in part 2 of [HMP02] (cf. proposition 1.5.3 and lemma 1.5.4). Thus the monoid $\mathcal{F}(X)$ consists of planar trees compatibly labeled by symbols of X, matching arity to the children of a given node, and matching types across edges. The distributive law λ is defined inductively, as follows:

$$\begin{aligned} T &= \langle (x,\sigma), t_i \rangle \in \mathcal{FS}(X) \\ \lambda(1_o, id) &= (1_o, id) \\ \lambda(T) &= (\lambda_0(T), \lambda_1(T)) \in \mathcal{F}(X) \times S_n \\ \lambda(T) &= (\langle x, \lambda_0(t_{\sigma^{-1}(i)}) \rangle, \sigma * (\lambda_1(t_i)) \rangle), \end{aligned}$$

where $*$ again refers to composition in the operad of symmetries. This formula follows by direct calculation, from the inductive definition of multiplication in $\mathcal{F}$ (and hence the counit of the corresponding monad), and the explicit form of the correspondence between distributive laws and lifings of monads to algebras. With this description we can prove the following important result.

Theorem 6.2.6. *The combing distributive law is a cartesian natural transformation in the category of functors* $\mathbf{Set}/O^\dagger \to \mathbf{Set}/O^\dagger$*.*

Note that $\mathbf{Sig}/O = \mathbf{Set}/O^\dagger$, so that the statement makes sense. The intuition for this theorem is simple – if we are given a combed tree with appropriate decorations and all the permutations it has been combed with (as additional decorations), then we can uniquely uncomb it.

Proof. By induction on tree height. For any morphism $f\colon X \to Y$ in $\mathbf{Sig}/O$ we must show that the naturality square

$$\begin{array}{ccc} \mathcal{FS}(X) & \xrightarrow{\lambda_X} & \mathcal{SF}(X) \\ {\scriptstyle \mathcal{FS}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{SF}(f)} \\ \mathcal{FS}(Y) & \xrightarrow{\lambda_Y} & \mathcal{SF}(Y) \end{array}$$

is a pullback in $\mathbf{Sig}/O$. Since all the objects under consideration consist of decorated planar trees (possibly with a permutation), and all the morphisms preserve tree height (for λ this follows from the inductive definition above), we show that this square is a pullback for trees of height at most n by induction. Trees of height 0 are the identity symbols 1_o, for $o \in O$, with possibly a unique permutation attached (since they are unary). For them the claim is obvious.

The inductive step: we are given $(T, \tau) \in \mathcal{SF}(X)$ and $T' = \langle (y, \sigma), t'_i \rangle \in \mathcal{FS}(Y)$, which satisfy

$$\begin{aligned} \mathcal{F}(f)(T) = \lambda_0(T') &= \langle y, t'_{\sigma^{-1}(i)} \rangle \\ \tau = \lambda_1(T') &= \sigma * (\lambda_1(t_i)). \end{aligned}$$

By unique readability we have $T = \langle x, t_i \rangle$ for some unique $x \in X$ and $t_i \in \mathcal{F}(X)$. Thus $f(x) = y$ and $\mathcal{F}(f)(t_i) = t'_{\sigma^{-1}(i)}$. By the inductive hypothesis, there are unique $\tilde{t}_i \in \mathcal{FS}(X)$ such that $\mathcal{SF}(f)(\tilde{t}_i) = t'_i$, $\lambda_0(\tilde{t}_{\sigma^{-1}(i)}) = t_i$, and $\lambda_1(\tilde{t}_i) = \lambda_1(t'_i)$. Since permutations are invertible, the unique tree we are looking for is $\langle (x, \sigma), \tilde{t}_i \rangle$. □

6.2.2 Existence of the Operad of Operads

Theorem 6.2.7 (Existence of the Operad of Operads). *For any set of types $O \in$ $\mathbf{Set}$, the forgetful functor $Mon(\mathbf{Sig}_s)/O \to \mathbf{Sig}/O = \mathbf{Set}/O^\dagger$ is monadic. The resulting monad is finitary, preserves wide pullbacks, and its structure maps are cartesian natural transformations. Therefore it corresponds to a multicategory with nonstandard amalgamation.*

Proof. We have already seen in corollary 6.2.4 that this functor is monadic. One may also use crude monadicity directly, as in the next theorem.

To show finitarity note that the left adjoint is the free functor for the composite monad $\mathcal{SF}$, as we have seen in the discussion above. The resulting monad acts as this functor on ordinary signatures. We know that $\mathcal{S}$ preserves filtered colimits since it consists of multiplication of the n-ary function symbols by the symmetric group S_n. To see this for the functor $\mathcal{F}$ note the monoidal product $\otimes$ for $\mathbf{Sig}$ preserves filtered colimits in both variables, and that the free monoid formula is a colimit of a diagram of such monoidal products. Since colimits commute the claim is proven.

Preservation of wide pullbacks: $\mathcal{S}$ preserves them, since it is multiplication by the symmetric groups. The functor $\mathcal{F}$ is given by the free monoid formula 1.5.2 as usual, and so the same argument as in subsection 2.7.2 applies to it – a compatible family of trees can only sit over a tree of the same shape.

Cartesianness of the structure maps: since they arise from a cartesian distributive law, we can treat each monad $\mathcal{S}$ and $\mathcal{F}$ separately. The composite monad will be cartesian by the pullback lemma. For $\mathcal{S}$ the necessary diagrams can be checked directly. For $\mathcal{F}$ this follows from an isomorphism of monads $\mathcal{W}(O^\dagger) \star_a (-) \simeq \mathcal{F}(-)$ (given below), the first of which is explicitly cartesian. This also gives another argument showing that $\mathcal{F}$ preserves wide pullbacks.

To see this isomorphism argue as follows: by our construction of colimits in $\mathbf{Sig}_a$ and its monoidal structure, the formula for $\mathcal{F}$ is the same for ordinary signatures and for signatures with nonstandard amalgamation (one can also see this by appealing to the Kleisli lift coming from the combing distributive law). The tautologous actions are also given by the same formula. By theorem 3.2.4, the category of $(\mathbf{Sig}_a/O)/O^\dagger$ consisting of signatures with maps to $O^\dagger$ is adjoint equivalent to $\mathbf{Sig}/O$ – ordinary signatures over O. Since this equivalence is monoidal (by theorem 3.3.1), multicategories with nonstandard amalgamation over $O^\dagger$, with the obvious multicategory structure, are equivalent to ordinary multicategories. The free functors for both therefore coincide, and so by theorem 6.1.4 we have $\mathcal{W}(O^\dagger) \star_a (-) \simeq \mathcal{F}(-)$. □

Remark 6.2.8. *The auxiliary operad F in the proof of theorem 14 in [BD98] is defined as $\mathcal{F}_s(O^\dagger)$, where $\mathcal{F}_s$ is the free symmetric operad functor, and $O^\dagger$ is the terminal symmetric signature over O (which is also the terminal symmetric operad over O). This operad cannot be isomorphic to $\mathcal{SF}(O^\dagger)$, since the symmetric actions for $O^\dagger$ are not free. If we had $\mathcal{SF}(O^\dagger) \simeq \mathcal{F}_s(O^\dagger)$, then the unit of the $\mathcal{F}_s$ adjunction would give a morphism of symmetric signatures $O^\dagger \to \mathcal{SF}(O^\dagger)$. However, for $O = \{*\}$ the domain has trivial symmetric actions, and the codomain has free actions – there can be no such morphism.*

In fact $\mathcal{F}_s(O^\dagger)$ consists of equivalence classes of trees (trees with a permutation of their leaves, i.e. elements of $\mathcal{SF}(O^\dagger)$) arising from the relation which identifies trees which have been combed by permutations from the isotropy groups of the symmetric actions of $O^\dagger$.

Definition 6.2.9. *The operad for operads over O will be denoted by $\mathcal{M}^O$.*

By the definition of the tautologous action for $\mathbf{Sig}_a$, the set of function symbols of $\mathcal{M}^O$, as a multicategory with nonstandard amalgamation, is $\mathcal{SF}(O^\dagger)$. This is the value of the corresponding monad on the terminal object in $\mathbf{Set}/O^\dagger = \mathbf{Sig}/O$. The function symbols of $\mathcal{M}^O$ as a symmetric operad include all the permutations of these symbols, according to their arity, which is the number of nodes in the corresponding tree.

Theorem 6.2.10 (Existence of Slice Operads)**.** *For any monoid $M \in Mon(\mathbf{Sig}_s)$ and an algebra $A \in Act(M)$, there is a monoid $M_A \in Mon(\mathbf{Sig}_s)/A$ whose category of algebras is equivalent to $Act(M)/A$.*

Proof. The free M-algebra over A can be easily seen to be given as follows: let $d\colon X \to A \in \mathbf{Set}/A$. Then the free M-algebra over A on $X \to A$ is given by

$$M(X) \xrightarrow{M(d)} M(A) \xrightarrow{\xi} A.$$

The requisite universal property is easily verified using the freeness of $M(X)$ in ordinary M-algebras. The resulting monad will be denoted M_A.

We must show that $Act(M)/A$ is monadic over $\mathbf{Set}/A$. For this we use the crude monadicity theorem. The functor in question obviously reflects isomorphisms – the inverse of a homomorphism of algebras is a homomorphism. We are left with showing that we have reflexive coequalizers in $Act(M)/A$ and the forgetful functor preserves them.

The definition of an M-algebra over A can be restated in terms of sets and functions between finite products of sets, making certain diagrams commute. To see a complete definition of this sort, for an untyped operad, and over the terminal algebra, see [May72]. The general case is more complicated, but it is clear that this can be done. For example, the structure map can be expressed as a series of maps

$$f\colon X(\alpha_1) \times \ldots X(\alpha_n) \to X(\alpha_0),$$

where $f \in M$ is a function symbol of M with typing $\alpha \in O^\dagger$, and $X(\alpha_i)$ is the fiber of X over $\alpha_i \in O$. Morphisms of algebras over A can also be restated in this way. Since reflexive coequalizers commute with finite products in $\mathbf{Set}$, we can construct the data for an algebra over A from the data of a reflexive pair in $Act(M)/A$, restated in $\mathbf{Set}$, by taking coequalizers in $\mathbf{Set}$, and it will satisfy the requisite properties, since all of them are stated in terms of maps between finite products in $\mathbf{Set}$. The resulting algebra will be a coequalizer in the category of algebras over A, since homomorphisms of algebras over A can also be expressed as maps between finite products in $\mathbf{Set}$ making certain diagrams commute. This process is essentially the construction of a quotient algebra from a congruence – we use reflexivity to connect n-tuples of elements of X by the appropriate equivalence relation on 1-tuples, that is, reflexivity ensures that the equivalence relation on elements is compatible with forming products (for example, lists of elements of X on which M can act).

Since we constructed everything in $\mathbf{Set}$, the preservation of these coequalizers by the forgetful functor to $\mathbf{Set}/A$ is obvious.

We must prove that the resulting monad is finitary, weakly preserves wide pullbacks, and its structure maps are weakly cartesian. For this we analyze limits and colimits in $Act(M)/A$.

Since M as a monad is finitary, colimits preserved by M are created in M-algebras by the forgetful functor, and $\mathbf{Set}/O$ is cocomplete, it follows that the category of M-algebras has filtered colimits and all limits, and these are computed in $\mathbf{Set}/O$.

It thus follows that $\mathit{Act}(M)/A$ has filtered colimits, since the projection to $\mathit{Act}(A)$ creates them. The projection $\mathit{Act}(M)/A \to \mathit{Act}(M)$ creates wide pullbacks. It follows that wide pullbacks in $\mathit{Act}(M)/A$ can be computed in $\mathbf{Set}/A$, due to the specific shape of wide pullback diagrams.

The projection $\mathbf{Set}/A \to \mathbf{Set}/O$ therefore creates all the limits and colimits under consideration, and M_A acts as M on the (domains of) objects of $\mathbf{Set}/A$. Thus M_A preserves everything that M does, and has the same types of structure maps. It is therefore a symmetric operad. □

What this proof really shows is that the category $\mathit{Act}(M)/A$ is the category of models of a multisorted equational theory – this is what writing down an explicit definition of an algebra over A in terms of sets, products and functions comes down to. Monadicity is then an easy corollary.

Note that this proof does not explicitly determine the set of function symbols of the resulting operad. The statement given in definition 12 of [BD98], that this set is (in our notation) $M \star_a A \simeq M(A)$, is true if M is a multicategory with nonstandard amalgamation (not just a symmetric one). This follows from the definition of the tautologous actions – the set of function symbols of a multicategory with nonstandard amalgamation is the value of the corresponding monad on the terminal object. Fortunately the operad of operads is always a multicategory with nonstandard amalgamation.

6.2.3 Comparison with the Web Monoid

The following theorem dictates the context of any comparison between the web monoid and the operad of operads.

Theorem 6.2.11. *For any multicategory with nonstandard amalgamation M we have an equivalence of categories*

$$\mathit{Act}(\mathcal{M}^O_{\mathcal{S}M}) \simeq \mathit{Act}(\mathcal{W}(M))$$

Proof. By definition, the algebras for $\mathcal{M}^O_{\mathcal{S}M}$ are symmetric operads over $\mathcal{S}M$ (with the same fixed set of types as $\mathcal{S}M$). They are freely symmetric, since the symmetric actions for $\mathcal{S}M$ are free. Hence these $\mathcal{S}\mathcal{F}$-algebras live in the Kleisli category of $\mathcal{S}$. Reformulating everything in that category we see that these algebras are multicategories with nonstandard amalgamation over M (with the same types as M, held fixed). By by remark 6.1.1 and theorem 6.1.4 these are exactly the algebras for $\mathcal{W}(M)$. □

There is no isomorphism $\mathcal{W}(M) \simeq \mathcal{M}^O_{\mathcal{S}M}$, since these monoids have different sets of types. The map $M \mapsto \mathcal{S}M$, while expressing the same structure, changes the set of function symbols. Our constructions are unfortunately sensitive to this difference.

For $M = O^\dagger$ there is a natural morphism $\mathcal{W}(O^\dagger) \to \mathcal{M}^O = \mathcal{M}^O_{O^\dagger}$. It is simply the homomorphism of monads $\mathcal{F} \to \mathcal{SF}$ coming from the distributive law. Recall from the proof of theorem 6.2.7 that $\mathcal{W}(O^\dagger) \simeq \mathcal{F}$. At the level of algebras it corresponds to forgetting the symmetric actions $Mon(\mathbf{Sig}_s)/O \to Mon(\mathbf{Sig})/O$. An analogous map can be constructed for any M with standard amalgamation, and again corresponds to forgetting the symmetric actions, this time for symmetric operads over M.

6.2.4 Comparison of the Resulting Categories

Theorem 6.2.12. *The category of* $\mathbf{O}$*-opetopic sets, as defined by Baez and Dolan, is a presheaf category.*

Proof. (Sketch) Since the assignment $O \mapsto \mathcal{M}^O = \mathcal{SF}(O^\dagger)$ preserves wide pullbacks, as does the slicing operation (for multicategories with nonstandard amalgamation – since it is given by $(-) \star_a X$ in that case, cf. definition 2.5.1 in and the results on wide pullbacks in signatures in chapter 2), the same argument, based on Artin gluing, as in section 5.2 can be repeated – the functor of top dimensional frames in a truncated opetopic set preserves wide pullbacks. Items 5.2.1 through 5.3.11 need only cosmetic changes to apply directly to the structures of Baez and Dolan. □

Theorem 6.2.13. *The category of Baez-Dolan opetopic sets is not equivalent to the category of opetopic sets defined above.*

Proof. (Sketch) Suppose there is an equivalence $\mathbf{OSet}_{BD} \simeq \mathbf{OSet}$. Since both categories are presheaf categories, whose exponents (Opt_{BD} and Opt, respectively) have no nontrivial endomorphisms, we obtain an equivalence $Opt_{BD} \simeq Opt$. These categories are skeletal (by our construction), and so the equivalence is actually an isomorphism – both Opt_{BD} and Opt are FOLDS signatures [M95]. This follows easily from the arguments in items 5.3.8-5.3.11. The isomorphism therefore identifies their respective levels of objects $Opt_{BD,k} \simeq Opt_k$, for $k \in \mathbb{N}$ (these consist precisely of the k-dimensional opetopes in each approach). Let $Opt_{BD,\leq n}$ and $Opt_{\leq n}$ be the respective full subcategories of Opt_{BD} and Opt consisting of objects of level at most n. Since levels of objects are preserved, the equivalence $Opt_{BD} \simeq Opt$ restricts to an equivalence $Opt_{BD,\leq n} \simeq Opt_{\leq n}$.

From this, by composition, we can construct a diagram

$$\begin{tikzcd} & Opt_{\leq 1} \arrow[dl] \arrow[dr] & \\ Opt_{\leq 2} \arrow[rr, "\simeq"] & & Opt_{BD,\leq 2} \end{tikzcd}$$

where the bottom arrow is an isomorphism. We are thus in the category of FOLDS signatures under $Opt_{\leq 1}$. A possible context for a FOLDS signature L is a contex arising from a kind in an extension $L \to L'$. The number of kinds (objects) of L', as a cardinal number, giving isomorphic possible contexts in L is an invariant of such an extension under isomorphism in the category of FOLDS signatures under L. However the possible context

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

of $Opt_{\leq 1}$ is realized by one kind in $Opt_{\leq 2}$ and two kinds in $Opt_{BD,\leq 2}$. Due to our construction of the exponent categories, this can be verified by direct calculation by analyzing the number of frames in the respective terminal objects consisting of exactly three 1-cells (making the frame binary). Such an arrangement of cells gives rise to a unique possible context for $Opt_{\leq 1}$, pictured informally above.

We have arrived at a contradiction, and so there can be no equivalence $\mathbf{OSet} \simeq \mathbf{OSet}_{BD}$. $\square$

Remark 6.2.14. *It is easy to see that the categories* $\mathbf{Set}^{Opt_{BD,\leq n}}$ *and* $\mathbf{Set}^{Opt_{\leq n}}$ *are exactly the truncations used in the proof of the presheaf property.*

6.3 Comparison to Hermida, Makkai, and Power

In this section we prove the following:

Theorem 6.3.1. *The category of opetopic sets is isomorphic to the category of multitopic sets.*

It is actually easier to prove isomorphism than equivalence. We adopt the notation for multitipic sets introduced in part 3 of [HMP02], with the following exceptions: we denote n-truncated multitopic sets by $\mathbf{MltSet}_n$, and the category of multitopes in this paper will be the opposite of the category of multitopes in [HMP02]. In particular we will freely alternate between C_n and X_n to refer to the set of n-cells of the structure under consideration. Before discussing the proof of this theorem, we note the most important corollary.

Corollary 6.3.2. *The category of opetopes is isomorphic to the category of multitopes.*

Proof. Since neither category has nontrivial idempotents, the equivalence of presheaf categories $\mathbf{Set}^{Opt(I)^{op}} \simeq \mathbf{OSet} \simeq \mathbf{MltSet} \simeq \mathbf{Set}^{\mathbf{Mlt}^{op}}$ implies an equivalence of exponent categories. Since both $Opt(I)$ and $\mathbf{Mlt}$ are skeletal and none of their objects have nontrivial endomorphisms, this equivalence is an isomorphism. $\square$

Remark 6.3.3. *In [HMP02] the category of multitopes was defined so that the category of multitopic sets was* $\mathbf{Set}^{\mathbf{Mlt}}$*. We have chosen to reverse this convention, that is multitopes in this paper are the opposite of multitopes in [HMP02].*

6.3.1 Preliminary Reductions

The comparison boils down to comparing the web monoid construction to the multicategory of function replacement from part 2 of [HMP02]. We will first show how this comes about, and then perform the rather technical comparison of the two constructions.

First recall that $\mathbf{OSet}_{n+1} \simeq \mathbf{Set}/S_n$ by lemma 5.2.3, and $\mathbf{OSet} \simeq \varprojlim \mathbf{OSet}_k$ by proposition 5.2.9. Likewise, it is evident, by the introductory discussion in part 3 of [HMP02], that $\mathbf{MltSet}_{n+1} \simeq \mathbf{Set}/Q_n$ (we will recall the definition of the Q_n functor below), and that $\mathbf{MltSet} \simeq \varprojlim \mathbf{MltSet}_k$.

To prove the above theorem, it therefore suffices to construct a compatible family of isomorphisms $\mathbf{OSet}_{n+1} \simeq \mathbf{MltSet}_{n+1}$, which in turn means constructing isomorphisms $\mathbf{Set}/S_n \simeq \mathbf{Set}/Q_n$. We will construct these isomorphisms inductively, starting with the obvious isomorphisms $\mathbf{OSet}_0 \simeq \mathbf{Set} \simeq \mathbf{MltSet}_0$, given by $X \mapsto X_0$ and $S \mapsto C_0(S)$, respectively.

For the inductive step we have the following lemma:

Lemma 6.3.4. *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G,H} \mathcal{C}$ *be functors. Then:*

1. *If F is an isomorphism, then there is a canonical isomorphism $\mathcal{C}/GF \simeq \mathcal{C}/G$.*

2. *If $\tau\colon G \to H$ is a natural isomorphism, then there is a canonical isomorphism $\mathcal{C}/G \simeq \mathcal{C}/H$.*

The proof is obvious – just compose with F and F^{-1} or τ and τ^{-1}.

The above lemma reduces our problem to showing an isomorphism between Q_n and S_n along some previously constructed isomorphism $\mathbf{OSet}_n \to \mathbf{MltSet}_n$. Note that this means that the sets of cells – X_n and $C_n(S)$ – will be unaffected by our isomorphism. The set maps $f_n\colon X_n \to Y_n$ comprising morphisms will also remain the same, since the isomorphisms indicated by the above lemma don't alter them. This can be summarized by saying that our isomorphism will commute with the obvious forgetful functors to $\mathbf{Set}^{\mathbb{N}}$ (or $\mathbf{Set}^n$ in the n-truncated case):

$$\begin{tikzcd} \mathbf{OSet} \arrow[rr, "\simeq"] \arrow[dr] & & \mathbf{MltSet} \arrow[dl] \\ & \mathbf{Set}^{\mathbb{N}} & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{OSet}_n \arrow[rr, "\simeq"] \arrow[dr] & & \mathbf{MltSet}_n \arrow[dl] \\ & \mathbf{Set}^{n} & \end{tikzcd}$$

In particular this means that we can talk about opetopic and multitopic structures on a sequence of sets X_n. By this we simply mean objects of **OSet** or **MltSet** (or their truncations) in the fiber over X_n. Given the above lemma, we see that it is enough to show that, given a multitopic or opetopic structure on a sequence $\{X_k\}_{k\leq n}$, both Q_n and S_n can be computed from it, and the results are naturally isomorphic.

The case $n = 0$, that is comparing Q_0 and S_0 over **Set** is easy – both definitions hardwire this case, and both are isomorphic to the set of ordered pairs of 0-cells. The map $X_1 \to Q_0 \simeq S_0$ gives 1-cells their domains and codomains.

The general case is described by the following theorem:

Theorem 6.3.5. *Given an opetopic structure on $\{X_k\}_{k\leq n}$ the generalized multicategory $\mathbf{D}_n$ for the associated multitopic structure on $\{X_k\}_{k\leq n}$ is given by the following pullback in* **Set***:*

$$\begin{tikzcd} \mathcal{W}(\mathbf{S}_{n-1}) \arrow[d] & \mathbf{D}_n \arrow[l] \arrow[d] \\ S^{\dagger}_{n-1} & S_{n-1} \times X^{*}_{n-1} \arrow[l, "1 \times \vartheta^{*}_{n}"'] \end{tikzcd}$$

The proof of this theorem is given in subsection 6.3.3. To give the proof, however, we must spend next subsection explaining how the above pullback gives $\mathbf{D}_n$ the structure of a generalized multicategory, that is, why the statement of the above theorem is even correctly typed. Note also that the proof of theorem 6.3.5 is still inductive. It is true by inspection for $n = 1$ and to prove it for a specific n we will assume it holds for $n-1$. In particular, we may assume that $Q_{n-1} \simeq S_{n-1}$.

Before any of this, let us use theorem 6.3.5 to construct an isomorphism $S_n \simeq Q_n$, and thus finish the proof of the comparison theorem 6.3.1. By the pullback lemma, the following diagram of iterated pullbacks is a pullback

$$\begin{tikzcd} \mathcal{W}(\mathbf{S}_{n-1}) \arrow[d] & \mathbf{D}_n \arrow[l] \arrow[d] & \mathbf{S}_n \arrow[l] \arrow[d] \\ S^{\dagger}_{n-1} & S_{n-1} \times X^{*}_{n-1} \arrow[l, "1 \times \vartheta^{*}_{n}"'] & X^{\dagger}_{n-1} \arrow[l, "\vartheta_n \times 1^{*}"'] \end{tikzcd}$$

Recall that the arrows (i.e. elements in our formalism) of $\mathbf{D}_n$ are exactly the pasting diagrams P_n. The functor Q_n is defined as a pullback (in **Set**)

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] & P_n \arrow[d, "{(d,c)}"] \\ C_n \arrow[r, "{(d,c)}"] & P_{n-1} \times C_{n-1} \end{tikzcd}$$

where d and c are the domain and codomain maps, respectively. We claim that this pullback, is equivalent to the right square in the pullback defining S_n above. To see this note that because of the special form of the morphism $\vartheta_n \times 1^*$ – the 1^* is simply the identity on the list of input types – the elements of S_n are simply the elements of $\mathbf{D}_n$ (i.e. elements of P_n) together with an output cell in $X_{n-1} = C_{n-1}$, which is parallel to the element of $\mathbf{D}_n$ it is paired up with. This means the output frame of the element in $\mathbf{D}_n$ matches the frame of its assigned output cell. This is also exactly the definition of Q_n given above. To see this, note that by the globularity of multitopic sets, that is the equalities $dd = dc$ and $cd = cc$ for both cells and pasting diagrams, both maps $C_n \xrightarrow{(d,c)} P_{n-1} \times C_{n-1}$ and $P_n \xrightarrow{(d,c)} P_{n-1} \times C_{n-1}$ actually take values in $Q_{n-1} \subset P_{n-1} \times C_{n-1}$. But by the inductive hypothesis $Q_{n-1} \simeq S_{n-1}$, and so parallelism in multitopic sets means equality of frames (in the associated opetopic structure). Thus $Q_n \simeq S_n$.

6.3.2 $\mathbf{D}_n$ as a Generalized Multicategory

We rely here in the comparison between 2-level signatures and generalized multicategories from section 6.5 of [Z10]

We first need to see how the construction in that theorem gives $\mathbf{D}_n$ the structure of a generalized multicategory. The set $\mathbf{D}_n = (1 \times \vartheta_n^*)^* \mathcal{W}(\mathbf{S}_{n-1})$ is the set of arrows of our multicategory, where $\mathcal{W}(\mathbf{S}_{n-1})$ is treated as a *set* (the set of function symbols in the web monoid). The pullback of the typing map gives us a morphism $\mathbf{D}_n \to S_{n-1} \times X_{n-1}^*$. Thus we define the upper level objects to be X_{n-1}, and the lower level objects to be S_{n-1} (considered as a set of function symbols). The map from upper level objects to lower level objects is ϑ_n, which assigns to each cell its frame. This discussion also defines an object of the category of 2-level signatures in a natural way.

We must now define composition. We will show that the functor of pulling back along $(1 \times \vartheta_n^*)$ maps monoids in signatures to generalized multicategories. For this, consider an arbitrary function $\vartheta \colon X \to S$. We will now show that the functor $(1 \times \vartheta^*)^* \colon \mathbf{Sig}_a/S \to \mathbf{Sig}_{2a}/\vartheta$ is almost lax monoidal, and its action on monoids lands in generalized multicategories.

The natural transformation (natural in A and B)

$$(1 \times \vartheta^*)^* A \otimes_\vartheta (1 \times \vartheta^*)^* B \to (1 \times \vartheta^*)^* (A \otimes_S B)$$

given by

$$\langle (a, y), (b_i, x_i) \rangle \mapsto (\langle a, b_i \rangle, x_i),$$

where the x_i and y are the new upper level typings of the inputs, explains that formal composites of retyped elements of two ordinary signatures can still be made naturally compatible in ordinary signatures. This allows us to transfer composition laws from ordinary signatures to two-level ones. Note that the upper level typing of the inputs we are filling (y) is forgotten by this map.

We have an obvious morphism of monoidal units $I_\vartheta \to (1 \times \vartheta^*)^* I_S$, defined using ϑ itself on the lower level objects and the identity on the upper level objects. Unfortunately this morphism is not contained in the fiber $\mathbf{Sig}_{2a}/\vartheta$ since it modifies lower level objects. For this reason our functor is not lax monoidal. It does, however suffice for our purposes. We define the identities as the image of this map.

Lemma 6.3.6. *With the above definitions the pullback functor* $(1 \times \vartheta^*)^* \colon \mathbf{Sig}_a/S \to \mathbf{Sig}_{2a}/\vartheta$ *maps monoids to generalized multicategories.*

Proof. The comparison of monoidal products gives us a composition law just like an ordinary lax monoidal functor does (we have defined the identities separately). This gives us a notion of simultaneous composition, as defined in part 2 of [HMP02]. It is associative, since the lax monoidal functor diagram for the associativity morphism actually holds for our comparison map, as a simple computation reveals. The only problem are the unit laws. These can be checked by hand, using the composition law. The strange form of the right unit law (as given in [HMP02]) comes from the fact that we delete y in our comparison map. □

6.3.3 Proof of Theorem 6.3.5

Let us see why $\mathbf{D}_n$ has a chance of being a multicategory of function replacement. First note that since we already know that $S_{n-1} \simeq Q_{n-1}$, and $Q_{n-1} \subset P_{n-1} \times C_{n-1}$, the lower level objects are correct – in our particular situation, due to globularity, not all of them will be codomains, and our construction of $\mathbf{D}_n$ merely reflects this restriction. Second, by the definition of the pullback action in section 2.5 and corollary 3.4.2, we have

$$\mathbf{D}_n = (1 \times \vartheta_n^*)^* \mathcal{W}(\mathbf{S}_{n-1}) = \mathcal{W}(\mathbf{S}_{n-1}) \star_a X_{n-1} = \mathcal{F}(X_{n-1}), \tag{6.1}$$

so that the arrows of $\mathbf{D}_n$ indeed coincide with the free multicategory on the set of $(n-1)$-cells, where the cells are typed according to the typing of their frame, that is by $(n-2)$-cells.

We must now show that the multiplication defined above coincides with the one constructed in part 2 of [HMP02]. Unfortunately there is no way to perform an abstract comparison – the multicategory of function replacement functor is defined on a category that is inequivalent to the domain of the web monoid functor. The two constructions overlap only in their specific application to the construction of the category of multitopic or opetopic sets. This forces us to examine the proof of the main theorem in part 2 of [HMP02], to see what multiplication it actually gives.

To begin, note that the identity arrows obviously coincide – in both cases they are the generating arrows of the free multicategory. For our construction of $\mathbf{D}_n$ this is part of corollary 3.4.2. So we only need to compare the composition maps. In the construction of [HMP02] it is done entirely within the free multicategory, using what are called "ample expansions". The defining properties of the multicategory of function replacement determine compositions uniquely for separated terms, that is those whose typing is injective. Then one shows that any term can be separated in a different free multicategory, that is all terms are images of separated terms under strict morphisms of free multicategories. This, along with the requirement of functoriality defines composition for all terms, and the construction is complete.

We will prove two things:

1. The two composition laws coincide for separated terms.
2. We can define natural ample expansions $\mathcal{W}(\pi)$ for the web monoid functor.

These two facts together imply that the construction of [HMP02] can be carried out inside the web monoid functor. Specifically it can be carried out in the ample expansion given by $(1 \times \vartheta_n)^*\mathcal{W}(\pi)$, which is provided by $\mathcal{W}(\pi)$ (with the natural transformation π defined below) and lemma 6.3.6. The domain of this map will be a free multicategory by a computation analogous to the one in equation 6.1. The projection maps which define function replacement composition for nonseparated terms are homomorphisms with respect to multiplication induced from web monoid. Since they are defined to be homomorphisms with respect to function replacement in [HMP02], and the compositions of separated terms agree, the two composition maps are equal for all terms. This argument relies on the fact that we can use arbitrary ample expansions to define function replacement, and the results will always agree. This is part of the argument for the existence of function replacement in [HMP02], and we will not repeat it here. It is also apparent in that proof that functoriality for just ample expansions is enough to complete the construction of the multiplication map for a single datum $(\mathcal{L}, \mathbf{C}, d)$. Thus we can use for our comparison only the convenient ample expansions of $\mathbf{D}_n$.

The proof of the second point is fairly trivial. Recall the separation principle 3.1. There, given a monoid $M \in Mon(\mathbf{Sig}_a)$, we constructed a new monoid with the same types, given by $M_{\mathbb{N}} = M \times \mathbb{N}$ together with an obvious strict projection homomorphism $\pi\colon M_{\mathbb{N}} \to M$. It is obvious that $\pi\colon (-)_{\mathbb{N}} \to 1_{Mon(\mathbf{Sig}_a)}$ is a natural transformation of functors. Then the natural transformation $\mathcal{W}(\pi)$ is the sought for natural ample expansion of $\mathcal{W}$. This is proven by induction on tree height (since $\mathcal{W}$ is $\mathcal{F}_{\odot}(I_{\otimes})$ – a set of terms, or trees) starting from the obvious fact that $M_{\mathbb{N}} \otimes M_{\mathbb{N}} \to M \otimes M$ is strict and can separate any finite set of formal composites. It then follows that $\mathcal{W}(\pi_M)$ is strict and can separate any finite set of function symbols (which are trees or terms). The argument here is essentially the same as in lemma 7 in part 2 of [HMP02].

The proof of the first point relies on the fact that the statement of lemma 4 in part 2 of [HMP02] is, in the separated context, equivalent to the main diagram defining multiplication in the web monoid. Lemma 4 states that, for separated terms, the operation of function replacement commutes with the free multiplication in $\mathcal{F}(X_{n-1})$. This, together with the unit laws (which we have already shown to coincide), defines function replacement uniquely for separated terms. Note that since $\mathcal{F}(X_{n-1})$ is an ordinary 1-level free multicategory, we may express its multiplication law using the monoidal structure for $\mathbf{Sig}_a$. The main diagram states ν commutes with the free multiplication for all terms. To see this consider the pullback of the main diagram by $(1 \times \vartheta_n^*)$. This computation, by equation 6.1, has already been done in the proof of theorem 6.1.7. Recall that the result is the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}^2(X_{n-1}) \otimes \mathcal{F}^2(X_{n-1}) \arrow[r, "(\nu \star_a X_{n-1}) \otimes (\nu \star_a X_{n-1})"] \arrow[d, "\mu_{\mathcal{F}(X_{n-1})}"'] & \mathcal{F}(X_{n-1}) \otimes \mathcal{F}(X_{n-1}) \arrow[d, "\mu_{X_{n-1}}"] \\
\mathcal{F}^2(X_{n-1}) \arrow[r, "\nu \star_a X_{n-1}"] & \mathcal{F}(X_{n-1})
\end{tikzcd}$$

In the above diagram we have identified $(\mathcal{W} \otimes_{S_{n-1}} \mathcal{W}) \star_a X_{n-1} \simeq \mathcal{W} \star_a (\mathcal{W} \star_a X_{n-1}) \simeq \mathcal{F}^2(X_{n-1})$ analogously to equation 6.1, and by that equation we also have $(1 \times \vartheta_n^*)^*\nu = \nu \star_a X_{n-1}$. It may be helpful to keep in mind that theorem 5.19 states, among others, that $\nu \star_a X_{n-1} = \varepsilon_{\mathcal{F}(X_{n-1})}$, the counit for the left adjoint $\mathcal{F}$. The map $\mu_{X_{n-1}}$ is the free multiplication in $\mathcal{F}(X_{n-1})$ and the map $\mu_{\mathcal{F}(X_{n-1})}$ is the free multiplication in $\mathcal{F}^2(X_{n-1})$ coming from the outermost application of $\mathcal{F}$. The monoidal structure $\otimes$ in the diagram above is the one in $\mathbf{Sig}_a/X_{n-2}$.

Using lemma 6.3.6 and the identification $\mathcal{F}(X_{n-1}) \simeq \mathbf{D}_n$ we obtain from this the following diagram:

$$\begin{tikzcd}
(\mathbf{D}_n \otimes_{\vartheta_n} \mathbf{D}_n) \otimes_{X_{n-2}} (\mathbf{D}_n \otimes_{\vartheta_n} \mathbf{D}_n) \arrow[r, "\tilde{\nu} \otimes \tilde{\nu}"] \arrow[d, "\tilde{\mu}_{\mathcal{F}(X_{n-1})}"'] & \mathbf{D}_n \otimes_{X_{n-2}} \mathbf{D}_n \arrow[d, "\mu_{X_{n-1}}"] \\
\mathbf{D}_n \otimes_{\vartheta_n} \mathbf{D}_n \arrow[r, "\tilde{\nu}"] & \mathbf{D}_n
\end{tikzcd}$$

Where $\tilde{\nu}$ is the multiplication on $\mathbf{D}_n$ induced from the web monoid, and $\tilde{\mu}_{\mathcal{F}(X_{n-1})}$ acts as $\mu_{\mathcal{F}(X_{n-1})}$ perserving the additional typing coming from using $\otimes_{\vartheta_n}$ instead of $\otimes_{S_{n-1}}$. This diagram states that $\tilde{\nu}$ commutes with the free multiplication in $\mathbf{D}_n$ for all terms. Converting it to single composition – as opposed to simultaneous composition – which means that all inputs get identities except for one (for details on how this is done see remark 2 at the end of section 6 in [Z10]), it states that

$$\tilde{\nu}(\mu(f,g,r),h,s) = \begin{cases} \mu(f,\tilde{\nu}(g,h,s),r) & \text{if } s \text{ belongs to } g \\ \mu(\tilde{\nu}(f,h,s),g,r) & \text{if } s \text{ belongs to } f \end{cases},$$

where r, s are numbers indicating the places of insertion, and $f, g, h \in \mathbf{D}_n$. The cases arise from considering which copy of $\tilde{\nu}$ in the map $\tilde{\nu} \otimes \tilde{\nu}$ will act on the single input. This is exactly the statement of lemma 4 in [HMP02]. Our $\tilde{\nu}$ thus coincides, by uniqueness, with function replacement for separated terms. This finishes the proof of theorem 6.3.5.

Chapter 7

The Relation to Logic

This chapter is a reflection on the various algebraic structures used by different authors to define opetopic sets. As we have seen in the previous chapters, they are almost the same, but differ in subtle details. The common ground for all these structures turns out to be equational logic.

The category of algebras of a (finitary) equational theory can be equivalently described as a category of models of a Lawvere theory or as a category of algebras of a finitary monad on the category **Set**. In some cases there are also two other descriptions available. Some categories of algebras can be described also as algebras for a symmetric operad and some can be described as algebras for a rigid[1] operad (c.f. [HMP02, Z10]). It is well known that the categories of equational theories **ET**, Lawvere theories **LT** and monads (on **Set**) **Mnd** are equivalent[2]. Its is also known that the categories of symmetric and rigid operads are equivalent to the categories of analytic and polynomial monads, respectively; see [Z10]. In this paper we give a description of the subcategories of **ET** and of **LT** that correspond to the categories of symmetric and rigid operads.

The equational theories corresponding to analytic monads are linear-regular theories. A linear-regular theory is an equational theory that can be axiomatized by equations having the same variables on both sides, each variable occurring exactly once. A linear-regular theory T is rigid iff whenever a linear-regular equation

$$t(x_1, \dots x_n) = t(x_{\sigma(1)}, \dots x_{\sigma(n)})$$

is provable in T then the permutation σ is the identity permutation. In the above

[1] In this chapter only, we call a "rigid operad" what was earlier called a "multicategory with non-standard amalgamation". This choice is motivated by the property of the equational theories that correspond to such operads. The category of rigid operads can be identified with the full subcategory of symmetric operads for which the symmetric actions are free.

[2] These correspondences preserves the notion of a "model", i.e. the corresponding categories of algebras in the suitable sense are canonically equivalent.

equation $t(x_1, \ldots x_n)$ denotes any term with n different variables $x_1, \ldots x_n$, each one occurring exactly once and $t(x_{\sigma(1)}, \ldots x_{\sigma(n)})$ denotes the same term t but with variables permuted according to σ. For example, the theory of commutative monoids is not rigid as it contains the equation

$$m(x_1, x_2) = m(x_2, x_1)$$

The category of polynomial monads **PolyMnd** corresponds to the category of rigid theories **RiET**. The notion of a linear-regular theory was considered in universal algebra but the notion of a rigid theory as well as that of a linear-regular interpretation seems to be new. If all the axioms of an equational theory are linear-regular, then the theory is linear-regular. However, the problem whether a finite set of linear-regular equations defines a rigid theory is undecidable, (cf. [BSZ]).

We also give a characterization of the categories of Lawvere theories that correspond to the categories of analytic and polynomial monads. The category $\mathbb{F}^{op}$, opposite of the skeleton of the category of finite sets is the initial Lawvere theory. Thus it has a unique morphism into any other Lawvere theory $\pi : \mathbb{F}^{op} \to \mathbf{T}$. The class of morphisms in the image of π closed under isomorphisms is called the class of structural morphisms in $\mathbf{T}$. The class of right orthogonal morphisms to the structural morphisms is the class of analytic morphisms in $\mathbf{T}$. A Lawvere theory $\mathbf{T}$ is analytic iff the classes of structural and analytic morphisms form a factorization system and the automorphisms of any object n in $\mathbf{T}$ are determined by the automorphisms of 1. A Lawvere theory is rigid if it is analytic and the symmetric group actions act freely on analytic operations. We show that the categories **AnLT** of analytic and **RiLT** of rigid Lawvere theories correspond to the categories of analytic and of polynomial monads.

The following diagram illustrates the relations between the categories mentioned above. The vertical lines denote adjoint equivalences. Thus up to equivalence of categories there are only four categories in it, one on each level. One equivalent to the category of all finitary monads on **Set**, the second equivalent to the category of semianalytic monads[3] on **Set**, the third equivalent to the category of analytic monads on **Set**, and the fourth equivalent to the category of polynomial monads on **Set**. These levels are denoted by letters f, r, a, and p, respectively. Thus all four columns of equational theories, Lawvere theories, monads and operads are levelwise equivalent. These columns are denoted by letters e, l, m and o, respectively. The vertical functors heading up are inclusions of subcategories. The lower functors are full inclusions and the upper are inclusions that are full on isomorphisms. The vertical functors heading down, the right adjoints to those

[3]This level is special, and not related to the main body of our work. We will not consider it in this chapter. It is described in detail in [SZ].

heading up, are monadic. All the squares in the diagram commute up to canonical isomorphisms.

$$
\begin{tikzcd}
e & o & l & m & \\
 & & \mathbf{LT} \arrow[dl, dash] \arrow[dr, dash] \arrow[ddd, shift left] & & \\
\mathbf{ET} \arrow[urr, dash] \arrow[dr, dash] \arrow[ddd, shift left] & & & \mathbf{Mnd} \arrow[ddd, shift left] & f \\
 & \mathbf{FOp} \arrow[urr, dash] \arrow[ddd, shift left] & & & \\
 & & \mathbf{RegLT} \arrow[dl, dash] \arrow[dr, dash] \arrow[uuu, shift left] \arrow[ddd, shift left] & & \\
\mathbf{RegET} \arrow[urr, dash] \arrow[dr, dash] \arrow[uuu, shift left] \arrow[ddd, shift left] & & & \mathbf{SanMnd} \arrow[uuu, shift left] \arrow[ddd, shift left] & r \\
 & \mathbf{RegOp} \arrow[urr, dash] \arrow[uuu, shift left] \arrow[ddd, shift left] & & & \\
 & & \mathbf{AnLT} \arrow[dl, dash] \arrow[dr, dash] \arrow[uuu, shift left] & & \\
\mathbf{LrET} \arrow[urr, dash] \arrow[dr, dash] \arrow[uuu, shift left] & & & \mathbf{AnMnd} \arrow[uuu, shift left] & a \\
 & \mathbf{SOp} \arrow[urr, dash] \arrow[uuu, shift left] & & & \\
 & & \mathbf{RiLT} \arrow[dl, dash] \arrow[dr, dash] \arrow[uuu] & & \\
\mathbf{RiET} \arrow[urr, dash] \arrow[dr, dash] \arrow[uuu] & & & \mathbf{PolyMnd} \arrow[uuu] & p \\
 & \mathbf{RiOp} \arrow[urr, dash] \arrow[uuu] & & &
\end{tikzcd}
$$

The notation concerning categories involved is displayed in the above diagram. The notation concerning functors is not on the diagram but it is meant to be systematically referring to the levels and columns they 'connect'. The horizontal functors are denoted using letters from both columns they connect; the codomain by the script letter, the domain by its subscript, and the level is denoted by superscript. Thus the functor **AnMnd** $\to$ **AnLT** will be denoted by $\mathcal{L}^a_m$. We usually drop superscripts and often subscripts when it does not lead to confusion. Thus we can write, for example, $\mathcal{E} = \mathcal{E}_o = \mathcal{E}^p_o : \mathbf{RiOp} \to \mathbf{RiET}$. The vertical functors heading up are denoted by script letter $\mathcal{P}$ with superscript indicating the column and subscript indicating the level of the codomain. The vertical functors heading down are denoted by script letter $\mathcal{Q}$ with subscript and superscript as those heading up. Thus we have, for example, functors $\mathcal{P} = \mathcal{P}^o = \mathcal{P}^o_a : \mathbf{RiOp} \to \mathbf{SOp}$ and $\mathcal{Q} = \mathcal{Q}_f = \mathcal{Q}^m_f : \mathbf{Mnd} \to \mathbf{AnMnd}$. We will also refer to various diagonal morphisms and then we need to extend the notation concerning vertical functors by specifying both the columns of the domain and the codomain. For example, we write $\mathcal{P}^{ol}_a : \mathbf{SOp} \to \mathbf{LT}$ to denote one such functor and its right adjoint will be denoted by $\mathcal{Q}^{lo}_f : \mathbf{LT} \to \mathbf{SOp}$. In principle this notation will leave the codomain not always uniquely specified but in practice it it sufficient, and in fact usually much less is needed and each time it is used it will be recalled on the spot.

In Section 7.1 we recall categories of equational theories, Lawvere theories, monads on **Set**, and operads[4]. We also discuss some of their subcategories. In Sections 7.2 we recall the correspondence between equational theories, Lawvere theories, and monads. In section 7.3 we study relations between Lawvere theories and operads. We define a functor $\mathcal{L}_o : \mathbf{SOp} \to \mathbf{LT}$ from the category of symmetric operads the category of Lawvere theories. We identify its image and we show that its right adjoint is monadic. We also identify the image of the category of the rigid operads **RiOp** in **LT**. In section 7.4 we relate the result from section 7.3 to monads. We note that finitary monads are monadic over analytic ones. But we also explain that this is a consequence of an even more fundamental fact that there is a lax monoidal monad on the category of analytic functors. This monoidal monad induces a distributive law. From this we obtain that finitary monads are monadic over analytic functors. This extends a result from [Ba70]. In section 7.5 we define the embedding **SOp** in **ET** and characterize the images of both **SOp** and **RiOp**. This gives the characterizations described at the beginning of the introduction that solves a problem stated in [CJ04]. We end this chapter with a section where we give some examples.

[4]We do not discuss the full and regular operads (the two top categories in the operadic column). For that, we refer the reader to [SZ].

Notation Reminder

For $n \in \mathbb{N}$, we have $n = \{0, \ldots, n-1\}$, $[n] = \{0, \ldots, n\}$, $(n] = \{1, \ldots, n\}$. The set X^n is interpreted as $X^{(n]}$ and it has a (natural) right action of the permutations group S_n by composition. The skeletal category equivalent to the category of finite sets will be denoted by $\mathbb{F}$. The objects of $\mathbb{F}$ are sets $(n]$, for $n \in \mathbb{N}$. The subcategories of $\mathbb{F}$ with the same objects as $\mathbb{F}$ but having as morphisms bijections, surjections, and injections will be denoted by $\mathbb{B}$, $\mathbb{S}$, $\mathbb{I}$, respectively. When S_n acts on the set A on the right and on the set B on the left, the set $A \otimes_n B$ is the usual tensor product of S_n-sets, namely the coend $\int^{S_n} A \times B$.

7.1 Presentations of Categories of Algebras

In this section we collect several categories whose objects describe (some) categories of algebras of finitary equational theories and whose morphisms induce functors between such categories of algebras.

7.1.1 Equational Theories

By an *equational theory* we mean a pair of sets $T = (L, A)$, $L = \bigcup_{n \in \mathbb{N}} L_n$ and L_n is the set of n-ary operations of T. The sets of operations of different arities are disjoint. The set $\mathcal{T}r(L, \vec{x}^n)$ of terms of L in context $\vec{x}^n = \langle x_1, \ldots, x_n \rangle$ is the usual set of terms over L build with the help of variables from $\vec{x}^n$. We write $t : \vec{x}^n$ for the term t in context $\vec{x}^n$. Thus all the variables occurring in t are among those in $\vec{x}^n$. The set A is a set of equations in context $t = s : \vec{x}^n$, i.e. both $t : \vec{x}^n$ and $s : \vec{x}^n$ are terms in context.

A morphism of equational theories, an *interpretation*, $I \colon (L, A) \to (L', A')$ is given by a set of functions $I_n \colon L_n \to \mathcal{T}r(L', \vec{x}^n)$, for $n \in \mathbb{N}$. I_n's extend to functions $\bar{I}_n \colon \mathcal{T}r(L, \vec{x}^n) \to \mathcal{T}r(L', \vec{x}^n)$, for $n \in \mathbb{N}$ as follows. We drop index n in $\bar{I}_n$ when it does not lead to confusion.

$$\bar{I}(x_i : \vec{x}^n) = x_i : \vec{x}^n$$

for $i = 1, \ldots, n$ and

$$\bar{I}(f(t_1, \ldots, t_k) : \vec{x}^n) = I(f)(x_1 \setminus \bar{I}(t_1), \ldots, x_k \setminus \bar{I}(t_k)) : \vec{x}^n$$

for $f \in L_k$ and $t_i \in \mathcal{T}r(L, \vec{x}^n)$ for $i = 1, \ldots, k$. On the right-hand side we have a simultaneous substitution of terms t_i's for variables x_i's. Moreover, for I to be an interpretation we require that the equations are preserved, i.e. for any $t = s : \vec{x}^n$ in A we have

$$A' \vdash \bar{I}(t) = \bar{I}(s) : \vec{x}^n$$

where $A' \vdash$ is the provability in the equational logic from axioms in the set A'. We identify two such interpretations I and $I'\colon (L, A) \to (L', A')$ iff they interpret all function symbols as provably equivalent terms, i.e.

$$A' \vdash I(f) = I'(f) : \vec{x}^n$$

for any $n \in \mathbb{N}$ and $f \in L_n$. In this way we have defined a category of equational theories **ET**.

A term in context $t : \vec{x}^n$ is *regular* if every variable in $\vec{x}^n$ occurs in t at least once. A term in context $t : \vec{x}^n$ is *linear* if every variable in $\vec{x}^n$ occurs in t at most once. A term in context $t : \vec{x}^n$ is *linear-regular* if it is both linear and regular. An equation $s = t : \vec{x}^n$ is *regular* (*linear-regular*) iff both $s : \vec{x}^n$ and $t : \vec{x}^n$ are regular (linear-regular) terms in contexts.

A *simple ϕ-substitution of a term in context* $t : \vec{x}^n$ along a function $\phi\colon (n] \to (k]$ is a term in context denoted $\phi \cdot t : \vec{x}^k$ such that every occurrence of the variable x_i is replaced by the occurrence of $x_{\phi(i)}$. An *α-conversion of a term in context* $t : \vec{x}^n$ is a simple ϕ-substitution of a term in context along a monomorphism $\phi\colon (n] \to (k]$.

An equational theory $T = (L, A)$ is a *regular* (*linear-regular*) *theory* iff every equation $s = t : \vec{x}^n$ that is a consequence of the theory T is a consequence of the set of regular (linear-regular) consequences of T. An interpretation is a *regular (linear-regular) interpretation* iff it interprets function symbols as regular (linear-regular) terms.

A theory $T = (L, A)$ is a *rigid theory* iff it is linear-regular and for any linear-regular term in context $t : \vec{x}^n$ whenever $A \vdash t = \tau \cdot t : \vec{x}^n$ then τ is the identity permutation. $\tau \cdot t$ is the simple τ-substitution of a term in context $t : \vec{x}^n$ along a permutation $\tau\colon (n] \to (n] \in S_n$.

Note that it is not assumed that the axioms of linear-regular theories are linear-regular. This is to keep the notion invariant under isomorphism of theories. In particular, if $T = (L, A)$ is a linear-regular theory and A' is the set of all equational consequences of the axioms from A in the language L then the $T' = (L, A')$ is also linear-regular. Of course T' is isomorphic to T. On the other hand, if the theory has linear-regular axioms then it is linear-regular. Thus if we find a linear-regular set of axioms of an equational theory T we can be sure that T is linear-regular. However, it is not so easy to decide whether a theory is rigid. In fact, even if we have a finite linear-regular presentation of a theory it is still undecidable whether this theory is rigid or not (cf. [BSZ]).

Remark. We could also consider here strongly regular theories (c.f. [CJ95]) that correspond to monotone monads (cf. [Z10]). They are more specific than rigid theories. However this part of correspondence is of a bit different kind. The monotone monads are not just monads with certain additional properties but also with certain additional structure. The forgetful functor from monotone monads to monads (on **Set**) is not full on isomorphisms. These theories and some other theories of this kind will be treated elsewhere. We give the definition just to show the difference between the notions in the examples below. A term in context $t : \vec{x}^n$ is a *strongly regular* iff it is linear-regular and the variables in the term t occur in the same order as in the sequence $\vec{x}^n$. An equation is $s = t : \vec{x}^n$ is a *strongly regular equation* iff both terms $s : \vec{x}^n$ and $t : \vec{x}^n$ are strongly regular. An equational theory $T = (L, A)$ is a *strongly regular theory* iff every equation $s = t : \vec{x}^n$ that is a consequence of the theory T is a consequence of the set of strongly regular consequences of T. An interpretation is a *strongly regular interpretation* iff it interprets n-ary function symbols as strongly regular terms. One can easily see that any strongly regular theory is rigid but the 'embedding' functor **SregET** $\longrightarrow$ **RiET** is not even full on isomorphisms, where **SregET** denotes the category of strongly regular theories and strongly regular interpretations. The examples below show that there are rigid theories that are not strongly regular.

We denote by **LrET** the subcategory of **ET** consisting of linear-regular theories and linear-regular interpretations. **RiET** denotes the full subcategory of **LrET** whose objects are rigid theories. **RegET** is a category of regular theories and regular interpretations. We have three inclusion functors

$$\mathbf{RiET} \longrightarrow \mathbf{LrET} \longrightarrow \mathbf{RegET} \longrightarrow \mathbf{ET}$$

with the first inclusion being full and the other two being full on isomorphisms (cf. [Z10]).

Examples.

1. The theory of monoids has two operations m and e, of arity 2 and 0, respectively and equations

$$m(x_1, m(x_2, x_3)) = m(m(x_1, x_2), x_3), \quad m(x_1, e) = x_1 = m(e, x_1)$$

By the form of these equations, this theory is strongly regular and hence it is rigid as well.

2. The theory of monoids with anti-involution has an additional unary operation s and additional two axioms

$$m(s(x_1), s(x_2)) = s(m(x_2, x_1)), \quad s(s(x_1)) = x_1$$

This theory is not strongly regular but it can be shown that it is rigid.

3. The theory of commutative monoids is the theory of monoids with an additional axiom
$$m(x_1, x_2) = m(x_2, x_1)$$
Thus is it linear-regular by the form of the axioms but it is obviously not rigid.

4. The theory of sup-lattices has two operations $\vee$ and $\perp$, of arity 2 and 0, respectively and equations
$$x_1 \vee (x_2 \vee x_3) = (x_1 \vee x_2) \vee x_3, \quad x_1 \vee e = x_1 = e \vee x_1$$
$$x_1 \vee x_1 = x_1, \quad x_1 \vee x_2 = x_2 \vee x_1$$
This theory is regular but not linear.

5. The theory of groups is not regular.

7.1.2 Lawvere Theories

By a *Lawvere theory*, (cf. [Lw04], [KR77]), we mean a category whose objects are natural numbers $\mathbb{N}$, so that n is a product 1^n with chosen projections $\pi_i^n\colon n \to 1$, for $n \in \mathbb{N}$ and $i \in (n]$. An interpretation (or a morphism) of Lawvere theories is a functor constant on objects, preserving the chosen projections. Lawvere theories and their morphisms form a category that is denoted by $\mathbf{LT}$.

The initial object in the category $\mathbf{LT}$ is the category $\mathbb{F}^{op}$ with the obvious inclusions as projections, see introduction. The unique morphism from $\mathbb{F}^{op}$ into any Lawvere theory $\mathbf{T}$ will be denoted by $\pi\colon \mathbb{F}^{op} \longrightarrow \mathbf{T}$. Thus for $\phi\colon (n] \to (m]$ in $\mathbb{F}$ we have $\pi_\phi = \langle \pi_{\phi(i)} \rangle\colon m \to n$ in $\mathbf{T}$.

Every equational theory has a model, and hence there is no inconsistent equational theory. But there are two equational theories that are nearly so. The terminal Lawvere theory $\mathbb{1}$ has exactly one morphism between any two objects. It has unique (up to isomorphism) one-element model. $\mathbb{1}$ is not a regular theory. It also has some equivalent internal characterizations as the Lawvere theory (unique up to an isomorphism) which is a groupoid or where $0 \cong 1$. The functor $\pi\colon \mathbb{F}^{op} \longrightarrow \mathbb{1}$ is not faithful. There is yet another Lawvere theory with this property. It is a subtheory of $\mathbb{1}$ in which there is no morphism $0 \to 1$. These categories are the only two Lawvere theories in which 2 is the initial object.

The class of *structural morphisms* in $\mathbf{T}$ is the closure under isomorphism of the image under π of all morphisms in $\mathbb{F}$. A morphism in $\mathbf{T}$ is *analytic* iff it is right orthogonal to all structural morphisms.

By a factorization system in a category $\mathcal{C}$ we mean the factorization system in the sense of [FK72], see [CJKP97] sec 2.8, i.e. it consists of two classes of morphisms in $\mathcal{C}$ closed under isomorphisms, say $\mathcal{E}$ and $\mathcal{M}$, such that morphisms in $\mathcal{E}$ are left orthogonal to those in $\mathcal{M}$, and each morphism f in $\mathcal{C}$ factors as $f = m \circ e$ where $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

$Aut(n)$ is the set of automorphisms of n in $\mathbf{T}$. As in any Lawvere theory $\mathbf{T}$, for $n \in \mathbb{N}$, n is canonically isomorphic to 1^n we always have a function

$$\rho_n \colon S_n \times Aut(1)^n \longrightarrow Aut(n)$$

such that

$$(\sigma, a_1, \ldots, a_n) \mapsto a_1 \times \ldots \times a_n \circ \pi_\sigma$$

i.e. ρ_n sends a permutation σ and n isomorphisms of 1 to an isomorphism of n in $\mathbf{T}$. We say that $\mathbf{T}$ has *simple automorphisms* iff ρ_n is a bijection, for $n \in \mathbb{N}$. Clearly, if $\mathbf{T}$ has simple automorphisms then 2 is not initial in $\mathbf{T}$.

A Lawvere theory $\mathbf{T}$ is *analytic* iff structural morphisms and analytic morphisms form a factorization system in $\mathbf{T}$ and $\mathbf{T}$ has simple automorphisms. A Lawvere theory $\mathbf{T}$ is *rigid* iff it is analytic and the symmetric groups S_n acting on $\mathbf{T}(n, 1)$ by permuting factors act freely on analytic morphisms, for $n \in \mathbb{N}$.

An *analytic interpretation* of Lawvere theories is an interpretation of Lawvere theories that preserves analytic morphisms. Thus we have a non-full subcategory of analytic Lawvere theories and analytic interpretations $\mathbf{AnLT}$. The latter has as a full subcategory the category $\mathbf{RiLT}$ of rigid Lawvere theories. We have inclusion functors

$$\mathbf{RiLT} \longrightarrow \mathbf{AnLT} \longrightarrow \mathbf{LT}$$

with the first one being a full inclusion.

We have an easy

Lemma 7.1.1. *In any analytic Lawvere theory* $\mathbf{T}$ *any morphism* $f \colon n \to m$ *has a factorization*

$$\begin{tikzcd} n \arrow[rr, "f"] \arrow[dr, "\pi_\phi"'] & & m \\ & k \arrow[ur, "a"'] & \end{tikzcd}$$

with a *being an analytic morphism in* $\mathbf{T}$ *and* $\phi \colon (k] \to (n]$ *a function. Such a factorization is unique up to a permutation, that is if* $f = a' \circ \pi_{\phi'}$ *is another such factorization there is* $\sigma \in S_k$ *such that*

$$\phi \circ \sigma = \phi', \quad a = a' \circ \pi_\sigma.$$

Proof. When $\mathbf{T}$ has simple automorphisms any structural morphism $s \colon n \to m$ in $\mathbf{T}$ can be presented as $(a_1 \times \ldots, a_m) \circ \pi_\phi$ for some function $\phi \colon (m] \to (n]$ and $a_i \in Aut(1)$ for $i \in (m]$. Thus if $f = a \circ s$ is a structural-analytic factorization of f, then $f = (a \circ (a_1 \times \ldots, a_m)) \circ \pi_\phi$ is also one. $\square$

7.1.3 Monads

We shall consider three categories of finitary monads on **Set**. The category of all finitary monads with usual morphisms of monads will be denoted by **Mnd**. A morphism of monads $\tau\colon (M,\eta,\mu) \to (M',\eta',\mu')$ is a natural transformation $\tau\colon M \to M'$ such that $\tau \circ \eta^M = \eta^{M'}$ and $\tau \circ \mu^M = \mu^{M'} \circ \tau_{M'} \circ M(\tau)$.

Recall that a finitary monad (M,η,μ) on **Set** is *analytic* iff M weakly preserves wide pullbacks and both η and μ are weakly cartesian natural transformations. A *morphism of analytic monads* on **Set** $\tau\colon (M,\eta,\mu) \to (M',\eta',\mu')$ is a weakly cartesian natural transformation τ that is a morphism of monads, (cf. [Jo86], [Z10]). Recall that a finitary monad (M,η,μ) is a *polynomial monad* on **Set** iff M preserves wide pullbacks and both η and μ are cartesian natural transformations. Both types of functors and monads have a much more explicit description (cf. [Jo86], [Z10]).

The categories of analytic and polynomial monads with the suitable morphisms will be denoted by **AnMnd** and **PolyMnd**, respectively. We have two inclusion functors

$$\mathbf{PolyMnd} \longrightarrow \mathbf{AnMnd} \longrightarrow \mathbf{Mnd}$$

the first one being full (cf. [Z10]), the second full on isomorphisms.

7.1.4 Operads

The symmetric operads provide yet another way of presenting models of equational theories. This kind of presentation is usually very convenient, but the models defined by such operads are more specific. The precise characterization of this more specific situation is the main objective of this chapter.

Recall that a *symmetric operad* $\mathcal{O}$ consists a family of sets $\mathcal{O}_n$, for $n \in \mathbb{N}$, a unit element $\iota \in \mathcal{O}_1$, for any $k, n, n_1, \ldots, n_k \in \mathbb{N}$ with $n = \sum_{i=1}^k n_i$, a composition operation

$$*\colon \mathcal{O}_{n_1} \times \ldots \times \mathcal{O}_{n_k} \times \mathcal{O}_k \longrightarrow \mathcal{O}_n$$

a left action of the symmetric groups

$$\cdot\colon S_n \times \mathcal{O}_n \longrightarrow \mathcal{O}_n$$

for $n \in \mathbb{N}$, such that the composition is associative with unit ι and compatible with group actions. A *morphism of symmetric operads* $f\colon \mathcal{O} \to \mathcal{O}'$ is a function that respects arities of operations, unit, compositions, and group actions. For more on symmetric operads and their history one can consult for example [Le04], but there the symmetric groups act on the right.

Recall that the symmetric operad of symmetries **S** is defined as follows. The set of n-ary operations of **S** is the symmetric group S_n on which S_n act on the left by multiplication. The composition

$$\star\colon S_{n_1} \times \ldots \times S_{n_k} \times S_k \longrightarrow S_n$$

for $(\sigma_1, \ldots, \sigma_k; \tau) \in S_{n_1} \times \ldots \times S_{n_k} \times S_k$ the permutation

$$\langle \sigma_1, \ldots, \sigma_k \rangle \star \tau\colon n = \sum_{i=1}^{k} n_{\tau(i)} \longrightarrow n = \sum_{i=1}^{k} n_i$$

is given by

$$\langle i, r \rangle \mapsto \langle \tau(i), \sigma_{\tau(i)}(r) \rangle$$

where we consider the obvious lexicographic order on both $\sum_{i=1}^{k} n_{\tau(i)}$ and $\sum_{i=1}^{k} n_{\tau(i)}$. Note that even if composition is a function between groups it is not a homomorphism of groups in general.

The category of rigid operads[5] **RiOp** can be identified with the full subcategory of symmetric operads whose objects are those operads that have all the actions of symmetric groups free. We recall their definition below; see [HMP02], [Z10] for more.

A *rigid operad* $\mathcal{O}$ consists of a family of sets $\mathcal{O}_n$, for $n \in \mathbb{N}$, a unit element $\iota \in \mathcal{O}_1$,for any $k, n, n_1, \ldots, n_k \in \mathbb{N}$ with $n = \sum_{i=1}^{k} n_i$, a composition operation

$$\langle *, \alpha \rangle\colon \mathcal{O}_{n_1} \times \ldots \times \mathcal{O}_{n_k} \times \mathcal{O}_k \longrightarrow \mathcal{O}_n \times S_n$$

such that the composition is associative with unit ι. The second part of the operation of composition is called the *amalgamation*. We spell out the definition in detail. Let $a \in \mathcal{O}_n$, $b_i \in \mathcal{O}_{k_i}$ for $i \in (n]$, $k = \sum_{i=1}^{n} k_i$, $c_j \in \mathcal{O}_{m_j}$ for $j \in (k]$, $m = \sum_{j=1}^{k} m_j$. The fact that ι is the unit for the composition means that

$$(\iota, \ldots, \iota) * a = a = a * \iota$$

and that the amalgamations in these compositions are identity permutations $1_{(n]}$. To explain associativity we need to name amalgamations for various compositions. We write

$$\tau = \alpha(b_1, \ldots, b_n; a) \ \in \ S_k$$

$$\rho = \alpha(c_{\tau(1)}, \ldots, c_{\tau(k)}; (\langle b_1, \ldots, b_n \rangle * a)) \ \in \ S_m$$

$$\sigma_i = \alpha(c_{\bar{k}_i+1}, \ldots, c_{\bar{k}_{i+1}}; b_i) \ \in \ S_{\check{m}_i}$$

$$\xi = \alpha(\langle c_1, \ldots, c_{\bar{k}_1} \rangle * b_1, \ldots, \langle c_{\bar{k}_{n-1}+1}, \ldots, c_{\bar{k}_n} \rangle * b_n; a) \ \in \ S_m$$

[5]The (colored) rigid operads were called "multicategories with non-standard amalgamation" in the previous chapters. This name change is motivated by the characterization we are going to prove in Theorem 7.5.4.

for $i \in (n]$. Where $\bar{k}_0 = 0$, $\bar{k}_i = \sum_{r=1}^{i} k_r$, for $i \in (k]$ and $\check{m}_i = \sum_{j=\bar{k}_i+1}^{\bar{k}_{i+1}} m_j$. We have $m = \sum_{i=1}^{n} \check{m}_i$. Then the associativity reads

$$\langle\langle c_1, \ldots, c_{\bar{k}_1}\rangle * b_1, \ldots, \langle c_{\bar{k}_{n-1}+1}, \ldots, c_{\bar{k}_n}\rangle * b_n\rangle * a = \langle\langle c_{\tau(1)}, \ldots, c_{\tau(k)}\rangle * (\langle b_1, \ldots, b_n\rangle * a)$$

and

$$(\sigma_1 + \ldots + \sigma_n) \circ \xi = \rho.$$

A *morphism of rigid operads* $(h, \sigma)\colon \mathcal{O} \to \mathcal{O}'$ is a family of functions $\langle f, \sigma\rangle\colon \mathcal{O}_n \to \mathcal{O}'_n \times S_n$ for $n \in \mathbb{N}$ that respects the unit and compositions. In detail, for the unit $\iota \in \mathcal{O}_1$ we have

$$h(\iota) = \iota, \qquad \sigma_\iota = id_1$$

and for $b_i \in \mathcal{O}_{n_i}$, $a \in \mathcal{O}_k$ we have

$$h(\langle b_1, \ldots, b_k\rangle * a) = \langle h(b_{\sigma_a(1)}), \ldots, h(b_{\sigma_a(k)})\rangle * h(a)$$

and

$$\alpha_{(b_1,\ldots,b_k;a)} \circ \sigma_{(\langle b_1,\ldots,b_k\rangle \star a)} = (\langle \sigma_{b_1}, \ldots, \sigma_{b_k}\rangle \star \sigma_a) \circ \alpha_{(h(b_{\sigma_a(1)}),\ldots,h(b_{\sigma_a(k)});h(a))}$$

For more see [Z10].

We denote by **SOp**, **RiOp** the categories of symmetric and rigid operads, respectively. We have the symmetrization functor

$$\mathcal{P}\colon \mathbf{RiOp} \longrightarrow \mathbf{SOp}$$

such that the set on n-ary operations of the symmetric operad $\mathcal{P}(\mathcal{O})$ associated to the rigid operad $\mathcal{O}$ is $\mathcal{O}_n \times S_n$. $\mathcal{P}$ is full and faithful.

7.2 The Equivalence of The Three Approaches

We shall recall the functors that exhibit equivalences of the following three categories **ET**, **LT** and **Mnd**:

$$\mathbf{ET} \underset{\mathcal{E}_l}{\overset{\mathcal{L}_e}{\rightleftarrows}} \mathbf{LT} \underset{\mathcal{L}_m}{\overset{\mathcal{M}_l}{\rightleftarrows}} \mathbf{Mnd}$$

As we described it in the introduction, the names of the functors are so chosen to remember their codomains with indices remembering their domains. We often drop the indices when it does not lead to a confusion.

Each of the above categories comes equipped with a semantic functor associating to objects of those categories their categories of models. As all monads in **Mnd** are defined on **Set** only, we consider the models only in **Set**. It is well known that the equivalences that we describe below respect those semantic functors.

7.2.1 The Functor $\mathcal{L}_e = \mathcal{L}\colon \mathbf{ET} \longrightarrow \mathbf{LT}$

Let $T = (L, A)$ be an equational theory. A morphism $n \to m$ in $\mathcal{L}(T)$ is an m-tuple $\langle [t_1 : \vec{x}^n], \ldots, [t_m : \vec{x}^n] \rangle \colon n \to m$ where $[t_i : \vec{x}^n]$ is an equivalence class of terms is context $\vec{x}^n$ modulo provable equivalence from axioms in A. The identity on n is $\langle [x_1 : \vec{x}^n], \ldots, [x_n : \vec{x}^n] \rangle \colon n \to n$. The composition is given by simultaneous substitution as follows

$$\begin{tikzcd} n \arrow[r, "\langle [t_i : \vec{x}^n] \rangle_{i \in (m]}"] \arrow[rr, bend right=10, "\langle [s_j(\langle x_i \backslash t_i \rangle) : \vec{x}^m] \rangle_{j \in (k]}"'] & m \arrow[r, "\langle [s_j : \vec{x}^m] \rangle_{j \in (k]}"] & k \end{tikzcd}$$

The i-th projection on 1 is $\pi_i^n = \langle [x_i : \vec{x}^n] \rangle$.

Let $I \colon T \to T'$ be an interpretation. The functor $\mathcal{L}(I)$ is defined on a morphism

$$\langle [t_1 : \vec{x}^n], \ldots, [t_m : \vec{x}^n] \rangle \colon n \to m$$

in $\mathcal{L}(T)$ as

$$\langle [\bar{I}(t_1) : \vec{x}^n], \ldots, [\bar{I}(t_m) : \vec{x}^n] \rangle \colon n \longrightarrow m$$

A routine verification shows that $\mathcal{L}$ is indeed a functor into $\mathbf{LT}$.

7.2.2 The Functor $\mathcal{E}_l = \mathcal{E}\colon \mathbf{LT} \longrightarrow \mathbf{ET}$

Let $\mathbf{T}$ be a Lawvere theory. Then $\mathbf{T}(n, 1)$ is the set of n-ary operations of the theory $\mathcal{E}(\mathbf{T})$, for $n \in \mathbb{N}$. The set of axioms $\mathcal{E}(\mathbf{T})$ contains a linear-regular axiom

$$g(x_1, \ldots, x_n) = f(f_1(x_1, \ldots, x_{n_1}), \ldots, f_k(x_{1+\sum_i^{k-1} n_i}, \ldots, x_n)) : \vec{x}^n$$

for any morphisms f, f_i, g in $\mathbf{T}$ such that $f \circ (f_1 \times \ldots \times f_k) = g$ holds in $\mathbf{T}$, and a linear axiom

$$x_i = \pi_i^n(x_1, \ldots, x_n) : \vec{x}^n$$

for any $n \in \mathbb{N}$ and $i \in (n]$. An interpretation of Lawvere theories $F \colon \mathbf{T} \to \mathbf{T}'$ induces an interpretation of equational theories $\mathcal{E}(F)$ such that

$$\mathcal{E}(F)(f) = F(f)(x_1, \ldots, x_n) : \vec{x}^n$$

for $f \colon n \to 1$ in $\mathbf{T}$.

7.2.3 The Functor $\mathcal{L}_m = \mathcal{L}\colon \mathbf{Mnd} \longrightarrow \mathbf{LT}$

For a monad $M = (M, \eta, \mu)$, the category $\mathcal{L}(M)$ is the dual of the full subcategory of the Kleisli category for M, spanned by the natural numbers. In detail, for a monad M we define the hom's in the category $\mathcal{L}(M)$ as

$$\mathcal{L}(M)(n, m) = \mathbf{Set}((m], M((n]))$$

for $n, m \in \mathbb{N}$. The compositions and identities are like in Kleisli category. The projection

$$\pi_i^n\colon (1] \longrightarrow M((n])$$

sends 1 to $\eta_{(n]}(i)$, for $n \in \mathbb{N}$ and $i \in (n]$.

For a morphism of monads $\tau\colon (M, \eta, \mu) \longrightarrow (M', \eta', \mu')$ and a morphism $f\colon n \to m$ in $\mathcal{L}(M)$ we put

$$\mathcal{L}(\tau)(f)(i) = \tau_{(n]}(f(i))$$

for $i \in (m]$.

7.2.4 The Functor $\mathcal{M}_l = \mathcal{M}\colon \mathbf{LT} \longrightarrow \mathbf{Mnd}$

For a Lawvere theory $\mathbf{T}$, we define the monad $\mathcal{M}(\mathbf{T})$ using coends. We put

$$\mathcal{M}(\mathbf{T})(X) = \int^{n \in \mathbb{F}} X^n \times \mathbf{T}(n, 1)$$

for $X \in \mathbf{Set}$. The unit of $\mathcal{M}(\mathbf{T})$ is

$$\eta_X^{\mathbf{T}}\colon X \to \mathcal{M}(\mathbf{T})(X)$$

sends $x \in X$ to the class of the element $\langle id_1, \bar{x} \rangle$ where id_1 is the identity on 1 in $\mathbf{T}$ and $\bar{x}\colon (1] \to X$ is the function picking x, i.e. $\bar{x}(1) = x$. The iterated functor $\mathcal{M}^2(\mathbf{T})$ is given, for X in $\mathbf{Set}$ by

$$\mathcal{M}^2(\mathbf{T})(X) = \int^{m, n_1, \ldots, n_m \in \mathbb{F}} X^n \times \mathbf{T}(n_1, 1) \times \ldots \times \mathbf{T}(n_m, 1) \times \mathbf{T}(m, 1)$$

where $n = \sum_{i=1}^m n_i$. The multiplication of the monad $\mathcal{M}(\mathbf{T})$

$$\mu_X^{\mathbf{T}}\colon \mathcal{M}^2(\mathbf{T})(X) \longrightarrow \mathcal{M}(\mathbf{T})(X)$$

is defined on components

$$X^n \times \mathbf{T}(n_1, 1) \times \ldots \mathbf{T}(n_m, 1) \times \mathbf{T}(m, 1) \longrightarrow X^n \times \mathbf{T}(n, 1)$$

by composition, i.e. for $f\colon m \to 1, f_1\colon n_1 \to 1, \ldots, f_m\colon n_m \to 1$ in $\mathbf{T}$ and $\vec{x}\colon (n] \to X$

$$\mu_X^{\mathbf{T}}(\vec{x}, f_1, \ldots, f_m, f) = \langle \vec{x}, f \circ (f_1 \times \ldots \times f_m) \rangle$$

where again $n = \sum_{i=1}^m n_i$.

7.3 Lawvere Theories vs Operads

In this section we study the relations between Lawvere theories and operads, both symmetric and rigid. We shall describe the adjunction $\mathcal{P}_a \dashv \mathcal{Q}_f$ and the properties of the embeddings $\mathcal{P}_a$ and $\mathcal{P}_p$.

$$\begin{tikzcd} \mathbf{RiOp} \arrow[r, "\mathcal{P}"] \arrow[rr, bend left, "\mathcal{P}_p"] & \mathbf{SOp} \arrow[r, shift left, "\mathcal{P}_a"] & \mathbf{LT} \arrow[l, shift left, "\mathcal{Q}_f"] \end{tikzcd}$$

7.3.1 The Functor $\mathcal{P}_a \colon \mathbf{SOp} \to \mathbf{LT}$

Let $\mathcal{O}$ be a symmetric operad: ι, $\cdot$, $*$ denote the unit, symmetric groups actions, and compositions in $\mathcal{O}$, respectively. We define a Lawvere theory $\mathcal{P}_a(\mathcal{O})$ as follows. The set of objects of $\mathcal{P}_a(\mathcal{O})$ is the set of natural numbers $\mathbb{N}$. A morphism from n to m is an equivalence class of spans

$$\begin{tikzcd} & r \arrow[dl, "\phi"'] \arrow[dr, "\langle f, g_i \rangle_{i \in m}"] & \\ n & & m \end{tikzcd}$$

such that $\phi\colon (r] \to (n]$ is a function, $f\colon (r] \to (m]$ is a monotone function, $r_i = |f^{-1}(i)|$ and we have $g_i \in \mathcal{O}_{r_i}$ for $i \in (m]$ and $r = \sum_{i=1}^{m} r_i$. Two spans $\langle \phi, f, g_i \rangle$ and $\langle \phi', f', g'_j \rangle$ are equivalent iff $f = f'$, and there are permutations $\sigma_i \in S_{r_i}$ for $i \in (m]$

$$\begin{tikzcd} & r \arrow[dl, "\phi"'] \arrow[dr, "\langle f, g_i \rangle_{i \in m}"] & \\ n & & m \\ & r' \arrow[ul, "\phi'"] \arrow[ur, "\langle f', g'_i \rangle_{i \in m}"'] \arrow[uu, "\Sigma_i \sigma_i"'] & \end{tikzcd}$$

such that

$$g_i = \sigma_i \cdot g'_i, \quad \phi \circ \sum_i \sigma_i = \phi'$$

By $\sum_i \sigma_i \colon r \to r$ we mean the permutation that is formed by placing permutations σ_i ‘one after another’. Thus, it respects the fibers of f, i.e. $f \circ \sum_i \sigma_i = f$. Clearly, we shall deal with the spans when we perform constructions on morphisms in $\mathcal{P}_a(\mathcal{O})$, but when we consider equalities between spans we shall invoke the above equivalence relation.

The composition $\langle \phi'', f'', g''_j \rangle \colon n \to k$ of two morphism $\langle \phi, f, g_i \rangle \colon n \to m$ and $\langle \phi', f', g'_j \rangle \colon m \to k$ is defined as follows. In the diagram

$$
\begin{tikzcd}
 & & r'' \arrow[dl, "\bar{\phi}"'] \arrow[dr, "\bar{f}"] \arrow[dll, bend right, "\phi''"'] \arrow[drr, bend left, "{\langle f'', g''_j \rangle}"] & & \\
 & r \arrow[dl, "\phi"'] \arrow[dr, "{\langle f, g_i \rangle}"'] & & r' \arrow[dl, "\phi'"'] \arrow[dr, "{\langle f', g'_j \rangle}"'] & \\
n & & m & & k
\end{tikzcd}
\tag{7.1}
$$

the square is a pullback of f along ϕ'. The function $\bar{f}$ is chosen so that it is monotone. We put $f'' = f' \circ \bar{f}$, $\phi'' = \phi \circ \bar{\phi}$, and $g''_j = g'_j * \langle g_{\phi(l)} \rangle_{l \in f^{-1}(j)}$.

The identity on n is the span

$$
\begin{tikzcd}
 & n \arrow[dl, "id_n"'] \arrow[dr, "\langle id_n \iota \rangle"] & \\
n & & n
\end{tikzcd}
$$

As S_1 contains the identity permutation only, any span equivalent to an identity span is actually equal to it.

The projection $\pi^n_i \colon n \to 1$ on i-th coordinate is the span

$$
\begin{tikzcd}
 & 1 \arrow[dl, "\bar{i}"'] \arrow[dr, "{\langle id, \iota \rangle}"] & \\
n & & 1
\end{tikzcd}
$$

where $i \in (n]$ and $\bar{i}(1) = i$.

For a morphism of symmetric operads $h \colon \mathcal{O} \to \mathcal{O}'$ we define a functor

$$\mathcal{P}_a(h) \colon \mathcal{P}_a(\mathcal{O}) \longrightarrow \mathcal{P}_a(\mathcal{O}')$$

so that for a morphism $\langle \phi, f, g_i \rangle \colon n \to m$ in $\mathcal{P}_a(\mathcal{O})$ we define a morphism

$$\mathcal{P}_a(h)(\langle \phi, f, g_i \rangle) = \langle \phi, f, h(g_i) \rangle \colon n \to m$$

in $\mathcal{P}_a(\mathcal{O}')$.

This ends the definition of the functor $\mathcal{P}_a$.

7.3.2 The Functor $\mathcal{Q}_f \colon \mathbf{LT} \longrightarrow \mathbf{SOp}$

Let $\mathbf{T}$ be a Lawvere theory. The operad $\mathcal{Q}_f(\mathbf{T})$ consists of operations of $\mathbf{T}$, i.e. morphisms to 1. In detail it can be described as follows. The set of n-ary operations $\mathcal{Q}_f(\mathbf{T})_n$ is the set of n-ary operations $\mathbf{T}(n,1)$ of $\mathbf{T}$, for $n \in \mathbb{N}$. The action

$$\cdot \colon S_n \times \mathcal{Q}_f(\mathbf{T})_n \longrightarrow \mathcal{Q}_f(\mathbf{T})_n$$

is given, for $f \in \mathbf{T}(n,1)$ and $\sigma \in S_n$, by

$$\sigma \cdot f = f \circ \pi_\sigma$$

The identity of $\mathcal{Q}_f(\mathbf{T})$ is $\iota = id_1 \in \mathbf{T}(1,1)$. The composition

$$* \colon \mathcal{Q}_f(\mathbf{T})_{n_1} \times \ldots \times \mathcal{Q}_f(\mathbf{T})_{n_k} \times \mathcal{Q}_f(\mathbf{T})_k \longrightarrow \mathcal{Q}_f(\mathbf{T})_n$$

is given, for $f \in \mathcal{Q}_f(\mathbf{T})_k$ and $f_i \in \mathcal{Q}_f(\mathbf{T})_{n_i}$, where $i \in (k]$, $n = \sum_{i \in k} n_i$, by

$$\langle f_1, \ldots, f_k \rangle * f = f \circ (f_1 \times \ldots, \times f_k)$$

where $f_1 \times \ldots, \times f_k$ is defined using the chosen projections in $\mathbf{T}$ and $\circ$ is the composition in $\mathbf{T}$.

If $F \colon \mathbf{T} \to \mathbf{T}'$ is a morphism of Lawvere theories then the map of symmetric operads

$$\mathcal{Q}_f(F) \colon \mathcal{Q}_f(\mathbf{T}) \to \mathcal{Q}_f(\mathbf{T}')$$

is defined, for $f \in \mathcal{Q}_f(\mathbf{T})_n$, by

$$\mathcal{Q}_f(F)(f) = F(f)$$

This ends the definition of the functor $\mathcal{Q}_f$.

7.3.3 The Adjunction $\mathcal{P}_a \dashv \mathcal{Q}_f$ and the Properties of the Functor $\mathcal{P}_a$

We note for the record

Proposition 7.3.1. *The functors $\mathcal{P}_a\colon \mathbf{SOp} \longrightarrow \mathbf{LT}$ and $\mathcal{Q}_f\colon \mathbf{LT} \to \mathbf{SOp}$ are well defined.*

We have an easy

Lemma 7.3.2. *Let $\mathcal{O}$ be a symmetric operad and $n \in \mathbb{N}$. An automorphism on n in $P_a(\mathcal{O})$ is represented by a span of the following form*

$$\begin{tikzcd} & n \arrow[dl, "\phi"'] \arrow[dr, "\langle id_n, a_i\rangle"] & \\ n & & n \end{tikzcd}$$

where $\phi\colon (n] \to (n]$ is a bijection, $a_i \in \mathcal{O}_1$ is an invertible operation, i.e. there is $b_i \in \mathcal{O}_1$ such that $a_i * b_i = \iota = b_i * a_i$ for $i \in (n]$. It is the unique span in its equivalence class.

Proof. Consider a pair of morphisms in $\mathcal{P}_a(\mathcal{O})$

$$\begin{tikzcd} & r \arrow[dl, "\phi"'] \arrow[dr, "\langle f, g_j\rangle"] & & r \arrow[dl, "\phi'"] \arrow[dr, "\langle f', h_i\rangle"] & \\ n & & n & & n \end{tikzcd}$$

that are inverse one to the other. As the above composition is an identity it follows that ϕ and f' are epi. Thus, because of the other composition ϕ' and f are surjections, as well. As pulling back along a surjection reflects injections, all functions ϕ, f, ϕ' and f' must be also injective and hence bijective. Then it is easy to see that $g_{\phi'(j)}$ is an inverse of h_j for $j \in (n]$. $\square$

Proposition 7.3.3. *We have an adjunction $\mathcal{P}_a \dashv \mathcal{Q}_f$. The functor $\mathcal{P}_a$ is faithful.*

Proof. We first show that $\mathcal{P}_a \dashv \mathcal{Q}_f$. For a symmetric operad $\mathcal{O}$ the unit is

$$\eta_{\mathcal{O}}\colon \mathcal{O} \longrightarrow \mathcal{Q}_f(\mathcal{P}_a(\mathcal{O}))$$

$$\mathcal{O}_n \ni g \mapsto \langle id_n, !, g\rangle$$

For Lawvere theory $\mathbf{T}$ the counit is

$$\varepsilon_{\mathbf{T}}\colon \mathcal{P}_a\mathcal{Q}_f(\mathbf{T}) \longrightarrow \mathbf{T}$$

$$\langle \phi, f, g_i\rangle \mapsto (g_1 \times \ldots \times g_m) \circ \pi_\phi$$

We verify the triangular equalities. For $g \in \mathcal{Q}_f(\mathbf{T})_n = \mathbf{T}(n,1)$ we have

$$\mathcal{Q}_f(\varepsilon_{\mathbf{T}}) \circ \eta_{\mathcal{Q}_f(\mathbf{T})}(g) =$$

$$= \mathcal{Q}_f(\varepsilon_{\mathbf{T}})(\langle id_n, !, g\rangle) =$$

$$= g \circ \pi_{id_n} = g$$

For $\langle \phi, f, g_i\rangle \in \mathcal{P}_a(\mathcal{O})$ we have

$$\varepsilon_{\mathcal{P}_a(\mathcal{O})} \circ \mathcal{P}_a(\eta_{\mathcal{O}})(\langle \phi, f, g_i\rangle) =$$

$$= \varepsilon_{\mathcal{P}_a(\mathcal{O})}(\langle \phi, f, \langle id_{r_i}, !, g_i\rangle\rangle) =$$

$$= (\langle id_{r_1}, !, g_1\rangle \times \ldots \times \langle id_{r_m}, !, g_m\rangle) \circ \pi_\phi =$$

$$= \langle \phi, f, g_i\rangle$$

As the unit $\eta_{\mathcal{O}}$ is mono, $\mathcal{P}_a$ is faithful. $\square$

Proposition 7.3.4. *The functor $\mathcal{P}_a$ is faithful, full on isomorphisms and its essential image is the category of analytic Lawvere theories* **AnLT** *i.e. it factorizes as an equivalence of categories $\mathcal{L}_o$ followed by $\mathcal{P}_a^l$*

$$\begin{tikzcd}
\mathbf{LT} & \\
\mathbf{AnLT} \arrow[u, "\mathcal{P}_a^l"] & \mathbf{SOp} \arrow[l, "\mathcal{L}_o"] \arrow[ul, "\mathcal{P}_a"']
\end{tikzcd}$$

Proof. Recall that we have a unique morphism of Lawvere theories from the initial theory $\pi\colon \mathbb{F}^{op} \to \mathcal{P}_a(\mathcal{O})$. For a function $\phi\colon (m] \to (n]$, π_ϕ the morphism π_ϕ is represented by the span of the form

$$\begin{tikzcd}
& m \arrow[dl, "\phi"'] \arrow[dr, "\langle id_m, \iota\rangle"] & \\
n & & m
\end{tikzcd}$$

The class of the structural morphisms in $\mathcal{P}_a(\mathcal{O})$ is the closure under isomorphism of the class of morphisms $\{\pi_\phi : \phi \in \mathbb{F}\}$. It is easy to see that the structural morphisms in $\mathcal{P}_a(\mathcal{O})$ are (represented by) the spans of the form

$$\begin{tikzcd}
& m \arrow[dl, "\phi"'] \arrow[dr, "{\langle id_n, a_i \rangle}"] & \\
n & & m
\end{tikzcd}$$

where ϕ is any function and a_i is an invertible unary operation, for $i \in (m]$. Thus by Lemma 7.3.2, a morphism is an isomorphism in $\mathcal{P}_a(\mathcal{O})$ iff it is represented by a span as above with ϕ being a bijection.

The analytic morphisms in $\mathcal{P}_a(\mathcal{O})$ are (represented by) the spans of the form

$$\begin{tikzcd}
& n \arrow[dl, "\phi"'] \arrow[dr, "{\langle f, g_i \rangle}"] & \\
n & & m
\end{tikzcd}$$

where ϕ is a bijection.

Clearly, both classes contain isomorphisms and are closed under composition.

Any morphism $\langle \phi, f, g_i \rangle \colon n \to m$ in $\mathcal{P}_a(\mathcal{O})$ has a structural-analytic factorization as follows

$$\begin{tikzcd}
& r \arrow[dl, "\phi"'] \arrow[dr, "{\langle id_r, \iota \rangle}"] & & r \arrow[dl, "id_r"] \arrow[dr, "{\langle f, g_i \rangle}"] & \\
n & & r & & n
\end{tikzcd}$$

Thus to show that structural and analytic morphisms form a factorization system it remains to show that structural morphisms are left orthogonal to the analytic morphisms. Let

$$\begin{tikzcd}
n & r \arrow[l, "\psi"'] \arrow[r, "{\langle f, h_i \rangle}"] \arrow[dd, "\sigma"] & m \\
k \arrow[u, "\phi"] \arrow[d, "{\langle id_k, a_j \rangle}"'] & & m \arrow[u, "id_m"'] \arrow[d, "{\langle !, g \rangle}"] \\
k & r' \arrow[l, "\phi'"'] \arrow[r, "{\langle !, g' \rangle}"] & 1
\end{tikzcd}$$

be a commutative square in $\mathcal{P}_a(\mathcal{O})$ with left vertical morphism $\langle \phi, id_r, a_i \rangle$ being a structural map and right vertical morphism $\langle id_m, !, g \rangle$ an analytic map. We have chosen the right bottom to be 1 to simplify notation but the general case is similar. The commutation means that $r = r'$ and there is a permutation $\sigma \in S_r$ such that

$$\psi = \phi \circ \phi' \circ \sigma$$

and

$$\langle a_{\phi'(1)}, \ldots, a_{\phi'(r)} \rangle * g' = \sigma \cdot (\langle h_1, \ldots, h_m \rangle * g)$$

Putting into the square a diagonal morphism $\langle \phi' \circ \sigma, f, \bar{h_i} \rangle$

$$\begin{tikzcd}
n & r \arrow[l, "\psi"'] \arrow[r, "\langle f, h_i \rangle"] \arrow[d, "id_r"] & m \\
k \arrow[u, "\phi"] \arrow[d, "\langle id_k, a_j \rangle"'] & r \arrow[ur, "\langle f, \bar{h_i} \rangle"'] \arrow[dl, "\phi' \circ \sigma"'] \arrow[d, "\sigma"] & m \arrow[u, "id_m"'] \arrow[d, "\langle !, g \rangle"] \\
k & r' \arrow[l, "\phi'"'] \arrow[r, "\langle !, g' \rangle"] & 1
\end{tikzcd}$$

where

$$\bar{h_i} = \langle a^{-1}_{\phi' \circ \sigma(l)} \rangle_{l \in f^{-1}(i)} * h_i$$

we see that the permutations id_r and σ show that both triangles commute. It is not difficult to see that this diagonal filling is unique. Thus analytic morphisms are indeed right orthogonal to the structural ones and $\mathcal{P}_a(\mathcal{O})$ is an analytic Lawvere theory.

From the description of the functor $\mathcal{P}_a(h) \colon \mathcal{P}_a(\mathcal{O}) \to \mathcal{P}_a(\mathcal{O}')$ and the description of the structure of $\mathcal{P}_a(\mathcal{O})$ it is clear that $\mathcal{P}_a(h)$ sends the analytic (structural) morphisms to the analytic (structural) ones. Thus $\mathcal{P}_a(h)$ is an analytic interpretation of Lawvere theories.

Now let $\mathbf{T}$ be any Lawvere theory. As the class of analytic morphisms in $\mathbf{T}$ is right orthogonal to a class of morphisms, it is closed under finite products and isomorphisms. In particular, a composition of an analytic morphism $f \colon n \to 1$ in $\mathbf{T}$ with a permutation morphism π_σ with $\sigma \in S_n$ is again an analytic morphism. Thus the analytic operations of any Lawvere theory $\mathbf{T}$ form a symmetric operad. The composition $\langle f_1, \ldots, f_n \rangle * f$ is defined to be $f \circ (f_1 \times \ldots \times f_n)$ and the action of $\sigma \in S_n$ on an analytic morphism $f \colon n \to 1$ is $\sigma \cdot f = f \circ \pi_\sigma$. The unit is the identity morphism on 1. So defined the symmetric part of the operad $\mathbf{T}$ will be denoted as $\mathbf{T}^s$. We have an inclusion morphism of symmetric operads

$$\mathbf{T}^s \to \mathcal{Q}_f(\mathbf{T})$$

By adjunction we get a morphism

$$\psi_{\mathbf{T}} \colon \mathcal{P}_a(\mathbf{T}^s) \longrightarrow \mathbf{T}$$

Clearly, $\psi_{\mathbf{T}}$ is bijective on objects. If $\mathbf{T}$ is analytic then $\psi_{\mathbf{T}}$ is full (faithful) since the structural-analytic factorization exists (is unique and $\pi \colon \mathbb{F} \to \mathbf{T}$ is faithful), see Lemma 7.1.1.

If $I \colon \mathbf{T} \to \mathbf{T}'$ is an analytic interpretation between any Lawvere theories, then the diagram

$$\begin{tikzcd}
\mathcal{P}(\mathbf{T}^s) \arrow[r, "\mathcal{P}(I^s)"] \arrow[d, "\psi_{\mathbf{T}}"'] & \mathcal{P}(\mathbf{T}'^s) \arrow[d, "\psi_{\mathbf{T}'}"] \\
\mathbf{T} \arrow[r, "I"] & \mathbf{T}'
\end{tikzcd}$$

commutes, where I^s is the obvious restriction of I to $\mathbf{T}^s$. Thus the essential image of $\mathcal{P}_a$ is indeed the category of analytic Lawvere theories and analytic interpretations. An isomorphic interpretation of Lawvere theories is always analytic. Therefore $\mathcal{P}_a$ is full on isomorphisms. $\square$

We have

Proposition 7.3.5. *The functor* $\mathcal{Q}_f \colon \mathbf{LT} \to \mathbf{SOp}$ *is monadic.*

Proof. We shall verify that $\mathcal{Q}_f$ satisfies the assumptions of Beck monadicity theorem. By Proposition 7.3.3, $\mathcal{Q}_f$ has a left adjoint. It is easy to see that $\mathcal{Q}_f$ reflects isomorphisms. We shall verify that $\mathbf{LT}$ has and $\mathcal{Q}_f$ preserves $\mathcal{Q}_f$-contractible coequalizers.

Let $I, I' \colon \mathbf{T}' \to \mathbf{T}$ be a pair of interpretations between Lawvere theories so that

$$\begin{tikzcd}
\mathcal{Q}_f(\mathbf{T}') \arrow[r, bend left, "\mathcal{Q}_f(I)"] \arrow[r, bend right, "\mathcal{Q}_f(I')"'] & \mathcal{Q}_f(\mathbf{T}) \arrow[l, "r"'] \arrow[r, shift left, "q"] & \mathcal{O} \arrow[l, shift left, "s"]
\end{tikzcd}$$

is a split coequalizer in **SOp**. We define a Lawvere theory $\mathbf{T}_{\mathcal{O}}$ so that a morphism from n to m in $\mathbf{T}_{\mathcal{O}}$ is an m-tuple $\langle g_1, \ldots, g_m \rangle$ with $g_i \in \mathcal{O}_n$, for $i = 1, \ldots, m$. The compositions and the identities in $\mathbf{T}_{\mathcal{O}}$ are defined in the obvious way from the compositions and the unit in $\mathcal{O}$. The projections $\bar{\pi}_i^n$ in $\mathbf{T}_{\mathcal{O}}$ are the images of the projections π_i^n in $\mathbf{T}$, i.e. $\bar{\pi}_i^n = q(\pi_i^n)$.

The functor $\tilde{q} \colon \mathbf{T} \to \mathbf{T}_{\mathcal{O}}$ is defined, for $f \colon n \to m$ in $\mathbf{T}$, as

$$\tilde{q}(f) = \langle q(\pi_1^m \circ f), \ldots, q(\pi_m^m \circ f) \rangle$$

First we verify, that $\mathbf{T}_{\mathcal{O}}$ has finite products. For this, it is enough to verify that $\langle f_1, \ldots, f_n \rangle * \bar{\pi}_i^n = f_i$, where $*$ is the composition in the operad $\mathcal{O}$. The uniqueness of the morphism into the product is obvious from the construction. We have routine calculations

$$\langle f_1, \ldots, f_n \rangle * \bar{\pi}_i^n =$$

$$q \circ s(\langle f_1, \ldots, f_n \rangle * q(\pi_i^n)) =$$

$$\langle q \circ s(f_1), \ldots, q \circ s(f_n) \rangle * (q \circ s \circ q(\pi_i^n)) =$$

$$\langle q \circ s(f_1), \ldots, q \circ s(f_n) \rangle * (q \circ Op(I) \circ r(\pi_i^n)) =$$

$$\langle q \circ s(f_1), \ldots, q \circ s(f_n) \rangle * (q \circ Op(I') \circ r(\pi_i^n)) =$$

$$\langle q \circ s(f_1), \ldots, q \circ s(f_n) \rangle * (q(\pi_i^n)) =$$

$$q(\langle s(f_1), \ldots, s(f_n) \rangle * \pi_i^n) =$$

$$q(s(f_i)) = f_i$$

It is obvious that $\tilde{q}$ is a morphism of Lawvere theories and that $\mathcal{Q}_f(\tilde{q}) = q$. It remains to verify that $\tilde{q}$ is a coequalizer in **LT**. Let $p \colon \mathbf{T} \to \mathbf{S}$ be a morphism in **LT** coequalizing I and I'

$$\begin{tikzcd}
\mathbf{T}' \arrow[r, shift left, "I"] \arrow[r, shift right, "I'"'] & \mathbf{T} \arrow[r, "\tilde{q}"] \arrow[dr, "p"'] & \mathbf{T}_{\mathcal{O}} \arrow[d, "\tilde{k}"] \\
 & & \mathbf{S}
\end{tikzcd}$$

The morphism $\mathcal{Q}_f(p)$ coequalizes $\mathcal{Q}_f(I)$ and $\mathcal{Q}_f(I')$ in **SOp**. Thus there is a unique morphism k in **SOp** making the triangle on the right

$$\begin{tikzcd} \mathcal{Q}_f(\mathbf{T}') \arrow[r, shift left, "\mathcal{Q}_f(I)"] \arrow[r, shift right, "\mathcal{Q}_f(I')"'] & \mathcal{Q}_f(\mathbf{T}) \arrow[r, "q"] \arrow[dr, "\mathcal{Q}_f(p)"'] & \mathbf{T}_{\mathcal{O}} \arrow[d, "k"] \\ & & \mathcal{Q}_f(\mathbf{S}) \end{tikzcd}$$

commute. We define the functor $\tilde{k}$ so that

$$\tilde{k}(\langle f_1, \ldots, f_n \rangle) = \langle k(f_1), \ldots, k(f_n) \rangle$$

for any morphism $\langle f_1, \ldots, f_n \rangle$ in $\mathbf{T}_{\mathcal{O}}$. The verification that $\tilde{k}$ is the required unique functor is left to the reader. □

7.3.4 The Functor $\mathcal{P}_p \colon \mathbf{RiOp} \to \mathbf{LT}$

The composition of the functors $\mathcal{P}_a \circ \mathcal{P}$ has a simpler description than $\mathcal{P}_a$ alone. We shall denote this composition by $\mathcal{P}_p \colon \mathbf{RiOp} \to \mathbf{LT}$. For a rigid operad $\mathcal{O}$ we define the Lawvere theory as follows. The morphism are spans

$$\begin{tikzcd} & r \arrow[dl, "\phi"'] \arrow[dr, "\langle f, g_i \rangle"] & \\ n & & m \end{tikzcd}$$

such that ϕ is a function, f is a monotone function, $r_i = |f^{-1}(i)|$ and we have $g_i \in \mathcal{O}_{r_i}$, for $i \in m$ and $r = \sum_{i \in m} r_i$. We do not make any identifications of spans! The compositions of spans are defined like the compositions of representees of the morphisms in case of symmetric operads but twisted by an amalgamating permutation as follows.

Let $\langle \phi, f, g_i \rangle \colon n \to m$ and $\langle \phi', f', g'_j \rangle \colon m \to k$ be two spans. Their composition is the span $\langle \phi'', f'', g''_j \rangle \colon n \to k$ defined from the diagram below

(7.2)

$$\begin{tikzcd} & & r'' \arrow[dl, "\bar{\phi}"'] \arrow[dr, "\bar{f}"] \arrow[dll, bend right, "\phi''"'] \arrow[drr, bend left, "\langle f'', g''_j \rangle"] & & \\ & r \arrow[dl, "\phi"'] \arrow[dr, "\langle f, g_i \rangle"'] & & r' \arrow[dl, "\phi'"'] \arrow[dr, "\langle f', g'_j \rangle"'] & \\ n & & m & & k \end{tikzcd}$$

where the square is a pullback of f along ϕ'. The function $\bar{f}$ is so chosen that it is monotone. We put $f'' = f' \circ \bar{f}$, $\phi'' = \phi \circ \bar{\phi} \circ (\sigma_1 + \ldots + \sigma_k)$, and $g''_j = \langle g_{\phi'(l)} \rangle_{l \in f'^{-1}(j)} * g'_j$. The permutation σ_i is the amalgamating permutation for the composition $\langle g_{\phi'(l)} \rangle_{l \in f'^{-1}(j)} * g'_j$, for $i \in (k]$.

Proposition 7.3.6. *The functor $\mathcal{P}_p \colon \mathbf{RiOp} \longrightarrow \mathbf{LT}$ is isomorphic to $\mathcal{P}_a \circ \mathcal{P}$ and its essential image contains rigid Lawvere theories and analytic morphisms between them.*

Proof. The first statement should be clear. As $\mathcal{P}$ is full and faithful, $\mathcal{P}_p$ is faithful and full on analytic morphisms. The image of $\mathcal{P}$ consists of those symmetric operads for which the symmetric group actions are free. Thus the image of $\mathcal{P}_p$ consists of those analytic Lawvere theories in which the symmetric actions are free on analytic operations, i.e. it consists of the rigid Lawvere theories. $\square$

We end this section pointing out to yet another property of analytic Lawvere theories. Let $\mathbf{T}$ be a category with finite products. A morphism $p \colon n \to m$ in $\mathbf{T}$ is a projection iff there is a morphism $p' \colon n \to m'$ so that the diagram

$$m \xleftarrow{\;p\;} n \xrightarrow{\;p'\;} m'$$

is a product in $\mathcal{T}$. We call such a diagram a *decomposition* of n. A decomposition is *trivial* iff m or m' is the terminal object (i.e. 0 if $\mathbf{T}$ is a Lawvere theory), otherwise it is non-trivial. An object is *indecomposable* if it does not have a non-trivial decomposition.

Proposition 7.3.7. 1 *is indecomposable in any analytic Lawvere theory.*

Proof. It is enough to show that for any symmetric operad $\mathcal{O}$, 1 is indecomposable in $\mathcal{P}_a(\mathcal{O})$. Consider the following diagram

$$
\begin{tikzcd}
 & & m+m' & & \\
 & & s \arrow[u, "\bar{\phi}"'] \arrow[d, "\langle !, g\rangle"] & & \\
m \arrow[uurr, "i_m"] \arrow[dd, "\langle id_m, \iota\rangle"'] & & 1 & & m' \arrow[uull, "i_{m'}"'] \arrow[dd, "\langle id_{m'}, \iota\rangle"] \\
 & r \arrow[ur, "\phi"'] \arrow[dl, "\langle f, g_i\rangle"'] & & r' \arrow[ul, "\phi'"] \arrow[dr, "\langle f', g'_i\rangle"] & \\
m & & & & m'
\end{tikzcd}
$$

We assume that the morphisms $\langle\phi, f, g_i\rangle_i$, $\langle\phi', f', g'_j\rangle_j$ are projections making 1 into a product in $\mathcal{P}_a(\mathcal{O})$. We also have two canonical projections from $m+m'$ to m and m'. The morphism $\langle\bar{\phi}, !, g\rangle$ is a morphism into the product making both triangle commute.

From the commutations of the triangles easily follows that

$$g_i * \langle g, \ldots, g\rangle = \iota = g'_j * \langle g, \ldots, g\rangle$$

for $i \in (m]$ and $j \in (m']$. This means that $g_i = g'_j = g^{-1} \in \mathcal{O}_1$ for $i \in (m]$ and $j \in (m']$ and hence $r = m$, $r' = m'$, $f = id_m$, $f' = id_{m'}$. Moreover, $s = 1$ and $! = id_1$. Now commutativity says that there are $\sigma \in S_m$ and $\sigma' \in S_{m'}$ such that $i_m \circ \sigma = \bar{\phi} \circ \phi$ and $i_{m'} \circ \sigma' = \bar{\phi} \circ \phi'$. This is possible only if $m + m' = 1$. □

It follows immediately from this proposition that the Lawvere theory of Jonsson-Tarski algebras is not analytic.

7.4 Finitary Monads vs Operads

First we explain the diagram

$$\begin{tikzcd}
\mathbf{LT} \arrow[r, "\mathcal{M}_l"] & \mathbf{Mnd} \\
\mathbf{SOp} \arrow[u, "\mathcal{P}_a = \mathcal{P}_a^{ol}"] \arrow[r, "\mathcal{M}_o^a"] & \mathbf{AnMnd} \arrow[u, "\mathcal{P}_a^m"'] \\
\mathbf{RiOp} \arrow[u, "\mathcal{P} = \mathcal{P}_p^o"] \arrow[r, "\mathcal{M}_o^p"] & \mathbf{PolyMnd} \arrow[u, "\mathcal{P}_p^m"']
\end{tikzcd}$$

commuting up to isomorphisms, with $\mathcal{P}_a^m$ and $\mathcal{P}_p^m$ being inclusions and $\mathcal{M}_l$ is the equivalence of categories defined in 7.2. The remaining two horizontal functors are also equivalences of categories. We recall them below (cf. [Z10]).

For a set X we consider X^n as the set of functions $X^{(n]}$. Then the permutation group S_n acts naturally of X^n on the right by composition. For a symmetric operad $\mathcal{O}$, the monad $\mathcal{M}_o^a(\mathcal{O})$ on a set X is defined as

$$\mathcal{M}_o^a(\mathcal{O})(X) = \sum_{n\in\mathbb{N}} X^n \otimes_n \mathcal{O}_n$$

Thus in $X^n \otimes_n \mathcal{O}_n$ we identify $\langle \vec{x}\circ\sigma, f\rangle$ with $\langle \vec{x}, \sigma\cdot f\rangle$ for $f \in \mathcal{O}_n$, $\vec{x}\colon (n] \to X$ and $\sigma \in S_n$.

For a rigid operad $\mathcal{O}$ the monad $\mathcal{M}_o^p(\mathcal{O})$ on a set X is defined as

$$\mathcal{M}_o^p(\mathcal{O})(X) = \sum_{n\in\mathbb{N}} X^n \times \mathcal{O}_n$$

For more detailed description see for example [Z10]. One can also find there the commutation of the lower square in the above diagram.

The commutation of the upper square is the content of the following proposition.

Proposition 7.4.1. *The square of categories and functors*

$$\begin{tikzcd}
\mathbf{LT} \arrow[r, "\mathcal{M}_l"] & \mathbf{Mnd} \\
\mathbf{SOp} \arrow[u, "\mathcal{P}_a = \mathcal{P}_a^{ol}"] \arrow[r, "\mathcal{M}_o^a"] & \mathbf{AnMnd} \arrow[u, "\mathcal{P}_a^m"']
\end{tikzcd}$$

commutes up to an isomorphism.

Proof. Let $\mathcal{O}$ be a symmetric operad. We need to define a natural isomorphism κ, so that

$$\kappa^{\mathcal{O}} \colon \mathcal{M}^a_o(\mathcal{O}) \longrightarrow \mathcal{M}_l\mathcal{P}_a(\mathcal{O})$$

is an isomorphism of monads natural in $\mathcal{O}$. The component of $\kappa^{\mathcal{O}}$ at a set X

$$\kappa^{\mathcal{O}}_X \colon \sum_{n\in\mathbb{N}} X^n \otimes \mathcal{O}_n \longrightarrow \int^{n\in\mathbb{F}} X^n \times \mathcal{P}(\mathcal{O})(n,1)$$

is given by

$$[\vec{x}, a] \mapsto [\vec{x}, (id_n, !, a)]$$

where $\vec{x} \colon (n] \to X$, $a \in \mathcal{O}_n$ and $(id_n, !, a)$ is a span

$$\begin{tikzcd} & n \arrow[dl, "id_n"'] \arrow[dr, "\langle !, a\rangle"] & \\ n & & 1 \end{tikzcd}$$

The verification that so defined κ is indeed a natural isomorphism is left for the reader. □

7.4.1 The Functor $\mathcal{Q}^m_f \colon \mathbf{Mnd} \to \mathbf{AnMnd}$

As the horizontal functors in the above diagram are equivalences of categories it follows from Proposition 7.3.5 that the embedding functor $i \colon \mathbf{AnMnd} \to \mathbf{Mnd}$ has a right adjoint

$$\mathcal{Q}^m_f \colon \mathbf{Mnd} \to \mathbf{AnMnd}$$

which is monadic. In other words, any finitary monad on **Set** is an algebra for a monad on the category of analytic monads. We could define the functor $\mathcal{Q}^m_f$ and the related monad $\bar{\mathbb{V}}$ on **AnMnd** directly, but we shall derive it from the more fundamental situation.

Let $\beta \colon \mathbb{B} \to \mathbb{F}$ be the inclusion functor. It induces the following diagram of categories and functors that we describe below

$$
\begin{tikzcd}
\mathbf{Mnd} \arrow[d, shift left, "\mathcal{Q}^m_f"] & = & Mon(\mathbf{End}) \arrow[rr, "\hat{U}"] \arrow[d, shift left, "Mon((-)^a)"] & & \mathbf{End} \arrow[d, shift left, "(-)^a"] & & \mathbf{Set}^{\mathbb{F}} \arrow[ll, "i_{\mathbb{F}}"'] \arrow[d, shift left, "\beta^*"] & \mathbb{F} \\
\mathbf{AnMnd} \arrow[u, shift left, "\mathcal{P}^m_a"] \arrow[loop below] & = & Mon(\mathbf{An}) \arrow[rr, "U"] \arrow[u, shift left] \arrow[loop below] & & \mathbf{An} \arrow[u, shift left, "i^a"] \arrow[loop below] & & \mathbf{Set}^{\mathbb{B}} \arrow[ll, "i_{\mathbb{B}}"'] \arrow[u, shift left, "Lan_\beta"] & \mathbb{B} \arrow[u, "\beta"'] \\
\bar{\mathbb{V}} & = & Mon(\mathbb{V}) & & \mathbb{V} & & &
\end{tikzcd}
$$

β^* is the functor of composing β. It has a left adjoint Lan_β, the left Kan extension along β. For $C \in \mathbf{Set}^{\mathbb{B}}$ it is given by the coend formula

$$
Lan_\beta(C)(X) = \int^{n\in\mathbb{F}} X^n \times C(n]
$$

The equivalences

$$
i_{\mathbb{F}} \colon \mathbf{Set}^{\mathbb{F}} \longrightarrow \mathbf{End}, \qquad i_{\mathbb{B}} \colon \mathbf{Set}^{\mathbb{B}} \longrightarrow \mathbf{An}
$$

are defined by left Kan extensions that might be given by the following formulas

$$
i_{\mathbb{F}}(G)(X) = \int^{n\in\mathbb{F}} X^n \times G(n], \qquad i_{\mathbb{B}}(C)(X) = \sum_{n\in\mathbb{N}} X^n \otimes_n C(n]
$$

where $G \in \mathbf{Set}^{\mathbb{F}}$ and $C \in \mathbf{Set}^{\mathbb{B}}$.

Then the functor $i^a \colon \mathbf{An} \to \mathbf{End}$ is just an inclusion and its right adjoint $(-)^a$ is given by the formula

$$
F^a(X) = \sum_{n\in\mathbb{N}} X^n \otimes_n F(n]
$$

for $F \in \mathbf{End}$. $(-)^a$ is associating to functors and natural transformations their 'analytic parts'.

Note that both $\mathbf{An}$ and $\mathbf{End}$ are strict monoidal categories with tensor given by composition, and i^a is a strict monoidal functor. Thus its right adjoint $(-)^a$ has a unique lax monoidal structure making the adjunction $i^a \dashv (-)^a$ a monoidal adjunction. This in turn gives us a monoidal monad $(\mathbb{V}, \eta, \mu)$ on $\mathbf{An}$.

We have (cf. [Z]) a 2-natural transformation $\mathcal{U}$

$$
\begin{tikzcd}
\mathbf{MonCat} \arrow[r, bend left=40, "Mon"{name=A}] \arrow[r, bend right=40, "|-|"'{name=B}] & \mathbf{Cat}
\arrow[from=A, to=B, phantom, "\Downarrow \mathcal{U}"]
\end{tikzcd}
$$

where **MonCat** is the 2-category of monoidal categories, lax monoidal functors, and monoidal transformations; Mon is the 2-functor associating monoids to monoidal categories, $|-|$ is the forgetful functor forgetting the monoidal structure, and $\mathcal{U}$ is a 2-natural transformation whose component at a monoidal category M is the forgetful functor from monoids in M to the underlying category of M: $\mathcal{U}_M \colon Mon(M) \to |M|$.

Applying $\mathcal{U}$ to the monoidal adjunction and $i^a \dashv (-)^a$ and monoidal monad $\mathbb{V}$ we get an adjunction between categories of monoids and a monad on $Mon(\mathbf{An})$. The unnamed arrow is $Mon(i^a)$. But the monoids in **End** and **An** are monads and hence we get the left most adjunction $\mathcal{Q}^m_f \dashv \mathcal{P}^m_a$ that we were looking for together with the monad $(\bar{\mathbb{V}}, \bar{\eta}, \bar{\mu})$ on the category of analytic monads.

There are free monads on finitary functors (cf. [Ba70]) and free analytic monads on analytic functors (cf. [Z10]). Therefore, the functors $\widehat{U}$ and U have left adjoints $\widehat{F}$ and F, respectively. The adjunctions $F \dashv U$ and $\widehat{F} \dashv \widehat{U}$ induce monads $\mathbb{M}$ and $\widehat{\mathbb{M}}$, respectively. $\widehat{\mathbb{M}}$ is the finitary version of what is called 'the monad for all monads' in [Ba70]. Putting this additional data to the above diagram and simplifying it at the same time we get a diagram

$$\begin{tikzcd}
\mathbf{Mnd} \arrow[rr, shift left, "\widehat{U}"'] \arrow[d, shift left, "\mathcal{Q}^m_f"] & & \mathbf{End} \arrow[ll, shift left, "\widehat{F}"'] \arrow[d, shift left, "(-)^a"] \arrow[loop right, "\widehat{\mathbb{M}}"] \\
\mathbf{AnMnd} \arrow[u, shift left, "\mathcal{P}^m_a"] \arrow[rr, shift right, "U"'] \arrow[loop below, "\bar{\mathbb{V}}"] & & \mathbf{An} \arrow[ll, shift right, "F"'] \arrow[u, shift left, "i^a"] \arrow[loop right, "\mathbb{M}"] \arrow[loop below, "\mathbb{V}"]
\end{tikzcd}$$

In the above diagram the square of the right adjoints commutes. Thus, the square of the left adjoint commutes as well. This shows in particular that the free monad on an analytic functor is analytic.

The monad $\bar{\mathbb{V}}$ is a lift of a monad $\mathbb{V}$ to the category of $\mathbb{M}$-algebras **AnMnd** and, by [Be69], we obtain

Theorem 7.4.2. *The monad $\mathbb{M}$ for analytic monads distributes over the monad $\mathbb{V}$ for finitary functors, i.e. we have a distributive law*

$$\lambda \colon \mathbb{M}\mathbb{V} \longrightarrow \mathbb{V}\mathbb{M}$$

The category of algebras of the composed monad $\mathbb{VM}$ *on* **AnMnd** *is equivalent to the category* **Mnd** *of all finitary monads on* **Set**.

Remark. We arrived at the above theorem with essentially no calculations at all. It has obvious positive aspects but it does not give an idea what the above distributive law is like. We shall present below explicit formulas how to calculate the values of some functors mentioned above and we shall also describe the coherence morphism φ on the monoidal monad $\mathbb{V}$. This coherence morphism generates the distributive law λ. λ is an analog of the combing distributive law 6.2.5.

First we describe the adjunction $i^a \dashv (-)^a$. We shall drop the inclusion i^a when possible. Let $A \in An$ and $G \in \mathbf{End}$ and X be a set. The analytic functor A is given by its coefficients. Its value at X is

$$A(X) = \sum_{n \in \mathbb{N}} X^n \otimes_n A_n$$

where A_n is an S_n-set for $n \in \mathbb{N}$. The value of G^a at X is

$$G^a(X) = \sum_{n \in \mathbb{N}} X^n \otimes_n G(n]$$

Thus

$$\mathbb{V}(A)(X) = A^a(X) = \sum_{n,m \in \mathbb{N}} X^n \otimes_n (n]^m \otimes_m A_m$$

The unit of the adjunction $i^a \dashv (-)^a$ at X

$$(\eta_A)_X \colon A(X) \longrightarrow A^a(X)$$

is given by

$$[\vec{x}, a] \mapsto [\vec{x}, id_n, a]$$

where $\vec{x} \colon (n] \to X$ and $a \in A_n$.

The counit of the adjunction at X

$$(\varepsilon_G)_X \colon \sum_{n \in \mathbb{N}} X^n \otimes_n G(n] \longrightarrow G(X)$$

is given by

$$[\vec{x}, t] \mapsto G(\vec{x})(t)$$

where $\vec{x} \colon (n] \to X$ and $t \in G(n]$.

The multiplication in the monad $\mathbb{V}$

$$(\mu_A)_X \colon \sum_{n,m,k \in \mathbb{N}} X^n \otimes_n (n]^m \otimes_m (m]^k \otimes_k A_k \longrightarrow \sum_{n,k \in \mathbb{N}} X^n \otimes_n (n]^k \otimes_k A_k$$

is given by composition

$$[\vec{x}, g, f, a] \mapsto [\vec{x}, g \circ f, a]$$

where $\vec{x}\colon (n] \to X$, $g\colon (m] \to (n]$, $f\colon (k] \to (m]$, and $a \in A_k$. This ends the definition of the monad $\mathbb{V}$.

Now we shall describe the monoidal structure on $\mathbb{V}$.

If B is another analytic functor, the n-th coefficient of the composition $A \circ B$ is given by

$$(A \circ B)_n = \sum_{m, n_1, \ldots, n_m \in \mathbb{N}, \sum_{i=1}^m n_i = n} (S_n \times B_{n_1} \times \ldots \times B_{n_m} \times A_m)_{/\sim_n}$$

where the equivalence relation $\sim_n$ is such that for $\sigma \in S_n$, $\sigma_i \in S_{n_i}$, $\tau \in S_m$, $b_i \in B_i$, for $i \in (m]$ and $a \in A_m$ we have

$$\langle \sigma, \sigma_1 \cdot b_1, \ldots, \sigma_m \cdot b_m, \tau \cdot a \rangle \sim_n \langle \sigma \circ (\langle \sigma_1, \ldots, \sigma_m \rangle \star \tau), b_{\tau(1)}, \ldots, b_{\tau(m)}, a \rangle$$

where $\star$ is the composition in the operad of symmetries $\mathcal{S}ym$.

The n-th coefficient of $\mathbb{V}(A) \circ \mathbb{V}(A)$ is given by

$$(\mathbb{V}(A) \circ \mathbb{V}(A))_n = \sum_{m, n_i, k_i \in \mathbb{N}, \sum_{i=1}^m n_i = n} (S_n \times (n_1]^{k_1} \otimes_{k_1} A_{k_1} \times \ldots \times (n_m]^{k_m} \otimes_{k_m} A_{k_m} \times A_m)_{/\sim_n}$$

and the n-th coefficient of $\mathbb{V}(A^2)$ is given by

$$(\mathbb{V}(A^2))_n = \sum_{m, k, k_i \in \mathbb{N}, \sum_{i=1}^m k_i = k} (n]^k \times A_{k_1} \times \ldots \times A_{k_m} \times A_m$$

The coherence morphism φ for $\mathbb{V}$ at the n-th coefficient of the functor A is

$$\varphi_n\colon (\mathbb{V}(A) \circ \mathbb{V}(A))_n \longrightarrow (\mathbb{V}(A^2))_n$$

is given by

$$\langle \sigma, [\sigma_1, a_1], \ldots, [\sigma_m, a_m], \tau, a \rangle \mapsto \langle \sigma \circ (\langle \sigma_1, \ldots, \sigma_m \rangle \star \tau), a_{\tau(1)} \ldots a_{\tau(m)}, a \rangle$$

Note that this map is well defined at the level of equivalence classes.

As the functor $(-)^a\colon \mathbf{End} \to \mathbf{An}$ is monadic every finitary functor is a $\mathbb{V}$-algebra on an analytic functor. For G in $\mathbf{End}$ the corresponding algebra map α_G at set X

$$\alpha_G(X)\colon \mathbb{V}((G)^a)(X) = \sum_{n, m \in \mathbb{N}} X^n \otimes_n (n]^m \otimes_m G(m) \longrightarrow \sum_{n \in \mathbb{N}} X^n \otimes_n G(n) = (G)^a(X)$$

is given by

$$(\vec{x}, f, t) \mapsto (\vec{x}, G(f)(t))$$

where $\vec{x}\colon (n] \to X$, $f\colon (m] \to (n]$, $t \in G(m)$.

7.5 Equational Theories vs Operads

In this section we study the relations between equational theories and operads, both symmetric and rigid. In particular, we shall describe the adjunction $\mathcal{Q}_f^{eo} \dashv \mathcal{P}_a^{oe}$ and the properties of the embeddings $\mathcal{E}_s$ and $\mathcal{E}_{ri}$.

$$\begin{tikzcd} \mathbf{RiOp} \arrow[r, "\mathcal{P}"] \arrow[rr, bend left, "\mathcal{P}_p^{oe}"] & \mathbf{SOp} \arrow[r, shift left, "\mathcal{P}_a^{oe}"] & \mathbf{ET} \arrow[l, shift left, "\mathcal{Q}_f^{oe}"] \end{tikzcd}$$

7.5.1 The Functor $\mathcal{P}_a^{oe}\colon \mathbf{SOp} \to \mathbf{ET}$

We start by defining the functor $\mathcal{P}_a^{oe}$. Let $\mathcal{O}$ be a symmetric operad. We define an equational theory $\mathcal{P}_a^{oe}(\mathcal{O}) = (L, A)$. As the set of n-ary function symbols we put $L_n = \mathcal{O}_n$ for $n \in \mathbb{N}$. The set of axioms A contains the following equations in context:

1. $\iota(x_1) = x_1 : \vec{x}^1$ where $\iota \in \mathcal{O}_1$ is the unit of the operad $\mathcal{O}$;

2. $f(f_1(x_1, \ldots, x_{k_1}), \ldots, f_m(x_{k_{m-1}+1}, \ldots, x_{k_m})) = (\langle f_1, \ldots, f_m \rangle * f)(x_1, \ldots x_k) : \vec{x}^k$

   where $f \in \mathcal{O}_m$, $f_i \in \mathcal{O}_{k_i}$ for $i \in 1, \ldots, m$, $k = \sum_{i=1}^m k_i$;

3. $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (\sigma \cdot f)(x_1, \ldots, x_n) : \vec{x}^n$ for all $f \in \mathcal{O}_n$ and $\sigma \in S_n$.

Clearly, all equations are linear-regular and hence the theory $\mathcal{P}_a^{oe}(\mathcal{O})$ is linear-regular.

Suppose that $h\colon \mathcal{O} \to \mathcal{O}'$ is a morphism of symmetric operads. We define the interpretation

$$\mathcal{P}_a^{oe}(h)\colon \mathcal{P}_a^{oe}(\mathcal{O}) \longrightarrow \mathcal{P}_a^{oe}(\mathcal{O}')$$

For $f \in \mathcal{O}_n$ we put

$$\mathcal{P}_a^{oe}(h)(f) = (h(f)(x_1, \ldots, x_n) : \vec{x}^n),$$

for $n \in \mathbb{N}$.

Proposition 7.5.1. *The following triangle*

$$
\begin{tikzcd}
\mathbf{ET} \arrow[rr, "\mathcal{L}_e"] & & \mathbf{LT} \\
& \mathbf{SOp} \arrow[ul, "\mathcal{P}_a^{oe}"] \arrow[ur, "\mathcal{P}_a"'] &
\end{tikzcd}
$$

commutes up to a natural isomorphism.

Proof. Let $\mathcal{O}$ be a symmetric operad. We define a functor

$$\psi_{\mathcal{O}} \colon \mathcal{P}_a(\mathcal{O}) \longrightarrow \mathcal{L}_e\mathcal{P}_a^{oe}(\mathcal{O})$$

by

$$[\phi, !, f] \colon n \to 1 \mapsto [f(x_{\phi(1)}, \dots x_{\phi(m)}) : \vec{x}^n]$$

where $\phi \colon (m] \to (n]$, $f \in \mathcal{O}_m$. The extension of this definition to morphisms with arbitrary codomains is obvious but it only complicates the notation.

$\psi_{\mathcal{O}}$ is clearly bijective on objects. Since every term in $\mathcal{P}_a^{oe}(\mathcal{O})$ is provably equal to a simple term (=operation applied to variables), $\psi_{\mathcal{O}}$ is full.

We shall show that $\psi_{\mathcal{O}}$ is faithful. This is where combinatorics meets equational logic. Suppose we have two morphisms $\langle \phi, !, g\rangle$, $\langle \phi', !, g'\rangle$ in $\mathcal{P}_a(\mathcal{O})$

$$
\begin{tikzcd}
& m \arrow[dl, "\phi"'] \arrow[d, "\sigma"] \arrow[dr, "\langle !, g\rangle"] & \\
n & m \arrow[l, "\bar{\phi}"'] & 1 \\
& m' \arrow[ul, "\phi'"] \arrow[u, "\sigma'"'] \arrow[ur, "\langle !, g'\rangle"'] &
\end{tikzcd}
$$

such that $\psi_{\mathcal{O}}(\phi, !, g) = \psi_{\mathcal{O}}(\phi', !, g')$. This means that the theory $\mathcal{P}_a^{oe}\mathcal{O})$ proves

$$g(x_{\phi(1)}, \dots, x_{\phi(m)}) = g'(x_{\phi'(1)}, \dots, x_{\phi'(m')}) : \vec{x}^n$$

Since $\mathcal{P}_a^{oe}(\mathcal{O})$ is linear-regular theory, $m = m'$ and there are permutations $\sigma, \sigma' \in S_m$ and a function $\bar{\phi} \colon (m] \to (n]$ such that $\mathcal{P}_a^{oe}(\mathcal{O})$ proves

$$g(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = g'(x_{\sigma'(1)}, \dots, x_{\sigma'(m)}) : \vec{x}^m$$

and

$$\phi = \bar{\phi} \circ \sigma, \quad \phi' = \bar{\phi} \circ \sigma'$$

Thus $\mathcal{P}_a^{oe}(\mathcal{O})$ proves

$$g(x_1, \ldots, x_m) = g'(x_{\sigma^{-1}\sigma'(1)}, \ldots, x_{\sigma^{-1}\sigma'(m)}) : \vec{x}^m$$

and

$$g(x_1, \ldots, x_m) = (\sigma^{-1}\sigma') \cdot g'(x_1, \ldots, x_m) : \vec{x}^m$$

The last equality holds only if

$$g = (\sigma^{-1}\sigma') \cdot g'$$

holds in $\mathcal{O}$. But this together with $\phi' = \phi \circ \sigma^{-1} \circ \sigma'$ means that

$$(\phi, !, g) = (\phi', !, g')$$

in $\mathcal{P}_a(\mathcal{O})$. Thus $\psi_{\mathcal{O}}$ is faithful as well. $\square$

Next we identify the image of the functor $\mathcal{P}_a^{oe}$.

Proposition 7.5.2. *The functor $\mathcal{P}_a^{oe}$ is faithful, full on isomorphisms and its essential image is the category of linear-regular theories* **LrET** *i.e. it factorizes as an equivalence of categories $\mathcal{E}_o$ followed by $\mathcal{P}_a^e$*

$$\begin{tikzcd}
\mathbf{ET} & \\
\mathbf{LrET} \arrow[u, "\mathcal{P}_a^e"] & \mathbf{SOp} \arrow[l, "\mathcal{E}_o"'] \arrow[ul, "\mathcal{P}_a^{oe}"']
\end{tikzcd}$$

Proof. As $\mathcal{L}_e$ is an equivalence of categories, the fact that $\mathcal{P}_a^{oe}$ is faithful and full on isomorphisms follows from Proposition 7.5.1 and the same properties of the functor $\mathcal{P}_a$ stated in Proposition 7.3.3.

Let $I\colon \mathcal{P}_a^{oe}(\mathcal{O}) \longrightarrow \mathcal{P}_a^{oe}(\mathcal{O}')$ be a linear-regular interpretation. We shall define $h_I\colon \mathcal{O} \longrightarrow \mathcal{O}'$ such that $\mathcal{P}_a^{oe}(h_I) = I$. For $f \in \mathcal{O}_n$, $I(f) : \vec{x}^n$ is a linear-regular term in $\mathcal{P}_a^{oe}(\mathcal{O}')$. As in $\mathcal{P}_a^{oe}(\mathcal{O}')$ every (linear-regular) term is provably equal to a simple (linear-regular) term (just one function symbol), we can assume that already

$$I(f) = \bar{f}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n$$

holds, where $\bar{f} \in \mathcal{O}'$. We put

$$h_I(f) = \sigma \cdot \bar{f}$$

The verification that $\mathcal{P}_a^{oe}(h_I) = I$ is left for the reader.

Let $T = (L, A)$ be a linear-regular theory. We shall define a symmetric operad $\mathcal{O}$ such that T is isomorphic to $\mathcal{E}_o(\mathcal{O})$. The set of n-ary operations $\mathcal{O}_n$ is the set of linear-regular terms in context $\vec{x}^n$ modulo provable equations from the set of axioms A. The group S_n acts of $\mathcal{O}_n$ by permuting variables

$$\sigma \cdot [t(x_1, \ldots, x_n) : \vec{x}^n] = [t(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n]$$

The unit in $\mathcal{O}_1$ is the term $[x_1 : \vec{x}^1]$. The composition in $\mathcal{O}$ is defined by 'disjoint substitution' i.e. before substituting terms we need to perform α-conversion to make the result of the substitution a linear-regular term. For example substituting terms in contexts $[t_1(x_1, x_2) : \vec{x}^2]$, $[t_2 : \vec{x}^0]$ and $[t_3(x_1, x_2, x_3) : \vec{x}^3]$ into the term $[t(x_1, x_2, x_3) : \vec{x}^3]$ we get

$$[t(t_1(x_1, x_2), t_2, t_3(x_3, x_4, x_5)) : \vec{x}^5]$$

We hope that this explains the composition in $\mathcal{O}$ better than a formal definition. It should be clear that $\mathcal{O}$ is a symmetric operad.

There is an interpretation $I\colon T \to \mathcal{P}^{oe}_a(\mathcal{O})$ sending an operation $f \in L_n$ to the term in context

$$[[f(x_1, \ldots, x_n) : \vec{x}^n)](\vec{x}^n) : \vec{x}^n)]$$

Note that the term in context $[f(x_1, \ldots, x_n) : \vec{x}^n]$ is just a symbol of the theory $\mathcal{P}^{oe}_a(\mathcal{O})$. There is also an interpretation $I'\colon \mathcal{P}^{oe}_a(\mathcal{O}) \to T$ sending an operation $[f(x_1, \ldots, x_n) : \vec{x}^n)] \in \mathcal{O}_n$ to the same thing but considered this time a term in context $[f(x_1, \ldots, x_n) : \vec{x}^n)]$ of the theory T. These two interpretations are mutually inverse. Thus T is isomorphic to $\mathcal{P}^{oe}_a(\mathcal{O})$ in **ET**, as required. $\square$

7.5.2 The Functor $\mathcal{P}^{oe}_p\colon \mathbf{RiOp} \to \mathbf{ET}$

Now we define the functor $\mathcal{P}^{oe}_p$. Let $\mathcal{O}$ be a rigid operad. We define an equational theory $\mathcal{P}^{oe}_p(\mathcal{O}) = (L, A)$. As the set of n-ary function symbols in $\mathcal{P}^{oe}_p(\mathcal{O})$ we put $L_n = \mathcal{O}_n$, for $n \in \mathbb{N}$. The set of axioms A contains the following equations in context:

1. $\iota(x_1) = x_1 : \vec{x}^1$ where $\iota \in \mathcal{O}_1$ is the unit of the operad $\mathcal{O}$;

2. $f(f_1(x_1, \ldots, x_{k_1}), \ldots, f_m(x_{k_{m-1}+1}, \ldots, x_{k_m})) = (\langle f_1, \ldots, f_m\rangle * f)(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ where $f \in \mathcal{O}_m$, $f_i \in \mathcal{O}_{k_i}$ for $i \in 1, \ldots, m$, $k = \sum_{i=1}^m k_i$, and $\sigma \in S_k$ is the amalgamation for this composition.

Clearly, all equations are linear-regular and hence the theory $\mathcal{P}^{oe}_p(\mathcal{O})$ is linear-regular.

Suppose that $(h, \sigma)\colon \mathcal{O} \to \mathcal{O}'$ is a morphism of rigid operads. We define the interpretation $\mathcal{P}_p^{oe}(h,\sigma)\colon \mathcal{P}_p^{oe}(\mathcal{O}) \longrightarrow \mathcal{P}_p^{oe}(\mathcal{O}')$. For $n \in \mathbb{N}$ and $f \in \mathcal{O}_n$ we put

$$\mathcal{P}_p^{oe}(h,\sigma)(f) = (h(f)(x_{\sigma_f(1)}, \ldots, x_{\sigma_f(n)}) : \vec{x}^n)$$

Proposition 7.5.3. *The functor $\mathcal{P}_p^{oe}$:* **RiOp** $\to$ **ET** *is faithful, full on isomorphisms and its essential image is the category of rigid theories* **RiET**.

Proof. As $\mathcal{L}_e$: **ET** $\to$ **LT** is an equivalence of categories and $\mathcal{P}$: **RiOp** $\to$ **SOp** is full and faithful, the fact that $\mathcal{P}_p^{oe}$ is faithful and full on analytic morphisms (and hence also on isomorphisms) follows from Proposition 7.5.1 and the same properties of the functor $\mathcal{P}_p^{ol}$: **RiOp** $\to$ **LT** stated in Proposition 7.3.6.

It remains to show that an equational theory is rigid iff it is of form $\mathcal{P}_p^{oe}(\mathcal{O})$ for a symmetric operad $\mathcal{O}$ whose actions on operations are free. In the equational theory $\mathcal{P}_p^{oe}(\mathcal{O})$ every term is equivalent to a function symbol $f \in \mathcal{O}_n$ applied to some variables. Let us fix $n \in \mathbb{N}$ and $f \in \mathcal{O}_n$. Assume that for some $\sigma \in S_n$, the theory $\mathcal{P}_p^{oe}(\mathcal{O})$ proves

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \vec{x}^n.$$

This means that in the Lawvere theory $\mathcal{P}_p^{ol}(\mathcal{O})(n,1)$ we have equalities of analytic morphisms

$$(id_n, !, f) = (\sigma, !, f) = (id_n, !, f) \circ \pi_\sigma.$$

But, by Proposition 7.3.6, the actions of S_n on analytic morphisms in $\mathcal{P}_p^{ol}(\mathcal{O})(n,1)$ are free, i.e. σ is the identity. Since f was arbitrary, $\mathcal{P}_p^{oe}(\mathcal{O})$ is indeed a rigid equational theory. $\square$

The following Corollary corrects a statement from [CJ95] (cf. [CJ04]) concerning the characterization of equational theories corresponding to polynomial monads.

Corollary 7.5.4. *The equivalence of categories*

$$\mathbf{ET} \xrightarrow{\mathcal{M}_l \circ \mathcal{L}_e} \mathbf{Mnd}$$

restricts to the equivalence between the category of rigid equational theories and the category of finitary polynomial monads on **Set**

$$\mathbf{RiET} \xrightarrow{\mathcal{M}_e} \mathbf{PolyMnd}$$

7.6 Comments and Examples

1. The "operad of operads" $\mathcal{M}^O$, proved to exist in theorem 6.2.7, by definition corresponds to the monad of symmetric operads with a fixed set of types O. The associated equational theory has as its models symmetric operads, i.e. other equational theories. Therefore we can say that the theory of linear-regular equational theories is a rigid equational theory.

   Similarly, the web monoid $\mathcal{W}(M)$ corresponds, by 6.1.1 and 6.1.4, to the rigid equational theory of rigid equational theories with the same types as M, with a linear-regular interpretation in (the theory corresponding to) M, holding the types fixed.

2. The equations expressing commutation of two operations are linear-regular. Therefore all operations in a theory T commute iff they do in its analytic part T^a. However, the analytic part of an equational theory (or its monad on **Set**) is usually much bigger than the original equational theory. For example, if $\mathbf{T}$ is a finitary monad on **Set** then the value of its analytic part on one element set is the coproduct of the symmetrized free $\mathbf{T}$ algebras on finitely many generators

$$\mathbf{T}^a(1) = \sum_{n\in\mathbb{N}} 1^n \otimes_n T(n) = \sum_{n\in\mathbb{N}} T(n)_{/S_n}$$

   Thus it is not so surprising that theories arising in this way might be of interest only in special circumstances, preferably when the theory we start with is very small.

3. The categories **SOp** and **LT** are complete and cocomplete. The functor $\mathcal{P}_a\colon \mathbf{SOp} \to \mathbf{LT}$ preserves all colimits as a left adjoint and it also preserves all connected limits. However, it does not preserve the terminal object. The terminal object is the value of $\mathcal{Q}_f\colon \mathbf{LT} \to \mathbf{SOp}$ on the terminal Lawvere theory. We describe it below.

4. Recall that $\mathbb{1}$ denotes the terminal equational theory. It has one constant, say e, and can be axiomatized by a single axiom: $x_1 = e : \vec{x}^1$. As a Lawvere theory it is the category that has exactly one morphism between any two objects. As $\mathcal{Q}^e_f\colon \mathbf{ET} \to \mathbf{LrET}$ is a right adjoint, it preserves the terminal object. Hence $\mathcal{Q}^e_f(\mathbb{1})$, the linear-regular part $\mathbb{1}$, is the terminal linear-regular theory. It is the theory of commutative monoids. It is best seen at the level of Lawvere theories. Both theories, $\mathcal{Q}^e_f(\mathbb{1})$ and the theory of commutative monoids, are linear-regular and, for any n, have exactly one analytic morphism

$$a\colon n \to 1$$

In case of the theory of monoids it is given by

$$x_1, \dots, x_n \mapsto x_1 \cdot \ldots \cdot x_n$$

5. As we mentioned in Section 2, $\mathbb{1}$ considered as a Lawvere theory has a proper subtheory, in which $0 \ncong 1$. As equational theory it has no function symbols, and can be axiomatized by a single axiom: $x_1 = x_2 : \vec{x}^2$. The linear-regular part of this theory is the theory of commutative semigroups.

6. The embedding of the strongly regular theories into all equational theories has a right adjoint, $\mathcal{Q}$, as well. The values $\mathcal{Q}$ on the terminal equational theory $\mathbb{1}$ is the terminal strongly regular theory, i.e. the theory of monoids.

7. As we saw above the Lawvere theory for monoids $\mathbf{T}_{mon}$ is analytic. Thus it is an image under $\mathcal{L}_o\colon \mathbf{SOp} \to \mathbf{AnLT}$ of a symmetric operad. It can be easily shown that any analytic morphism

$$a\colon n \to 1$$

in $\mathbf{T}_{mon}$ is of the form

$$x_1, \dots, x_n \mapsto x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}$$

where $\sigma \in S_n$, i.e. it is a multiplication of all variables in any order. Thus the operad $\mathbf{T}^s_{mon}$ (see the proof of Proposition 7.3.4 for notation $(-)^s$) is the operad of symmetries, (cf. [Le04]), and hence the theory of monoids $\mathbf{T}_{mon}$ is the image of the operad of symmetries under $\mathcal{L}_o$.

8. The Lawvere theory for monoids with anti-involution $\mathbf{T}_{mai}$ is analytic, as well. Any analytic morphism

$$a\colon n \to 1$$

in $\mathbf{T}_{mai}$ is of form

$$x_1, \dots, x_n \mapsto s^{\varepsilon_1}(x_{\sigma(1)}) \cdot \ldots \cdot s^{\varepsilon_n}(x_{\sigma(n)})$$

where $\sigma \in S_n$, and $\varepsilon_i \in \{0, 1\}$, for $i = 1, \dots, n$, and $s^0(x) = x$, $s^1(x) = s(x)$.

Bibliography

[A74] J. Adamek, *Free algebras and automata realizations in the language of categories*, Commentationes Math. Universitatis Carolinae 15, 1974, pp. 589-602 (avaiable online: http://dml.cz/dmlcz/105583).

[BD95] J. C. Baez, J. Dolan, *Higher-dimensional Algebra and Topological Quantum Field Theory*, J. Math. Phys. 36 (1995), pp. 6073-6105. Available as arXiv:q-alg/9503002.

[BD98] J. C. Baez, J. Dolan, *Higher-Dimensional Algebra III: n-Categories and the Algebra of Opetopes*, Adv. Math. 135 (1998), pp. 145-206. available as arXiv:q-alg/9702014v1.

[Ba70] M. Barr, *Coequalizers and Free Triples*, Math. Z. 116 (1970), pp. 307-322.

[BJT97] H-J Baues, M. Jibladze, A. Tonks, *Cohomology of Monoids in Monoidal Categories*, article in *Operads: Proceedings of Renaissance Conferences*, Contemporary Mathematics 202, American Mathematical Society, Providence 1997.

[Be69] J. Beck, *Distributive Laws*, Lecture Notes in Mathematics 80, pp. 119–140, (1969).

[BSZ] M. Bojanczyk, S. Szawiel, M. Zawadowski, *Rigidity is Undecidable*, preprint.

[BHS11] R. Brown, P. J. Higgins, R. Siviera, *Nonabelian Algebraic Topology*, EMS Tracts in Mathematics 15, European Mathematical Society, Zürich 2011.

[BH81] R. Brown, P. J. Higgins, *The equivalence of ∞-groupoids and crossed complexes*, Cahiers Topologie Géom. Différentielle 22 (1981), pp. 371-386.

[CJ95] A. Carboni, P. T. Johnstone, *Connected Limits, Familial Representability and Artin Glueing*, Mathematical Structures in Computer Science 5 (1995), pp. 441-459.

[CJ04] A. Carboni, P. T. Johnstone: Corrigenda for 'Connected limits, familial representability and Artin glueing'. Mathematical Structures in Computer Science 14 (2004), pp. 185-187.

[CJKP97] A. Carboni, G. Janelidze, G. M. Kelly, P. Paré, *On Localization and Stabilization for Factorization Systems*, Applied Categorical Structures 5 (1997), pp. 1-58.

[Ch03] E. Cheng, *The Category of Opetopes and the Category of Opetopic Sets*, Theory and Applications of Categories 11 (2003), pp. 353-374.

[Ch04] E. Cheng, *Weak n-categories: opetopic and multitopic foundations*, Journal of Pure and Applied Algebra 186 (2004), pp. 109-137.

[D05] C. Deninger, *Arithmetic Geometry and Analysis on Foliated Spaces*, available as arXiv:math/0505354 [math.NT].

[FGHW08] M. Fiore, N. Gambino, M. Hyland, G. Winskel, *The Cartesian Closed Bicategory of Generalised Species of Structures*, J. London Math. Soc. 77 (2008), pp. 203-220.

[G64] J. Giraud, *Méthode de la descente*, Mémoires de la Société Mathématique de France 2 (1964), pp. III1-VIII150.

[HMZ08] V. Harnik, M. Makkai, M. Zawadowski, *Computads and Multitopic Sets*, available as arXiv:0811.3215 [math.CT].

[Ka] A. Kapustin, *Topological Field Theory, Higher Categories, and Their Applications*, available as arXiv:1004.2307 [math.QA]

[FK72] P. Freyd, G. M. Kelly, *Categories of Continous Functors I*, J. Pure Appl. Algebra 2 (1972), pp. 169-191.

[GJ99] P. G. Goerss, J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser, Boston 1999.

[GG76] M. F. Gouzou, R. Gounig, *Fibration relative*, Seminaire de Theorie des Categories, dirige par J. Benabou, 1976.

[G07] N. Gurski, *An algebraic theory of tricategories*, available at http://users.math.yale.edu/∼mg622/tricats.pdftext

[HMP02] C. Hermida, M. Makkai, J. Power, *On weak higher dimensional categories I*, part 1: J. Pure Appl. Alg. 154 (2000), 221-246, part 2: J. Pure Appl. Alg. 157 (2000), 247-277, part 3: J. Pure Appl. Alg. 166 (2002), pp. 83-104.

[J99] P. T. Johnstone, *A Note On Discrete Conduché Fibrations*, Theory and Applications of Categories 5, pp. 1-11.

[J02] P. T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*, Oxford Logic Guides 43 & 44, Clarendon Press, Oxford University Press 2002.

[Jo86] A. Joyal, *Foncteurs Analytiques et Espéces de Structures*, Lecture Notes in Mathematics 1234, Springer (1986), pp. 126-159.

[Ke80] G. M. Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. 22 (1980), pp. 1-84.

[Ke82] G. M. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series 64, Cambridge University Press, New York 1982.

[KJBM10] J. Kock, A. Joyal, M. Batanin, J-F Mascari, *Polynomial Functors and Opetopes*, Adv. Math. 224 (2010) pp. 2690-2737. Available as arXiv:0706.1033 [math.QA].

[KR77] A. Kock, G. E. Reyes, *Doctrines in Categorical Logic*, article in *Handbook of Mathematical Logic*, Studies in Logic and The Foundations of Mathematics 90, North-Holland, (1977), pp. 283-313.

[Lw04] F. W. Lawvere, *Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories*, Ph.D. thesis, Columbia University, 1963. Reprinted in Theory and Applications of Categories 5 (2004) pp. 1-121.

[Le04] T. Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Note Series 298, Cambridge University Press, Cambridge 2004. Preprint available as arXiv:math/0305049v1 [math.CT].

[CWM98] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics 5, Springer-Verlag, New York 1998.

[M95] M. Makkai, *First Order Logic with Dependent Sorts, with Applications to Category Theory*, 1995.

[M04] M. Makkai, *The multitopic ω-category of all multitopic ω-categories*, announcement, available at

http://www.math.mcgill.ca/makkai/mltomcat04/mltomcat04.pdf.

[MZ08] M. Makkai, M. Zawadowski, *The category of 3-computads is not cartesian closed*, Journal of Pure and Applied Algebra 212 (2008), pp. 2543-2546.

[Ma95] Y. Manin, *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*, Astrisque 228 (1995), pp. 121–163.

[Mar] A. A. Markov, *On Impossibility of Certain Algorithms in the Theory of Associative Systems*, (in Russian) Dokl. Akad. Nauk SSSR 55, pp. 587-590. [English translation in C. R. Acad. Sci. URSS, 55, pp. 533-586]

[Mo14] G. W. Moore, *Physical Mathematics and the Future*, talk at Strings 2014, notes available at

http://www.physics.rutgers.edu/∼gmoore/PhysicalMathematicsAndFuture.pdf.

[May72] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture notes in mathematics 271, Springer-Verlag, New York 1972.

[P04] T. Palm, *Dendrotopic Sets*, article in *Galois Theory, Hopf Algebras, and Semiabelian Categories*, Fields Institute Communications 43 (American Mathematical Society, Providence, 2004), pp. 393-443.

[Po47] E. Post, *Recursive Unsolvability of a Problem of Thue*, J. Symbolic Logic 12 (1947) pp. 1-11.

[Sa72] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics 265, Springer-Verlag, 1972.

[Sh08] M. Shulman, *Framed bicategories and monoidal fibrations*, Theory and Applications of Categories 20 (2008), pp. 650-738.

[St08] T. Streicher, *Fibered Categories à la Jean Bénabou*, available at http://www.mathematik.tu-darmstadt.de/∼streicher/FIBR/FibLec.pdf.

[SZ13] S. Szawiel, M. Zawadowski, *The Web Monoid and Opetopic Sets*, J. Pure Appl. Alg. 217 (2013), pp. 1105-1140.

[SZ10] S. Szawiel, M. Zawadowski, *Representing Multicategories as Monads*, talk at CT 2010, Genova.

http://www.mimuw.edu.pl/∼zawado/Talks/GenovaSlides.pdf

[SZ] S. Szawiel, M. Zawadowski, *Monads of Regular Theories*, preprint.

[T06] T. Trimble, *Notes on Tetracategories*, webpage,

http://math.ucr.edu/home/baez/trimble/tetracategories.html.

[W74] G. Wraith, *Arting Glueing*, Journal of Pure and Applied Algebra, 4 (1974), pp. 345-348.

[Z10] M. Zawadowski, *Lax Monoidal Fibrations*, Models, Logics, and Higher-Dimensional Categories: A Tribute to the Work of Mihály Makkai - B. Hart, T. G. Kucera, A. Pillay, P. J. Scott, and R. A. G. Seely, Editors - AMS | CRM, 2011, pp. 341-426. Available at http://www.mimuw.edu.pl/∼zawado/papers.htm.

[Z] M. Zawadowski, *Formal Theory of Monoidal Monads*, JPAA in print.