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Doctoral dissertation

Mathematical analysis of obstacle approximation strategies for incompressible flow

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Supervisors' statement
This dissertation is ready to be reviewed.

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Author's statement
Hereby I declare that the presented thesis was prepared by me and none of its contents was obtained by means that are against the law. The thesis has never before been a subject of any procedure of obtaining an academic degree.

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Abstract

This dissertation addresses the approximation of a flow around solid obstacles in a fluid governed by the Navier-Stokes equations, with a focus on either stationary or time-dependent cases, in two or three dimensions. The approach is mainly based on modeling the rigid obstacle as a highly viscous fluid. Such a penalization method has potential application in numerical schemes, especially when the obstacle is moving. While formally we recover the original problem when the viscosity inside the obstacle goes to infinity, mathematical analysis of the subject brings in several interesting challenges.

In the thesis, we improve regularity results for approximate solutions, specifically targeting pointwise estimates for the gradient of the velocity field. They are crucial for ensuring well-posedness and providing better convergence properties for penalized solutions.

First, we consider the approximation of a rigid obstacle for flows governed by the stationary Navier-Stokes system. The main contribution of this result lies in obtaining pointwise estimates for the velocity gradient. In addition, we include evidence that the method may indeed produce a plausible numerical approximation of the flow.

Therefore, next we investigate and compare several penalty-based obstacle approximation techniques, in the framework of the steady incompressible Navier-Stokes equations. In addition to the viscosity penalization mentioned above we explore a volume penalization method, as well as a combination of both approaches. We derive convergence rate estimates for all three approaches. Comprehensive numerical experiments have been conducted to evaluate their sharpness and quantitative influence of both penalty parameters on the approximation error.

Finally, the last chapter is dedicated to the approximation of the motion of a rigid obstacle in the time-evolutionary Navier-Stokes equations, we establish the existence of weak solutions to the approximate problem and demonstrate their convergence to the weak solution of the original problem. The main contribution of this part is a tangential regularity result for the velocity. This is the first step in proving a pointwise estimate for the gradient of the velocity.

Keywords: weak solutions, incompressible flow, Navier-Stokes equations, obstacle, volume penalization, viscosity penalization, tangential regularity, numerical simulations

Streszczenie

Niniejsza rozprawa doktorska dotyczy aproksymacji przepływu płynu opisywanego równaniami Naviera-Stokesa wokół zanurzonego w nim ciała stałego, koncentrując się na przypadkach stacjonarnych i zależnych od czasu, w dwóch lub trzech wymiarach przestrzennych. Głównym przedmiotem zainteresowania pracy jest podejście polegające na modelowaniu niepodlegającej odkształceniom przeszkody jako płynu o bardzo dużej lepkości. Metoda ta, oparta na penalizacji, ma również znaczenie w schematach numerycznych, zwłaszcza w przypadku poruszających się przeszkód. Formalnie otrzymujemy oryginalny problem, gdy lepkość wewnątrz przeszkody dąży do nieskończoności, jednak dokładna matematyczna analiza zadania niesie ze sobą wiele interesujących problemów.

W pracy uzyskane zostały wyniki dotyczące regularności rozwiązań przybliżonych, w szczególności punktowe oszacowania gradientu pola prędkości. Te ostatnie są kluczowe dla zapewnienia poprawności zagadnienia oraz lepszych własności zbieżności rozwiązań uzyskanych metodą penalizacji.

Pierwsza część pracy poświęcona jest aproksymacji sztywnej przeszkody w przepływach opisywanych przez stacjonarny układ Naviera-Stokesa. Głównym wynikiem tej części jest uzyskanie punktowych oszacowań gradientu prędkości. Ponadto, przedstawione zostały wyniki symulacji numerycznych wskazujące, że metoda ta może być podstawą wiarygodnej aproksymacji numerycznej przepływu wokół przeszkody.

Następnie badamy i porównujemy kilka opartych na penalizacji technik aproksymacji przeszkód w ramach stacjonarnych, nieściśliwych równań Naviera-Stokesa. Oprócz wspomnianej penalizacji lepkościowej badamy metodę penalizacji objętościowej oraz kombinację obu podejść. Dla wszystkich trzech przypadków wyprowadzamy teoretyczne oszacowania szybkości zbieżności. Analiza uzupełniona została kompleksowymi eksperymentami numerycznymi. Służą one ocenie ostrości otrzymanych oszacowań teoretycznych i ilościowego wpływu parametrów penalizacji na błąd aproksymacji.

Ostatni rozdział poświęcony jest aproksymacji problemu opływu ruchomej, sztywnej przeszkody przez płyn, którego dynamika opisana jest układem Naviera-Stokesa zależnym od czasu. Wykazujemy istnienie słabych rozwiązań dla problemu przybliżonego oraz ich zbieżność do słabego rozwiązania oryginalnego problemu. Głównym rezultatem tej części jest wynik dotyczący tak zwanej stycznej regularności pola prędkości, co stanowi pierwszy krok do uzyskania punktowych oszacowań gradientu prędkości.

Słowa kluczowe: rozwiązania słabe, przepływ nieściśliwy, równania Naviera-Stokesa, przeszkoda, penalizacja objętościowa, penalizacja lepkościowa, regularność styczna, symulacje numeryczne

Declaration of Authorship for Chapter 3

Piotr Krzyżanowski : supervision, review, and draft editing of numerical experiments.

Signed: __________________________________

Sadokat Malikova: part of the mathematical analysis related to the rate of convergence, writing software, numerical simulations and their analysis, visualization of results.

Signed: __________________________________

Piotr Bogusław Mucha: statement of the problem, mathematical analysis related to the question of the issue of the existence and control of the error.

Signed: __________________________________

Tomasz Piasecki: mathematical analysis related to convergence rates, writing part of the original draft.

Signed: __________________________________

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Contents

List of Figures

Chapter 1

Introduction

1.1 Background of the problem

The flow around solid bodies immersed in a viscous fluid presents both a theoretical and computational challenge, especially given that their boundaries may have complex geometry. The higher regularity of weak solutions is essential for ensuring the stability and uniqueness of solutions to this kind of problems. The time evolution of the velocity field $u = u(t, x)$ of a viscous incompressible fluid is governed by the Navier-Stokes equations in time varying domain $\Omega_F(t)$

$$\begin{aligned} \partial_t u - \nu \operatorname{div} \mathbb{D}u + (u \cdot \nabla_x)u + \nabla_x p = f \quad &\text{in } Q_F, \\ \operatorname{div}_x u = 0 \quad &\text{in } Q_F, \end{aligned} \tag{1.1.1}$$

where $Q_F := (0, T) \times \Omega_F(t)$ and $\mathbb{D}u$ is the strain rate tensor with components:

$$(\mathbb{D}u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The viscosity penalization method is a technique used to approximate weak solutions for the problem of rigid obstacle motion in a fluid. This approach treats the rigid obstacle as a fluid with infinitely high viscosity, effectively eliminating the obstacle and filling the domain that should be occupied by the rigid object with a viscous fluid. This method is one of the methods for establishing the existence of weak solutions to the problem of rigid obstacle motion in a fluid, developed by Starovoitov [28], who introduces the concept of "*solidification*". This approach involves first constructing a sequence of approximate solutions that account for the presence of a solidifying region within the fluid. As the sequence is refined, it converges to the target system of equations in the limit. The method has been effectively employed in Hoffmann and Starovoitov [14]; and San Martin, Starovoitov, and Tucsnak [26], Wróblewska-Kamińska [32], particularly for the case where the fluid domain is bounded.

On other hand, achieving higher regularity of the approximate solution is quite challenging task due to the presence of a jump function in the diffusive term. However, it is possible to define tangential regularity. It means that the velocity is smoother along specific fixed directions determined by a nondegenerate family of vector-fields. The method of tangential regularity was first introduced by Bony in [6] while investigating hyperbolic equations, and later developed by Chemin [7]. In recent years, this method is widely used by Danchin *et al*(refer to [8]) for nonlinear partial differential equations, where authors introduce the concept of *striated regularity*.

However, the most well-known and studied method is the volume penalization method (see (1.2.9) below), which models solid bodies as porous media with permeability approaching zero and it is sometimes called Brinkman penalization, or simply the L^2 penalization. Here we mention a number of works based on a volume penalization method such as [1], [2], [3],[15]. In works of Angot [2], and Angot, Bruneau and Fabrie [3], authors established the strong convergence of the solutions and derived some error estimates for the approximate and exact problem in the steady Stokes system and unsteady Navier-Stokes with homogeneous boundary data. The further analysis of error estimates for steady systems with inhomogeneous boundary conditions were done recently by Aguayo and Lincopi [1]. Penalization of mixed type - that is, a combination of viscosity and volume penalizations, has been considered among others in [2], [3].

1.2 Overview of the results and organization of the thesis

The primary aim of this dissertation is to investigate certain properties of approximate solutions resulting from penalization methods. The thesis is structured around three topics.

The first result, presented in **Chapter 2** and published in [23] concerns the viscosity penalized stationary Navier-Stokes system in $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} (u_m \cdot \nabla)u_m - \text{div } [\nu_m(x)\mathbb{D}u_m] + \nabla p = f \quad &\text{in} \quad \Omega, \\ \text{div}\, u_m = 0 \quad &\text{in} \quad \Omega, \\ u_m = 0 \quad &\text{in} \quad \partial\Omega, \end{aligned} \tag{1.2.1}$$

where the kinematic viscosity $\nu(x)$ is a discontinuous function:

$$\nu_m(x) = \begin{cases} 1, & x \in \Omega \setminus \Omega_S, \\ m, & x \in \Omega_S, \end{cases} \tag{1.2.2}$$

where Ω_S denotes the domain occupied by the rigid obstacle. The problem (1.2.1) approximates the original problem, defined by the stationary Navier-Stokes equations:

$$\begin{aligned} (u \cdot \nabla)u - \nu\text{div } [\mathbb{D}u] + \nabla p = f \quad &\text{in} \quad \Omega_F, \\ \text{div}\, u = 0 \quad &\text{in} \quad \Omega_F, \\ u = 0 \quad &\text{in} \quad \partial\Omega_F, \end{aligned} \tag{1.2.3}$$

The result of existence of a weak solution of approximate problem (1.2.1) is established using a standard energy approach. This involves applying the Galerkin method and employing compactness arguments, specifically the Rellich-Kondrachov theorem. It is shown that the penalized solution converges to the target system of equations as the penalization parameter approaches infinity. This result is derived using energy estimates and compactness arguments.

The main and novel part of this result is the L^∞ bound for the velocity gradient. The presence of a jump function in the diffusive term disrupts the standard methods for demonstrating improved regularity, such as maximal regularity. Because of that, the key tool to achieve improved regularity of approximate solutions is the tangential regularity approach. This method relies on that the velocity is smooth along extension of a vector field which is tangential to the boundary of the obstacle. Under assumption that there exists a smooth vector field $X \in C^2$ which satisfies following

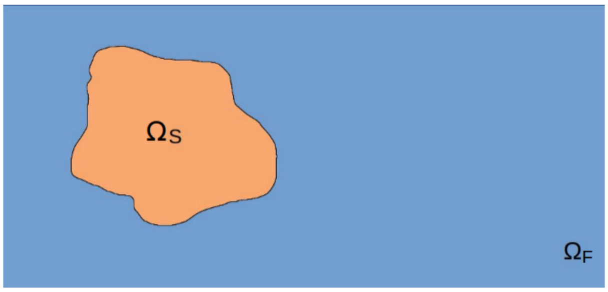


FIGURE 1.1: Fluid domain Ω_F (blue) and obstacle domain Ω_S (yellow). They both make up the compound domain $\bar{\Omega} = \bar{\Omega}_F \cup \bar{\Omega}_S$

conditions

$$X \cdot \tau = 1, \; X \cdot n = 0 \quad \text{on } \partial\Omega_S \cup \partial\Omega, \tag{1.2.4}$$

where the differentiation along X means

$$\partial_X u_m := X^1 \partial_{x_1} u_m + X^2 \partial_{x_2} u_m,$$

the results on tangential regularity $\nabla \partial_X u_m \in L^2(\Omega)$ and higher tangential regularity $\nabla \partial_X^2 u_m \in L^2(\Omega)$ are obtained by using energy approach. To do this, we take the first and second derivatives along X of the system (1.2.1), then use $\partial_X u_m \in H_0^1(\Omega)$ and $\partial_X^2 u_m \in H_0^1(\Omega)$ as test functions.

Due to technical difficulties, the analysis is divided into two parts. The first part focuses on deriving tangential regularity for the approximate Stokes system:

$$\begin{aligned} &- \operatorname{div}\, [\nu_m(x)\mathbb{D}(\partial_X u_m)] + \frac{1}{2}\operatorname{div}\, (\nu_m(x)\nabla^T X \nabla^T u_m + \nu_m(x)\nabla u_m \nabla X) \\ &+ \sum_i X^i_{,k} \partial_{x_i} (\nu_m(x)\mathbb{D}u_m) + \nabla(\partial_X p) - \sum_i X^i_{,k} \partial_{x_i} p = \partial_X f. \end{aligned} \tag{1.2.5}$$

In the second part we analyse the approximate Navier-Stokes system (1.2.1).

Another issue encountered during the application of tangential regularity is that div $\partial_X u_m$:

$$\operatorname{div}\, (\partial_X u_m) = \sum_{k,j=1}^{2} \partial_{x_k} X^j \partial_{x_j} u_m^k,$$

is no longer zero. Due to this, a problematic term $\partial_X p$ appears in the weak formulation of (1.2.5). To address this issue, we use Bogovskii type approach [11, Lemma III.3.1]. This method, previously employed by Galdi in Lemma IV.1.1 [11], is used to derive a similar result for the pressure $p \in L^2(\Omega)$ in the system (1.2.1). Additionally, adapting this approach enables us to establish bounds for $\partial_X p \in L^2(\Omega)$ and $\partial_X^2 p \in L^2(\Omega)$.

To establish the L^∞ bound for the velocity gradient, in addition to tangential regularity, it is essential to show the regularity along normal vector field. However, taking derivative along the normal vector field poses challenges, as the normal derivative of a jump function is not well-defined. To overcome this issue, we introduce a model scenario where the Stokes equation is considered in the entire $\mathbb{R}^2$ space. In this model, the interior of the obstacle domain Ω_S is represented as the half-space where $x_2 < 0$, while the exterior corresponds to the region where $x_2 > 0$,

$$-\operatorname{div}\, [\nu_m(x)\mathbb{D}u_m] + \nabla p = f \quad \text{in} \quad \mathbb{R}^2, \tag{1.2.6}$$

where

$$\nu_m(x_2) = \begin{cases} 1, & x_2 > 0, \\ m, & x_2 < 0, \end{cases} \tag{1.2.7}$$

and the tangential vector field takes form

$$\partial_X = \partial_{x_1}.$$

We reduce the problem (1.2.6) to one dimension in each direction x_1 and x_2 by splitting the equation coordinate-wise. Then by applying the results on tangential regularity, we obtain bounds for each component of the velocity ∇u_m in L^∞.

We reformulate the Navier-Stokes system using curvilinear coordinates and extend the system to the entire space $\mathbb{R}^2$. In this new coordinate system, the tangent and normal vectors are used to define the new coordinate axes. Then we follow the idea of the model case and obtain $\nabla u_m \in L^\infty(\Omega)$.

In addition, we conduct numerical experiments to show that approximate problem (1.2.1) has a potential for application in practice. Specifically, we examine a rectangular channel flow problem within the domain Ω, which contains a fixed rigid obstacle Ω_S touching the boundary of Ω. The results indicate that the viscosity penalized solutions can effectively approximate the limiting case.

Chapter 3, which is based on joint work presented in [17], focuses on the quality of viscosity or volume penalized solutions as approximations to the flow in domain $\Omega_F \subset \mathbb{R}^d$, where $d = 2$ or 3, around an obstacle Ω_S governed by the stationary Navier-Stokes system

$$\begin{aligned} -\nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad & \text{in } \Omega_F, \\ \text{div } u = 0 \quad & \text{in } \Omega_F, \\ u = 0 \quad & \text{on } \partial\Omega_F. \end{aligned} \tag{1.2.8}$$

The key innovation of this work lies in the thorough examination of the effectiveness of a standard volume penalization approach, an approximation method involving high viscosity within the obstacle region, and the combination of these techniques.

Below we briefly introduce each of them. The volume penalization method considers approximate solutions u_n satisfying:

$$\begin{aligned} -\nu\Delta u_n + (u_n \cdot \nabla)u_n + \nabla p_n + \eta_n u_n = f \quad & \text{in } \Omega, \\ \text{div } u_n = 0 \quad & \text{in } \Omega, \\ u_n = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.2.9}$$

posed in a compound domain $\bar{\Omega} = \bar{\Omega}_F \cup \bar{\Omega}_S$. When the penalization source term η_n in the momentum equation tends to infinity, this method, at least formally, leads to the vanishing of the L^2-norm of the solution at the obstacle Ω_S. Consequently, some behaviors related to the shape of the body immersed in the fluid may not be captured.

For the stationary version of the Navier-Stokes system (1.2.8), in general, we cannot guarantee the uniqueness of solutions. This property is only valid in specific, limited scenarios, e.g. when the external force is sufficiently small relative to the viscosity. Consequently, in the case of large data, our approximation identifies the original solution along a particular subsequence. Our analysis establish the following bounds for the convergence rate

$$\|u_n\|_{L^2(\Omega_S)} \leq Cn^{-3/4}, \tag{1.2.10}$$

$$\|u - u_n\|_{H^1(\Omega_F)} \leq Cn^{-1/4}. \tag{1.2.11}$$

For the viscosity penalization, i.e. applying a very large artificial viscosity μ_m within Ω_S - though this approach is viable only if the obstacle is pinned to the boundary of Ω, otherwise, we would only achieve the constant flow inside the obstacle in the limit, not necessarily zero - the approximate solution (u_m, p_m) is defined on Ω by the Navier–Stokes system

$$\begin{aligned} -\operatorname{div}(\mu_m \nabla u_m) + (u_m \cdot \nabla)u_m + \nabla p_m &= f \quad \text{in } \Omega, \\ \operatorname{div} u_m &= 0 \quad \text{in } \Omega, \\ u_m &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2.12}$$

For this type of approximation we obtain new estimates, ensuring better convergence rates:

$$\|u_m\|_{H^1(\Omega_S)} \leq C(\nu m)^{-1}, \tag{1.2.13}$$

$$\|u - u_m\|_{H^1(\Omega_F)} \leq C\nu^{-1}m^{-1/2}. \tag{1.2.14}$$

The third approach we consider here is a straightforward combination of the two previously mentioned methods, resulting in so called mixed penalization method, governed by a system

$$\begin{aligned} -\operatorname{div}(\mu_m \nabla u_{m\vee n}) + (u_{m\vee n} \cdot \nabla)u_{m\vee n} + \nabla p_{m\vee n} + \eta_n u_{m\vee n} &= f \quad \text{in } \Omega, \\ \operatorname{div} u_{m\vee n} &= 0 \quad \text{in } \Omega, \\ u_{m\vee n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2.15}$$

for which we obtain convergence rates

$$\|u_{m\vee n}\|_{H^1(\Omega_S)} \leq C(\nu m)^{-3/4}n^{-1/4}, \tag{1.2.16}$$

$$\|u_{m\vee n}\|_{L^2(\Omega_S)} \leq C(\nu m)^{-1/4}n^{-3/4}, \tag{1.2.17}$$

$$\|u - u_{m\vee n}\|_{H^1(\Omega_F)} \leq C(\nu m n)^{-1/4}. \tag{1.2.18}$$

In proving these results, we make use of the Poincaré inequality, which requires the assumption that the obstacle touches the boundary. However, it is important to note that this assumption is not necessary for establishing the convergence of the mixed approximation—it is only needed to derive the new estimates. To our knowledge, these are the first results of this kind for mixed penalization with $m \neq n$.

In Section 3.3 of Chapter 3, we conduct numerical experiments to explore how the convergence rate of the approximate solutions, as introduced above, depends on the penalizing parameters m and n, and compare them with the theoretical ones.

To achieve this, we analyze a two-dimensional flow around an obstacle within a fixed channel. To gain a comprehensive understanding, we not only vary m and n, but also modify the shape and position of the obstacles.

We proved (see (1.2.12)) that viscosity penalization converges at a linear rate inside the obstacle when it touches the boundary of the domain Ω. The numerical experiments suggest that this result is sharp.

The experiments seem to indicate that the presence of several sharp corners in the obstacle does not affect the convergence rate. Moreover, they show that for the same penalty parameter, viscosity penalization generally produces smaller approximation errors (both in Ω_S and Ω) compared to volume penalization. However, note that the latter property is valid only for obstacles touching the boundary of Ω.

Chapter 4 concerns the approximation of time dependent system (1.1.1) using the mixed penalty approximation, that is

$$\begin{aligned} \partial_t u_m - \operatorname{div}_x(\mathbb{D}_x u_m + \eta_m \mathbb{D}_x(u_m - v)) + (u_m \cdot \nabla_x)u_m + \nabla_x p + \eta_m(u_m - v) = f \quad &\text{in } Q^T, \\ \operatorname{div}_x u_m = 0 \quad &\text{in } Q^T, \\ u_m = 0 \quad &\text{on } \partial Q^T \\ u_m(0,\cdot) = u_0 \quad &\text{in } \Omega. \end{aligned} \tag{1.2.19}$$

where $Q^T := (0,T) \times \Omega$, $\partial Q^T := (0,T) \times \partial\Omega$ and $\eta_m := m\chi(t,x)$ is a non negative piece wise constant function depending on a penalty parameter $m \geq 0$, where

$$\chi(t,x) = \begin{cases} 0, & x \in \Omega_F(t), \\ 1, & x \in \Omega_S(t). \end{cases}$$

We denote $\Omega_S(t)$ as the time-varying domain occupied by the rigid obstacle, and $\Omega_F(t)$ as the time-varying domain occupied by the fluid. The corresponding domain notations are as follows: $Q_S := (0,T) \times \Omega_S(t)$, and $Q_F := (0,T) \times \Omega_F(t)$.

The reader may also refer to the webpage [24] for an animation of a moving obstacle, which illustrates the time-evolutionary simulations of flow around obstacles using the penalized approximation (1.2.19). This simulation was implemented by the author in FEniCS, with further details provided in [17].

We assume that the obstacle moves with a sufficiently smooth velocity given v. The characteristic function χ is defined in terms of a given vector field v, which corresponds to the weak solution of the transport equation

$$\begin{aligned} \partial_t \chi + v\nabla\chi = 0 \\ \chi(0,\cdot) = \mathbf{1}_\Omega - \mathbf{1}_{\Omega_F^0}. \end{aligned} \tag{1.2.20}$$

The first result establishes a priori estimates, where we obtain

$$u_m \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

regularity for penalized solutions u_m of (1.2.19). For the time derivative, we show $\partial_t u_m \in L^2(0,T;H^{-1}(\Omega))$ in $d = 2$, and $\partial_t u_m \in L^2(0,T;H^{-3/2}(\Omega))$ in $d = 3$.

Also, we provide the existence of weak solutions to the penalized problem using the Lions-Aubin compactness lemma.

Finally, we provide a convergence result for the penalized problem, as the penalization parameter m tends to infinity. We have derived uniform a priori estimates for the solutions of the penalized problem, which are independent of the penalty parameter m. However, the m-independent estimate for the time derivative $\partial_t u_m$ holds only in the time-dependent fluid domain. To address this issue we use the auxiliary compactness result gained from [14], to show the strong convergence of the penalized solution u_m of (1.2.19) to the solution u of the original problem (1.1.1).

One of the key results of this chapter is to achieve $\partial_X u_m \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$, where X satisfies (1.2.4). To accomplish this, we again make use of the concept of tangential regularity. For this purpose we generalize the tangential regularity results of Chapter 2 to time dependent case.

Notably, in the presence of a jump function within the diffusive term, we can establish the tangential regularity of the weak solution $\partial_X u_m$, as $\partial_X \eta_m = 0$. We show that the time evolution of X_0 along the velocity of rigid obstacle is the solution of the transport equation:

$$\begin{cases} (\partial_t + v \cdot \nabla)X = \partial_X v, \\ X_{|t=0} = X_0. \end{cases} \tag{1.2.21}$$

The next result addresses the regularity of the vector field X, establishing that

$$\nabla X,\ \nabla^2 X \in L^\infty(Q^T).$$

To derive the estimate for ∇X, we take the first derivative of the transport equation (1.2.21). Next, we multiply the resulting expression by $|\nabla X|^{p-2}\partial_{x_j} X$ and integrate over the spatial domain. After dividing both sides by $\|\nabla X\|^{p-1}_{L^p}$, we apply Gronwall's inequality to complete the estimate. Since the estimates are uniform in p, we can finally pass to the limit $p = \infty$. The estimate for $\nabla^2 X$ is obtained in a similar way by taking the second derivative of (1.2.21) and multiplying the resulting equation by $|\nabla X|^{p-2}\partial_{x_j} X$.

We then provide the proof of the tangential regularity result in a manner similar to the approach discussed in Chapter 2.

Chapter 2

Approximation of rigid obstacle by highly viscous fluid

The content of this chapter, where we study the approximation of a flow around a rigid obstacle for a fluid governed by the stationary Navier-Stokes equations in the two-dimensional case, was published [23]. Here we only slightly adapt the natation to make it compatible with the rest of the thesis. The idea is to consider a highly viscous fluid in the place of the obstacle. Formally, as the fluid viscosity goes to infinity inside the region occupied by the obstacle, we obtain the original problem in the limit. The main goal is to establish a better regularity of approximate solutions. In particular, the pointwise estimate for the gradient of the velocity is proved. In addition, we give numerical evidence that the penalized solution can reasonably approximate the problem, even for relatively small values of the penalty parameter.

2.1 Introduction

The viscosity penalty approach treats the rigid obstacle as a fluid with infinitely high viscosity, effectively eliminating the obstacle and filling the domain that should be occupied by the rigid objects with a viscous fluid. We demonstrate that the weak solution to the approximate problem exhibits higher regularity. Additionally, we assert that the approximate problem converges to a rigid obstacle problem as viscosity tends to infinity. We support our claims with numerical simulations, illustrating the practical applicability of our approach. This method is particularly effective for approximating rough obstacles. We present numerical results for such scenarios, with a detailed theoretical analysis to be conducted in future work

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2. Suppose that $\Omega \setminus \Omega_S$ is filled with a homogeneous viscous incompressible fluid with an obstacle inside (Fig.1.1). Denote by $\Omega_S \subset \Omega$ the domain occupied by the rigid obstacle, f is a given vector function. To simplify calculations we assume the density $\rho = 1$.

The viscosity penalized approximate stationary Navier-Stokes system reads

$$(u_m \cdot \nabla)u_m - \operatorname{div}\,[\nu_m(x)\mathbb{D}u_m] + \nabla p = f \quad \text{in} \quad \Omega, \tag{2.1.1}$$

$$\operatorname{div} u_m = 0 \quad \text{in} \quad \Omega, \tag{2.1.2}$$

$$u_m = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1.3}$$

where the kinematic viscosity $\nu_m(x)$ is a discontinuous function that has the following structure

$$\nu_m(x) = \begin{cases} 1, & x \in \Omega \setminus \Omega_S, \\ m, & x \in \Omega_S, \end{cases} \tag{2.1.4}$$

and $\mathbb{D}u$ is the deformation rate tensor with the components:

$$(\mathbb{D}u)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Since we are interested in weak solutions, we introduce integral formulation of the problem: we say that a vector function $u_m \in V(\Omega)$ is a weak solution to the problem (2.1.1)-(2.1.3) if the following integral identity

$$\int_\Omega ((u_m \cdot \nabla)u_m)w\,dx + \int_\Omega \nu_m \mathbb{D}(u_m) : \mathbb{D}(w)\,dx = \int_\Omega f \cdot w\,dx \tag{2.1.5}$$

holds for arbitrary $w \in V(\Omega)$, where $V(\Omega)$ is defined at the beginning of Section 2.2.

In the rest of the chapter, we prove that the gradient of the velocity field of the approximate problem (2.1.1)-(2.1.2) has a pointwise estimate in L^∞ norm.

2.2 Notations and main results

In this section, we introduce some notations. In order to define spaces of divergence-free vector functions we introduce

$$\mathcal{V}(\Omega) = \{v \in C_0^\infty(\Omega, \mathbb{R}^2) \,|\, \operatorname{div} v = 0\} \tag{2.2.1}$$

$$V(\Omega) = \text{ the closure of } \mathcal{V}(\Omega) \text{ in } H_0^1(\Omega). \tag{2.2.2}$$

According to classical result [30] for Ω an open Lipschitz set we have

$$V(\Omega) = \{v \in H_0^1(\Omega) \,|\, \operatorname{div} v = 0\}.$$

Next we define

$$H(\Omega) = \{v \in L^2 \,|\, \operatorname{div} v = 0 \text{ in } \mathcal{D}'(\Omega),\ v \cdot n = 0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega)\}.$$

The space H is equipped by scalar product $(\cdot,\cdot)$, and the space V is a Hilbert space with the scalar product

$$((u,v)) = \sum_{i=1}^{n} (\nabla_i u, \nabla_i v).$$

By ":" we denote the scalar product of two tensors,

$$\xi : \eta = \sum_{i,j=1}^{n} \xi_{i,j}\eta_{i,j},$$

for $\xi = (\xi_{i,j})_{i=1,\dots,n, j=1,\dots,n} \in \mathbb{R}^{n\times n}$ and $\eta = (\eta_{i,j})_{i=1,\dots,n, j=1,\dots,n} \in \mathbb{R}^{n\times n}$.
Moreover, $f \in L^2(\mathbb{R}_{x_2}; L^2(\mathbb{R}_{x_1}))$ stands notation of 1-dimensional space in x_1 and x_2 directions respectively, and $f \in H^1(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1}))$ means that $f, f_{x_2} \in L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1}))$ i.e.

$$\|f\|_{L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))} := \left[\int_{\mathbb{R}_{x_2}} \left(\int_{\mathbb{R}_{x_1}} |f(x_1,x_2)|^2\,dx_1\right) dx_2\right]^{\frac{1}{2}} \tag{2.2.3}$$

$$\|f\|_{L^2(\mathbb{R}_{x_2};H^k(\mathbb{R}_{x_1}))} := \left[\int_{\mathbb{R}_{x_2}} \left(\int_{\mathbb{R}_{x_1}} \left(|f(x_1,x_2)|^2 + \sum_{j=1}^{k} |\frac{\partial^j f(x_1,x_2)}{\partial {x_1}^j}|^2\right) dx_1\right) dx_2\right]^{\frac{1}{2}}. \tag{2.2.4}$$

Due to assumed regularity of Ω and Ω_S, there exists a vector field $X \in C^2$, such that

$$X \cdot \tau = 1,\ X \cdot n = 0 \quad \text{on } \partial\Omega_S \cup \partial\Omega. \tag{2.2.5}$$

The first result ensures the existence of weak solutions to the penalized Navier-Stokes equations (2.1.1) and is established using the Galerkin method [30]. The approximations are proven to satisfy the weak formulation of the problem (2.1.5). To pass to the limit and obtain a solution, the Rellich-Kondrachov compactness lemma is applied, ensuring convergence to a weak solution of the penalized Navier-Stokes equations.

THEOREM 2.2.1 (Existence)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, let f be given in $L^2(\Omega)$. Then Problem (2.1.5) has at least one solution $u_m \in V(\Omega)$ and there exist a function $p \in L^2(\Omega)$ such that (2.1.1) are satisfied.*

The next result concerns the limit of the penalty parameter $m \to \infty$.

THEOREM 2.2.2 (The limit)**.** *Let assumptions of Theorem 2.2.1 hold and let $m \to \infty$, then $u_m \to u$ in $L^2(\Omega)$, where $\mathbb{D}u = 0$ in the domain Ω_S. In particular, u is a solution to (1.2.3).*

When the symmetric gradient $\mathbb{D}u = 0$ in the obstacle domain Ω_S, this condition characterizes the rigid motion of the obstacle. Rigid motion means that the fluid inside the obstacle domain moves as a solid body without deformation. This implies that the velocity field u inside Ω_S must be such that it satisfies certain regularity conditions. Specifically, the trace of the velocity field u on the boundary of the obstacle $\partial\Omega_S$, denoted as $u|_{\partial\Omega_S} = \phi$, belongs to the Sobolev space $H^{\frac{1}{2}}(\partial\Omega_S)$, which indicates that ϕ is sufficiently smooth.

In situations where the obstacle Ω_S touches the boundary of the domain Ω, the boundary condition (2.1.3) together with Theorem 2.2.2 has significant implications. The theorem implies that as the penalty parameter tends to infinity, the velocity of the fluid on the boundary of the obstacle tends to zero. This means that the norm $\|\phi\|_{H^{\frac{1}{2}}(\partial\Omega_S)} \to 0$ as the penalty parameter increases. Consequently, in the limit, the obstacle effectively becomes stationary relative to the fluids. This result highlights the effectiveness of the penalization approach in enforcing the rigid boundary condition in the limit.

Higher regularity in the approximate solutions of the penalized Navier-Stokes equations is crucial for ensuring their robustness and accuracy. Regularity refers to how smooth and well-behaved the solutions are. When solutions are more regular, it's easier to accurately estimate the difference between the approximate and real solutions, leading to better error control.

Moreover, higher regularity helps to keep the nonlinear terms in the equations well-behaved, preventing any unexpected jumps or discontinuities in the solution. This smoothness is also key in proving that solutions exist globally in time and in showing that these solutions are unique within a given function space. The main result of this chapter is stated as follows

THEOREM 2.2.3. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, $\Omega_S \subset \Omega$ with the boundary $\partial\Omega_S \in C^2$, and let $f \in H^2(\Omega)$, then for every solution u of Navier-Stokes equations (2.1.1)-(2.1.2) we have $\nabla u \in L^\infty(\Omega)$.*

The rest of the chapter is divided into sections, which contain the main steps of the proof of Theorem 2.2.3. The structure of the proof is as follows:

Due to technical difficulties, we split our analysis into two parts. We consider two approximate problems: Stokes and Navier-Stokes. The method of our proof relies on tangential regularity results for the approximate problem. The proof of tangential regularity differs from Danchin [8] and Chemin [7] works. The difficulty is that we propagate the whole approximate Navier-Stokes equations (2.1.1) along the given vector field.

In section 2.3, we establish tangential regularity for the Stokes system in Lemma 2.3.1, where we derive an energy estimate for the derivative $\partial_X u \in L^2(\Omega)$. In Lemma 2.3.2, we further extend this analysis to demonstrate higher tangential regularity. Specifically, we differentiate the entire system of equations twice along the given vector field X and prove that $\nabla\partial_X^2 u \in L^2(\Omega)$. Throughout the proof of these tangential regularity results, we employ a Bogovskii-type approach (see Appendix A.0.3), as outlined in [11], to effectively handle the 'pressure terms' $\partial_X p$ in paragraph 2.3.1, $\partial_X^2 p$ in paragraph 2.3.2.

In section 2.4, we explore a model scenario where the approximate Stokes problem is analyzed in $\mathbb{R}^2$ with the rigid obstacle occupying a half-space. To establish L^∞ regularity for the approximate solution, we simplify the problem by reducing it to one-dimensional functional spaces. The proof requires tangential regularity results and utilizes fundamental tools such as Hölder's inequality, Poincaré's inequality, and embedding theorems.

In section 2.5, we establish tangential and higher tangential regularity results for the approximate Navier-Stokes system (2.1.1)-(2.1.2) in Lemma 2.5.1 and Lemma 2.5.2. Proofs of these lemmas require results for Stokes system.

In Subsection 2.6, we present the proof of the main result. A key aspect of our approach involves the region where the viscosity jump occurs. Consequently, in proving Theorem 2.2.3, we focus our analysis on the domain Σ, which is the neighborhood surrounding the approximate obstacle boundary. We extend the problem to the entire space $\mathbb{R}^2$ by transitioning to a curvilinear coordinate system as described in [4]. In our case the new system is composed of tangential and normal vectors. The idea from the model case is also incorporated into the proof of the theorem. The tangential regularity results are crucial in establishing the pointwise estimate for the gradient of the velocity field in the L^∞ norm.

The proofs of Theorem 2.2.1 and Theorem 2.2.2 are more standard and rely on established results, which are discussed in Appendix A.

2.3 Tangential regularity for Stokes equations

In this section, we consider the approximate Stokes system

$$-\text{div}\ [\nu(x)\mathbb{D}u] + \nabla p = f \quad \text{in} \quad \Omega, \tag{2.3.1}$$
$$\text{div}\ u = 0 \quad \text{in} \quad \Omega \tag{2.3.2}$$
$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.3.3}$$

where $\nu(x)$ is defined in (2.1.4). Since m is fixed, we use the simplified notations ν and u instead of ν_m and u_m.

Definition 2.3.1. *A vector field $u : \Omega \longrightarrow \mathbb{R}^2$ is called a weak (or generalized) solution to the Stokes problem (2.3.1)- (2.3.3) if and only if $u \in V(\Omega)$ and it satisfies the identity*

$$(\nu \mathbb{D}u, \mathbb{D}\phi) = (f, \phi), \quad \forall \phi \in V(\Omega). \tag{2.3.4}$$

The idea behind analyzing the propagated Stokes problem along the tangential vector field is that, at the level of weak formulation, choosing the adjoint test function $(\partial_X)^* v$ in the Stokes system is equivalent to formulating the weak problem for the propagated system with the test function v.

2.3.1 First order derivative

The following lemma gives tangential regularity of the solution to Stokes problem, which will be useful in the proof of Lemma 2.5.1.

Lemma 2.3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, $\Omega_S \subset \Omega$ with the boundary $\partial S \in C^2$. Assume X satisfies (2.2.5) and $f, \partial_X f \in L^2(\Omega)$. Then for every weak solution u of (2.3.1)-(2.3.2) we have*

$$\|\nabla \partial_X u\|_{L^2} \leqslant C_1 \left(\|f\|_{L^2} + \|\partial_X f\|_{L^2} \right) \tag{2.3.5}$$

where $C_1 := C_1(\Omega, X)$.

Proof. Recall, that the vector field $X = (X^1, X^2)$ satisfies (2.2.5), and we are interested in the regularity of function u along X, i.e. in the quantity

$$\partial_X u := X^1 \partial_{x_1} u + X^2 \partial_{x_2} u.$$

We take the derivative along tangential vector field of Stokes equation, that is

$$-\partial_X \mathrm{div}\, [\nu(x) \mathbb{D}u] + \partial_X \nabla p = \partial_X f. \tag{2.3.6}$$

To derive an equation for $\partial_X u$, we will first rewrite (2.3.6). According to the definitions of the differential operators, the diffusive term takes the form:

$$\partial_X \mathrm{div}\, [\nu(x)\mathbb{D}u] = \frac{1}{2}(X^1\partial_{x_1} + X^2\partial_{x_2})(\partial_{x_1}, \partial_{x_2}) \left[\nu(x) \begin{bmatrix} 2u^1_{,1} & u^1_{,2} + u^2_{,1} \\ u^2_{,1} + u^1_{,2} & 2u^2_{,2} \end{bmatrix} \right]$$
$$= \frac{1}{2} \left(\begin{bmatrix} (X^1\partial_{x_1} + X^2\partial_{x_2}) \left[\partial_{x_1}(\nu(x) 2u^1_{,1}) + \partial_{x_2}(\nu(x)(u^1_{,2} + u^2_{,1})) \right] \\ (X^1\partial_{x_1} + X^2\partial_{x_2}) \left[\partial_{x_1}(\nu(x)(u^2_{,1} + u^1_{,2})) + \partial_{x_2}(\nu(x) 2u^2_{,2}) \right] \end{bmatrix} \right).$$

We apply the derivative ∂_X inside the divergence operator, resulting in the following remainder term

$$\begin{aligned}
\operatorname{div}\left[\partial_X\left(\nu(x)\mathbb{D}u\right)\right] =& \frac{1}{2}(\partial_{x_1},\partial_{x_2})\left[(X^1\partial_{x_1}+X^2\partial_{x_2})\nu(x)\begin{bmatrix} 2u^1_{,1} & u^1_{,2}+u^2_{,1} \\ u^2_{,1}+u^1_{,2} & 2u^2_{,2}\end{bmatrix}\right] \\
=& \frac{1}{2}(X^1\partial_{x_1}+X^2\partial_{x_2})(\partial_{x_1},\partial_{x_2})\left[\nu(x)\begin{bmatrix} 2u^1_{,1} & u^1_{,2}+u^2_{,1} \\ u^2_{,1}+u^1_{,2} & 2u^2_{,2}\end{bmatrix}\right] \\
&+ \begin{bmatrix} (X^1_{,1}\partial_{x_1}+X^2_{,1}\partial_{x_2}) \\ (X^1_{,2}\partial_{x_1}+X^2_{,2}\partial_{x_2})\end{bmatrix}\left(\nu(x)\mathbb{D}u\right).
\end{aligned}$$

Combining the last two expressions we obtain the formula

$$\partial_X \operatorname{div}\left[\nu(x)\mathbb{D}u\right] = \operatorname{div}\left[\nu(x)\partial_X\mathbb{D}u\right] - \sum_i X^i_{,k}\partial_{x_i}\left(\nu(x)\mathbb{D}u\right). \tag{2.3.7}$$

In particular

$$\nabla(\partial_X u) = \nabla u\nabla X + \partial_X\nabla u,$$

that follows from the precise calculations:

$$\begin{aligned}
\nabla(\partial_X u) =& \nabla\begin{bmatrix}(X^1\partial_{x_1}+X^2\partial_{x_2})u^1 \\ (X^1\partial_{x_1}+X^2\partial_{x_2})u^2\end{bmatrix} = \\
&\begin{bmatrix}\partial_{x_1}(X^1\partial_{x_1}u^1+X^2\partial_{x_2}u^1) & \partial_{x_2}(X^1\partial_{x_1}u^1+X^2\partial_{x_2}u^1) \\ \partial_{x_1}(X^1\partial_{x_1}u^2+X^2\partial_{x_2}u^2) & \partial_{x_2}(X^1\partial_{x_1}u^2+X^2\partial_{x_2}u^2)\end{bmatrix} \\
&= \begin{bmatrix} X^1_{,1}\partial_{x_1}u^1+X^2_{,1}\partial_{x_2}u^1 & X^1_{,2}\partial_{x_1}u^1+X^2_{,2}\partial_{x_2}u^1 \\ X^1_{,1}\partial_{x_1}u^2+X^2_{,1}\partial_{x_2}u^2 & X^1_{,2}\partial_{x_1}u^2+X^2_{,2}\partial_{x_2}u^2\end{bmatrix} \\
&+ \begin{bmatrix} X^1u^1_{,11}+X^2u^1_{,21} & X^1u^1_{,12}+X^2u^1_{,22} \\ X^1u^2_{,11}+X^2u^2_{,21} & X^1u^2_{,12}+X^2u^2_{,22}\end{bmatrix} \\
&= \nabla u\nabla X + \partial_X\nabla u,
\end{aligned}$$

and for the transposed gradient term, we get

$$\nabla^T(\partial_X u) = (\nabla u\nabla X)^T + \partial_X\nabla^T u.$$

By using derived identities we can rewrite the directional derivative of the symmetric tensor as follows

$$\partial_X(\mathbb{D}u) = \frac{1}{2}\left[\nabla(\partial_X u)+\nabla^T(\partial_X u) - \nabla u\nabla X - \nabla X^T\nabla u^T\right]. \tag{2.3.8}$$

Finally, by applying (2.3.8) and (2.3.7) to equation (2.3.6), we get

$$\begin{aligned}
&-\operatorname{div}\left[\nu(x)\mathbb{D}(\partial_X u)\right] + \frac{1}{2}\operatorname{div}\left(\nu(x)\nabla^T X\nabla^T u + \nu(x)\nabla u\nabla X\right) \\
&+ \sum_i X^i_{,k}\partial_{x_i}\left(\nu(x)\mathbb{D}u\right) + \nabla(\partial_X p) - \sum_i X^i_{,k}\partial_{x_i}p = \partial_X f.
\end{aligned} \tag{2.3.9}$$

Multiplying the equation (2.3.9) by the test function $\psi \in C_0^\infty(\Omega)$ and integrating by parts we obtain

$$\begin{aligned}&\int_\Omega \nu(x)\mathbb{D}(\partial_X u) : \mathbb{D}(\psi)dx - \frac{1}{2}\int_\Omega \nu(x)\left(\nabla^T X \nabla^T u + \nabla u \nabla X\right) : \nabla(\psi)dx\\ &- \int_\Omega (\nu(x)\mathbb{D}u)\,\partial_{x_i}(X^i_{,k}\,\psi)\,dx - \int_\Omega \partial_X\,(p)\ \mathrm{div}\,\psi\,dx\\ &+ \int_\Omega p\,\partial_{x_i}\left(X^i_{,k}\psi\right)\,dx = \int_\Omega \partial_X f\,\psi\,dx.\end{aligned} \tag{2.3.10}$$

We test equation (2.3.9) by function $\partial_X u \in H_0^1(\Omega)$ and get

$$\begin{aligned}&\int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2 dx - \frac{1}{2}\int_\Omega \nu(x)\left(\nabla^T X \nabla^T u + \nabla u \nabla X\right) : \nabla(\partial_X u)dx\\ &\quad - \int_\Omega (\nu(x)\mathbb{D}u)\,\partial_{x_i}(X^i_{,k}\,\partial_X u)\,dx - \int_\Omega \partial_X\,(p)\ \mathrm{div}\,(\partial_X u)dx\\ &\quad - \int_\omega X^i_{,k}\partial_{x_i} p\,\partial_X u\,dx = \int_\Omega \partial_X f\,\partial_X u\,dx.\end{aligned} \tag{2.3.11}$$

Note that $\mathrm{div}\,u = 0$, however, when differentiating the Stokes system along the given vector field X, we obtain the following result:

$$0 = \partial_X \mathrm{div}\,u = \mathrm{div}\,(\partial_X u) - \sum_{i,j=1}^{2} \partial_{x_i} X^j \partial_{x_j} u^i. \tag{2.3.12}$$

From the above equality we deduce that $\mathrm{div}\,(\partial_X u) = \sum_{k,j=1}^{2} \partial_{x_k} X^j \partial_{x_j} u^k$.

The pressure term. We are going to show that $\partial_X p \in L^2(\Omega)$. In order to estimate $\partial_X p$ we will use the Bogovskii type approach (see Lemma A.0.3). Our goal is to prove the following estimate

$$\begin{aligned}\|\partial_X p\|_{L^2} \leqslant C_{b_1}\Big(&\|\nu(x)\mathbb{D}\partial_X u\|_{L^2} + \frac{1}{2}\|\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X)\|_{L^2}\\ &+ ((\|\nu(x)\mathbb{D}u\|_{L^2} + \|p\|_{L^2})\,(c_p c_{X''} + c_{X'}) + c_p\|\partial_X f\|_{L^2}\Big),\end{aligned} \tag{2.3.13}$$

here, c_p is the constant from the Poincaré inequality, and $c_{X''}$, and $c_{X'}$ are constants that depend only on X. In general, $\partial_X p$ satisfies the following inequality

$$\left|\int_\Omega \partial_X p\,dx\right| = \left|-\int_\Omega p\,\mathrm{div}\,X\,dx + \int_{\partial\Omega} X^i n^i p\,ds\right| \leqslant \|\mathrm{div}\,X\|_{L^2}\|p\|_{L^2}. \tag{$\star$}$$

In the above inequality we implement integration by parts and the boundary condition

$$X^i n^i = 0 \ on\ \partial\Omega.$$

From (2.3.10) we have the functional

$$\begin{aligned}\mathcal{F}(\psi) = (\nu(x)\mathbb{D}(\partial_X u), \mathbb{D}\psi) - \frac{1}{2}\left(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X), \nabla\psi\right) \\ - \left(\nu(x)\mathbb{D}u, \partial_{x_i}(X^i_{,k}\,\psi)\right) + (p, \partial_{x_i}(X^i_{,k}\psi)) - (\partial_X f,\, \psi)\end{aligned} \tag{2.3.14}$$

for any $\psi \in H^1_0(\Omega)$. If take a test function $\partial^*_X\psi \in C^\infty_0(\Omega)$ to the Stokes problem (2.3.1), then we get

$$(p, \operatorname{div} \partial^*_X\psi) = -(\partial_X p\,, \operatorname{div}\psi) + \left(p\,, \partial_{x_i}\left(X^i_{,k}\psi\right)\right)$$

where $\partial^*_X\psi = -\partial_{x_i}(X^i\,\psi)$. Therefore, by Lemma A.0.3 there exists a uniquely determined $\partial_X p \in L^2(\Omega)$ that $\frac{1}{|\Omega|}\int_\Omega \partial_X p$ is bounded, and such that

$$\mathcal{F}(\psi) = (\partial_X p, \operatorname{div}\psi). \tag{2.3.15}$$

Consider the problem

$$\begin{aligned}\operatorname{div}\psi = \partial_X p - \frac{1}{|\Omega|}\int_\Omega \partial_X p = g \\ \psi \in H^1_0(\Omega) \\ \|\psi\|_{H^1} \leqslant C_b\|\partial_X p\|_{L^2},\end{aligned} \tag{2.3.16}$$

with Ω bounded and satisfying the cone condition. Since

$$\int_\Omega g = 0,\ g \in L^2(\Omega),$$

from ([11, Theorem III.3.1]) we deduce the existence of ψ which satisfies the equation (2.3.16). We use such ψ as a test function in (2.3.15) to obtain

$$\begin{aligned}\|\partial_X p\|^2_{L^2} = (\nu(x)\mathbb{D}(\partial_X u), \mathbb{D}\psi) - \frac{1}{2}\left(\nu(x)(\nabla^T X\nabla^T u + \nabla u\nabla X), \nabla\psi\right) \\ - \left(\nu(x)\mathbb{D}u, \partial_{x_i}(X^i_{,k}\,\psi)\right) + (p, \partial_{x_i}(X^i_{,k}\psi)) - (\partial_X f,\, \psi).\end{aligned} \tag{2.3.17}$$

By applying the Hölder and Poincaré inequalities to the above equation, we get

$$\begin{aligned}\|\partial_X p\|^2_{L^2} \leqslant& \|\nu(x)\mathbb{D}\partial_X u\|_{L^2}\|\mathbb{D}\psi\|_{L^2} + \frac{1}{2}\|\nu(x)(\nabla^T X\nabla^T u + \nabla u\nabla X)\|_{L^2}\|\nabla\psi\|_{L^2} \\ &+ \|\nu(x)\mathbb{D}u\|_{L^2}\|X^i_{,ki}\,\psi + X^i_{,k}\,\nabla\psi\|_{L^2} + \|p\|_{L^2}\|X^i_{,ki}\,\psi + X^i_{,k}\,\nabla\psi\|_{L^2} \\ &+ \|\partial_X f\|_{L^2}\|\psi\|_{L^2} \\ &\leqslant \|\nu(x)\mathbb{D}\partial_X u\|_{L^2}\|\nabla\psi\|_{L^2} + \frac{1}{2}\|\nu(x)(\nabla^T X\nabla^T u + \nabla u\nabla X)\|_{L^2}\|\nabla\psi\|_{L^2} \\ &+ (\|\nu(x)\mathbb{D}u\|_{L^2} + \|p\|_{L^2})(c_p c_{X''} + c_{X'})\|\nabla\psi\|_{L^2} + c_p\|\partial_X f\|_{L^2}\|\nabla\psi\|_{L^2}.\end{aligned} \tag{2.3.18}$$

Using inequality (2.3.16), we divide both sides of the above expression by $\|\partial_X p\|_{L^2}$and obtain the desired estimate in (2.3.13).

Now, we will examine the remaining terms of the (2.3.11), in order to estimate them. Recall that, $X \cdot n = 0$, $X \cdot \tau = 1$. We rewrite the directional derivative in the following form

$$\partial_X u = X\cdot\tau\,\partial_\tau u + X\cdot n\,\partial_n u.$$

By assumptions and boundary condition (2.3.3), we have

$$\partial_X u|_{\partial\Omega} = \partial_\tau u|_{\partial\Omega} = 0. \tag{2.3.19}$$

In this case, Korn inequality (A.0.21) holds

$$\int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2 dx \geqslant C\int_\Omega |\nabla\partial_X u|^2 dx. \tag{2.3.20}$$

By applying Hölder and Young's inequalities to the second term of equation (2.3.11), we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \nu(x)\left(\nabla^T X\nabla^T u + \nabla u\nabla X\right) : \nabla(\partial_X u)dx\\
&\quad\leqslant \int_\Omega \nu(x)\left(|\nabla u||\nabla X|\right)|\nabla(\partial_X u)|dx\\
&\quad\leqslant C\|\nu(x)\nabla u\|_{L^2}\|\nabla(\partial_X u)\|_{L^2}\\
&\quad\leqslant C_1(\epsilon)\|\nu(x)\nabla u\|_{L^2}^2 + C\epsilon\|\nabla(\partial_X u)\|_{L^2}^2.
\end{aligned} \tag{2.3.21}$$

To the 5^{th} term of the LHS of (2.3.11) we apply integration by parts to get

$$V = -\int_\Omega X^i_{,k}\partial_{x_i} p\, X^l\partial_{x_l} u^k\, dx = 2\left[\int_\Omega p\, X^i_{,k}\partial_{x_i}\partial_X u^k + \int_\Omega p\, X^i_{,ki}\partial_X u^k\right]. \tag{2.3.22}$$

Combining all estimates for terms of (2.3.11), we obtain

$$\begin{aligned}
\int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2\, dx &\leqslant \int_\Omega |\partial_X p||X^i_k\partial_{x_i} u|\, dx\\
&+ \int_\Omega \nu(x)\left(|\nabla u||\nabla X|\right)|\nabla(\partial_X u)|dx\\
&+ \int_\Omega |\nu(x)\mathbb{D}u||X^i_{,ki}\,\partial_X v + X^i_{,k}\,\nabla\partial_X u|\, dx\\
&+ \int_\Omega |p||X^i_{,ki}\,\partial_X u + X^i_{,k}\,\nabla\partial_X u|\, dx\\
&+ \int_\Omega |\partial_X f||\partial_X u|\, dx\\
&\leqslant C_{X'}\|\partial_X p\|_{L^2}\|\nabla u\|_{L^2}\\
&+ c_{X'}\|\nu(x)\nabla u\|_{L^2}\|\nabla\partial_X u\|_{L^2}\\
&+ (c_p c_{X''} + c_{X'})\|\nu(x)\mathbb{D}u\|_{L^2}\|\nabla\partial_X u\|_{L^2}\\
&+ (c_p c_{X''} + c_{X'})\|p\|_{L^2}\|\nabla\partial_X u\|_{L^2}\\
&+ \|\partial_X f\|_{L^2}\|\partial_X u\|_{L^2}.
\end{aligned} \tag{2.3.23}$$

By applying (2.3.13) together with Young's inequality, using a small ϵ, in (2.3.23) we obtain

$$\begin{aligned}\int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2 dx \leqslant & C_{b_1}\epsilon_1\Big(\|\nu(x)\mathbb{D}\partial_X u\|_{L^2}^2 + c_2\|\nu(x)\nabla u\|_{L^2}^2 \\ & + \|p\|_{L^2}^2 + c_p\|\partial_X f\|_{L^2}^2\Big) + C(\epsilon_1)\|\nabla u\|_{L^2}^2 \\ & + C(\epsilon_2)[c_2\|\nu(x)\nabla u\|_{L^2}^2 + \|p\|_{L^2}^2 + c_p\|\partial_X f\|_{L^2}^2] \\ & + \epsilon_2\|\nabla\partial_X u\|_{L^2}^2\end{aligned} \tag{2.3.24}$$

where $(c_p c_{X''} + c_{X'}) := c_3$, $c_{X'}c_\nu =: c_2$ are positive constants. We apply the basic energy estimates (A.0.18) and (A.0.15) to (2.3.1)

$$\begin{aligned}\int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2 dx \leqslant & C_{b_1}\epsilon_1\|\nu(x)\mathbb{D}\partial_X u\|_{L^2}^2 + (C_{b_1}\epsilon_1 + C(\epsilon_2))\Big(c_2\|\nu(x)\nabla u\|_{L^2}^2 \\ & + c_4\|f\|_{L^2}^2 + c_p\|\partial_X f\|_{L^2}^2\Big) + C(\epsilon_1)\|\nabla u\|_{L^2}^2 \\ & + \epsilon_2\|\nabla\partial_X u\|_{L^2}^2 \\ & \leqslant C_{b_1}\epsilon_1\|\nu(x)\mathbb{D}\partial_X u\|_{L^2}^2 + c_5\|f\|_{L^2}^2 + c_6\|\partial_X f\|_{L^2}^2 \\ & + \epsilon_2\|\nabla\partial_X u\|_{L^2}^2.\end{aligned} \tag{2.3.25}$$

The first term on the RHS of (2.3.25) with ϵ is absorbed by the LHS, which leads to

$$\int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2 dx \leqslant A\Big(c_5\|f\|_{L^2}^2 + c_6\|\partial_X f\|_{L^2}^2 + \epsilon_2\|\nabla\partial_X u\|_{L^2}^2\Big), \tag{2.3.26}$$

where $A = 1/(1 - C_{b_1}\epsilon)$.

By applying Korn inequality (2.3.20) to (2.3.26), we obtain

$$\begin{aligned}C\|\nabla\partial_X u\|_{L^2}^2 \leqslant & \int_\Omega \nu(x)|\mathbb{D}(\partial_X u)|^2 dx \\ & \leqslant A(c_5\|f\|_{L^2}^2 + c_6\|\partial_X f\|_{L^2}^2 + \epsilon_2\|\nabla\partial_X u\|_{L^2}^2).\end{aligned} \tag{2.3.27}$$

The last term on the RHS is absorbed by the LHS of (2.3.27) and we obtain desired inequality (2.3.5).

□

2.3.2 Second order derivative

In this subsection, we introduce a lemma that gives the higher tangential regularity of the solution to Stokes problem. The result is stated as follows:

Lemma 2.3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, $\Omega_S \subset \Omega$ with the boundary $\partial\Omega_S \in C^2$, and let X be a vector field satisfying (2.2.5). Assume $f, \partial_X f, \partial_X^2 f \in L^2(\Omega)$, and let u solve (2.3.1)-(2.3.2). Then $\partial_X^2 u \in H_0^1(\Omega)$, with an estimate*

$$\|\nabla\partial_X^2 u\|_{L^2} \leqslant C_2\left(\|f\|_{L^2} + \|\partial_X f\|_{L^2} + \|\partial_X^2 f\|_{L^2}\right) \tag{2.3.28}$$

where $C_2 := C_2(\Omega, X)$.

Proof. Let us take the second derivative along tangential vector field of the Stokes equation, it means we take derivative ∂_X from (2.3.9):

$$\begin{aligned}
&-\mathrm{div}\left[\nu(x)\mathbb{D}(\partial_X^2 u)\right] + \frac{1}{2}\mathrm{div}\left(\nu(x)\nabla^T X \nabla^T \partial_X u + \nu(x)\nabla \partial_X u \nabla X\right) \\
&\quad + \sum_i X^i_{,k}\partial_{x_i}\left(\nu(x)\mathbb{D}(\partial_X u)\right) \\
&\quad + \partial_X\left[\frac{1}{2}\mathrm{div}\left(\nu(x)\nabla^T X \nabla^T u + \nu(x)\nabla u \nabla X\right) + \sum_i X^i_{,k}\partial_{x_i}\left(\nu(x)\mathbb{D}u\right)\right] \\
&\quad + \partial_X\left[\nabla(\partial_X p) - \sum_i X^i_{,k}\partial_{x_i} p\right] = \partial_X^2 f.
\end{aligned} \tag{2.3.29}$$

We will rewrite the above expression term by term. Straightforward calculations of the 4^{th} term of the LHS of (2.3.29) gives

$$\begin{aligned}
&\partial_X\left[\frac{1}{2}\mathrm{div}\left(\nu(x)\nabla^T X \nabla^T u + \nu(x)\nabla u \nabla X\right)\right] = \frac{1}{2}\mathrm{div}\left(\nu(x)\partial_X(\nabla^T X \nabla^T u + \nabla u \nabla X)\right) \\
&\quad - \frac{1}{2}X^s_{,k}\partial_{x_s}\left(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X)\right) = \frac{1}{2}\mathrm{div}\left\{\nu(x)(\nabla^T X \nabla^T \partial_X u\right. \\
&\quad + \nabla \partial_X u \nabla X) + \nu(x)(X^s X^j_{,is}\,\nabla^T u + \nabla u\, X^s X^i_{,js}) \\
&\quad \left. - \nu(x)(X^j_{,i} X^s_{,i}\,\nabla^T u + \nabla u\, X^i_{,j} X^s_{,j})\right\} - \frac{1}{2}X^s_{,k}\partial_{x_s}\left(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X)\right).
\end{aligned} \tag{2.3.30}$$

For the 5^{th} term we have

$$\begin{aligned}
\partial_X\left[X^i_{,k}\partial_{x_i}\left(\nu(x)\mathbb{D}u\right)\right] =& X^s X^i_{,ks}\partial_{x_i}\left(\nu(x)\mathbb{D}u\right) + X^i_{,k}\partial_{x_i}\left(\nu(x)\partial_X \mathbb{D}u\right) \\
& - X^i_{,k} X^s_{,i}\partial_{x_s}\left(\nu(x)\mathbb{D}u\right) \\
=& X^s X^i_{,ks}\partial_{x_i}\left(\nu(x)\mathbb{D}u\right) - X^i_{,k} X^s_{,i}\partial_{x_s}\left(\nu(x)\mathbb{D}u\right) \\
& + X^i_{,k}\partial_{x_i}\left(\nu(x)\mathbb{D}\partial_X u\right) - X^i_{,k}\partial_{x_i}\left(\nu(x)(\nabla^T X \nabla^T u\right. \\
& \left. + \nabla u \nabla X)\right).
\end{aligned} \tag{2.3.31}$$

Also, the 6^{th} term with pressure

$$\begin{aligned}
\partial_X(\nabla \partial_X p) - X^s\partial_{x_s}(X^i_{,k}\partial_{x_i} p) =& \nabla \partial_X^2 p - X^s_{,k}\partial_{x_s}\partial_X p \\
& - X^s X^i_{,ks}\partial_{x_i} p - X^i_{,k}\partial_{x_i}\partial_X p + X^s_{,i} X^i_{,k}\partial_{x_s} p.
\end{aligned} \tag{2.3.32}$$

By multiplying the equation (2.3.29) with $\psi \in C_0^\infty(\Omega)$ and integrating, we get

$$
\begin{aligned}
&(\nu(x)\mathbb{D}(\partial_X^2 u), \mathbb{D}(\psi)) + \Big[(\nu(x)(\nabla^T X \nabla^T \partial_X u + \nabla \partial_X u \nabla X), \nabla \psi)\Big] \\
&\quad + \frac{1}{2}(\nu(x)(X^s X^j_{,is}\, \nabla^T u + \nabla u\, X^s X^i_{,js}), \nabla \psi) \\
&\quad - \frac{1}{2}(\nu(x)(X^j_{,i} X^s_{,i}\, \nabla^T u + \nabla u\, X^i_{,j} X^s_{,j}), \nabla \psi) \\
&\quad - (\nu(x)\mathbb{D}(\partial_X u), \partial_{x_i}(X^i_{,k}\psi)) + \frac{3}{2}(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X), \partial_{x_s}(X^s_{,k}\psi)) \\
&\quad - (\partial_X^2 p, \operatorname{div} \psi) + 2(\partial_X p, \partial_{x_s}(X^s_{,k}\psi)) + (p, \partial_{x_i}(X^s X^i_{,ks}\psi)) \\
&\quad - (p, \partial_{x_i}(X^s_{,i} X^i_{,k}\psi)) = (\partial_X^2 f, \psi).
\end{aligned}
\tag{2.3.33}
$$

The weak formulation of equation (2.3.29), tested with $\partial_X^2 u \in H_0^1(\Omega)$ gives

$$
\begin{aligned}
&\int_\Omega \nu(x)|\mathbb{D}(\partial_X^2 u)|^2\, dx + \int_\Omega \Big(\nu(x)(\nabla^T X \nabla^T \partial_X u + \nabla \partial_X u \nabla X) \\
&\quad + \frac{1}{2}\nu(x)(X^s X^j_{,is}\, \nabla^T u + \nabla u\, X^s X^i_{,js}) \\
&\quad - \frac{1}{2}\nu(x)(X^j_{,i} X^s_{,i}\, \nabla^T u + \nabla u\, X^i_{,j} X^s_{,j})\Big) : \nabla \partial_X^2 u\, dx \\
&\quad + \int_\Omega X^i_{,k} \partial_{x_i}\, (\nu(x)\mathbb{D}(\partial_X u))\, \partial_X^2 u\, dx \\
&\quad - \frac{1}{2}\int_\Omega X^s_{,k} \partial_{x_s}(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X))\partial_X^2 u\, dx \\
&\quad - \frac{3}{2}\int_\Omega X^i_{,k} \partial_{x_i}(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X))\partial_X^2 u\, dx \\
&\quad - \int \partial_X^2 p \operatorname{div} \partial_X^2 u\, dx + \int_\Omega \partial_X p \partial_{x_s}(X^s_{,k}\partial_X^2 u)\, dx + \int_\Omega p\, \partial_{x_i}(X^s X^i_{,ks}\partial_X^2 u)\, dx \\
&\quad + \int_\Omega \partial_X p\, \partial_{x_i}(X^i_{,k}\partial_X^2 u)\, dx - \int_\Omega p\, \partial_{x_i}(X^s_{,i} X^i_{,k}\partial_X^2 u)\, dx = \int_\Omega \partial_X^2 f \partial_X^2 u\, dx.
\end{aligned}
\tag{2.3.34}
$$

The pressure term: Bogovskii type estimate. We are going to estimate the pressure term $\partial_X^2 p$ in the view of Lemma A.0.3. Moreover, the pressure term is well defined:

$$
\left|\int_\Omega \partial_X^2 p\, dx\right| = \left|-\int_\Omega \operatorname{div} X\, \partial_X p\, dx + \int_{\partial\Omega} X^i n^i \partial_X p\, ds\right| \leqslant \|\operatorname{div} X\|_{L^2} \|\partial_X p\|_{L^2} \quad (\star\star).
$$

We derived the above inequality by leveraging the fact that $\partial_X p$ is bounded in $L^2(\Omega)$ as established in the proof of Lemma 2.3.1, combined with integration by parts and the application of Hölder's inequality.

Let us consider the functional from (2.3.33):

$$
\begin{aligned}
\mathcal{F}(\psi) &= (\nu(x)\mathbb{D}(\partial_X^2 u), \mathbb{D}(\psi)) + (\nu(x)(\nabla^T X \nabla^T \partial_X u + \nabla \partial_X u \nabla X), \nabla \psi) \\
&+ \frac{1}{2}(\nu(x)(X^s X^j_{,is}\, \nabla^T u + \nabla u\, X^s X^i_{,js}), \nabla \psi) \\
&- \frac{1}{2}(\nu(x)(X^j_{,i} X^s_{,i}\, \nabla^T u + \nabla u\, X^i_{,j} X^s_{,j}), \nabla \psi) \\
&- (\nu(x)\mathbb{D}(\partial_X u), \partial_{x_i}(X^i_{,k}\psi)) + \frac{3}{2}(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X), \partial_{x_s}(X^s_{,k}\psi)) \\
&+ 2(\partial_X p, \partial_{x_s}(X^s_{,k}\psi)) + (p, \partial_{x_i}(X^s X^i_{,ks}\psi)) \\
&- (p, \partial_{x_i}(X^s_{,i} X^i_{,k}\psi)) - (\partial_X^2 f, \psi),
\end{aligned}
\tag{2.3.35}
$$

for any $\psi \in H^1_0(\Omega)$. Thinking on the level of the weak formulation of Stokes system with a test function $(\partial_X^2)^* \psi \in C_0^\infty(\Omega)$ we get

$$
(p, \operatorname{div}(\partial_X^2)^* \psi) = \langle \partial_X^2 \nabla p, \psi \rangle
$$

where $(\partial_X^2)^* \psi := \partial_{x_k}(X^k\, \partial_{x_i}(X^i\, \psi))$. We deduce by Lemma A.0.3 there exists a uniquely determined $\partial_X^2 p \in L^2(\Omega)$ with bounded $\frac{1}{|\Omega|}\int_\Omega \partial_X^2 p$, such that

$$
\mathcal{F}(\psi) = (\partial_X^2 p, \operatorname{div} \psi) \tag{2.3.36}
$$

for all $\psi \in H^1_0(\Omega)$. Consider the problem

$$
\begin{aligned}
\operatorname{div} \psi = \partial_X^2 p - \frac{1}{|\Omega|}\int_\Omega \partial_X^2 p = g. \\
\psi \in H^1_0(\Omega) \\
\|\psi\|_{H^1} \leqslant C_{b_2} \|\partial_X^2 p\|_{L^2}
\end{aligned}
\tag{2.3.37}
$$

with Ω bounded and satisfying the cone condition. Since $\frac{1}{|\Omega|}\int_\Omega \partial_X^2 p$ is bounded and

$$
\int_\Omega g = 0,\; g \in L^2(\Omega),
$$

from Theorem III.3.1 ([11]) we deduce the existence of ψ solving the equation (2.3.37), using such a ψ as test function in the equation (2.3.33), we have

$$
\begin{aligned}
\|\partial_X^2 p\|_{L^2}^2 &= (\nu(x)\mathbb{D}(\partial_X^2 u), \mathbb{D}(\psi)) + (\nu(x)(\nabla^T X \nabla^T \partial_X u + \nabla \partial_X u \nabla X), \nabla \psi) \\
&+ \frac{1}{2}(\nu(x)(X^s X^j_{,is}\, \nabla^T u + \nabla u\, X^s X^i_{,js}), \nabla \psi) \\
&- \frac{1}{2}(\nu(x)(X^j_{,i} X^s_{,i}\, \nabla^T u + \nabla u\, X^i_{,j} X^s_{,j}), \nabla \psi) \\
&- (\nu(x)\mathbb{D}(\partial_X u), \partial_{x_i}(X^i_{,k}\psi)) + \frac{3}{2}(\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X), \partial_{x_s}(X^s_{,k}\psi)) \\
&+ 2(\partial_X p, \partial_{x_s}(X^s_{,k}\psi)) + (p, X^s \partial_{x_i}(X^i_{,ks}\psi)) \\
&- (p, X^i_{,k} \partial_{x_i}(X^s_{,i}\psi)) - (\partial_X^2 f, \psi).
\end{aligned}
\tag{2.3.38}
$$

By applying Hölder, Poincaré inequalities to the above equation, we get

$$\begin{aligned}\|\partial_X^2 p\|_{L^2}^2 \leqslant \{&\|\nu(x)\mathbb{D}(\partial_X^2 u)\|_{L^2} + c_p\|\nu(x)(\nabla^T X \nabla^T \partial_X u + \nabla \partial_X u \nabla X)\|_{L^2} \\ &+ \frac{1}{2}c_p\|\nu(x)(X^s X^j_{,is}\,\nabla^T u + \nabla u\, X^s X^i_{,js})\|_{L^2} \\ &+ \frac{1}{2}c_p\|\nu(x)(X^j_{,i} X^s_{,i}\,\nabla^T u + \nabla u\, X^i_{,j} X^s_{,j})\|_{L^2} \\ &+ (c_{X''}c_p + c_{X'})\|\nu(x)\mathbb{D}(\partial_X u)\|_{L^2} \\ &+ \frac{3}{2}(c_{X''}c_p + c_{X'})\|\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X)\|_{L^2} \\ &+ 2(c_{X''}c_p + c_{X'})\|\partial_X p\|_{L^2} + (c_X c_{X''} c_p + c_X c_{X''} + c_{X'} c_{X''} c_p + c_{X'}^2)\|p\|_{L^2} \\ &+ c_p\|\partial_X^2 f\|_{L^2}\}\|\nabla \psi\|_{L^2}.\end{aligned} \tag{2.3.39}$$

Then using the inequality from (2.3.37), we could reduce both sides of the above inequality by $\|\partial_X^2 p\|$. Then by applying (2.3.5) and (A.0.15) we obtain the estimate for the pressure term

$$\|\partial_X^2 p\|_{L^2} \leqslant C'_{b_2}\{\|\nu(x)\mathbb{D}(\partial_X^2 u)\|_{L^2} + B_1\|f\|_{L^2} + F_1\|\partial_X f\|_{L^2} + c_p\|\partial_X^2 f\|_{L^2}\}. \tag{2.3.40}$$

Let us take the second directional derivative from the divergence equation (2.3.12):

$$\begin{aligned}\partial_X^2 \mathrm{div}\, u &= \mathrm{div}\,\partial_X^2 u - X^s_{,k}\partial_X u - X^s X^i_{,ks}\partial_{x_i} u - X^i_{,k} X^s \partial_{x_s}\partial_{x_i} u \\ &= \mathrm{div}\,\partial_X^2 u - X^s_{,k}\partial_X u - X^s X^i_{,ks}\partial_{x_i} u - X^i_{,k}\partial_{x_i}\partial_X u + X^i_{,k} X^s_{,i}\partial_{x_s} u\end{aligned} \tag{2.3.41}$$

from the above equality we get the relation:

$$\mathrm{div}\,\partial_X^2 u = X^s_{,k}\partial_X u + X^s X^i_{,ks}\partial_{x_i} u + X^i_{,k}\partial_{x_i}\partial_X u - X^i_{,k} X^s_{,i}\partial_{x_s} u. \tag{2.3.42}$$

By applying Hölder inequality to (2.3.34), we get

$$\begin{aligned}\int_\Omega &|\nu(x)|\mathbb{D}(\partial_X^2 u)|^2\,dx \leqslant \|\partial_X^2 p\|_{L^2}\|\mathrm{div}\,\partial_X^2 u\|_{L^2} \\ &+ \Big[\|\nu(x)(\nabla^T X \nabla^T \partial_X u + \nabla \partial_X u \nabla X)\|_{L^2} + \frac{1}{2}\|\nu(x)(X^s X^j_{,is}\,\nabla^T u + \nabla u\, X^s X^i_{,js})\|_{L^2} \\ &+ \frac{1}{2}\|\nu(x)(X^j_{,i} X^s_{,i}\,\nabla^T u + \nabla u\, X^i_{,j} X^s_{,j})\|_{L^2}\Big]\|\partial_X^2 u\|_{L^2} \\ &+ \|\nu(x)\mathbb{D}(\partial_X u)\|_{L^2}\|\partial_{x_i}(X^i_{,k}\partial_X^2 u)\|_{L^2} \\ &+ \frac{3}{2}\|\nu(x)(\nabla^T X \nabla^T u + \nabla u \nabla X)\|_{L^2}\|\partial_{x_s}(X^s_{,k}\partial_X^2 u)\|_{L^2} \\ &+ 2\|\partial_X p\|_{L^2}\|\partial_{x_s}(X^s_{,k}\partial_X^2 u)\|_{L^2} + \|p\|_{L^2}\|X^s \partial_{x_i}(X^i_{,ks}\partial_X^2 u)\|_{L^2} \\ &+ \|p\|_{L^2}\|X^i_{,k}\partial_{x_i}(X^s_{,i}\partial_X^2 u)\|_{L^2} + \|\partial_X^2 f\|_{L^2}\|\partial_X^2 u\|_{L^2}.\end{aligned} \tag{2.3.43}$$

We use (2.3.42), (2.3.40) and apply Young's inequality, with small ϵ, to the above inequality to obtain

$$\begin{aligned}\int_{\Omega} |\nu(x)\mathbb{D}(\partial_X^2 u)|^2\,dx \leqslant & C'_{b_2}\epsilon\|\nu(x)\mathbb{D}(\partial_X^2 u)\|_{L^2}^2\\ &+ (C'_{b_2}\epsilon + C(\epsilon))\{B_1\|f\|_{L^2}^2 + F_1\|\partial_X f\|_{L^2}^2 + \|\partial_X^2 f\|_{L^2}^2\}\\ &+ C(\epsilon)\{(c_{X'}c_p + c_{X'})(B_1\|f\|_{L^2}^2 + F_1\|\partial_X f\|_{L^2}^2)\}\\ &+ \epsilon\|\nabla\partial_X^2 u\|_{L^2}^2.\end{aligned} \tag{2.3.44}$$

The 1^{st} term of the RHS of the above inequality is absorbed by the LHS:

$$\int_{\Omega} |\nu(x)\mathbb{D}(\partial_X^2 u)|^2\,dx \leqslant B_2\|f\|_{L^2}^2 + F_2\|\partial_X f\|_{L^2}^2 + A_2\|\partial_X^2 f\|_{L^2}^2 + \epsilon\|\nabla\partial_X^2 u\|_{L^2}^2. \tag{2.3.45}$$

Similarly to the approach used in (2.3.19), we deduce that $\partial_X^2 u|_{\partial\Omega} = 0$. Then the Korn inequality holds as follows:

$$\int_{\Omega} \nu(x)|\mathbb{D}(\partial_X^2 u)|^2 dx \geqslant C\int_{\Omega} |\nabla\partial_X^2 u|^2 dx. \tag{2.3.46}$$

Implementing (2.3.46) to (2.3.45) gives required inequality (2.3.28).

□

2.4 The model case

In this section, we analyse a model problem in the entire space $\mathbb{R}^2$, which is given by

$$\begin{aligned} -\mathrm{div}\ [\nu(x)\mathbb{D}u] + \nabla p = f \quad &\text{in} \quad \mathbb{R}^2\\ \mathrm{div}\ u = 0 \quad &\text{in} \quad \mathbb{R}^2. \end{aligned} \tag{2.4.1}$$

We consider the domain of the obstacle Ω_S as a half-space. Specifically, the interior of Ω_S is half-space $x_2 < 0$, while the exterior is defined by the region where $x_2 > 0$:

$$\Omega_S := \{x \in \mathbb{R}^2 \mid x_2 < 0\}.$$

In this case, the derivative along the tangential vector field takes the following form

$$\partial_X := \partial_{x_1},$$

and the viscosity (2.1.4) becomes a jump function along the direction x_2 :

$$\nu(x_2) = \begin{cases} 1, & x_2 > 0\\ m, & x_2 < 0. \end{cases} \tag{2.4.2}$$

To establish higher regularity results, in addition to tangential regularity, we must also demonstrate regularity along the normal vector field, which poses significant challenges. To address this, we can reduce the problem (2.4.1) to a one-dimensional space. Another difficulty in proving higher regularity for the case where $n = p = 2$ is the lack of the embedding $H^1 \nrightarrow L^\infty$. Furthermore, the jump function $\nu(x)$ does not belong to the space H^1.

To overcome these challenges, we work within the spaces defined in (2.2.3)-(2.2.4), effectively reducing our problem to a one-dimensional space where the embedding $H^1 \hookrightarrow L^\infty$ holds. In this context, we can derive a formal estimate for this particular case. This subsection provides the key ideas for the proof of Theorem 2.2.3.

Lemma 2.4.1. *Assume that domain $\Omega = \mathbb{R}^2$ and let $\partial_{x_1}^k f \in L^2(\mathbb{R}^2)$ for some $k \in \mathbb{N}$. Then for every solution u of Stokes equations (2.3.1)-(2.3.2) we have $\partial_{x_1} u \in H^1(\mathbb{R}_{x_2}, H^k(\mathbb{R}_{x_1}))$. Moreover, $\nabla u \in L^\infty(\mathbb{R}^2)$.*

Proof. Let us propagate the Stokes problem (2.4.1) over the given vector field

$$-\text{div}\ [\nu(x)\mathbb{D}\partial_{x_1} u] + \nabla \partial_{x_1} p = \partial_{x_1} f. \tag{2.4.3}$$

Multiplying by the test function $v = \partial_{x_1} u \in H_0^1(\mathbb{R}^2)$, integrating by parts in (2.4.3) and using Korn inequality (A.0.21), we get

$$c\|\nabla \partial_{x_1} u\|_{L^2}^2 \leqslant \int_{\mathbb{R}^2} \nu(x_2)|\mathbb{D}\partial_{x_1} u|^2,$$

and we have

$$\|\nabla \partial_{x_1} u\|_{L^2} \leqslant c_1 \|\partial_{x_1} f\|_2.$$

So, we have that $\partial_{x_1} u \in H^1(\mathbb{R}^2)$.

Now, we repeat the previous steps for the second derivative of (2.4.3) along the tangential vector field, assuming that $\partial_{x_1}^2 f \in L^2(\mathbb{R}^2)$. Differentiating (2.4.3) with respect to x_1, we obtain

$$-\text{div}\ \left[\nu(x)\mathbb{D}\partial_{x_1}^2 u\right] + \nabla \partial_{x_1}^2 p = \partial_{x_1}^2 f, \tag{2.4.4}$$

and we have

$$\|\nabla \partial_{x_1}^2 u\|_{L^2} \leqslant c_2 \|\partial_{x_1}^2 f\|_2.$$

From the above estimates, we deduce that the derivatives along x_1 exhibit higher regularity:

$$u_{,1}^2 \in H^1(\mathbb{R}_{x_2}; H^2(\mathbb{R}_{x_1})),\ u_{,1}^1 \in H^1(\mathbb{R}_{x_2}; H^2(\mathbb{R}_{x_1})). \tag{2.4.5}$$

If $\partial_{x_1}^k f \in L^2(\mathbb{R}^2)$, by iterating the same procedure as described above, it follows that

$$\nabla \partial_{x_1}^k u \in L^2(\mathbb{R}^2) \tag{2.4.6}$$

for some $k \in \mathbb{N}$. From standard theory [30] (Ch.I, Proposition 2.2.), for this weak solution u, we can deduce the existence of a pressure p that exhibits higher regularity in the x_1 direction.

Let's rewrite the first row of the Stokes equation in the following form:

$$-\nu(x_2)u_{,11}^1 - \frac{1}{2}\partial_{x_2}\left[\nu(x)(u_{,2}^1 + u_{,1}^2)\right] + \partial_{x_1} p = f^1.$$

We transfer well defined terms of the above equation on the RHS

$$-\partial_{x_2}\left[\nu(x_2)(u_{,2}^1 + u_{,1}^2)\right] = 2f^1 - 2p_{x_1} + 2\nu(x_2)u_{,11}^1. \tag{2.4.7}$$

Taking the L^2 norm in the direction x_1, we obtain:

$$\begin{aligned}\int_{\mathbb{R}_{x_1}} \left|\partial_{x_2}\left[\nu(x_2)(u^1_{,2}+u^2_{,1})\right]\right|^2 dx_1 \leqslant C\Big(&\int_{\mathbb{R}_{x_1}} |f^1|^2\,dx_1 + \int_{\mathbb{R}_{x_1}} |p_{x_1}|^2\,dx_1 \\ &+ \int_{\mathbb{R}_{x_1}} |\nu(x_2)\partial^2_{x_1}u^1|^2\,dx_1\Big).\end{aligned} \tag{2.4.8}$$

By differentiating (2.4.7) with respect to x_1 and taking the $L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))$ norm, we obtain:

$$\begin{aligned}\left\|\partial_{x_2}\left[\nu(x)(u^1_{,21}+u^2_{,11})\right]\right\|_{L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))} \leqslant C_1\Big(&\|\partial_{x_1}f^1\|_{L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))} \\ &+ \|\partial_{x_1}p_{x_1}\|_{L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))} \\ &+ \|\nu(x_2)\partial^3_{x_1}u\|_{L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))}\Big).\end{aligned} \tag{2.4.9}$$

From the above considerations, we obtain the following estimate

$$\begin{aligned}\|\nu(x_2)(u^1_{,2}+u^2_{,1})\|_{H^1(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \leqslant C_2\Big(&\|f^1\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ &+ \|p_{x_1}\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ &+ \|\nu(x_2)\partial^2_{x_1}u^1\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))}\Big).\end{aligned} \tag{2.4.10}$$

We have the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the following inequality holds

$$\|u\|_{L^\infty(\mathbb{R})} \leqslant C\|u\|_{H^1(\mathbb{R})}. \tag{2.4.11}$$

Using this embedding, we can provide a lower bound for the left-hand side of equation (2.4.10). This allows us to further refine our estimate and strengthen the regularity results obtained for u

$$\begin{aligned}C\|\nu(x_2)(u^1_{,2}+u^2_{,1})\|_{L^\infty(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \leqslant &\|\nu(x_2)(u^1_{,2}+u^2_{,1})\|_{H^1(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ &\leqslant C_2\Big(\|f^1\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ &+ \|p_{x_1}\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ &+ \|\nu(x_2)\partial^2_{x_1}u^1\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))}\Big).\end{aligned} \tag{2.4.12}$$

We know that $u^2_{,1} \in H^1(\mathbb{R}_{x_2}, H^k(\mathbb{R}_{x_1}))$, which implies $u^2_{,1} \in L^\infty(\mathbb{R}_{x_2}, L^\infty(\mathbb{R}_{x_1}))$, so we can bound $u^1_{,2}$ using the triangle inequality as follows:

$$\begin{aligned}\|\nu(x_2)u^1_{,2}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))} \leqslant &\|\nu(x_2)(u^1_{,2}+u^2_{,1})\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))} \\ &+ \|\nu(x_2)u^2_{,1}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))}.\end{aligned} \tag{2.4.13}$$

We can use the following inequality to further bound $u^1_{,2}$:

$$\|u^1_{,2}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))} \leqslant \left\|\frac{1}{\nu(x_2)}\right\|_{L^\infty(\mathbb{R}_{x_2})} \|\nu(x_2)u^1_{,2}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))}. \tag{2.4.14}$$

Using the triangle inequality and the inequality above, we obtain the following bound:

$$\begin{aligned}\|u^1_{,2}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1})))} \leqslant & \Big\|\frac{1}{\nu(x_2)}\Big\|_{L^\infty(\mathbb{R}_{x_2})}\{\|\nu(x_2)(u^1_{,2}+u^2_{,1})\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))} \\ & + \|\nu(x_2)u^2_{,1}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))}\} \\ \leqslant & C_3\Big\|\frac{1}{\nu(x_2)}\Big\|_{L^\infty(\mathbb{R}_{x_2})}\{\|f^1\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ & + \|p_{x_1}\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} + c\|\nu(x_2)\partial^2_{x_1}u^1\|_{L^2(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \\ & + \|\nu(x_2)u^2_{,1}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))}\}.\end{aligned} \tag{2.4.15}$$

From (2.4.6) $u^1_{,1}$ has higher regularity in x_1 direction, such that

$$\partial_{x_2}\partial_{x_1}u^1_{,1} \in L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))$$

$$\|\partial_{x_2}\partial_{x_1}u^1_{,1}\|_{L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1}))} \leqslant c_2\|\partial_{x_1}f\|_2$$

that is

$$\|u^1_{,1}\|_{H^1(\mathbb{R}_{x_2};H^1(\mathbb{R}_{x_1}))} \leqslant c_2\|\partial_{x_1}f\|_2$$

by embedding we get

$$\|u^1_{,1}\|_{L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1}))} \leqslant c_2\|\partial_{x_1}f\|_2.$$

Recall, that $u^1_{,1} = -u^2_{,2}$, which implies that $u^2_{,2} \in L^\infty(\mathbb{R}_{x_2};L^\infty(\mathbb{R}_{x_1})))$. From estimates on the derivatives of u, we can conclude that

$$\nabla u \in L^\infty(\mathbb{R}^2).$$

□

2.5 Tangential regularity for the nonlinear system

In this section, we finally approach the Navier-Stokes problem, focusing initially on the tangential regularity of the weak solution u. The concept of tangential regularity plays a key role in the analysis of nonlinear systems, particularly in the Navier-Stokes equations. It refers to the smoothness of the velocity field along specific, predetermined directions - typically aligned with a family of nondegenerate vector fields. This refined regularity, focusing on tangential components, helps address challenges in establishing uniqueness and stability of weak solutions in nonlinear PDEs.

In cases involving a jump function within the diffusive term of the nonlinear system, the tangential regularity of weak solutions becomes particularly valuable. By leveraging the condition $\partial_X\nu(x) = 0$, where $\nu(x)$ represents a jump in viscosity, we can still obtain tangential regularity for the weak solution $\partial_X u$.

2.5.1 First order derivative

Tangential regularity result of approximate Navier-Stokes equations (2.1.1) reads

Lemma 2.5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, $\Omega_S \subset \Omega$ with the boundary $\partial\Omega_S \in C^2$, and let X be a vector field satisfying (2.2.5). Assume $f, \partial_X f \in L^2(\Omega)$, and let u solve (2.1.1)-(2.1.2). Then $\partial_X u \in H^1(\Omega)$, with an estimate*

$$\|\nabla \partial_X u\|_{L^2} \leqslant C \left(\|f\|_{L^2} + \|\partial_X f\|_{L^2} + \|f\|^2_{L^2}\right) \tag{2.5.1}$$

where $C := C(\Omega, X)$.

Proof. Let us first differentiate the stationary Navier-Stokes equation along the vector field X

$$\partial_X \left[(u \cdot \nabla)u - \operatorname{div}\, [\nu(x)\mathbb{D}u] + \nabla p\right] = \partial_X f \tag{2.5.2}$$
$$\partial_X \operatorname{div} u = 0.$$

We are going to show estimates for the nonlinear part of the equation tested by $\partial_X u$, as the rest has been proved in Lemma 2.3.1. We have

$$\partial_X \left[(u \cdot \nabla)u\right] = (\partial_X u \cdot \nabla)u + (u \cdot \nabla)\partial_X u - u^i \partial_{x_i} X^l \partial_{x_l} u^k. \tag{2.5.3}$$

Let us separately test the nonlinear part by $\partial_X u$. We have

$$I = \int_\Omega (\partial_X u \cdot \nabla)u \, \partial_X u \, dx \leqslant \|\partial_X u\|^2_{L^4} \|\nabla u\|_{L^2}. \tag{2.5.4}$$

In our case $p = n = 2$, $q = 4$ we use Ladyzhenskya inequality (B.0.10)

$$\|\partial_X u\|_{L^4} \leqslant C\|\partial_X u\|^{\frac{1}{2}}_{L^2} \|\partial_X \nabla u\|^{\frac{1}{2}}_{L^2}, \tag{2.5.5}$$

where

$$\int |\partial_X u|^2 \, dx \leqslant \int |X|^2 |\nabla u|^2 \, dx \leqslant C\|\nabla u\|^2_{L^2}.$$

We implement (2.5.5) to (2.5.4) and get that

$$I \leqslant C\|u\|_{H^1_0} \|\partial_X u\|_{L^2} \|\nabla \partial_X u\|_{L^2} \leqslant C_1 \|u\|^2_{H^1_0} \|\nabla \partial_X u\|_{L^2} \leqslant C_1 \|u\|^2_{H^1_0} \|\partial_X u\|_{H^1_0}. \tag{2.5.6}$$

From the property of trilinear form (A.0.3), for the second of (2.5.3) term we obtain

$$II = \int_\Omega (u \cdot \nabla)\partial_X u \, \partial_X u \, dx = 0. \tag{2.5.7}$$

To the next term we apply general Hölder and Poincaré inequalities to get

$$III = \int_\Omega u^i \partial_{x_i} X^l \partial_{x_l} u^k X^s \partial_{x_s} u^k dx \leqslant \int_\Omega |u \nabla X \nabla u \partial_X u| dx \leqslant C\|u\|_{L^4} \|\nabla u\|_{L^2} \|\partial_X u\|_{L^4} \tag{2.5.8}$$

by Sobolev embedding, we have

$$III \leqslant C_2 \|u\|^2_{H^1_0} \|\partial_X u\|_{H^1_0}. \tag{2.5.9}$$

Summing up all the above estimates and using Hölder, Young' inequalities we get

$$\left| \int \partial_X \left[(u \cdot \nabla)u\right] \partial_X u \, dx \right| \leqslant C_3 \|u\|^2_{H^1} \|\partial_X u\|_{H^1} \leqslant C_4 \|f\|^4_{L^2} + C_5 \epsilon \|\partial_X u\|^2_{H^1}. \tag{2.5.10}$$

From Lemma IX 1.2 [11] and the same way as in (2.3.15) we deduce that there exists a uniquely determined $\partial_X p \in L^2(\Omega)$ for Navier-Stokes system (2.5.2).

Lemma 2.3.1 for propagated Stokes equation combined with (2.5.2) give us an estimate

$$C\|\nabla\partial_X u\|_{L^2}^2 \leqslant C_6\|f\|_{L^2}^2 + C_7\|\partial_X f\|_{L^2}^2 + C_4\|f\|_{L^2}^4 + C_8\epsilon\|\nabla\partial_X u\|_{L^2}^2. \tag{2.5.11}$$

Taking ϵ in such a way that $C > C_8\epsilon$, we have (2.5.1).

□

2.5.2 Second order derivative

The higher tangential regularity is a crucial component in proving Theorem 2.2.3, as it provides enhanced regularity in X direction for the solutions of the Navier-Stokes problem (2.1.1)-(2.1.2). The following lemma states the result

Lemma 2.5.2. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, $\Omega_S \subset \Omega$ with the boundary $\partial\Omega_S \in C^2$, and let X be a vector field satisfying (2.2.5). Assume $f, \partial_X f, \partial_X^2 f \in L^2(\Omega)$, and let u solve (2.1.1)-(2.1.2). Then $\partial_X^2 u \in H^1(\Omega)$, with an estimate*

$$\|\nabla\partial_X^2 u\|_{L^2} \leqslant C\Big(\|f\|_{L^2} + \|\partial_X f\|_{L^2} + \|\partial_X^2 f\|_{L^2} + \|f\|_{L^2}^2 + \|\partial_X f\|_{L^2}^2 + \|f\|_{L^2}^4\Big), \tag{2.5.12}$$

where $C := C(\Omega, X)$.

Proof. Consider

$$\partial_X^2\left[(u\cdot\nabla)u - \operatorname{div}\left[\nu(x)\mathbb{D}u\right] + \nabla p\right] = \partial_X^2 f \tag{2.5.13}$$

$$\partial_X^2 \operatorname{div} u = 0. \tag{2.5.14}$$

In order to prove the main estimate we just take the second directional derivative from the nonlinear term of (2.1.1), the rest has been proved in Lemma 2.3.2.

$$\begin{aligned}\partial_X^2\left[(u\cdot\nabla)u\right] &= \partial_X\left[(\partial_X u\cdot\nabla)u + (u\cdot\nabla)\partial_X u - u^i\partial_{x_i}X^l\partial_{x_l}u^k\right]\\ &= (\partial_X^2 u\cdot\nabla)u + 2(\partial_X u\cdot\nabla)\partial_X u + (u\cdot\nabla)\partial_X^2 u\\ &\quad - 2\partial_X u\cdot\nabla X\cdot\nabla u - u\cdot X\nabla^2 X\cdot\nabla u - 2u\cdot\nabla X\cdot\nabla\partial_X u + u(\nabla X)^2\nabla u.\end{aligned} \tag{2.5.15}$$

We test the above expression by $\partial_X^2 u$ and consider more precisely the most problematic term:

$$I = \int_\Omega (\partial_X^2 u\cdot\nabla)u\,\partial_X^2 u\,dx \leqslant \|\partial_X^2 u\|_{L^4}^2\|\nabla u\|_{L^2}. \tag{2.5.16}$$

By Ladyzhenskaya inequality (B.0.10), we have

$$\|\partial_X^2 u\|_{L^4} \leqslant C\|\partial_X^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_X^2 u\|_{L^2}^{\frac{1}{2}}, \tag{2.5.17}$$

where

$$\|\partial_X^2 u\|_{L^2} \leqslant C\|\nabla\partial_X u\|_{L^2}.$$

We implement these estimates in (2.5.16) and get that

$$
\begin{aligned}
I &\leqslant C_1\|u\|_{H_0^1}\|\nabla\partial_X u\|_{L^2}\|\nabla\partial_X^2 u\|_{L^2} \leqslant C_2\|u\|_{H_0^1}\|\partial_X u\|_{H_0^1}\|\nabla\partial_X^2 u\|_{L^2} \\
&\leqslant C_2\|u\|_{H^1}\|\partial_X u\|_{H_0^1}\|\partial_X^2 u\|_{H_0^1}.
\end{aligned}
\tag{2.5.18}
$$

The term $\int_\Omega (u\cdot\nabla)\partial_X^2 u\,\partial_X^2 u\,dx = 0$ will vanish by the property of trilinear form (A.0.3). The vector field X is smooth, and we have $u \in H_0^1(\Omega)$, $\partial_X u \in H_0^1(\Omega)$. Therefore, without loss of generality, can estimate the remaining terms in (2.5.15) as follows:

$$
\begin{aligned}
\int_\Omega \partial_X^2\,[(u\cdot\nabla)u]\,\partial_X^2 u\,dx &\leqslant C_3\|u\|_{H^1}\|\partial_X u\|_{H^1}\|\partial_X^2 u\|_{H_0^1} + C_4\|u\|_{H_0^1}^2\|\partial_X^2 u\|_{H_0^1} \\
&\quad + C_5\|\partial_X u\|_{H_0^1}^2\|\partial_X^2 u\|_{H_0^1} \\
&\leqslant (C_6\|u\|_{H_0^1}^2 + C_7\|\partial_X u\|_{H_0^1}^2)\|\partial_X^2 u\|_{H_0^1}.
\end{aligned}
\tag{2.5.19}
$$

Using Young's inequality with small ϵ and Lemma 2.5.1 to the above inequality, we get

$$
\int_\Omega \partial_X^2\,[(u\cdot\nabla)u]\,\partial_X^2 u\,dx \leqslant C_8\|f\|_{L^2}^4 + C_9\|\partial_X f\|_{L^2}^4 + C_{10}\|f\|_{L^2}^8 + \epsilon\|\partial_X^2 u\|_{H^1}^2. \tag{2.5.20}
$$

From Lemma IX 1.2 [11] and using the same approach as in (2.3.36) we deduce that there exists a uniquely determined $\partial_X^2 p \in L^2(\Omega)$ for Navier-Stokes system (2.5.13).

The estimate (2.3.45) from the final step of the proof of Lemma 2.3.2, combined with the above estimate for the nonlinear term, provides us with an estimate for (2.5.13)

$$
\begin{aligned}
C\|\nabla\partial_X u\|_{L^2}^2 &\leqslant B_2\|f\|_{L^2}^2 + F_2\|\partial_X f\|_{L^2}^2 + A_2\|\partial_X^2 f\|_{L^2}^2 + \epsilon\|\nabla\partial_X u\|_{L^2}^2 \\
&\quad + C_8\|f\|_{L^2}^4 + C_9\|\partial_X f\|_{L^2}^4 + C_{10}\|f\|_{L^2}^8.
\end{aligned}
\tag{2.5.21}
$$

By choosing ϵ in (2.5.21) such that $C > \epsilon$, we obtain (2.5.12).

□

2.6 Proof of Theorem 2.3

To prove Theorem 2.2.3, we rely on results stated from subsection 2.5.1. Specifically, we assume that the approximate solution to problem (2.1.1)-(2.1.2) possesses tangential regularity as stated in Lemma 2.5.1, and higher tangential regularity as outlined in Lemma 2.5.2.

Proof. In general, we aim to transform the global coordinate system (x_1, x_2) to a local coordinate system (τ, n) on $\partial\Omega_S$. The tangent vector aligns with the direction y_1 and the normal vector aligns with the direction y_2. The primary challenge lies in adapting the Navier-Stokes equations from a closed domain to the entire space.

Thus consider, that Navier-Stokes equation is given in the neighbourhood Σ of the boundary $\partial\Omega_S$ s.t. there are open domains $\Omega_S \subset \Omega_S'$ and $\Omega_S'' \subset \Omega_S$ that is $dist(x, x') = \delta$ for $x \in \partial\Omega_S$, $x' \in \partial\Omega_S'$, and also $dist(x, x'') = \delta$ for $x \in \partial\Omega_S$, $x'' \in \partial\Omega_S''$. Let us denote the neighbourhood of $\partial\Omega_S$ that is $\Omega_S' \setminus \Omega_S'' := \Sigma$, and $\Omega^{(1)} := \Omega \setminus (\Omega_S \cup \Sigma)$, $\Omega^{(2)} := \Omega_S \setminus \Sigma$ (Fig.2.1). Let us fix ϵ and introduce notations

$$
\Omega_\epsilon^{(1)} := \{x \in \Sigma \cup \Omega^{(1)} :\ dist(x, \Omega^{(1)}) < \epsilon\} \tag{2.6.1}
$$

$$\Omega_\epsilon^{(2)} := \{x \in \Sigma \cup \Omega^{(2)} : \; dist(x, \Omega^{(2)}) < \epsilon\}. \tag{2.6.2}$$

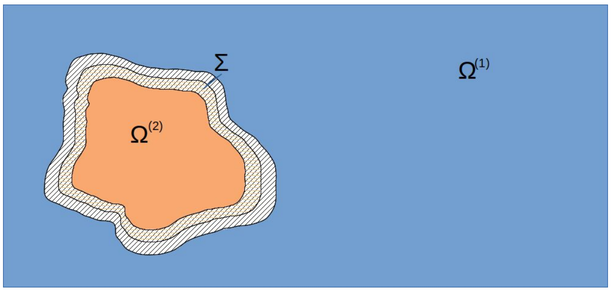


FIGURE 2.1: Σ neighborhood of the boundary of Ω_S.

There exists a Φ - C^2 diffeomorphism from Σ onto itself in the curvilinear system of coordinates. In this regard, we apply a classical change of variables for the curvilinear system of coordinates (Fig.2.2). Let $\Sigma = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1, \mathcal{O}_2$ - open sets. Consider one part $\mathcal{O}_1$ with its image $V = \Phi(\mathcal{O}_1)$.

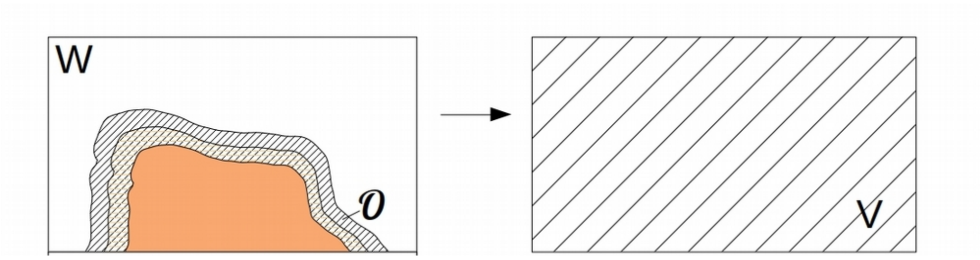


FIGURE 2.2: Change of the system of coordinates.

Change of variables takes form

$$y_i = \Phi_i(x), \; x_i = \Psi_i(y)$$

such that

$$J_\Psi = \left(\frac{\partial \Psi_i}{\partial y_j}\right)_{i,j}$$

$$J_\Phi = \left(\frac{\partial \Phi_i}{\partial x_j}\right)_{i,j},$$

with $\det J_\Psi(y) = 1, \;\; \forall y \in \mathbb{R}^2$ [29].

$$u(x) = J_\Psi(\Phi(x))U(\Phi(x))$$

$$U(y) = J_\Phi(\Psi(y))u(\Psi(y)), \tag{2.6.3}$$

i.e.,

$$U_i(y) = \sum_{j=1}^{2} \frac{\partial \Phi_i}{\partial x_j} u_j(\Psi(y))$$

$$P(y) = p(\Psi(y)).$$

The derivative along tangential field in the curvilinear system takes form

$$\partial_Y := \partial_{y_1}.$$

The jump of the viscosity field is transferred along y_2 direction, so we get the dependence

$$\nu(y_2) = \begin{cases} 1, & y_2 > 0 \\ m, & y_2 < 0. \end{cases} \tag{2.6.4}$$

We define the second order derivative operator as

$$\begin{aligned}[][\mathcal{L}U]_i =& \frac{\partial}{\partial y_j}\left\{\nu(y_2)g^{jk}\mathbb{D}_{ik}(U) + \nu(y_2)g^{kj}\Gamma^i_{lk}U_l\right\} + \nu(y_2)g^{kl}\Gamma^i_{lk}\mathbb{D}_{lk}(U) \\ &+ \nu(y_2)g^{jk}\Gamma^m_{lk}\Gamma^i_{jm}U_l \end{aligned} \tag{2.6.5}$$

gradient of the pressure

$$[\mathcal{G}P]_i = \sum_j g^{ij}\frac{\partial P}{\partial y_j}.$$

and the convection term

$$[\mathcal{N}U]_i = \sum_j U_j \frac{\partial U_i}{\partial y_j} + \sum_{j,k=1} \Gamma^i_{jk}U_jU_k.$$

Above, Γ^i_{jk} are Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left\{\frac{\partial g_{il}}{\partial g_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right\},$$

with contravariant vectors tensor

$$g^{ij} = \sum_k \frac{\partial \Phi_i}{\partial x_k}\frac{\partial \Phi_j}{\partial x_k},$$

and covariant vectors tensor

$$g_{ij} = \sum_k \frac{\partial \Psi_k}{\partial y_i}\frac{\partial \Psi_k}{\partial y_j}.$$

So, we have

$$\mathrm{div}_y\, U = 0$$

by Corollary A.3. in [9] .

Extending all the fields by Sobolev extension s.t. $EU = U$ in V and $EU = 0$ in $\mathbb{R}^2 \setminus V'$, $V \subset V'$, by Theorem II.3.3 in [11], we get Navier-Stokes equations in the curvilinear coordinates

$$-\mathcal{L}U + \mathcal{N}U + \mathcal{G}P = F \quad \text{in} \quad \mathbb{R}^2, \tag{2.6.6}$$

$$\mathrm{div}_y\, U = 0 \quad in \quad \mathbb{R}^2. \tag{2.6.7}$$

Note that in our case, we are using an orthogonal curvilinear coordinate system, so the metric tensor is diagonal

$$g_{ii} = h_i^2,$$

where h_i is scale factor. To replicate the proof concept from subsection 2.4, we will express equation equation (2.6.6) in terms of the physical components of vectors and tensors. According to [4], the physical component of a vector A is related to its contravariant and covariant components, A^i, A_i respectively, by the relation

$$A(i) := g_{ii}^{1/2} A^i = g_{ii}^{1/2} g^{ij} A_i, \tag{2.6.8}$$

we denote $U(i) := U_i$. In the orthogonal system of coordinates the differential operators takes form (see [4])

$$\begin{aligned}
[\mathcal{L}U]_i =& \frac{h_i}{h_1 h_2}\frac{\partial}{\partial y_j}\left\{\nu(y_2)\frac{h_1 h_2}{h_i h_j}\left(\frac{1}{2}(\frac{1}{h_j}\frac{\partial U_i}{\partial y_j}+\frac{1}{h_i}\frac{\partial U_j}{\partial y_i})+\frac{1}{h_k}\Gamma^i_{lk}U_l\right)\right\}\\
&+\nu(y_2)\frac{h_1 h_2}{h_j h_k}\Gamma^i_{jk}\left(\frac{1}{2}(\frac{1}{h_k}\frac{\partial U_j}{\partial y_k}+\frac{1}{h_j}\frac{\partial U_k}{\partial y_j})+\nu(y_2)\Gamma^m_{lk}\Gamma^i_{jm}U_l\right),
\end{aligned} \tag{2.6.9}$$

$$\Gamma^k_{pq} = \begin{cases} \frac{1}{h_i}\left(\frac{\partial h_i}{\partial x^j}\right), & \text{for } k = p = q = i = j, \text{ or } k = p \text{ and } p \neq q, \\ -\frac{h_i}{h_j^2}\left(\frac{\partial h_i}{\partial x^j}\right), & \text{for } p = q = i, k = j \\ 0, & \text{otherwise.} \end{cases} \tag{2.6.10}$$

$$[\mathcal{N}U]_i = \frac{h_i}{h_j}U_j\frac{\partial}{\partial y_j}\frac{U_i}{h_i} + \frac{h_i}{h_j h_k}\Gamma^i_{jk}U_jU_k.$$

$$[\mathcal{G}P]_i = \frac{1}{h_i}\frac{\partial P}{\partial y_i}.$$

Recalling that the space dimension is 2 and the curvilinear coordinate system is orthogonal, we obtain much simpler remainder terms from (2.6.9). By splitting the equation coordinate-wise, the first row of the resulting equation takes the form:

$$\begin{aligned}
&-\nu(y_2)\partial_{y_1}\left\{\frac{h_2}{h_1^2}\frac{\partial U_1}{\partial y_1}+\frac{h_2}{h_1^2}(\Gamma^1_{21}U_2)\right\}\\
&\quad-\partial_{y_2}\left\{\nu(y_2)\frac{1}{2}\left(\frac{1}{h_2}\frac{\partial U_1}{\partial y_2}+\frac{1}{h_1}\frac{\partial U_2}{\partial y_1}+(\frac{1}{h_1}\Gamma^2_{11}U_1+\frac{1}{h_2}\Gamma^1_{22}U_2)\right)\right\}\\
&\quad+\frac{h_1h_2}{h_j}(U_j\frac{\partial U_1}{\partial y_j}+\frac{1}{h_1}\Gamma^1_{11}U_1^2+\frac{2}{h_2}\Gamma^1_{12}U_1U_2+\frac{1}{h_2}\Gamma^1_{22}U_2^2)+\frac{h_2}{h_1}\frac{\partial P}{\partial y_1}=h_2\overline{F}_1.
\end{aligned} \tag{2.6.11}$$

where $\overline{F_1} = F_1 + \nu(y_2)\frac{h_1h_2}{h_jh_k}\Gamma^i_{jk}\left(\frac{1}{2}(\frac{1}{h_k}\frac{\partial U_j}{\partial y_k}+\frac{1}{h_j}\frac{\partial U_k}{\partial y_j})+\nu(y_2)\Gamma^m_{lk}\Gamma^i_{jm}U_l\right)$. Next we differentiate the above equation along tangential direction y_1

$$\begin{aligned}
&-\nu(y_2)\partial^2_{y_1}\left\{\frac{h_2}{h_1^2}\frac{\partial U_1}{\partial y_1}+\left[\frac{h_2}{h_1^2}(\Gamma^1_{21}U_2)\right]\right\}\\
&\quad-\partial_{y_1}\partial_{y_2}\left\{\nu(y_2)\frac{1}{2}\left(\frac{1}{h_2}\frac{\partial U_1}{\partial y_2}+\frac{1}{h_1}\frac{\partial U_2}{\partial y_1}+\frac{1}{h_1}\Gamma^2_{11}U_1+\frac{1}{h_2}\Gamma^1_{22}U_2\right)\right\}\\
&\quad+\partial_{y_1}(\frac{h_1h_2}{h_j}(U_j\frac{\partial U_1}{\partial y_j}+\frac{1}{h_1}\Gamma^1_{11}U_1^2+\frac{1}{h_2}\Gamma^1_{12}U_1U_2+\frac{1}{h_2}\Gamma^1_{22}U_2^2))\\
&\quad+\partial_{y_1}(\frac{h_2}{h_1}\frac{\partial P}{\partial y_1})=\partial_{y_1}(h_2\overline{F}_1).
\end{aligned} \tag{2.6.12}$$

Since $\partial\Omega_S \in C^2$ and compact, we have

$$|g^{ij}| \leqslant K_0, \qquad |\frac{1}{h_j}| \leqslant K_1$$

$$\left|\frac{\partial h_i}{\partial y_s}\right| \leqslant K_2, \qquad |\Gamma^i_{jk}| \leqslant K_3.$$

Recall, the space dimension $n = 1$ and $p = 2$, we use an estimate for the convection term as follows

$$\|U\nabla U\|_{L^2(\mathbb{R})} \leqslant C\|U\|^2_{H^1(\mathbb{R})}. \tag{2.6.13}$$

We obtained $u \in H^1(\Omega)$, which implies $U \in H^1(\mathbb{R}^2)$. By Lemma 2.5.1 and (2.6.3) we have $\nabla\partial_{y_1}U \in L^2(\mathbb{R}_{y_2}; L^2(\mathbb{R}_{y_1}))$. We take $L^2(\mathbb{R}_{y_2}; L^2(\mathbb{R}_{y_1}))$ norm of (2.6.12), and noting that differentiation in the y_1 direction is well-defined, we move all these terms to the RHS except the second term on the LHS. By applying the Cauchy-Schwarz, Young and Poincaré inequalities, we obtain:

$$\begin{aligned}
&\Big[\int_{\mathbb{R}_{y_2}}\int_{\mathbb{R}_{y_1}}\Big|\partial_{y_1}\partial_{y_2}\Big\{\nu(y_2)\Big(\frac{1}{2h_2}\frac{\partial U_1}{\partial y_2}+\frac{1}{2h_1}\frac{\partial U_2}{\partial y_1}+R\Big)\Big\}\Big|^2 d\,y_1 d\,y_2\Big]^{\frac{1}{2}}\\
&\quad\leqslant \|\nu(y_2)\partial^2_{y_1}\Big\{\frac{h_2}{h_1^1}(\frac{\partial U_1}{\partial y_1}+\Gamma^1_{21}U_2)\Big\}\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\quad+\|\partial_{y_1}\Big(\nu(y_2)\frac{h_1h_2}{h_jh_k}\Gamma^i_{jk}\Big(\frac{1}{2}(\frac{1}{h_k}\frac{\partial U_j}{\partial y_k}+\frac{1}{h_j}\frac{\partial U_k}{\partial y_j})+\nu(y_2)\Gamma^m_{lk}\Gamma^i_{jm}U_l\Big)\Big)\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\quad+\|\partial_{y_1}(h_2F_1)\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}+\|\partial_{y_1}(\frac{h_1h_2}{h_j}U_j\frac{\partial U_1}{\partial y_j})\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\quad+(1+K_3)\Big(\|U_1\|^2_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}+\|U_2\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\|U_1\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\Big)\\
&\quad+K_0\|\partial_{y_1}(\frac{h_2}{h_1}\partial_{y_1}P)\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}+K_3\|U_2\|^2_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))},
\end{aligned} \tag{2.6.14}$$

where $R := \Big[\frac{1}{h_1}\Gamma^2_{11}U_1 + \frac{1}{h_2}\Gamma^1_{22}U_2\Big]$. Estimating the remainder term

$$\begin{aligned}
&\|\partial_{y_1}\Big(\nu(y_2)\frac{h_1(h_2)^2}{h_jh_k}\Gamma^i_{jk}\Big(\mathbb{D}_{jk}U)+\nu(y_2)\Gamma^m_{lk}\Gamma^i_{jm}U_l\Big)\Big)\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\qquad\leqslant K_2K_3\|\nu(y_2)\mathbb{D}_{jk}U\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}+K_1K_3\|\nu(y_2)\partial_{y_1}\mathbb{D}_{jk}U\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\qquad+K_2K_3\|\nu(y_2)U_l\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}+K_1K_3\|\nu(y_2)\partial_{y_1}U_l\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\qquad\leqslant \overline{K}(\|\nu(y_2)\nabla U\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}+\|\nu(y_2)U_l\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}).
\end{aligned} \tag{2.6.15}$$

In the above inequality, we applied the Hölder's inequality in the y_1 direction, considering that the vector U is smooth in y_1. Additionally, we bounded the symmetric gradient by the gradient and combined the constants.

For given $\partial^2_{y_1}F \in L^2(\mathbb{R}_{y_2}; L^2(\mathbb{R}_{y_1}))$ we have $\partial^2_{y_1}P \in L^2(\mathbb{R}_{y_2}; L^2(\mathbb{R}_{y_1}))$ (see par.2.3.2) and from Lemma 2.5.2 $\nabla\partial^2_{y_1}U \in L^2(\mathbb{R}^2)$, which means the last three lines of (2.6.14) remains bounded.

Observe that

$$\begin{aligned}
&\Big\|\nu(y_2)(\frac{1}{2h_2}U_{1,2}+\frac{1}{2h_1}U_{2,1}+R)\Big\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\\
&\quad\leqslant \overline{K}\Big(\Big\|\partial_{y_1}\partial_{y_2}\Big\{\nu(y_2)\left(\frac{1}{2h_2}\frac{\partial U_1}{\partial y_2}+\frac{1}{2h_1}\frac{\partial U_2}{\partial y_1}+R\right)\Big\}\Big\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\quad+\|\nu(y_2)U_1\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}+\|\nu(y_2)U_2\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}\\
&\quad+\|\nu(y_2)\nabla U\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}+\|\nu(y_2)U\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}+C\Big),
\end{aligned} \tag{2.6.16}$$

where $\overline{K}$ is a constant that depends on S, n, and the constant C obtained in Lemmas 2.5.1 and 2.5.2. It is important to note that neither constant depends on the penalty parameter. In particular, using triangle and Poincaré inequalities we get

$$\|\nu(y_2)U\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\leqslant\|\nu(y_2)U_1\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}+\|\nu(y_2)U_2\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}$$

We bound the above inequality from below, than splitting the above norm into two parts using the triangle inequality

$$\begin{aligned}
\|\nu(y_2)\frac{1}{2h_2}U_{1,2}\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\leqslant&\Big\|\nu(y_2)\left(\frac{1}{2h_1}U_{2,1}+\frac{1}{2h_2}U_{1,2}+R\right)\Big\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\\
&+\Big\|\nu(y_2)\left(\frac{1}{2h_1}U_{2,1}+R\right)\Big\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))},
\end{aligned} \tag{2.6.17}$$

We've established that $U_{2,1}\in H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))$, from (2.5.12) we get

$$\|\partial_{y_1}\partial_{y_2}U_{2,1}\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\leqslant C(F), \tag{2.6.18}$$

it gives

$$\|U_{2,1}\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\leqslant C(F) \tag{2.6.19}$$

and by embedding, we have

$$\|U_{2,1}\|_{L^\infty(\mathbb{R}_{y_2};L^\infty(\mathbb{R}_{y_1}))}\leqslant C(F). \tag{2.6.20}$$

Using (2.6.17) in the inequality (2.6.16), we obtain

$$\begin{aligned}
&\|\nu(y_2)\frac{1}{2h_2}U_{1,2}\|_{L^\infty(\mathbb{R}_{y_2};L^\infty(\mathbb{R}_{y_1}))}\\
&\quad\leqslant \overline{K}\Big(\Big\|\partial_{y_1}\partial_{y_2}\Big\{\nu(y_2)\left(\frac{1}{2h_2}\frac{\partial U_1}{\partial y_2}+\frac{1}{2h_1}\frac{\partial U_2}{\partial y_1}+R\right)\Big\}\Big\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}\\
&\quad+\|\nu(y_2)U_1\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}\\
&\quad+\|\nu(y_2)U_2\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}+\|\nu(y_2)\nabla U\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}+C\\
&\quad+\|\nu(y_2)(\frac{1}{2h_1}U_{2,1}+R)\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))}\Big).
\end{aligned} \tag{2.6.21}$$

Using the following inequality

$$\|U_{1,2}\|_{L^\infty(\mathbb{R}_{y_2};L^\infty(\mathbb{R}_{y_1}))}\leqslant\Big\|\frac{2h_2}{\nu(y_2)}\Big\|_{L^\infty(\mathbb{R}_{y_2};L^\infty(\mathbb{R}_{y_1}))}\|\nu(y_2)\frac{1}{2h_2}U_{1,2}\|_{L^\infty(\mathbb{R}_{y_2};L^\infty(\mathbb{R}_{y_1}))}, \tag{2.6.22}$$

we obtain an estimate for $U_{1,2}$

$$\|U_{1,2}\|_{L^\infty(\mathbb{R}^2)} \leqslant \Big\| \frac{2h_2}{\nu(y_2)} \Big\|_{L^\infty(\mathbb{R}^2)} \{ \overline{K} \Big(\Big\| \partial_{y_1}\partial_{y_2} \Big\{ \nu(y_2) \Big(\frac{1}{2h_2}\frac{\partial U_1}{\partial y_2} + \frac{1}{2h_1}\frac{\partial U_2}{\partial y_1} + R \Big) \Big\} \Big\|_{L^2(\mathbb{R}_{y_2};L^2(\mathbb{R}_{y_1}))}$$
$$+ \|\nu(y_2)U_1\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))} + \|\nu(y_2)U_2\|_{L^2(\mathbb{R}_{y_2};H^2(\mathbb{R}_{y_1}))}$$
$$+ \|\nu(y_2)\nabla U\|_{L^2(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))} + C$$
$$+ \|\nu(y_2)U_{2,1}\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))} + \|\nu(y_2)(h_{1,2}U_1 + h_{2,1}U_2)\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))} \Big) \}. \tag{2.6.23}$$

So, we have that $U_{1,2} \in L^\infty(\mathbb{R}_{y_2}; L^\infty(\mathbb{R}_{y_1}))$.

From (2.5.12), we obtain

$$\|U_{1,1}\|_{H^1(\mathbb{R}_{y_2};H^1(\mathbb{R}_{y_1}))} \leqslant C(F), \tag{2.6.24}$$

and using embedding, we get

$$\|U_{1,1}\|_{L^\infty(\mathbb{R}_{y_2};L^\infty(\mathbb{R}_{y_1}))} \leqslant C(F). \tag{2.6.25}$$

Recall that, by (2.6.7) we have $U_{1,1} = -U_{2,2}$, which gives that

$$U_{2,2} \in L^\infty(\mathbb{R}_{y_2}; L^\infty(\mathbb{R}_{y_1})).$$

Combining all estimates we have that $\nabla U \in L^\infty(\mathbb{R}_{y_2}; L^\infty(\mathbb{R}_{y_1}))$. By extension theorem and using (2.6.3) that is $C_1\|\nabla U\| \leqslant \|\nabla u\| \leqslant C_2\|\nabla U\|$, we conclude that $\nabla u \in L^\infty(\Sigma)$. The conclusion follows.

□

The next lemma gives a higher regularity of u in $\Omega^{(1)} \cup \Omega^{(2)}$.

Lemma 2.6.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2. And let u satisfy Navier-Stokes system of equations (2.1.1)-(2.1.2), $q > 2$. If $f \in L^q(\Omega)$ then $u \in W^{2,q}(\Omega^{(1)} \cup \Omega^{(2)})$ and $p \in W^{1,q}(\Omega^{(1)} \cup \Omega^{(2)})$. In particular, $\nabla u \in L^\infty(\Omega^{(1)} \cup \Omega^{(2)})$ and there exists $p \in W^{1,q}(\Omega^{(1)} \cup \Omega^{(2)})$ such that (2.1.1) is satisfies a.e.*

Proof. We have

$$\nu(x) = \begin{cases} 1, & x \in \Omega^{(1)} \\ m, & x \in \Omega^{(2)}. \end{cases} \tag{2.6.26}$$

We localize the problem, by using a "cut off" function $\eta \in C_0^\infty(\mathbb{R}^2)$ such that

$$\eta^{(1)}(x) = \begin{cases} 1, & \mathbf{x} \in \Omega^{(1)} \\ 0, & \mathbf{x} \in \mathbb{R}^2 \setminus \Omega_\epsilon^{(1)}. \end{cases} \tag{2.6.27}$$

Putting $w = u\eta^{(1)}$, $\pi = p\eta^{(1)}$, and taking into account (2.1.4) we have that w and π satisfies the following problem

$$-\text{div } [\mathbb{D}w] + u \cdot \nabla w + \nabla \pi = F \quad \text{in } \Omega_\epsilon^{(1)} \tag{2.6.28}$$

$$\text{div } w = g \tag{2.6.29}$$

and

$$w = 0 \quad \text{in} \quad \partial\Omega_\epsilon^{(1)},$$

where

$$\begin{aligned} F =& \eta^{(1)} f + p\nabla\eta^{(1)} - u \operatorname{div}(\nabla\eta^{(1)}) - (\mathbb{D}u)(\nabla\eta^{(1)})^\intercal - (\nabla\eta^{(1)} \cdot \nabla)u \\ & - (u\cdot\nabla)(\nabla\eta^{(1)}) + u\cdot\nabla\eta^{(1)} u \end{aligned} \tag{2.6.30}$$

that is

$$\begin{aligned} F =& \eta^{(1)} f + p\nabla\eta^{(1)} - u\,\Delta\eta^{(1)} - u\nabla^2\eta^{(1)} - (\nabla^T u)(\nabla\eta^{(1)}) - 2\nabla u \nabla\eta^{(1)} \\ & + u\cdot\nabla\eta^{(1)} u \end{aligned} \tag{2.6.31}$$

$$g = \nabla\eta^{(1)} \cdot u.$$

Recalling the property of η that is

$$|\nabla\eta^{(1)}| \leqslant C.$$

In some sense, we will repeat the proof according to Galdi [11]. We want to show that

$$(u, p) \in W^{2,q}(\Omega_\epsilon^{(1)}) \times W^{1,q}(\Omega_\epsilon^{(1)}). \tag{2.6.32}$$

We know that there is

$$u \in W^{1,2}(\Omega_\epsilon^{(1)}) \tag{2.6.33}$$

that satisfies Navier-Stokes system (2.1.1). By the Lemma IX.2.1 ([11]) and Lemma 2.1., we deduce there is $p \in L^2(\Omega_\epsilon^{(1)})$.

$$\|F\|_q \leqslant c_1\|f\|_q + c_2\|p\|_q + c_3\|\nabla u\|_q + c_3\|u\|_q^2. \tag{2.6.34}$$

So that by Lemma IX.5.1 [11] and (2.6.33) we want to prove (2.6.32). Assume next $q > 2$. Then, we have

$$u \in W^{1,q}(\Omega_\epsilon^{(1)}), \quad p \in L^q(\Omega_\epsilon^{(1)}),$$

yielding,

$$u \cdot \nabla u \in L^q(\Omega_\epsilon^{(1)}).$$

From the interior estimates for the Stokes problem proved in Theorem IV.4.1 [11] follows that $u \in W^{2,q}(\Omega_\epsilon^{(1)})$ and $p \in W^{1,q}(\Omega_\epsilon^{(1)})$.

Denote $F' = F - u\cdot\nabla w$. By the Hölder inequality and recalling the properties of η it follows that

$$\|u\cdot\nabla w\|_q \leqslant C\|u\|_\infty\|\nabla u\|_q \leqslant C \quad \text{in} \quad \Omega_\epsilon^{(1)}.$$

So, we have $F' \in L^q(\Omega_\epsilon^{(1)})$, using Lemma IX.5.1 [11], we deduce that $w \in W^{2,q}(\Omega_\epsilon^{(1)})$, $\pi \in W^{1,q}(\Omega_\epsilon^{(1)})$ and satisfy an estimate

$$\|w\|_{2,q,\Omega_\epsilon^{(1)}} + \|\pi\|_{1,q,\Omega_\epsilon^{(1)}} \leqslant C(\|F\|_{q,\Omega^{(1)}} + \|g\|_{1,q,\Omega_\epsilon^{(1)}}) \tag{2.6.35}$$

also, using the Theorem IX.5.1 [11] and properties of η we derive

$$\|u\|_{2,q,\Omega_\epsilon^{(1)}} + \|p\|_{1,q,\Omega_\epsilon^{(1)}} \leqslant C_1(\|F\|_{q,\Omega_\epsilon^{(1)}} + \|g\|_{1,q,\Omega_\epsilon^{(1)}}), \tag{2.6.36}$$

For the case $q > 2$, we have embedding $W^{1,q}(\Omega_\epsilon^{(1)}) \hookrightarrow L^\infty(\Omega_\epsilon^{(1)})$, thus $\nabla u \in L^\infty(\Omega_\epsilon^{(1)})$,

$$\|\nabla u\|_{L^\infty(\Omega_\epsilon^{(1)})} + \|p\|_{1,q,\Omega_\epsilon^{(1)}} \leqslant C_2(\|F\|_{q,\Omega_\epsilon^{(1)}} + \|g\|_{1,q,\Omega_\epsilon^{(1)}}). \tag{2.6.37}$$

Let us now consider the another "cut off" function $\eta^{(2)}$ such that,

$$\eta^{(2)} = \begin{cases} 1, & \mathbf{x} \in \Omega^{(2)} \\ 0, & \mathbf{x} \in \mathbb{R}^2 \setminus \Omega_\epsilon^{(2)}. \end{cases} \tag{2.6.38}$$

We localize the problem (4.2)-(4.3), and put $v = u\eta^{(2)}$, $P = p\eta^{(2)}$, using the properties of the function $\eta^{(2)}$ we get

$$-m \operatorname{div} [\mathbb{D}v] + u \cdot \nabla v + \nabla \pi = F \quad \text{in } \Omega_\epsilon^{(2)} \tag{2.6.39}$$

$$\operatorname{div} v = g, \tag{2.6.40}$$

and

$$w = 0 \quad \text{in} \quad \partial\Omega_\epsilon^{(2)}.$$

From the bounds of the first case, using that $F \in L^q(\Omega_\epsilon^{(2)})$, $g \in W^{1,q}(\Omega_\epsilon^{(2)})$, and using Lemma IX.5.1, Theorem IX.5.1 ([11]) we get an estimate

$$\|\nabla u\|_{1,q,\Omega_\epsilon^{(2)}} + \frac{1}{m}\|p\|_{1,q,\Omega_\epsilon^{(2)}} \leqslant \frac{C_1}{m}(\|F\|_{q,\Omega_\epsilon^{(2)}} + \|g\|_{1,q,\Omega_\epsilon^{(2)}}) \tag{2.6.41}$$

using the embedding to $W^{1,q}(\Omega_\epsilon^{(2)}) \hookrightarrow L^\infty(\Omega_\epsilon^{(2)})$,

$$\|\nabla u\|_{L^\infty(\Omega_\epsilon^{(2)})} + \frac{1}{m}\|p\|_{1,q,\Omega_\epsilon^{(2)}} \leqslant \frac{C_3}{m}(\|F\|_{q,\Omega_\epsilon^{(2)}} + \|g\|_{1,q,\Omega_\epsilon^{(2)}}). \tag{2.6.42}$$

So, from (2.6.34) and (2.6.37) we conclude that $\nabla u \in L^\infty(\Omega_\epsilon^{(1)} \cup \Omega_\epsilon^{(2)})$.

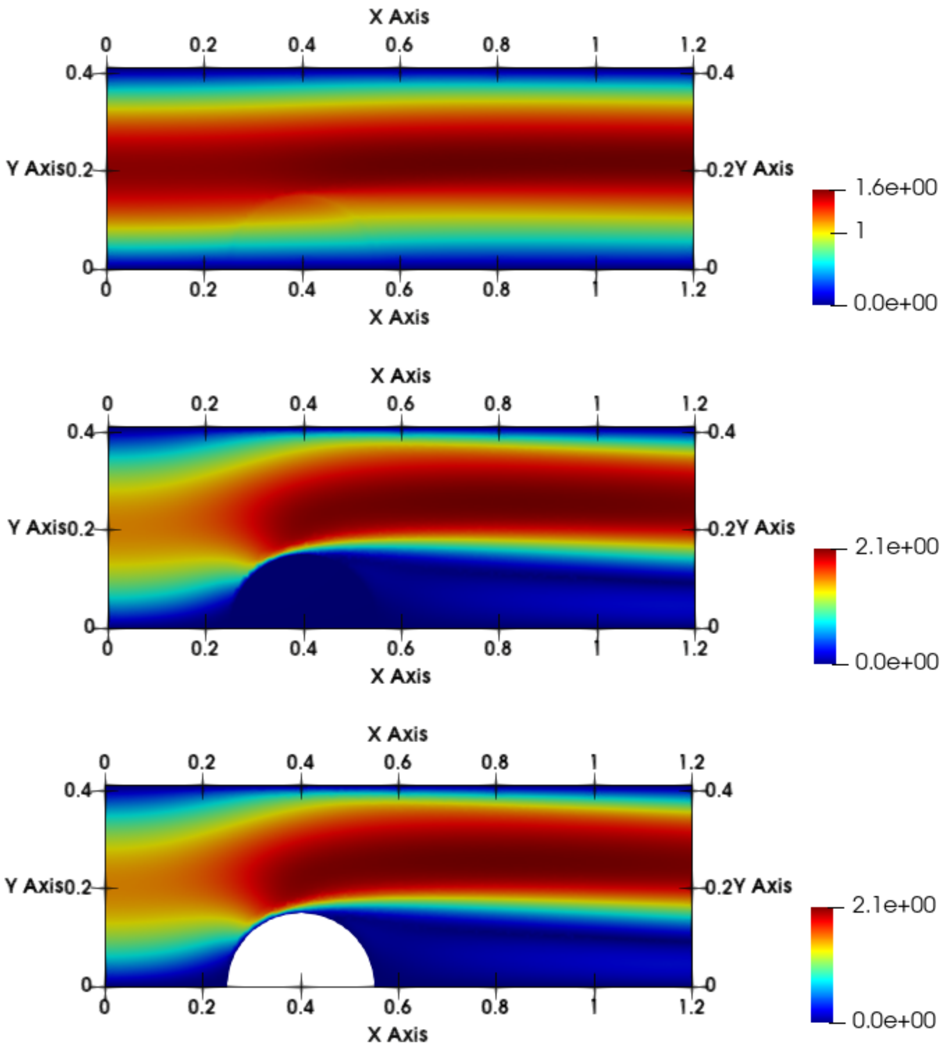
□

2.7 Numerical simulations

In this section, we will illustrate Theorem 2.2.2 with some numerical simulations, and show that the approximate problem (2.1.1-2.1.4) has a potential application in practice.

We conduct two numerical experiments, where we consider a smooth obstacle (a half ball) and an obstacle with sharp corners (a wall). We do a number of tests with increasing value of the penalizing viscosity m (see 2.1.4). It turns out that for moderately high values of m the approximate flow is very similar to the actual flow around the obstacle. We consider a rectangular channel flow problem in $\Omega \setminus \Omega_S$ with fixed rigid obstacle Ω_S touching the boundary of Ω ([27], [19]). We refer to it as a "real obstacle" problem and denote the velocity by u. The experimental data and the geometry of channel flow are inherited from the Turek's benchmark [27], test case 2D-2, with two exceptions; the channel's length equals $L = 1.2$, and the obstacles are of different shape and touch the boundary (Fig.1.1-2.1]). The experiments with half ball obstacle are illustrated in Fig.2.3 with radius $R = 0.15$ and center at $(0.4; 0.0)$. And the experiments with the wall obstacle are in Fig.2.4 with height $h = 0.16$, width $w = 0.1$ and center of symmetry $x = 0.4$. We assume a parabolic velocity profile at the inlet and Dirichlet boundary condition on the boundary. We compare the solution of the real obstacle problem with the solution of the approximate problem (2.1.1)-(2.1.4), where the fixed obstacle domain Ω_S is filled with highly viscous fluid of viscosity m.

The results have been computed with the FEniCS package [22] using the incremental pressure correction scheme to solve the problem ([13]).



(a)

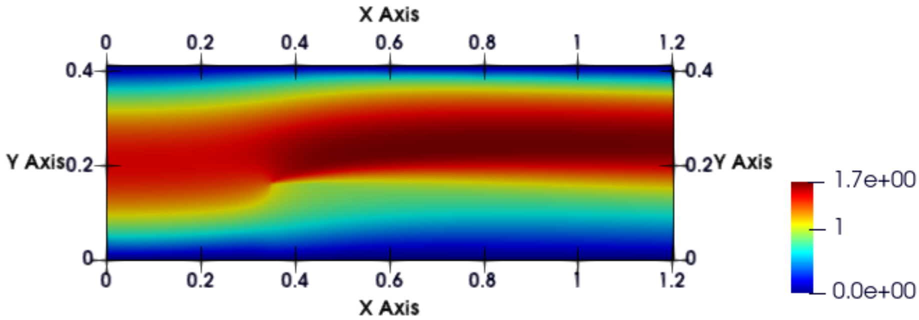
(b)

(c)

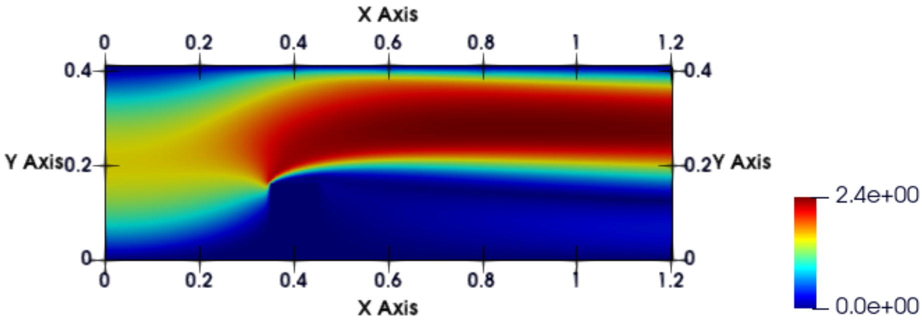
FIGURE 2.3: The magnitude of the velocity for the half ball test problem. The (a),(b) cases correspond to parameters with $k = 1$ or 4, respectively. The (c) correspond to the real obstacle test case.

We compare solutions in a cross-section y at $x = 0.4$, along the vertical axis of symmetry of the obstacle. The comparison graphs of extracted solution of each test are given in Fig.2.5-2.6. In these graphs, the velocity u_m represents the viscosity approximate solution for the penalty parameter $m = 10^k$, corresponding to test case number k. Fig.2.5-2.6 show that with increasing penalty, the gradient of approximate solutions diminishes, as predicted by Theorem 2.2.2. Moreover, the flow inside Ω_F becomes indistinguishable from the true flow around a real obstacle, as visualized in Figures 2.3-2.6.

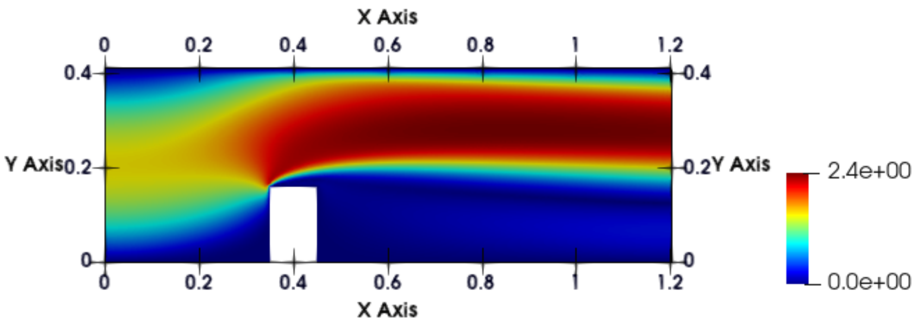
Additional simulations for the time-dependent case are available on the [24] web-page, where a qualitative side-by-side comparison illustrates two time-evolutionary



(a)



(b)



(c)

FIGURE 2.4: The magnitude of the velocity for the wall test problem. The (a), (b) cases correspond to parameters with $k = 2$ or 5, respectively. The (c) correspond to the real obstacle test case.

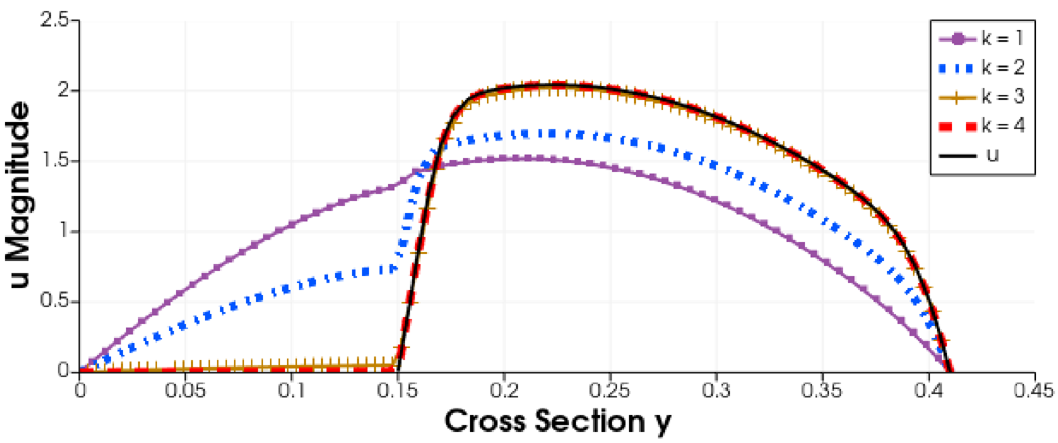


FIGURE 2.5: Comparison of the velocity magnitude for the half ball case. The velocity u_m represents the viscosity approximate solution corresponding to test cases $k := 1, .., 4$. The plot u(black) corresponds to the solution for the real obstacle problem.

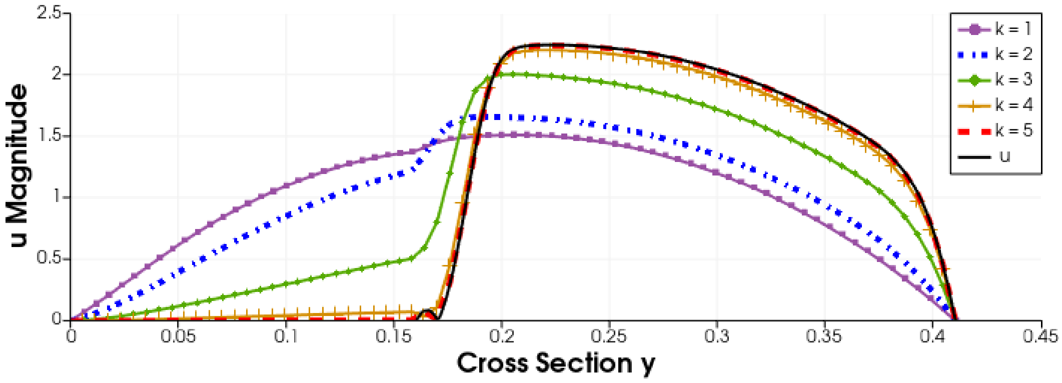


FIGURE 2.6: Comparison of velocity magnitude for the wall case. The velocity u_m represents the viscosity approximate solution corresponding to test cases $k := 1, .., 5$. The plot u corresponds to the solution for the real obstacle problem.

simulations of flow around obstacles: one with a "real obstacle" (1.1.1) and the other with its penalized approximation (1.2.19).

Chapter 3

Comparative analysis of obstacle approximation strategies for the steady incompressible Navier-Stokes equations

This chapter is entirely based on [17], except for minor notational improvements. It aims to compare and evaluate various obstacle approximation techniques employed in the context of the steady incompressible Navier-Stokes equations. Specifically, we investigate the effectiveness of a standard volume penalization approximation and an approximation method utilizing high viscosity inside the obstacle region, as well as their composition. Analytical results concerning the convergence rate of these approaches are provided, and extensive numerical experiments are conducted to validate their performance.

3.1 Introduction

Let us consider a domain $\Omega \subset \mathbb{R}^d$ where $d = 2$, or 3, in which a solid obstacle Ω_S is immersed. Let us assume that the remaining part of the region, $\Omega_F = \Omega \setminus \bar{\Omega}_S$, is occupied by a viscous incompressible fluid, whose motion is governed by the Navier–Stokes equations:

$$\begin{aligned} -\nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad & \text{in } \Omega_F, \\ \operatorname{div} u = 0 \quad & \text{in } \Omega_F, \\ u = 0 \quad & \text{on } \partial\Omega_F. \end{aligned} \tag{3.1.1}$$

Here, $u : \Omega_F \to \mathbb{R}^d$ is the velocity of the fluid and $p : \Omega_F \to \mathbb{R}$ denotes the pressure. Positive constant ν is the kinematic viscosity and $f : \Omega_F \to \mathbb{R}^d$ is the external force. We assume a no-slip boundary condition on the fluid–solid interface $\Sigma = \bar{\Omega}_F \cap \bar{\Omega}_S$ (cf. Figure 3.1) and, for simplicity, the same homogeneous Dirichlet boundary condition on the fluid velocity u on other parts of $\partial\Omega_F$ as well. We will refer to equation (3.1.1) as the "real obstacle" problem with constant viscosity.

The central inquiry under consideration in this chapter revolves around the effectiveness of approximating the problem (3.1.1) through the utilization of a suitable system defined over the entire domain Ω, rather than confining it solely to Ω_F. The proposed concept is rather straightforward: within the solid region Ω_S, a penalization term is introduced, which, for a pertinent parameter value, renders the system comparable to the original (3.1.1). The volume penalization method stands out as the most widely known and scrutinized technique in this regard, as it incorporates

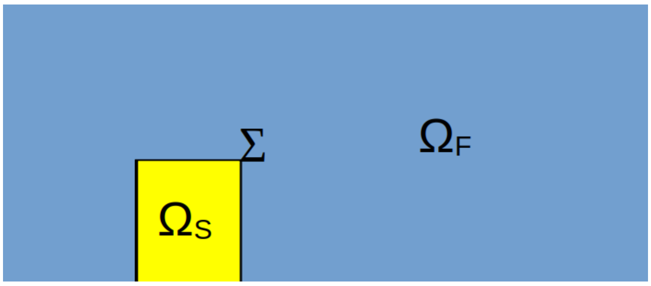


FIGURE 3.1: Decomposition of Ω into fluid domain Ω_F (blue region) and solid obstacle Ω_S (yellow region). The interface between the fluid and the solid is denoted by Σ and marked with a solid black line.

a friction term over the set Ω_S. Consequently, as the friction parameter tends towards infinity, we attain, at least formally, the equivalent of (3.1.1). Nonetheless, a notable concern arises regarding the convergence of such an approach, as it results in the diminution of the L^2-norm of the solution at the obstacle. Consequently, certain phenomena associated with the shape of the body immersed in the fluid may not be accurately captured. To address this issue, we undertake a more comprehensive examination of the viscosity penalization method. In this case, a high viscosity coefficient is imposed inside the region Ω_S, which ultimately yields the equivalent of (3.1.1) in the limit. From the perspective of weak solutions, this method exhibits a more natural behavior. However, it necessitates meticulous attention due to the intricate convergence analysis that is sought.

Consequently, we investigate three distinct types of such approximations (where, with a slight abuse of notation, $f : \Omega \to \mathbb{R}^d$ represents the force term of equation (3.1.1), extended by zero outside Ω_F).

Volume penalization One popular approach augments the momentum equation with a penalization term, aiming at slowing down the fluid inside the obstacle region. With this approach, one aims at suppressing the motion of the fluid in the obstacle region by introducing inside Ω_S damping, controlled by (large enough) parameter $n \geq 0$, so that the approximate solution (u_n, p_n) satisfies

$$\begin{aligned} -\nu\Delta u_n + (u_n \cdot \nabla)u_n + \nabla p_n + \eta_n u_n &= f \quad \text{in } \Omega, \\ \operatorname{div} u_n &= 0 \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1.2}$$

where η_n is a nonnegative piecewise constant function depending on a penalty parameter $n \geq 0$,

$$\eta_n(x) = \begin{cases} 0, & x \in \Omega_F, \\ n, & x \in \Omega_S. \end{cases}$$

Since the method essentially treats the obstacle as a porous medium whose permeability coefficient is proportional to $1/n$, it is sometimes called Brinkman penalization; in other papers it is referred to as the L^2 penalization. Here we mention a number of works based on a volume penalization method such as [1], [2], [3],[15]. In works of Angot [2], and Angot, Bruneau and Fabrie [3], authors established the strong convergence of the solutions and derived some error estimates for the approximate and exact problem in the steady Stokes system and unsteady Navier-Stokes with homogeneous boundary data. The further analysis of error estimates for steady systems with inhomogeneous boundary conditions were done recently by Aguayo and Lincopi [1].

Viscosity penalization Similar effect may be realized by introducing a very large artificial viscosity in Ω_S instead — though this approach is viable only if the obstacle is pinned to the boundary of Ω. The approximate solution (u_m, p_m) is defined on Ω by the Navier–Stokes system

$$\begin{aligned} -\operatorname{div}(\mu_m \nabla u_m) + (u_m \cdot \nabla)u_m + \nabla p_m &= f \quad \text{in } \Omega, \\ \operatorname{div} u_m &= 0 \quad \text{in } \Omega, \\ u_m &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1.3}$$

where μ_m is a positive, piecewise constant function depending on a penalty parameter $m > 0$,

$$\mu_m(x) = \begin{cases} \nu, & x \in \Omega_F, \\ m\,\nu, & x \in \Omega_S. \end{cases}$$

The viscosity penalization method was developed by Hoffmann and Starovoitov [14], and San Martin *et al* [26]. This method was used in works of Wróblewska-Kamińska [32], and Starovoitov [28] to construct solutions to systems describing motion of rigid bodies immersed in incompressible fluid.

Mixed penalization The third possibility that we will consider here is simply a combination of the two above mentioned approaches, leading to a system of the form

$$\begin{aligned} -\operatorname{div}(\mu_m \nabla u_{m\vee n}) + (u_{m\vee n} \cdot \nabla)u_{m\vee n} + \nabla p_{m\vee n} + \eta_n u_{m\vee n} &= f \quad \text{in } \Omega, \\ \operatorname{div} u_{m\vee n} &= 0 \quad \text{in } \Omega, \\ u_{m\vee n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1.4}$$

where ν_m, η_n are defined as above and $(u_{m\vee n}, p_{m\vee n})$ denote the approximate solution. Clearly, this formulation covers the previous ones as special cases: if $n = 0$, then $u_{m\vee n} \equiv u_m$ and similarly, for $m = 1$ there holds $u_{m\vee n} \equiv u_n$. Penalization of mixed type has been considered among others in [2], [3] for $m = n$; the second includes also penalization of the time derivative. The authors provide theoretical results on the rate of convergence and their numerical validation.

The primary objective of our study is to advance in this research direction by offering analytical insights into the convergence rate of the aforementioned obstacle approximation methods, including their combined approach. Our analysis encompasses both two-dimensional and three-dimensional scenarios.

To validate the theoretical findings, we conduct a comprehensive numerical investigation to assess the convergence rate in the two-dimensional case. The numerical experiments are designed to encompass all above mentioned penalization approaches and diverse obstacle shapes as well, ensuring a robust evaluation of the proposed approximation methods.

By undertaking this analytical and numerical analysis, we aim to contribute to a deeper understanding of the performance and efficacy of these approximation techniques in capturing the behavior of fluid flow around obstacles. This research has the potential to inform and guide practitioners in selecting the most suitable approach based on the Reynolds number and the specific geometric characteristics of the obstacles encountered in practical applications.

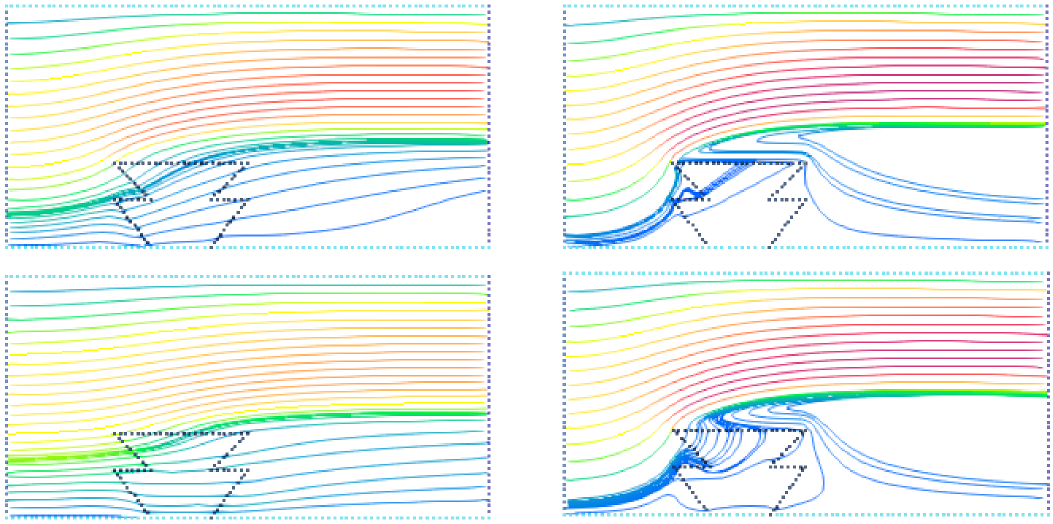


FIGURE 3.2: Streamlines plots, with color corresponding to velocity magnitude, for viscosity penalization (top row) and for volume penalization (bottom row) and varying penalty parameter. Panels on the left correspond to penalty parameter $m = n = 10^3$; those on the right, to $m = n = 10^5$. (For details regarding the experimental setting, see Section 3.3.2)

As a practical application, we present a demonstrative example (Figure 3.2) to showcase the performance of two approximation methods considered here. From this experiment it follows that while viscosity penalization with $m = 10^5$ already results in a quite plausible flow, for the same penalty value $n = 10^5$ one still gets visible non-physical artifacts when using volume penalization. From our subsequent analysis, it becomes evident that the approach based on the viscous penalization method exhibit favorable error characteristics. Further experiments, presented in Section 3.3, show that a combination of both methods — i.e., (3.1.4) — leads to further improvement of the accuracy.

This observation highlights the effectiveness and suitability of obstacle approximation techniques based on the incorporation of high viscosity within the obstacle region. Numerical results provide evidence supporting the practical viability of these methods in accurately capturing fluid flow behavior around obstacles. Clearly, to develop a robust numerical solver based on the penalization idea, one would need to put a significant amount of further effort. In particular, the ill-conditioning resulting from very high contrast in the coefficients would require one to employ an efficient preconditioner; otherwise the overall performance of the solver will suffer. Several promising preconditioners for high contrast Stokes equations have recently been developed (see [31] for an example) and may possibly be adapted to fit the present framework. However, in this paper we do not investigate such practical issues, and use numerical simulations only as a mean to verify the quality of the approximation and sharpness of theoretical estimates.

3.1.1 Weak formulation

We shall assume that Ω, Ω_F and Ω_S are bounded, open domains in $\mathbb{R}^d$, and that their boundaries are sufficiently regular. As mentioned above, $\bar{\Omega} = \bar{\Omega}_F \cup \bar{\Omega}_S$, while $\Omega_F \cap \Omega_S = \emptyset$. We also assume that the interface $\Sigma = \bar{\Omega}_F \cap \bar{\Omega}_S$ has a positive measure.

As we are working with weak solutions, we shall recall the weak formulation of the original and approximate problems. For this purpose we define

$$V_F = \{v \in H_0^1(\Omega_F) : \text{div } v = 0\} \qquad \text{and} \qquad V = \{v \in H_0^1(\Omega) : \text{div } v = 0\}. \quad (3.1.5)$$

Assume $f \in V_F'$. A weak solution to (3.1.1) is $u \in V_F$ such that

$$\int_{\Omega_F} (u \cdot \nabla) u \, v + \nu \nabla u : \nabla v \, dx = \langle f, v \rangle_{V_F', V_F} \qquad (3.1.6)$$

holds for any $v \in V_F$, where $\langle \cdot, \cdot \rangle$ is the duality pairing.

In order to define weak solutions to approximate problems we assume $f \in V'$. Then a weak solution to the viscosity penalization equation (3.1.3) is $u_m \in V$ such that

$$\int_{\Omega} (u_m \cdot \nabla) u_m \, v + \mu_m \nabla u_m : \nabla v \, dx = \langle f, v \rangle_{V', V} \qquad (3.1.7)$$

holds for any $v \in V$.

Next, by a weak solution to the volume penalization problem (3.1.2) we mean $u_n \in V$ s.t.

$$\int_{\Omega} (u_n \cdot \nabla) u_n \, v + \nu \nabla u_n : \nabla v + \eta_n u_n v \, dx = \langle f, v \rangle_{V', V} \qquad (3.1.8)$$

holds for every $v \in V$.

Finally, a weak solution to the mixed penalization system equation (3.1.4) is $u_{m \vee n} \in V$ such that

$$\int_{\Omega} (u_{m \vee n} \cdot \nabla) u_{m \vee n} v + \mu_m \nabla u_{m \vee n} : \nabla v + \eta_n u_{m \vee n} v \, dx = \langle f, v \rangle_{V', V} \qquad (3.1.9)$$

holds for every $v \in V$.

3.2 Theoretical bounds on the convergence rate

In this section we prove convergence and upper bounds on the approximation error with respect to penalty parameters for all three approximation schemes introduced above.

3.2.1 Volume penalization

First we recall the error estimates for the approximation of the flow by means of volume penalization. For inflow condition on the velocity and $f = 0$ they have been proved recently in [1, Theorems 5 and 6]. In order to understand the statement of results we shall recall that for stationary version of the Navier-Stokes system (3.1.1) we are not able to require the uniqueness of solutions. This feature holds only for some restrictive cases like smallness of the external force. For that reason in the large data case, our approximation defines the original solution on a certain subsequence. The proof in our setting requires only minor modification, therefore we skip it.

THEOREM 3.2.1. *Assume $f \in H^{-1}(\Omega)$ and Ω is a Lipschitz domain. Let u_n denote a weak solution to equation (3.1.2). Then*

$$\|u_n\|_{L^2(\Omega_S)} \leq C n^{-1/2}. \qquad (3.2.1)$$

Moreover, for a subsequence u_{n_k}, denoted in what follows u_n,

$$\lim_{n\to\infty} \|u - u_n\|_{H^1(\Omega)} = 0, \tag{3.2.2}$$

where u is a weak solution to equation (3.1.1). Assuming additionally that $\partial\Omega \in C^2$, $f \in L^2(\Omega)$ and $\|f\|_{H^{-1}(\Omega)}$ is small enough with respect to ν we have

$$\|u_n\|_{L^2(\Omega_S)} \le Cn^{-3/4}, \tag{3.2.3}$$

$$\|u - u_n\|_{H^1(\Omega_F)} \le Cn^{-1/4}. \tag{3.2.4}$$

3.2.2 Viscosity penalization

It turns out that for the approximation by means of viscosity penalization we are able to obtain better bounds for the convergence rate, however we have to assume that the obstacle touches the boundary of Ω to ensure the convergence of the approximate solution on the obstacle domain to zero (otherwise we would only obtain a constant flow). A result of this kind is expected, but to our knowledge has not been proved so far in the stationary case.

THEOREM 3.2.2. *Assume Ω and Ω_F are Lipschitz domains and $f \in H^{-1}(\Omega)$. Assume moreover that and*

$$\mathit{int}\,\overline{\partial\Omega_S \cap \partial\Omega} = \partial\Omega_S \cap \partial\Omega \quad \text{and} \quad \lambda_S(\partial\Omega_S \cap \partial\Omega) > 0, \tag{3.2.5}$$

where λ_S is the surface Lebesgue measure. Let u be a weak solution to equation (3.1.1) and u_m a weak solution to (3.1.3). Then

$$\|u_m\|_{H^1(\Omega_S)} \le Cm^{-1/2}\nu^{-1}, \tag{3.2.6}$$

Moreover, there exists a subsequence u_{m_k}, which we will denote again by u_m, such that

$$\lim_{m\to\infty} \|u - u_m\|_{H^1(\Omega_F)} = 0, \tag{3.2.7}$$

where u is a weak solution to equation (3.1.1). If additionally Ω and Ω_F are C^2 domains, while $f \in L^2(\Omega)$ and $\|f\|_{H^{-1}(\Omega)}$ is sufficiently small with respect to ν, then

$$\|u_m\|_{H^1(\Omega_S)} \le C(\nu m)^{-1}, \tag{3.2.8}$$

$$\|u - u_m\|_{H^1(\Omega_F)} \le C\nu^{-1}m^{-1/2}. \tag{3.2.9}$$

Proof. Let $(u, p) \in V(\Omega) \times L^2(\Omega)$ be the solution of stationary Navier–Stokes equations

$$\begin{aligned} -\nu\,\Delta u_F + (u_F \cdot \nabla)u_F + \nabla p_F &= f_F \quad \text{in } \Omega_F, \\ \operatorname{div} u_F &= 0 \quad \text{in } \Omega_F, \\ u_F &= 0 \quad \text{on } \partial\Omega_F. \end{aligned} \tag{3.2.10}$$

with the prolongation $(u_S, p_S) = (0, 0)$ in Ω_S (see [2]). Taking u_m as a test function in equation (3.1.7) we get

$$\int_{\Omega_F} \nu|\nabla u_m|^2\,dx + \nu m \int_{\Omega_S} |\nabla u_m|^2\,dx = \langle f, u_m\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \le \|f\|_{H^{-1}(\Omega)}\|\nabla u_m\|_{L^2(\Omega)}. \tag{3.2.11}$$

Using Young inequality, we have

$$\frac{\nu}{2}\|\nabla u_m\|^2_{L^2(\Omega_F)} + \frac{\nu m}{2}\|\nabla u_m\|^2_{L^2(\Omega_S)} \leq \frac{1}{2\nu}\|f\|^2_{H^{-1}(\Omega)}, \tag{3.2.12}$$

therefore we get

$$\|\nabla u_m\|_{L^2(\Omega_S)} \leq \frac{C}{m^{1/2}\nu}\|f\|_{H^{-1}(\Omega)}, \tag{3.2.13}$$

which proves (3.2.6). The estimate (3.2.12) implies that there exists a subsequence, which we denote again by u_m, s.t.

$$u_m \to \tilde{u} \in L^p(\Omega), \quad u_m \rightharpoonup \tilde{u} \in H^1(\Omega) \tag{3.2.14}$$

for $1 \leq p < \infty$ in case $d = 2$ and $1 \leq p < 6$ in case $d = 3$. Moreover, (3.2.13) and (3.2.5) gives $\tilde{u} = 0$ in Ω_S. Next, we can rewrite (3.1.7) as

$$\nu m \int_{\Omega_S} \nabla u_m : \nabla v \, dx = \langle f, v\rangle_{V',V} - \int_{\Omega_F} \nu \nabla u_m : \nabla v \, dx - \int_\Omega (u_m \cdot \nabla) u_m v \, dx.$$

This identity together with (3.2.12) implies that $-\operatorname{div}\,[m\chi_{\Omega_S}\nabla u_m]$ is bounded in V', therefore there exists $h \in V'$ such that

$$\lim_{m\to\infty} (-\operatorname{div}\, m\chi_{\Omega_S}\nabla u_m) = h \ \text{ weakly in } V'$$

and

$$\langle h, \phi\rangle = 0 \quad \forall \phi \in \mathcal{D}(\Omega_F). \tag{3.2.15}$$

The convergences (3.2.14) allow to pass with $m \to \infty$ in equation (3.1.7) to obtain

$$\int_\Omega (\tilde{u} \cdot \nabla)\tilde{u} v \, dx + \nu \int_\Omega \nabla\tilde{u} \cdot \nabla v \, dx + \langle h, v\rangle_{V',V} = \langle f, v\rangle_{V',V} \quad \forall v \in V. \tag{3.2.16}$$

From (3.2.15) and (3.2.16) we conclude

$$\int_{\Omega_F} (\tilde{u} \cdot \nabla)\tilde{u} v \, dx + \nu \int_{\Omega_F} \nabla\tilde{u} \cdot \nabla v \, dx = \langle f, v\rangle_{V_F',V_F} \quad \forall v \in V_F. \tag{3.2.17}$$

By continuity of trace operator, $u_m = 0$ on $\partial\Omega$ implies $\tilde{u}|_{\partial\Omega} = 0$. Therefore $\tilde{u}$ is indeed a weak solution to equation (3.1.1), so in what follows we write $u = \tilde{u}$. It remains to prove the strong convergence in $H^1(\Omega)$. For this purpose we subtract (3.1.7) and (3.2.16) taking $v = u_m - u$. We get

$$\begin{aligned} \nu \int_{\Omega_F} |\nabla(u_m - u)|^2 \, dx + \nu m \int_{\Omega_S} |\nabla(u_m - u)|^2 \, dx \\ = \langle h, (u_m - u)\rangle_{V',V} - \int_\Omega [(u_m \cdot \nabla)u_m - (u \cdot \nabla)u](u_m - u) \, dx. \end{aligned} \tag{3.2.18}$$

By the weak convergence of u_m in $H^1(\Omega)$, the first term on the RHS of the above expression tends to zero. The second can be decomposed as

$$\int_\Omega (u_m \cdot \nabla)(u_m - u)(u_m - u) \, dx + \int_\Omega ((u_m - u) \cdot \nabla)u(u_m - u) \, dx.$$

We have $u_m(u_m - u) \to 0$ in L_p for $p < 3$, which together with weak convergence of ∇u_m implies convergence of the first integral to zero. The second converges obviously by equation (3.2.14) and Hölder inequality. Therefore we have (3.2.7).

Now, assume that $f \in L^2(\Omega)$ and $\Omega, \Omega_F \in C^2$. The idea is to get rid of integrals over Ω_F on the RHS of the identity leading to the energy estimate, and keep there only Ω_S terms, which allow to take advantage of the large parameter m to show higher order of convergence. However, in case on nonlinear system we have also a term with difference of convective terms, which enforces the assumption of smallness of $\|f\|_{H^{-1}}$. Under the assumed regularity of f, Ω and Ω_F we have $(u_f, p_f) \in H^2(\Omega_F) \times H^1(\Omega_F)$, and therefore

$$\tau_f n := -pn + \nu \nabla u \cdot n \in H^{1/2}(\Sigma). \tag{3.2.19}$$

Testing the Navier-Stokes equation (3.1.1) with $\phi \in H_0^1(\Omega)$ not vanishing on Σ and assuming $u = 0$ in Ω_S we obtain

$$\int_{\Omega_F} (u \cdot \nabla) u \phi \, dx + \nu \int_{\Omega_F} \nabla u \cdot \nabla \phi \, dx - \int_\Sigma g \cdot \phi ds = \int_{\Omega_F} f \cdot \phi dx, \tag{3.2.20}$$

where $g := \tau_f n \in H^{1/2}(\Sigma)$. The identity (3.2.20) holds in particular $\forall \phi \in V$, which together with (3.2.16) gives

$$\langle h, \phi \rangle_{V', V} = \int_{\Omega_S} f \cdot \phi \, dx - \int_\Sigma g \cdot \phi \, ds \quad \forall \, \phi \in V. \tag{3.2.21}$$

Using (3.2.21) in (3.2.18) and taking $v_m := u_m - u$ we obtain

$$\begin{aligned} \nu \int_{\Omega_F} |\nabla v_m|^2 \, dx + \nu m \int_{\Omega_S} |\nabla v_m|^2 \, dx &= \int_{\Omega_S} f \cdot v_m \, dx - \int_\Sigma g \cdot v_m \, ds \\ &\quad - \int_\Omega ((u_m \cdot \nabla) u_m - (u \cdot \nabla) u) v_m \, dx. \end{aligned} \tag{3.2.22}$$

Now, we will estimate RHS of (3.2.22). For this purpose we use Hölder, Young and Poincaré inequalities and get

$$\int_{\Omega_S} f \cdot v_m \, dx \leq \frac{C_P}{2\nu m} \|f\|^2_{L^2(\Omega_S)} + \frac{\nu m}{2} \|\nabla v_m\|^2_{L^2(\Omega_S)}, \tag{3.2.23}$$

where C_P is the constant from the Poincaré inequality. Next, we use Hölder inequality, trace lemma and Young inequality to obtain

$$\int_\Sigma g \cdot v_m \, ds \leq C \|g\|_{L^2(\Sigma)} \|v_m\|_{L^2(\Sigma)} \leq \frac{C}{\nu m} \|g\|^2_{L^2(\Sigma)} + \frac{\nu m}{4} \|\nabla v_m\|^2_{L^2(\Omega_S)}. \tag{3.2.24}$$

For the last term on the RHS of (3.2.22) we have, again by Hölder and Poincaré inequalities,

$$\begin{aligned} \left| \int_\Omega [(u_m \cdot \nabla) u_m - (u \cdot \nabla) u] v_m \, dx \right| &\leq \int_\Omega (|(u_m \cdot \nabla v_m) v_m| + |(v_m \cdot \nabla u) v_m|) \, dx \\ &\leq C(\|\nabla u_m\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}) \|\nabla v_m\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.2.25}$$

Combining the above inequalities, we get

$$\nu\|\nabla v_m\|^2_{L^2(\Omega_F)} + \frac{\nu m}{4}\|\nabla v_m\|^2_{L^2(\Omega_S)} \\ \leq \frac{1}{\nu m}\Big(C(\Omega, f) + C\|g\|^2_{L^2(\Sigma)}\Big) + C(\|\nabla u_m\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)})\|\nabla v_m\|^2_{L^2(\Omega)}.$$

Assuming $\|f\|_{H^{-1}(\Omega)}$ sufficiently small w.r.t. ν we can absorb the last term on the RHS by the LHS to obtain

$$\nu\|\nabla v_m\|^2_{L^2(\Omega_F)} + \frac{\nu m}{4}\|\nabla v_m\|^2_{L^2(\Omega_S)} \leq \frac{1}{\nu m}\Big(C(\Omega, f) + C\|g\|^2_{L^2(\Sigma)}\Big). \tag{3.2.26}$$

Recalling that $v_m = u_m - u$ and $u \equiv 0$ on Ω_S we get

$$\|u_m - u\|_{H^1(\Omega)} \leq \nu^{-1}m^{-1/2}C_1(\Omega, g, f)$$

and

$$\|u_m\|_{H^1(\Omega_S)} \leq (\nu m)^{-1}C_2(\Omega, g, f).$$

□

3.2.3 Mixed penalization

It is obvious that the mixed approximation (3.1.4) at least satisfies the same estimates as both volume and viscosity approximations. However, it is possible to show additional estimates which involve both approximation parameters. In our proof, we will make use of the Poincaré inequality, therefore we will assume again that the obstacle touches the boundary (we recall however that this assumption is not necessary for the convergence of the mixed approximation — we only need it to obtain the novel estimates).

THEOREM 3.2.3. *Assume Ω and Ω_F are Lipschitz domains, $f \in H^{-1}(\Omega)$ and condition (3.2.5) holds. Let $u_{m\vee n}$ denote a weak solution to (3.1.4). Then $u_{m\vee n}$ satisfies (3.2.1) and (3.2.6). Moreover, there exists a subsequence, still denoted $u_{m\vee n}$, which satisfies the estimates (3.2.2) and (3.2.7), where u is a weak solution to equation (3.1.1). If we assume additionally that Ω and Ω_F are C^2 domains, $f \in L^2(\Omega)$ and $\|f\|_{H^{-1}(\Omega)}$ is sufficiently small with respect to ν then the estimates (3.2.3)-(3.2.4) and (3.2.8)-(3.2.9) hold. Moreover, we have*

$$\|u_{m\vee n}\|_{H^1(\Omega_S)} \leq C(\nu m)^{-3/4}n^{-1/4}, \tag{3.2.27}$$

$$\|u_{m\vee n}\|_{L^2(\Omega_S)} \leq C(\nu m)^{-1/4}n^{-3/4}, \tag{3.2.28}$$

$$\|u - u_{m\vee n}\|_{H^1(\Omega_F)} \leq C(\nu m n)^{-1/4}. \tag{3.2.29}$$

Proof. Taking $u_{m\vee n}$ as a test function in equation (3.1.9) we get

$$\int_{\Omega_F} \nu|\nabla u_{m\vee n}|^2\,dx + \nu m\int_{\Omega_S}|\nabla u_{m\vee n}|^2\,dx + n\int_{\Omega_S}|u_{m\vee n}|^2\,dx \leq \|f\|_{H^{-1}}\|\nabla u_{m\vee n}\|_{L^2(\Omega)}, \tag{3.2.30}$$

which gives the same estimates as in pure volume and viscosity approximations. The proof of strong convergence in $H^1(\Omega)$ is analogous to the viscosity case. We have, up to a subsequence,

$$u_{m\vee n} \to \tilde{u} \in L^p(\Omega), \quad u_{m\vee n} \rightharpoonup \tilde{u} \in H^1(\Omega) \tag{3.2.31}$$

for $1 \le p < \infty$ in case $d = 2$ and $1 \le p < 6$ in case $d = 3$, where $\tilde{u} = 0$ in Ω_S. Moreover, there exists $h \in V'$ such that

$$\lim_{m\to\infty} (-\operatorname{div} m\chi_{\Omega_S}\nabla u_{m\vee n} + n\chi_{\Omega_S} u_{m\vee n}) = h \text{ weakly in } V'$$

and

$$\langle h, \phi\rangle = 0 \quad \forall \phi \in \mathcal{D}(\Omega_F). \tag{3.2.32}$$

The convergences (3.2.31) allow to pass with $m \to \infty$ in equation (3.1.9) to obtain

$$\int_\Omega (\tilde{u}\cdot\nabla)\tilde{u}v\,dx + \int_\Omega \nabla\tilde{u}\cdot\nabla v\,dx + \langle h, v\rangle_{V',V} = \langle f, v\rangle_{V',V} \quad \forall v \in V. \tag{3.2.33}$$

From (3.2.32) and (3.2.33) we conclude that $\tilde{u}$ is indeed a weak solution to equation (3.1.1), so in what follows we write $u = \tilde{u}$ with $\tilde{u}|_{\partial\Omega} = 0$.

Next, we subtract (3.1.9) and (3.2.33) taking $v = u_{m\vee n} - u$ to obtain

$$\begin{aligned}\nu\int_{\Omega_F} |\nabla(u_{m\vee n}-u)|^2\,dx + \nu m\int_{\Omega_S} |\nabla(u_{m\vee n}-u)|^2\,dx + n\int_{\Omega_S} |(u_{m\vee n}-u)|^2\,dx\\ = \langle h, (u_{m\vee n}-u)\rangle_{V',V} - \int_\Omega [(u_{m\vee n}\cdot\nabla)u_{m\vee n} - (u\cdot\nabla)u](u_{m\vee n}-u)\,dx.\end{aligned} \tag{3.2.34}$$

Exactly as in the proof of equation (3.2.7) we verify that the RHS of the above expression tends to zero, which proves the strong convergence in $H^1(\Omega)$.
Now, assume that $f \in L^2(\Omega)$ and $\Omega, \Omega_F \in C^2$. Using (3.2.21) in (3.2.34) and taking $v_{m\vee n} := u_{m\vee n} - u$ we get

$$\begin{aligned}\int_{\Omega_F} |\nabla v_{m\vee n}|^2\,dx + m\int_{\Omega_S} |\nabla v_{m\vee n}|^2\,dx + n\int_{\Omega_S} |v_{m\vee n}|^2\,dx\\ = \int_{\Omega_S} f\cdot v_{m\vee n}\,dx - \int_\Sigma g\cdot v_{m\vee n}\,ds - \int_\Omega [(u_{m\vee n}\cdot\nabla)u_{m\vee n} - (u\cdot\nabla)u]v_{m\vee n}\,dx.\end{aligned} \tag{3.2.35}$$

Let us estimate the RHS of (3.2.35). By Hölder, Young and Poincaré inequalities we get

$$\begin{aligned}\int_{\Omega_S} f\cdot v_{m\vee n}\,dx &\le \frac{C}{(\nu mn)^{1/2}}\|f\|^2_{L^2(\Omega_S)} + \frac{1}{2}(\nu mn)^{1/2}\|\nabla v_{m\vee n}\|_{L^2(\Omega_S)}\|v_{m\vee n}\|_{L^2(\Omega_S)}\\ &\le \frac{C}{(\nu mn)^{1/2}}\|f\|^2_{L^2(\Omega_S)} + \frac{\nu m}{4}\|\nabla v_{m\vee n}\|^2_{L^2(\Omega_S)} + \frac{n}{4}\|v_{m\vee n}\|^2_{L^2(\Omega_S)}.\end{aligned} \tag{3.2.36}$$

Next, we use Hölder inequality, trace theorem, Young and interpolation inequalities to obtain

$$\begin{aligned}\int_\Sigma g\cdot v_{m\vee n}\,ds &\le \|g\|_{L^2(\Sigma)}\|v_{m\vee n}\|_{L^2(\Omega_S)} \le C\|g\|_{L^2(\Sigma)}\|v_{m\vee n}\|_{H^{1/2}(\Omega_S)}\\ &\le \frac{C}{(\nu mn)^{1/2}}\|g\|^2_{L^2(\Sigma)} + \frac{1}{2}(\nu mn)^{1/2}\|\nabla v_{m\vee n}\|_{L^2(\Omega_S)}\|v_{m\vee n}\|_{L^2(\Omega_S)}\\ &\le \frac{C}{(\nu mn)^{1/2}}\|g\|^2_{L^2(\Sigma)} + \frac{\nu m}{4}\|\nabla v_{m\vee n}\|^2_{L^2(\Omega_S)} + \frac{n}{4}\|v_{m\vee n}\|^2_{L^2(\Omega_S)}.\end{aligned}$$

For the last term on the RHS of (3.2.35) we have (3.2.25).

Combining the above inequalities and assuming $\|f\|_{H^{-1}(\Omega)}$ sufficiently small with respect to ν, which allows to repeat (3.2.25) and absorb the last term of the RHS of (3.2.35) by the LHS, we obtain

$$\|\nabla v_{m\vee n}\|^2_{L^2(\Omega_F)} + \nu m \|\nabla v_{m\vee n}\|^2_{L^2(\Omega_S)} + n\|v_{m\vee n}\|^2_{L^2(\Omega_S)} \le \frac{C}{(\nu m n)^{1/2}} \left(\|f\|^2_{L^2(\Omega_S)} + \|g\|^2_{L^2(\Sigma)} \right),$$

from which we conclude (3.2.27)-(3.2.29). □

3.3 Numerical simulations

In this section, we present numerical experiments to investigate the dependence of the convergence rate of approximate solutions introduced in (3.1.2)–(3.1.4) on the penalizing parameters m, n.

To this end, we compute a two-dimensional flow around an obstacle in a fixed channel. In order to get broader insight, in addition to varying m and n, we also change the shape and placement of obstacles. First, we consider a box-shaped obstacle touching the boundary in accordance with the theoretical results (see Section 3.3.1). Next, in Section 3.3.2 we consider an obstacle with more complicated geometry to check how well penalization methods can handle cases with less regular solutions. Finally, in Section 3.3.3 we relax the assumption that at least a part of each connected component of the interface Σ must touch $\partial\Omega$ and consider a flow around two obstacles, where one of them is fully immersed in a fluid.

The geometry and the input data closely follow [1], differing only in the number and shape of obstacles. Thus, all test cases are set up in a rectangular channel domain $\Omega = [0, L] \times [0, H] \subset \mathbb{R}^2$, with $L = 4$ and $H = 2$. We assume the kinematic viscosity $\nu = 1$, so that the Reynolds number $\mathrm{Re} = UH/\nu = 200$ with $U = 100$. Since we always assume no external forces, i.e. $f \equiv 0$ in (3.1.1), in our experiments we make a slight departure from the theoretical framework and consider, as in [1], a flow driven by nonhomogeneous boundary conditions. To be specific, on the left edge of Ω, that is on $\Gamma_{in} = \{0\} \times [0, H]$, we prescribe an inflow Dirichlet boundary condition,

$$u = (u_{in}, 0) \qquad \text{on } \Gamma_{in}$$

where u_{in} is a parabolic profile

$$u_{in}(x, y) = \frac{4U}{H^2} y\,(H - y).$$

On the right edge, $\Gamma_{out} = \{L\} \times [0, H]$, we prescribe the do-nothing boundary condition,

$$\nu \partial_n u - pn = 0 \qquad \text{on } \Gamma_{out},$$

where n denotes the outer normal. On all other parts of the boundary of the corresponding domain (i.e. Ω_F in the case of "real obstacle" flow, or Ω otherwise), the no-slip boundary condition is imposed, $u = 0$.

We have implemented our experimental framework in FEniCS package [22], discretizing Navier-Stokes equations with the $P_2 - P_1$ Taylor-Hood finite element [20] on a triangular mesh in Ω (or in Ω_F in the case of "real obstacle" flow), whose elements are aligned with the interface Σ. The (unstructured) meshes have their diameter set to $h = 0.05$ and consist of roughly 8000 triangular elements — precise number depending on the selection of the obstacle — and have been generated with the help

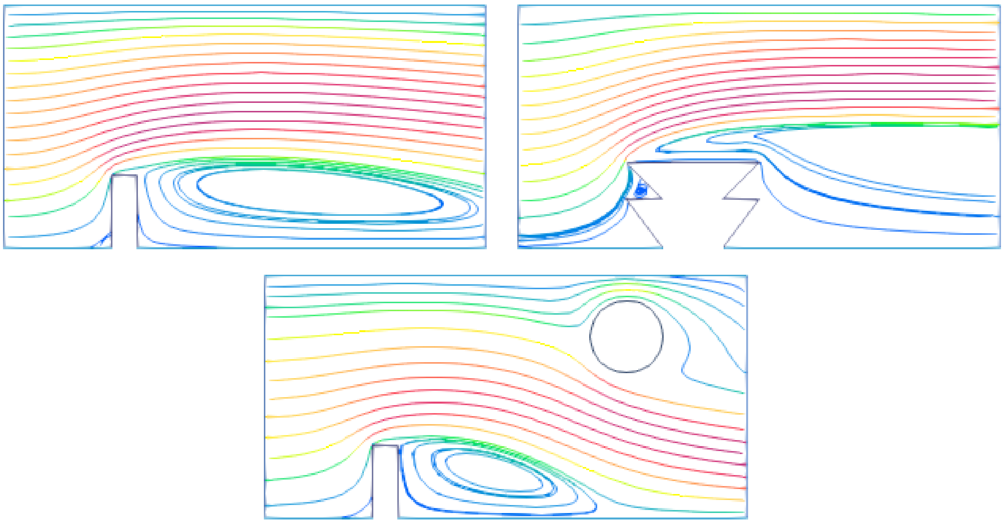


FIGURE 3.3: From left to right and down: „real obstacle” domains Ω_F considered in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, together with corresponding flow streamlines.

of Gmsh software [12]. The resulting nonlinear system of algebraic equations is then approximately solved by means of the Newton's method with accuracy 10^{-10}.

In each experiment we solve numerically equation (3.1.1) to compute the “reference” flow (u, p) in Ω_F around “real obstacle” Ω_S and then compare it with the velocity components u_m, u_n and $u_{m\vee n}$ of approximate penalized numerical solutions to (3.1.2), (3.1.3), (3.1.4), respectively. To get more insight, for each u_{APP}, where $\text{APP} \in \{n, m, m \vee n\}$, and for $10^1 \leq m, n \leq 10^{10}$, we compute separately L^2 norm and H^1 seminorm of errors in the compound domain Ω,

$$\|u - u_{\text{APP}}\|_{L^2(\Omega)}, \quad |u - u_{\text{APP}}|_{H^1(\Omega)}$$

and inside the obstacle domain Ω_S as well, i.e.

$$\|u_{\text{APP}}\|_{L^2(\Omega_S)}, \quad |u_{\text{APP}}|_{H^1(\Omega_S)}.$$

(If possible, we also conduct some tests when either $n = 0$ or $m = 1$, i.e. for pure viscosity or volume penalization, respectively.) Based on these measurements, we estimate empirical convergence rates as m or n increase towards the infinity in the above (semi)norms.

The reader may also refer to the web-page [24], which qualitatively compares side-by-side two - this time-evolutionary simulations of the flow around obstacles: a "real obstacle" vs its penalized approximation. Both have been implemented by the author in FEniCS; their details are specified in Section (3.3.3).

3.3.1 A box-shaped obstacle touching the boundary

In this section, we experiment with an obstacle touching the boundary of the domain Ω, as required in Theorem 3.2.2. Precisely, we place a box-shaped construction $\Omega_S = [0.9, 1.1] \times [0.0, 0.6]$ at the bottom of the channel, as shown in Figure 3.3 (so the geometry corresponds to [1] with the floating obstacle removed).

Figure 3.5 presents the errors between the penalized and “real obstacle” solutions as a function of penalization parameter for all three types of approximation: volume,

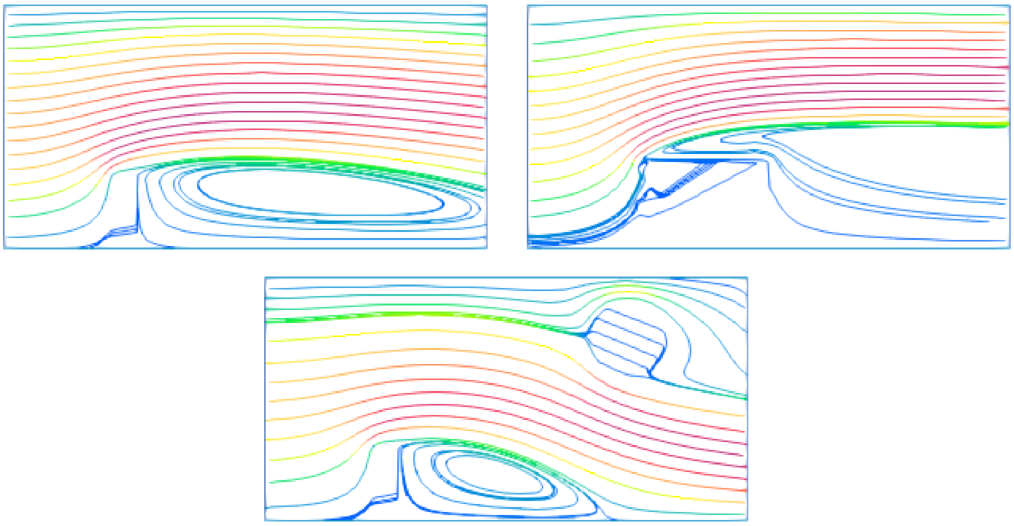


FIGURE 3.4: Penalized solution streamlines in Ω, approximating the flow depicted in Figure 3.3. The first two correspond to viscosity penalization with $m = 10^5$; the last one comes from mixed penalization with $m = n = 10^5$. Note how well the streamlines in the fluid domain Ω_F approximate the actual flow around the obstacles, cf. Figure 3.3.

viscosity and a mixed one. For the latter, we prescribe $n = 10^2 \cdot m$ and so treat m as the only independent parameter.

The graphs indicate that penalized solutions do converge to the "real obstacle" solution at an asymptotically linear rate in all considered norms. In particular, the results suggest that the $H^1(\Omega_S)$ estimate (3.2.8) is sharp. In all other combinations of norms and domains under consideration, experimental convergence rates are significantly better than predicted by Theorems 3.2.1, 3.2.2 and 3.2.3 giving a hint that they may be suboptimal — however, there is also some chance that the improved rates may be an artifact resulting from comparing numerical, not actual solutions.

Two other conclusions follow directly from convergence plots in Figure 3.5. Firstly, for identical value of penalization parameter, viscosity approximation seems to result in an error smaller than the corresponding volume approximation (i.e. the former has a smaller multiplicative constant in front of m^{-1}). If *both* penalization parameters are properly balanced, one can achieve an even smaller error when using mixed penalization. Secondly, the penalty parameter — especially in the volume penalization case — has to be large enough before the error starts to decrease at a linear speed; see e.g. the $H^1(\Omega_S)$ error plot in Figure 3.5.

To get a better impression on how these two penalty parameters interact with each other in the mixed penalization case, in Figure 3.6 we present logarithmic graphs of the errors for $10^1 \leq m, n \leq 10^{10}$, with level lines corresponding to the magnitude of the error. Obviously, to get a prescribed level of accuracy, at least one penalty parameter must be sufficiently large. Moreover the observed error turns out to be roughly inversely proportional to $\max\{m, Cn\}$ for some positive constant C. It also follows that for fixed m (or n), the other penalization parameter must be large enough to start contributing to error improvement in a significant way.

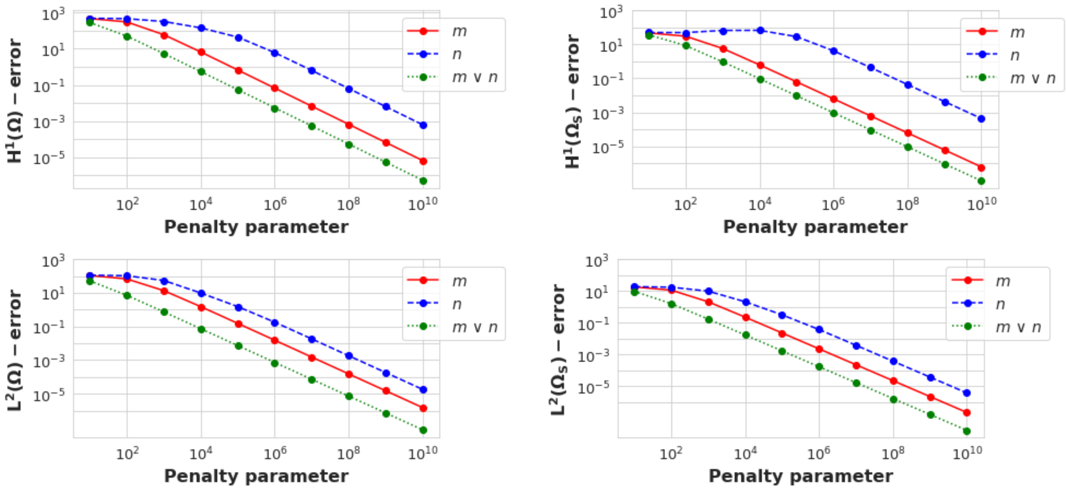


FIGURE 3.5: Approximation errors measured in H^1 seminorm (top) or L^2 norm (bottom) on Ω (left) and Ω_S (right), for the experiment from Section 3.3.1. The plots correspond to u_n (blue), u_m (red) and $u_{m \vee n}$ (green). For $u_{m \vee n}$, we set $n = 10^2 \cdot m$ and treat only m as the independent penalization parameter (the x-axis corresponds to values of m in this case).

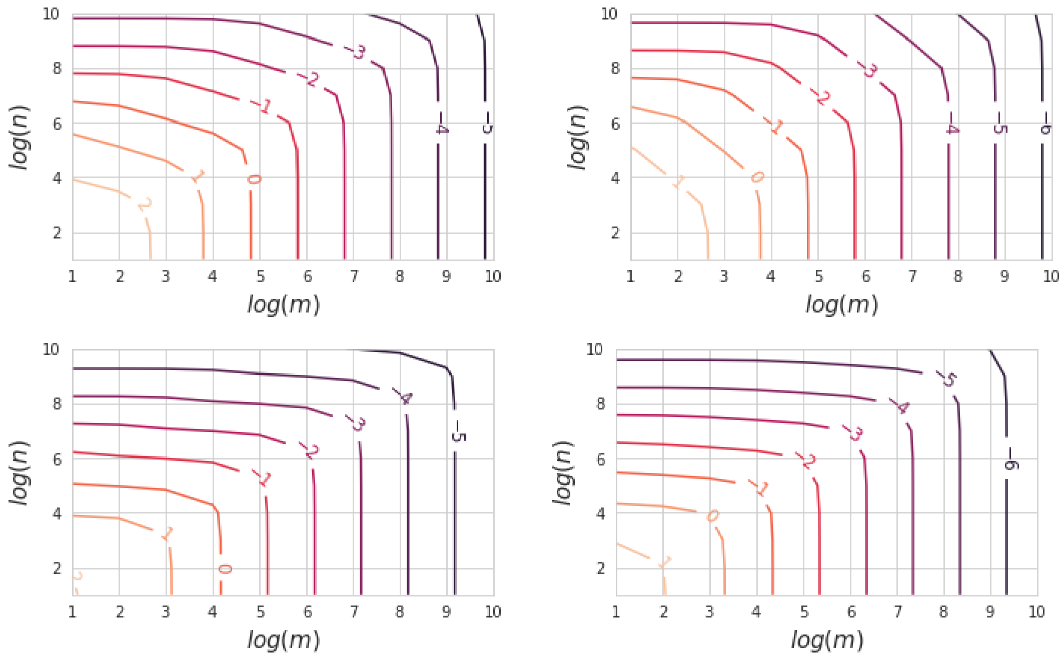


FIGURE 3.6: Contour graphs of the logarithm of errors in H^1 seminorm (top) or L^2 norm (bottom) measured on Ω (left) and Ω_S (right), for the mixed penalization solution $u_{m \vee n}$ from Section 3.3.1.

3.3.2 An obstacle with sharp corners

For complex geometry fluid-structure problems, penalization methods in a fictitious domain may have a potential to be easier to implement and more efficient than other approaches. Therefore, in this section, we consider an obstacle featuring many acute or obtuse angles (see Figure 3.3), while leaving all other parameters unchanged, in order to check if such a complex shape — which typically results in less regular solutions [10] — affects either the error level or the convergence rate.

Figure 3.7 shows the corresponding convergence histories. While we observe the same consistent pattern as in Section 3.3.1, the approximation errors are roughly two orders of magnitude higher than those in Figure 3.5, probably because the accurate solution is less regular. The level lines in Figure 3.8 are also qualitatively similar to Figure 3.6, suggesting — somewhat surprisingly — that the order of approximation may be only weakly dependent, or even independent, on the smoothness of the solutions.

3.3.3 Two obstacles, one surrounded by the fluid

Finally, we present a numerical experiment involving a domain containing two types of obstacles: the first obstacle, Ω_S^1, is attached to the boundary of the entire virtual domain, while the second obstacle, Ω_S^2, is completely embedded within the fluid. The experiment replicates (up to a horizontal symmetry) the setting from [1]. The first obstacle Ω_S^1 is defined as:

$$\Omega_S^1 := [0.9, 1.1] \times [0.0, 0.6]$$

and the second obstacle Ω_S^2 is defined by

$$\Omega_S^2 := \left\{ (x, y) \in \mathbb{R}^2 : (x - 3.0)^2 + (y - 1.5)^2 = (0.3)^2 \right\},$$

so that $\Omega_S = \Omega_S^1 \cup \Omega_S^2$. This setup is illustrated in Figure 3.3. Since Ω_S^2 does not touch the boundary of the domain Ω, pure viscosity penalization will, in principle, not result in a reasonable approximation. The reason is that in such case we only obtain $\|\nabla u\| \equiv 0$ in the limit, i.e. the solution is only constant (not necessarily vanishing) inside the immersed obstacle. Therefore in this case we restrict the comparison only to volume and mixed penalty approximations.

Error graphs in Figure 3.9 confirm linear convergence in n and the possibility of further reduction of the error when using mixed penalization. Figure 3.10, which presents logarithmic graphs of the errors[1] for $10^1 \leq m \leq 10^{10}$ and $10^7 \leq n \leq 10^{10}$, indicates that to obtain a prescribed level of accuracy, the volume penalty parameter must be large enough; and the influence of the viscosity penalty is limited by the value of n. This is in contrast to the case when all obstacles are touching $\partial\Omega$, when it is enough to increase only one of the penalties to improve the accuracy.

3.4 Conclusions

In this chapter, we considered approximation methods of a fluid flow around an obstacle, using an approach based on penalization of the fluid motion inside the

[1] We had to restrict ourselves to large volume penalty parameters, as our numerical solver struggled otherwise.

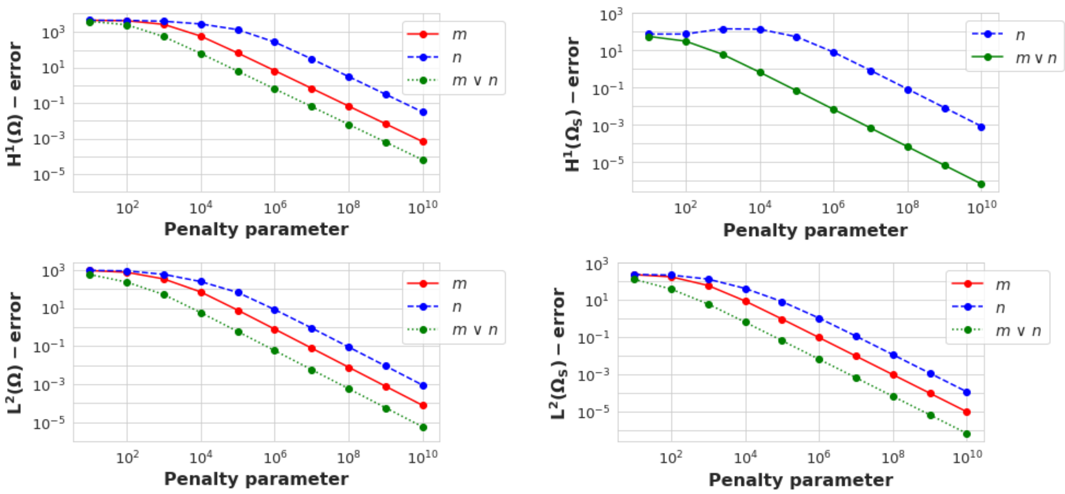


FIGURE 3.7: Approximation errors measured in H^1 seminorm (top) or L^2 norm (bottom) on Ω (left) and Ω_S (right), for the experiment from Section 3.3.2. The plots correspond to u_n (blue), u_m (red) and $u_{m\vee n}$ (green). For $u_{m\vee n}$, we set $n = 10^2 \cdot m$ and put the values of m on the x-axis, as before.

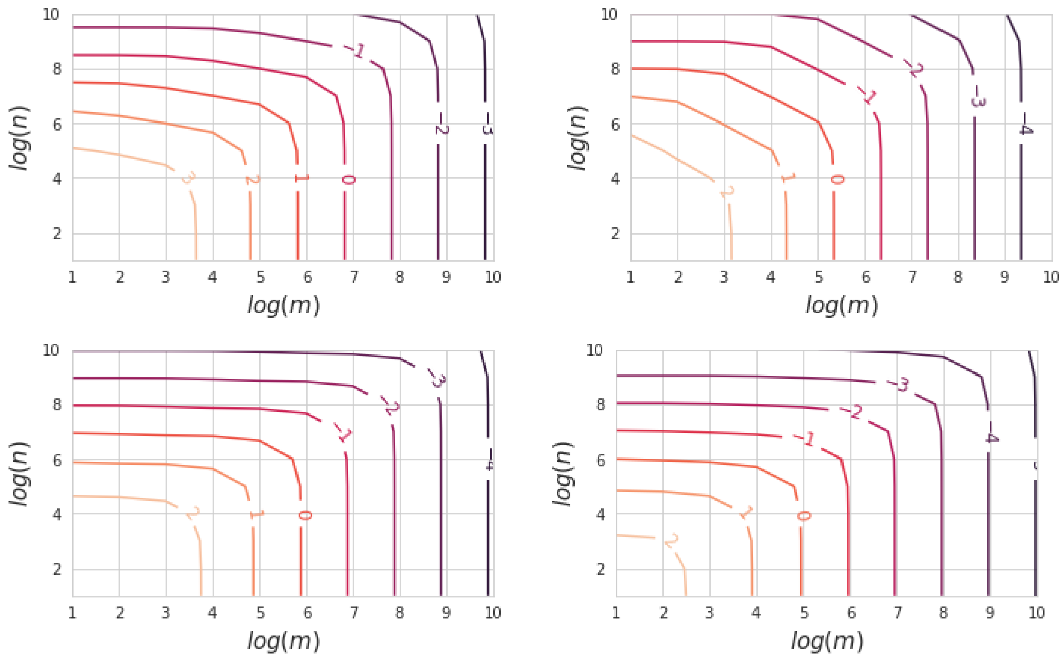


FIGURE 3.8: Contour graphs of the logarithm of errors in H^1 seminorm (top) or L^2 norm (bottom) measured on Ω (left) and Ω_S (right), for the mixed penalization solution $u_{m\vee n}$ from Section 3.3.2.

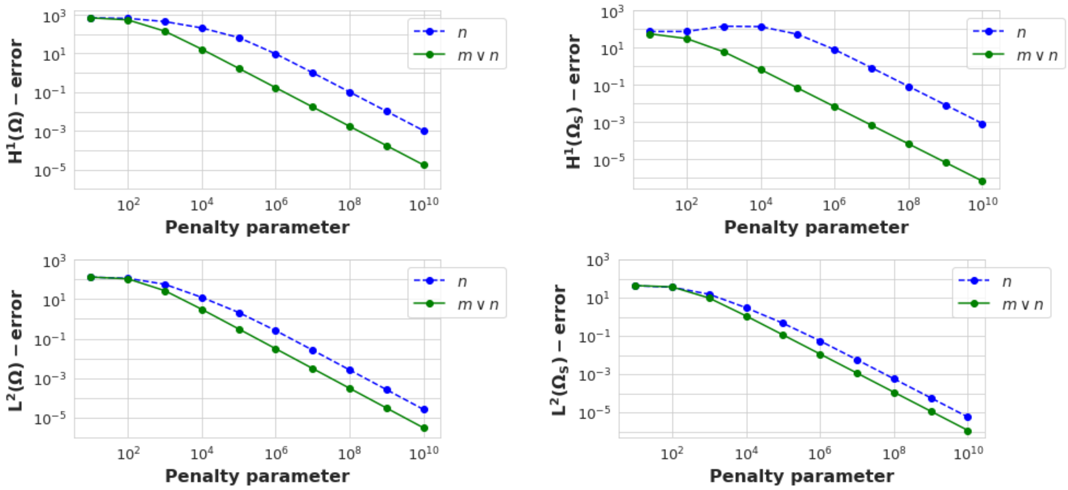


FIGURE 3.9: Approximation errors measured in H^1 seminorm (top) or L^2 norm (bottom) on Ω (left) and Ω_S (right), for the experiment from Section 3.3.3. The plots correspond to u_n (blue) and $u_{m\vee n}$ (green). For $u_{m\vee n}$, we set $n = m$.

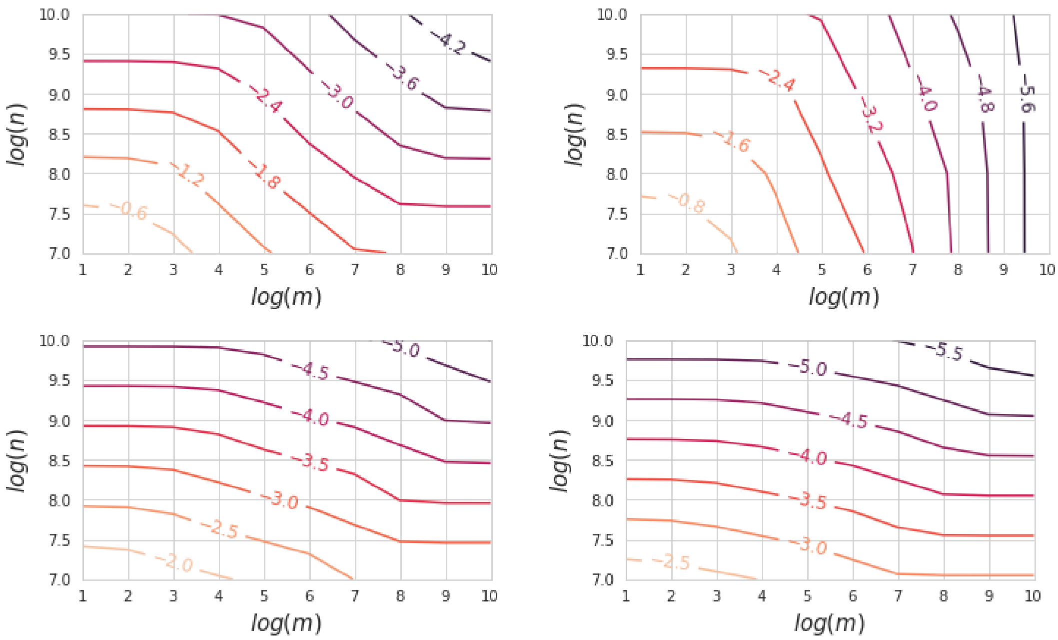


FIGURE 3.10: Contour graphs of the logarithm of errors in H^1 seminorm (top) or L^2 norm (bottom) measured on Ω (left) and Ω_S (right), for the mixed penalization solution $u_{m\vee n}$ from Section 3.3.3.

obstacle Ω_S by means of either very large viscosity or friction (with corresponding penalty parameters m, n, respectively).

Restricting ourselves to the case when the obstacle touches the boundary of the domain Ω we proved that, under certain regularity assumptions, viscosity penalization converges at a linear rate inside the obstacle — a result which our numerical experiments suggest is sharp. We also obtained bounds on the convergence speed in the mixed penalization case (i.e., when both volumetric and viscosity terms are present) which, in the special case $m = n$ reduces to an optimal, linear rate on Ω_S. Thanks to the aforementioned assumption, which stops the flow on a part of $\partial\Omega_S$, this bound also compares favorably to [2] and [1].

On the other hand, our other convergence rate bounds — in particular, those on entire Ω — seem suboptimal: from computer simulations it follows that the error is roughly inversely proportional to $\max\{m, Cn\}$ for some positive constant C. Interestingly, they also suggest that the convergence *rate* may be not influenced by the shape of the obstacle. In addition, experiments show that for the same value of penalty parameter, viscosity penalization leads, in general, to smaller approximation errors (both in Ω_S and Ω) than the corresponding volume penalization.

The overall picture significantly changes when the obstacle is fully immersed, so that it does not touch the boundary of Ω. The volume penalty parameter n then becomes the leading force reducing the error, while the influence of the viscosity penalty m is minor and limited by the value of n.

Chapter 4

Tangential regularity of approximate solutions to the Navier-Stokes equations of rigid obstacle motion

In this chapter, we explore the mixed penalized Navier-Stokes equations that approximates the motion of a rigid obstacle within an incompressible viscous flow. The approach involves modeling the rigid obstacle as a highly viscous fluid incorporating with fictitious terms in place of the obstacle. The main result of this chapter is to achieve tangential regularity for the approximate solutions, in the two dimensional case. This is an essential step in proving pointwise estimate for the gradient of the velocity.

4.1 Introduction

The problem of solid boundaries immersed in a viscous fluid flow presents an intriguing theoretical challenge, especially given that these boundaries may possess complex geometric structures and may be in motion.

This chapter specifically delves into the mathematical analysis of mixed penalization approximation of the flow around moving obstacle in a viscous incompressible fluid. By letting the penalized parameters approach infinity, we obtain the solutions to the target system of equations in the limit.

This work investigates the tangential regularity of the velocity field of the approximate problem, which constitutes the novel contribution of this study.

We denote by $\Omega_S(t)$ the domain occupied by the rigid obstacle and by $\Gamma(t)$ its boundary. The time evolution of the velocity field $u = u(t,x)$ of a viscous incompressible fluid is governed by the Navier-Stokes equations [14], [28]:

$$\begin{aligned} \partial_t u - \nu \operatorname{div} \mathbb{D}u + (u \cdot \nabla_x)u + \nabla_x p = f \quad & \text{in } Q_F, \\ \operatorname{div}_x u = 0 \quad & \text{in } Q_F, \\ u - v = 0 \quad & \text{on } \Gamma(t), \\ u = 0 \quad & \text{on } \partial Q^T, \\ u(0,\cdot) = u_0 \quad & \text{in } \Omega_F^0, \end{aligned} \tag{4.1.1}$$

with is the deformation rate tensor with the components:

$$\mathbb{D}_{ij}u = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

In this chapter, we focus on the two and three dimensional problems defined by the aforementioned system.

This work concerns the approximation of time dependent system (4.1.1) using the mixed penalty approximation, that is

$$\begin{aligned} \partial_t u_m - \text{div}\ _x(\mathbb{D}_x u_m + \eta_m \mathbb{D}_x(u_m - v)) + (u_m \cdot \nabla_x)u_m + \nabla_x p + \eta_m(u_m - v) = f \quad &\text{in } Q^T, \\ \text{div}\ _x u_m = 0 \quad &\text{in } Q^T, \\ u_m = 0 \quad &\text{on } \partial Q^T, \\ u_m(0,\cdot) = u_0 \quad &\text{in } \Omega, \end{aligned} \tag{4.1.2}$$

where $Q^T := (0,T) \times \Omega$, $\partial Q^T := (0,T) \times \partial\Omega$ and $\eta_m := m\chi(t,x)$ is a non negative piecewise constant function depending on the penalty parameter $m \geq 0$,

$$\chi(t,x) = \begin{cases} 0, & x \in \Omega_F(t), \\ 1, & x \in \Omega_S(t). \end{cases}$$

We denote $\Omega_S(t)$ as the time-varying domain occupied by the rigid obstacle, and $\Omega_F(t)$ as the time-varying domain occupied by the fluid. The corresponding domain notations are as follows: $Q_S := (0,T) \times \Omega_S(t)$, and $Q_F := (0,T) \times \Omega_F(t)$.

We assume that a rigid obstacle is moving with given, sufficiently smooth velocity field v which satisfies

$$\begin{aligned} \text{div}\ v = 0 \quad &\text{in} \quad Q^T, \\ v = 0 \quad &\text{on} \quad \partial Q^T, \\ v = 0 \quad &\text{on} \quad \Omega \setminus \Omega_\varepsilon, \end{aligned}$$

where

$$\Omega_\varepsilon = \{x \in \Omega | \quad dist(x, \partial\Omega) > \varepsilon\}.$$

Properties of v imply that $dist(\Omega, \Omega_\varepsilon) \geq \varepsilon \quad \forall t > 0$.

From the classical result of Liouville, see [5], there exists ψ - a measure-preserving Lagrange flow of v determined by

$$\begin{aligned} \partial_t \psi(x_0,t) &= v(\psi(x_0,t),t) \\ \psi(x_0,0) &= x_0. \end{aligned} \tag{4.1.3}$$

We define:

$$\Omega_F(t) = \{x \in \mathbb{R}^d \quad | \quad x = X(t,x_0) \text{ for a certain } x_0 \in \Omega_F^0\}, \tag{4.1.4}$$

where $\Omega_F^0 := \Omega_F(0)$ is the initial domain occupied by the fluid at time $t = 0$, contained within Ω.

The characteristic function χ is defined in terms of the vector field v, which corresponds to the weak solution of the transport equation

$$\begin{aligned} \partial_t \chi + v \nabla \chi = 0 \\ \chi(0, \cdot) = \mathbf{1}_{\Omega} - \mathbf{1}_{\Omega_F^0}. \end{aligned} \tag{4.1.5}$$

Existence and uniqueness of a distributional solution χ of (4.1.5) follows from properties of v.

As per the results in [14], the boundaries of Ω and the set $\Omega_S^0 := \Omega_S(0)$ satisfy the following assumption:

Assumption 1 (On the regularity of $\partial\Omega$ and $\partial\Omega_S^0$) The boundaries $\partial\Omega$ and $\partial\Omega_S^0$ are surfaces of class C^2.

Remark Due to the fact that v is given and sufficiently smooth function, for any $\delta > 0$ there is Ω_{S_δ}, a δ-neighbourhood of the set Ω_S, such that $\Omega_S \subset \Omega_{S_\delta}$.

Let us introduce the following notation of functional spaces

$$\mathcal{V}(\Omega) = \{ u \in \mathcal{C}_0^\infty(\Omega), \text{ div } u = 0\}, \tag{4.1.6}$$

$$\begin{aligned} V^1(\Omega) &= \text{the closure of } \mathcal{V} \text{ in } H^1(\Omega), \\ V^{3/2}(\Omega) &= \text{the closure of } \mathcal{V} \text{ in } H^{3/2}(\Omega), \end{aligned} \tag{4.1.7}$$

$$H(\Omega) = \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega), \tag{4.1.8}$$

$$V^{-1}(\Omega) = (V^1(\Omega))^*, \quad V^{-3/2}(\Omega) = (V^{3/2}(\Omega))^*, \quad (H(\Omega))^* = H(\Omega), \tag{4.1.9}$$

where X^* denotes the dual space to X. We denote $L^2(L^2) := L^2(0, T; L^2(\Omega))$ and $\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\Omega)}$.

Definition 4.1.1. *Assume $u_0 \in L^2(\Omega)$, $T > 0$, $f \in L^2(0, T; H^{-1}(\Omega))$. Function $u_m \in L^2(0, T; V^1(\Omega)) \cap L^\infty(0, T; H(\Omega))$ with $u_{m,t} \in L^2(0, T; H^{-1}(\Omega))$ in case $d = 2$ and $u_{m,t} \in L^2(0, T; H^{-3/2}(\Omega))$ in case $d = 3$ is called a weak solution of problem (4.1.2) if*

$$\begin{aligned} \langle u_{m,t}, \phi \rangle + (\mathbb{D}u_m, \mathbb{D}\phi) + (\eta_m \mathbb{D}(u_m - v), \mathbb{D}\phi) + ((u_m \cdot \nabla) u_m, \phi) + (\eta_m (u_m - v), \phi) \\ = (f, \phi) \end{aligned} \tag{4.1.10}$$

for any $\phi \in C_c^\infty([0, T] \times \Omega)$, such that div $\phi = 0$ *in* $[0, T] \times \Omega$.

The main result of the chapter addresses the two-dimensional case, i.e. d=2:

THEOREM 4.1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2. Assume $T > 0$ and X satisfies (4.3.1) and (4.3.9) and $f, \partial_X f \in L^2(Q^T)$. Then for every weak solution u_m of (4.1.2) we have*

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \| \partial_X u_m - \partial_X v \|_{L^2} + \| \mu_m^{1/2} \nabla (\partial_X u_m - \partial_X v) \|_{L^2(0,T;L^2(\Omega))} \\ &+ \| \eta_m^{1/2} (\partial_X u_m - \partial_X v) \|_{L^2(0,T;L^2(\Omega))} + \| \partial_t \partial_X u_m \|_{L^2(0,T;V^{-1}(\Omega))} \leqslant C \end{aligned} \tag{4.1.11}$$

where $C := C(\Omega, f, \partial_X f, X, v, m)$.

The rest of the chapter is devoted to the proof of the above theorem. Keeping in mind possible future extension of the theorem to d=3, some of the auxiliary lemmas are provided for both $d = 2$ or 3. Section 4.2 presents results related to a priori estimates. In subsection 4.2.1, we establish the existence of weak solutions to approximate problem (4.1.2), while subsection 4.2.2 covers the convergence of the approximate problem (4.1.2) to the weak solution of (4.1.1). Section 4.3.1 provides regularity results for the tangential vector field. Finally, in subsection 4.3.2, we prove the regularity result stated in Theorem 4.1.1. In Appendix B, we introduce essential inequalities and the Bogovskii-type approach used to obtain these results.

4.2 A priori estimates

To establish the existence of the presented approximate equations (4.1.2), we rely on a priori estimation. We use energy estimates to ensure that the approximate solutions are uniformly bounded in suitable function spaces. These bounds prevent the solutions from displaying unphysical behaviors such as blow-up or loss of regularity over time. By examining the energy functional, we can establish inequalities that the approximate solutions must adhere to. These inequalities enable us to manage the growth of the solutions and their derivatives. As a result, this control aids in the transition to the limit, thereby demonstrating the existence of solutions to the approximate equations.

Lemma 4.2.1. *Let* $\Omega \subset \mathbb{R}^d$ *be an open bounded domain of class* C^2*, with* $d = 2, 3$*. Assume* $T > 0$ *and let* f *be given in* $L^2(0,T;L^2(\Omega))$*,* $u_0 \in H(\Omega)$*,* $v \in L^\infty(0,T;H^1_0(\Omega))$ *and* $v_t \in L^2(0,T;L^2(\Omega))$*, then for every solution* u_m *of Navier-Stokes equations (4.1.2) we have* $u_m \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;H(\Omega))$ *with*

$$\|u_m - v\|_{L^\infty(L^2)} + \|\mu_m^{1/2}\nabla(u_m - v)\|_{L^2(L^2)} + \|\eta_m^{1/2}(u_m - v)\|_{L^2(L^2)} \leqslant C(d, \Omega, u_0, f, v) \tag{4.2.1}$$

where the constant does not dependent on the penalty parameter m*, where*

$$\mu_m := (1 + m\chi(t, x)).$$

Moreover, $u_{m,t} \in L^2(0,T;V^{-1}(\Omega))$ *in case* $d = 2$ *and* $u_{m,t} \in L^2(0,T;V^{-3/2}(\Omega))$ *in case* $d = 3$ *with bounds*

$$\begin{aligned} \|u_{m,t}\|_{L^2(0,T;H^{-1}(\Omega))} &\leqslant C(m^{1/2}, \Omega, u_0, f, v) \quad \text{in case } d = 2 \\ \|u_{m,t}\|_{L^2(0,T;H^{-3/2}(\Omega))} &\leqslant C(m^{1/2}, \Omega, u_0, f, v) \quad \text{in case } d = 3 \end{aligned} \tag{4.2.2}$$

and

$$\begin{aligned} \|u_{m,t}\|_{L^2(0,T;V^{-1}(\Omega\setminus\Omega_{S_\delta})} &\leqslant C(d, \Omega, u_0, f, v) \quad \text{in case } d = 2 \\ \|u_{m,t}\|_{L^2(0,T;V^{-3/2}(\Omega\setminus\Omega_{S_\delta})} &\leqslant C(d, \Omega, u_0, f, v) \quad \text{in case } d = 3 \end{aligned} \tag{4.2.3}$$

Proof. Since m is fixed we skip it in notation and write u instead of u_m. The proof relies on the standard energy approach. To implement this, we first need to reformulate the penalized problem (4.1.2) as follows

$$\begin{aligned} \partial_t(u - v) - \text{div}\,_x(\mu_m \mathbb{D}_x(u - v)) + (u \cdot \nabla_x)(u - v) + \nabla_x p + \eta_m(u - v) \\ = f - v_t + \text{div}\,(\mathbb{D}v) - (u \cdot \nabla_x)v \end{aligned} \tag{4.2.4}$$

By restructuring the problem, we can more effectively apply energy estimates and other analytical tools. Taking $u - v$ as a test function in (4.2.4) we have

$$\frac{1}{2}\frac{d}{dt}\|(u-v)\|_{L^2}^2 + \|\mu_m^{1/2}\mathbb{D}(u-v)\|_{L^2}^2 + \|\eta_m^{1/2}(u-v)\|_{L^2}^2 = \int_\Omega (f(u-v) - v_t(u-v) - \mathbb{D}v\mathbb{D}(u-v) - (u\cdot\nabla)v\,(u-v))\,dx \tag{4.2.5}$$

To analyze this equation, we begin by estimating the last term on the RHS in the case $d = 2$. We use Hölder's inequality to handle products of functions, Ladyzhenskaya's inequality (B.0.10) inequality to control norms in L^4 by norms in L^2 and H^1, and Young's (B.0.12) inequality to balance terms and avoid unbounded growth.

$$\begin{aligned}\int_\Omega (u\cdot\nabla)(u-v)\,v\,dx &\leqslant \|\nabla(u-v)\|_{L^2}\|u\|_{L^4}\|v\|_{L^4}\\ &\leqslant \|\nabla(u-v)\|_{L^2}^{3/2}\|u\|_{L^2}^{1/2}\|v\|_{L^4}\\ &\leqslant \epsilon_1\|\nabla(u-v)\|_{L^2}^2 + C_1(\epsilon_1)(\|u-v\|_{L^2}^2 + \|v\|_{L^2}^2)\|v\|_{L^4}^4\end{aligned} \tag{4.2.6}$$

Next, we consider the same term in the case $d = 3$. By applying Hölder, Ladyzhenskaya (B.0.11) and Young' (B.0.12) inequalities with $\lambda = 8/7, \quad \lambda' = 8$, we obtain

$$\begin{aligned}\int_\Omega (u\cdot\nabla)(u-v)\,v\,dx &\leqslant \|\nabla(u-v)\|_{L^2}\|u\|_{L^4(D)}\|v\|_{L^4}\\ &\leqslant \|\nabla(u-v)\|_{L^2}^{7/4}\|u\|_{L^2}^{1/4}\|v\|_{L^4}\\ &\leqslant \epsilon_1\|\nabla(u-v)\|_{L^2}^2 + C_1(\epsilon_1)(\|u-v\|_{L^2}^2 + \|v\|_{L^2}^2)\|v\|_{L^4}^8\end{aligned} \tag{4.2.7}$$

Note that the bounds in (4.2.6) and (4.2.7) differ only in the exponent of the last term. Therefore, we can unify the notation of the exponent and apply Hölder's inequality to the remaining terms in (4.2.5). This allows us to write a general estimate valid for both $d = 2$ and $d = 3$ as follows:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-v\|_{L^2}^2 + \|\mu_m^{1/2}\mathbb{D}(u-v)\|_{L^2}^2 + \|\eta_m^{1/2}(u-v)\|_{L^2}^2\\ &\leqslant \|f\|_{L^2}^2 + \|v_t\|_{L^2}^2 + (1 + C_1(\epsilon_1)\|v\|_{L^4}^r)\|u-v\|_{L^2}^2\\ &\quad + C_1(\epsilon_1)\|v\|_{L^4}^r\|v\|_{L^2}^2 + C_2(\epsilon_2)\|\nabla v\|_{L^2}^2 + (\epsilon_1+\epsilon_2)\|\nabla(u-v)\|_{L^2}^2\end{aligned} \tag{4.2.8}$$

where $\epsilon_1 + \epsilon_2 < 1$ and $r = 4$ and $r = 8$ in $d = 2, 3$ respectively. We apply the Korn inequality (A.0.21) to the LHS of (4.2.8), and the last term on the RHS is absorbed by the LHS. We obtain

$$\begin{aligned}&\frac{d}{dt}\|u-v\|_{L^2}^2 + \|\mu_m^{1/2}\nabla(u-v)\|_{L^2}^2 + \|\eta_m^{1/2}(u-v)\|_{L^2}^2\\ &\qquad\leqslant \|f\|_{L^2}^2 + \phi_1(v) + (1 + C_1(\epsilon_1)\|v\|_{L^4}^r)\|u-v\|_{L^2}^2,\end{aligned} \tag{4.2.9}$$

where

$$\phi_1(v) = \|v_t\|_{L^2}^2 + C_1(\epsilon_1)\|v\|_{L^4}^r\|v\|_{L^2}^2 + C_2(\epsilon_2)\|\nabla v\|_{L^2}^2$$

is an integrable function of time under assumed regularity of v. Therefore by Gronwall's inequality we obtain

$$\begin{aligned}\sup_{0\leqslant t\leqslant T} \|u-v\|_{L^2}^2 + &\|\mu_m^{1/2}\nabla(u-v)\|_{L^2(L^2)}^2 + \|\eta_m^{1/2}(u-v)\|_{L^2(L^2)}^2 \\ &\leqslant C\Big[\|u_0\|_{L^2}^2 + \int_0^T \psi(s)\,ds\Big]\exp\Big(\int_0^T \|v(s)\|_{L^4}^r\,ds\Big)\end{aligned} \tag{4.2.10}$$

where $\psi(t) = \|f(t)\|_{L^2}^2 + \phi_1(t)$ is integrable. We bound below ((4.2.10)) for fixed m, and use triangle inequality

$$\|u\| \leqslant \|u-v\| + \|v\|,$$

and obtain

$$\begin{aligned}\|u\|_{L^\infty(L^2)}^2 + &\|u\|_{L^2(H^1)}^2 \\ &\leqslant C\Big[\|u_0\|_{L^2}^2 + \int_0^T \psi(s)\,ds\Big]\exp\Big(\int_0^T \|v(s)\|_{L^4}^r\,ds\Big) \\ &\quad + \|v\|_{L^\infty(L^2)}^2 + \|v\|_{L^2(H^1)}^2,\end{aligned} \tag{4.2.11}$$

from which we conclude (4.2.1)

To obtain bound for u_t, we test the equation (4.1.2) with a divergence free function $w \in H_0^1(\Omega \setminus \Omega_{S_\delta})$, such that $w = 0$ on $\partial\Omega$ and on $\partial\Omega_{S_\delta}$

$$\begin{aligned}\langle u_t, w\rangle =&(f,w) - (\mu_m\mathbb{D}(u-v), \mathbb{D}w) - (\eta_m(u-v), w) + (u\otimes u, \mathbb{D}w) \\ &- (\mathbb{D}v, \mathbb{D}w) = (f,w) - (\mathbb{D}(u-v), \mathbb{D}w) + (u\otimes u, \mathbb{D}w) \\ &- (\mathbb{D}v, \mathbb{D}w)\end{aligned} \tag{4.2.12}$$

The above equality follows from

$$m\int_\Omega \chi\,\mathbb{D}(u) : \mathbb{D}(w)\,dx + m\int_\Omega \chi(u-v)w dx = 0.$$

In the expression above, the second equality arises because the support of the test function is in the fluid domain and away from the obstacle's boundary. Let's address the nonlinear term first. By applying Hölder inequality and Sobolev imbedding, we derive an estimate

$$|(u\otimes u, \mathbb{D}w)| \leqslant \|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}, \tag{4.2.13}$$

for any u, $w \in V^1(\Omega)$. By applying Hölder's inequality to the remaining terms in (4.2.12), we obtain

$$\begin{aligned}|\langle u_t, w\rangle| \leqslant&\|f\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|w\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|\nabla(u-v)\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla w\|_{L^2(\Omega\setminus\Omega_{S_\delta})} \\ &+ \|u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla w\|_{L^2(\Omega\setminus\Omega_{S_\delta})} \\ &+ \|v\|_{H^1(\Omega\setminus\Omega_{S_\delta})}\|w\|_{H_0^1(\Omega\setminus\Omega_{S_\delta})}.\end{aligned} \tag{4.2.14}$$

Recalling the definition of dual norm

$$\|u_t\|_{V'} := \sup_{w\in V,\,\|w\|\leqslant 1} \langle u_t, w\rangle_{V',V} \tag{4.2.15}$$

we get

$$\begin{aligned}\|u_t\|_{V^{-1}(\Omega\setminus\Omega_{S_\delta})} \leqslant &\|f\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|\nabla(u-v)\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\\ &+ \|u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla u\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|v\|_{H^1(\Omega\setminus\Omega_{S_\delta})}.\end{aligned} \tag{4.2.16}$$

If now $u \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;H(\Omega))$, then $u_t(t)$ belongs to V' for almost every t and we have the estimate

$$\begin{aligned}\|u_t(t)\|_{V^{-1}(\Omega\setminus\Omega_{S_\delta})} \leqslant &\|f(t)\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|\nabla(u(t)-v(t))\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\\ &+ \|u(t)\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla u(t)\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|v(t)\|_{H^1(\Omega\setminus\Omega_{S_\delta})}.\end{aligned} \tag{4.2.17}$$

Integrating the square of the above inequality in time we get

$$\begin{aligned}\|u_t(t)\|_{L^2(V^{-1}(\Omega\setminus\Omega_{S_\delta}))} \leqslant &\|f(t)\|_{L^2(L^2(\Omega\setminus\Omega_{S_\delta}))} + \|\nabla(u(t)-v(t))\|_{L^2(L^2(\Omega\setminus\Omega_{S_\delta}))}\\ &+ \|u(t)\|_{L^\infty(L^2(\Omega\setminus\Omega_{S_\delta}))}\|\nabla u(t)\|_{L^2(L^2(\Omega\setminus\Omega_{S_\delta}))}\\ &+ \|v(t)\|_{L^2(H^1(\Omega\setminus\Omega_{S_\delta}))},.\end{aligned} \tag{4.2.18}$$

which, by equation (4.2.1), implies the first inequality in equation (4.2.2).

Now, for $d = 3$, we choose the divergence free test function $w \in H_0^{3/2}(\Omega \setminus \Omega_{S_\delta})$ in (4.2.12) and consider the nonlinear term first

$$(u \otimes u, \mathbb{D}w) \leqslant \|u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|u\|_{L^6(\Omega\setminus\Omega_{S_\delta})}\|\nabla w\|_{L^3(\Omega\setminus\Omega_{S_\delta})}. \tag{4.2.19}$$

In the inequality above, we applied the Sobolev embedding theorem and the Hölder inequality. By applying Hölder's and (4.2.19) inequalities in (4.2.12), we get

$$\begin{aligned}|\langle u_t, w\rangle| \le &\|f\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|w\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|\nabla(u-v)\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla w\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\\ &+ \|u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|u\|_{L^6(\Omega\setminus\Omega_{S_\delta})}\|\nabla w\|_{L^3(\Omega\setminus\Omega_{S_\delta})}\\ &+ \|v\|_{H^1(\Omega\setminus\Omega_{S_\delta}))}\|\nabla w\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\\ &\le \Big(\|f\|_{L^2(\Omega\setminus\Omega_{S_\delta})} + \|\nabla(u-v)\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\| + \|u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\nabla u\|_{L^2(\Omega\setminus\Omega_{S_\delta})}\|\\ &+ \|v\|_{H^1(\Omega\setminus\Omega_{S_\delta}))}\Big)\|w\|_{H^{3/2}(\Omega\setminus\Omega_{S_\delta})}.\end{aligned} \tag{4.2.20}$$

From (4.2.15) and (4.2.20), we obtain

$$\begin{aligned}\|u_t\|_{L^2(V^{-3/2})} \leqslant &\|f\|_{L^2(L^2(\Omega\setminus\Omega_{S_\delta}))} + \|\nabla(u-v)\|_{L^2(L^2(\Omega\setminus\Omega_{S_\delta}))}\\ &+ \|u\|_{L^\infty(L^2)}\|\nabla u\|_{L^2(L^2(\Omega\setminus\Omega_{S_\delta}))}\\ &+ \|v\|_{L^2(H^1(\Omega\setminus\Omega_{S_\delta}))} \leqslant C(d,\Omega,f,v).\end{aligned} \tag{4.2.21}$$

To derive the above estimate, we used (4.2.1). Consequently, we have the second estimate in (4.2.3). The estimates (4.2.2) are obtained in the same way, the difference is that we obtain $m\|\nabla(u-v)\|$ on the RHS, and (4.2.1) gives bound in $\sqrt{m}\|\nabla(u-v)\|$ only, so factor $m^{1/2}$ appears on the RHS of (4.2.2). This completes the proof.

□

4.2.1 Existence of solutions to the penalized problem

In this subsection, we present a result regarding the existence of weak solutions to (4.1.2). The theorem establishes the existence of a weak solution to the penalized Navier-Stokes equations. The proof employs the Galerkin method, where approximate solutions are constructed using finite-dimensional subspaces. These approximations are shown to satisfy the weak formulation of the problem. To pass to the limit and obtain a solution in the infinite-dimensional setting, the Lions-Aubin compactness lemma is used. This lemma helps to demonstrate the compactness of the sequence of approximate solutions, ensuring convergence to a weak solution of the penalized Navier-Stokes equations.

THEOREM 4.2.2. *Let Ω be an open, bounded C^2 set in $\mathbb{R}^d$, $d \in \{2,3\}$. Assume $T > 0$ and let f, u_0, v satisfy the assumptions of Lemma 4.2.1. Then there exists at least one function u_m which satisfies equation (4.1.2) with*

$$u_m \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;H(\Omega)). \tag{4.2.22}$$

Proof. We construct an approximate solution u_m^N by the Galerkin method [30]. Since V^1 is separable and $\mathcal{V}$ is dense in V^1, there exists a sequence $w_1, ...w_N, ..$ of elements of $\mathcal{V}$, which is linearly independent and total in V^1. For each N we define an approximate solution u_m^N as follows

$$u_m^N = \sum_{i=1}^{N} c_{iN}(t) w_i, \tag{4.2.23}$$

$$\begin{aligned}(\partial_t u_m^N, w_j) + (\mathbb{D}u_m^N, \mathbb{D}w_j) &+ (\eta_m \mathbb{D}(u_m^N - v_N), \mathbb{D}w_j) + (\eta_m(u_m^N - v^N), w_j) \\ &+ (u_m^N \cdot \nabla u_m^N, w_j) = (f, w_j), \\ &u_m^N(0) = u_0^N, \quad t \in [0,T], \quad j = 1, ..., N.\end{aligned} \tag{4.2.24}$$

Let us rewrite the above equation in following form

$$\begin{aligned}&(\partial_t(u_m^N - v^N), w_j) + (\mu_m \mathbb{D}(u_m^N - v^N), \mathbb{D}w_j) + (\eta_m(u_m^N - v^N), w_j) \\ &+ (u_m^N \cdot \nabla u_m^N, w_j) = (f, w_j) - (\partial_t v^N, w_j) - (\mathbb{D}v^N, \mathbb{D}w_j).\end{aligned} \tag{4.2.25}$$

Inverting the nonsigular matrix with elements (w_i, w_j), we write the differential equations

$$\begin{aligned}c'_{iN}(t) + \sum_{j=1}^{N} \alpha_{ij}\mu_m(c_{jN}(t) - d_{jN}(t)) + \sum_{j,k=1}^{N} \alpha_{ijk} c_{iN}(t) c_{kN}(t) + \eta_m(c_{iN}(t) - d_{jN}(t)) \\ = \sum_{j=1}^{N} \beta_{ij}(f(t), w_j) + R(v),\end{aligned} \tag{4.2.26}$$

where

$$d_{jN}(t) = (v, w_j).$$

The equations (4.2.26) form a nonlinear differential system for the functions $c_{1N}, ..., c_{NN}$ that has at least one solution. We multiply (4.2.24) by $c_{iN}(t)$ and subtract (4.2.24) times d_{jN} to get following

$$\frac{1}{2}\frac{d}{dt}\|u_m^N - v^N\|_{L^2}^2 + \|\mu_m^{1/2}\mathbb{D}(u_m^N - v^N)\|_{L^2}^2 + \|\eta_m^{1/2}(u_m^N - v^N)\|_{L^2}^2$$
$$= \int_\Omega (\partial_t v^N + f + \operatorname{div}(\mathbb{D}v^N) - (u_m^N \cdot \nabla)v^N)(u_m^N - v^N)\,dx. \tag{4.2.27}$$

Applying the energy estimates from section 4.2 with $u_m = u_m^N$ and $v = v^N$ in (4.2.1) we have

$$\sup_{0\leqslant t\leqslant T} \|u_m^N\|_{L^2}^2 + \|\nabla u_m^N\|_{L^2(L^2)}^2 \leqslant C_1. \tag{4.2.28}$$

The estimate (4.2.28) enable us to assert the existence of an element $u_m \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;H(\Omega))$ and a sub-sequence $u_m^{N_k}$ such that

$$\begin{aligned} u_m^{N_k} &\to u_m \quad \text{in} \quad L^2(0,T;V^1(\Omega)) \quad \text{weakly}, \\ u_m^{N_k} &\to u_m \quad \text{in} \quad L^\infty(0,T;H(\Omega)) \quad \text{weak-star, as } N_k \to \infty. \end{aligned} \tag{4.2.29}$$

From (4.2.2) we deduce that for fixed m we have

$$\begin{aligned} \partial_t u_m^N &\in L^2(0,T;V^{-1}(\Omega)) \;\; \text{in case} \;\; d=2, \\ \partial_t u_m^N &\in L^2(0,T;V^{-3/2}(\Omega)) \;\; \text{in case} \;\; d=3. \end{aligned} \tag{4.2.30}$$

Therefore, by applying the Aubin-Lions compactness theorem (see [30], Theorem III.2.1), we also have that

$$u_m^{N_k} \to u_m \;\; \text{in} \;\; L^2(0,T;L^2(\Omega)) \;\; \text{strongly, as} \;\; N_k \to \infty. \tag{4.2.31}$$

Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T) = 0$. We multiply (4.2.24) by $\psi(t)$, and then integrate by parts

$$-\int_0^T (u_m^N, \psi'(t)w_j) + (\mu_m \mathbb{D}(u_m^N - v_N), \mathbb{D}w_j\psi(t)) + (\eta_m(u_m^N - v^N), w_j\psi(t))$$
$$+ (u_m^N \cdot \nabla u_m^N, w_j\psi(t))\,dt = (u_0^N, w_j)\psi(0) + \int_0^T (f, w_j\psi(t)) - (\nabla v^N, \nabla w_j\psi(t))\,dt. \tag{4.2.32}$$

Passing to the limit with N_k we obtain

$$-\int_0^T (u_m, \psi'(t)\phi) + (\mu_m \mathbb{D}(u_m - v), \mathbb{D}\phi\psi(t)) + (\eta_m(u_m - v), \phi\psi(t))$$
$$+ (u_m \cdot \nabla u_m, \phi\psi(t))\,dt = (u_0, \phi)\psi(0) + \int_0^T (f, \phi\psi(t)) - (\nabla v, \nabla\phi\psi(t))\,dt, \tag{4.2.33}$$

holds for any $\phi = w_1, w_2, ...,$ finite linear combination of the w_j. Convergence of the nonlinear part follows from convergences (4.2.29) and (4.2.31).

□

4.2.2 Convergence of the solutions of the penalized problem

In this paragraph, we will demonstrate the convergence of the penalized problem (4.2.34) as the penalization parameter m tends to infinity. At the limit, we recover the original Navier-Stokes equations. This result is crucial for confirming the penalized problem as a reliable approximation of the Navier-Stokes equations. Recall the penalized problem

$$\begin{aligned} \partial_t u_m - \operatorname{div}_x(\mathbb{D}_x u_m + \eta_m \mathbb{D}_x(u_m - v)) + (u_m \cdot \nabla_x) u_m & \\ + \nabla_x p + \eta_m (u_m - v) = f & \quad \text{in } Q^T, \\ \operatorname{div}_x u_m = 0 & \quad \text{in } Q^T, \\ u_m = 0 & \quad \text{on } \partial Q^T \\ u_m(0, \cdot) = u_0 & \quad \text{in } \Omega_0. \end{aligned} \tag{4.2.34}$$

We have obtained uniform a priori estimates (4.2) for the solutions of the penalized problem that are not dependent on m. However, the m-independent estimate for the time derivative, which is necessary to apply Aubin-Lions type compactness argument, holds only in the time-dependent fluid domain. To overcome this difficulty we use the auxiliary compactness result gained from [14], to show that a subsequence of the solutions to the penalized problem strongly converges to a limit function.

THEOREM 4.2.3. *Let Ω and $\Omega_S(t)$ satisfy the assumption of the Theorem 4.1.1. Assume $T > 0$ and let f, u_0, v satisfy the assumptions of Lemma 4.2.1. Then, the weak approximate solution u_m to system (4.2.34) converges strongly to the weak solution u of system (4.1.1) in $L^2(Q^T)$:*

$$u_m \longrightarrow u \quad \textit{strongly in} \quad L^2(Q^T). \tag{4.2.35}$$

Moreover, (4.2.1) gives

$$\begin{aligned} \|u_m - v\|_{L^2(Q_S)} &\leqslant \frac{C_1}{m^{1/2}} \\ \|\mathbb{D}(u_m)\|_{L^2(Q_S)} &\leqslant \frac{C_2}{m^{1/2}} \end{aligned} \tag{4.2.36}$$

with

$$u = v \ \textit{ in } \ Q_S. \tag{4.2.37}$$

Proof. The proof is based on the result from [14]. From the energy estimate (4.2.1) the approximate solution u_m is bounded in $L^2(Q^T)$, which allows us to extract a subsequence u_m that converges weakly to a function u. Let ψ be a smooth function vanishing close to $\partial\Omega$ such that $\mathbb{D}\psi = 0$ for $x \in \Omega_{S_\sigma}$, $\sigma > 0$. We have to prove that

$$\lim_{m\to\infty} \int_{Q^T} (u_m \otimes u_m) : \mathbb{D}(\psi) dx\, dt = \int_{Q^T} (u \otimes u) : \mathbb{D}(\psi) dx\, dt. \tag{4.2.38}$$

To pass to the limit with $m \to \infty$ we will utilize the compactness result presented in Lemma B.0.2. Let us introduce a rigid function γ, $\mathbb{D}\gamma \equiv 0$, such that $\gamma(x,t) = \psi(x,t)$ for $x \in \Omega_{S_\sigma}$. Then

$$\int_{Q^T} (u_m \otimes u_m) : \mathbb{D}(\psi) dx\, dt = \int_{Q^T} (u_m \otimes u_m) : \mathbb{D}(w) dx\, dt, \tag{4.2.39}$$

where $w = \psi + \gamma$, $w(x,t) = 0$ for $x \in \Omega_{S_\sigma}$, $w(x,t) = \gamma(x,t)$ for $x \in \partial\Omega$.

Since u_m is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$, by Lemma 7.1 (see [21], ch.1, p.103), for any $\delta > 0$ and any $\beta > 0$ there exists a function $g \in L^2(0,T;H^1(\Omega))$, div $g = 0$ such that $g(x,t) = 0$ for $x \in \Omega \setminus (\partial\Omega)_\delta$, $g(x,t) = \gamma(x,t)$ for $x \in \partial\Omega$ and

$$\Big| \int_{Q^T} (u_m \otimes u_m) : \mathbb{D}(g) dx\, dt \Big| \leqslant \beta \|u_m\|^2_{L^2(0,T;H^1(\Omega))}. \tag{4.2.40}$$

Therefore, it is enough to pass to the limit for the test functions w which equal to zero on $\partial\Omega_{S_\sigma}$ and on $\partial\Omega$.

Let $G_\sigma = \{(x,t) \in Q^T \mid x \in \Omega \setminus \Omega_{S_\sigma},\ t \in [0,T]\}$. We fix $\sigma > 0$ and take arbitrary cylinder $E := A \times [t_1,t_2]$, $t_1,\ t_2 \in [0,T]$, belonging to G_σ and based on the set A with the boundary ∂A. Let us test the equation (4.2.34) with a function w that is zero outside of E. Then we have

$$m \int_{Q^T} \chi \mathbb{D}(u_m) : \mathbb{D}(w) dx\, dt + m \int_{Q^T} \chi (u_m - v) w dx\, dt = 0.$$

Let us introduce the following notations

$$\tilde{V}(A) = \{u \in H^1(A) \mid \text{div } u = 0\}$$
$$J(A) = \{u \in L^2(A) \mid \text{div } u = 0\}.$$

Let P_A be the orthogonal projector in $J(A)$ on $H(A)$, which means $P_A J(A) = H(A)$. Any function $u \in \tilde{V}(A)$ (see (4.1.7), (4.1.8)) can be represented as follows [11]

$$u = P_A u + \nabla \zeta_A,$$

where ζ_A is a harmonic function.

As it follows from (4.2.3) the functions $\partial_t(P_A u_m)$ are uniformly bounded in $L^2(t_1,t_2;V'(A))$. Thus, using Lemma B.0.2 with $X_0 = \tilde{V}(A)$, $X_1 = V'(A)$, $X = J(A)$, $Y = J_0(A)$, $P = P_A$, we conclude that $P_A u_m \to P_A u$ in $L^2(E)$.

Additionally, we have $\nabla \zeta_A^m \to \nabla \zeta_A$ weakly in $L^2(E)$. Thus, we need to show that

$$\lim_{m\to\infty} \int_E (\nabla \zeta_A^m \otimes \nabla \zeta_A^m) : \mathbb{D}(w) dx\, dt = \int_E (\nabla \zeta_A \otimes \nabla \zeta_A) : \mathbb{D}(w) dx\, dt.$$

However, since $w = 0$ on ∂A, we even have

$$\int_E (\nabla \zeta_A^m \otimes \nabla \zeta_A^m) : \mathbb{D}(w) dx\, dt = \int_E (\nabla \zeta_A \otimes \nabla \zeta_A) : \mathbb{D}(w) dx\, dt = 0. \tag{4.2.41}$$

Namely, performing integration by parts twice, we arrive at the desired result

$$\begin{aligned} &\int_A (\nabla \zeta_A \otimes \nabla \zeta_A) : \mathbb{D}(w) dx = -\int_A \text{div}\, (\nabla \zeta_A \otimes \nabla \zeta_A) \cdot w\, dx = \\ &= -\int_A (\Delta \zeta_A \nabla \zeta_A + \frac{1}{2} \nabla |\nabla \zeta_A|^2) \cdot w\, dx = \int_A \frac{1}{2} |\nabla \zeta_A|^2 \text{div } w\, dx = 0, \end{aligned} \tag{4.2.42}$$

and the same holds for ζ^m, which gives (4.2.41).

Since E was chosen arbitrarily and G_σ can be approximated by a countable collection of such cylinders, we can conclude that (4.2.38) holds.

Therefore, the limit function u satisfies the equation

$$\int_{Q^T} (u_t \cdot \psi + (u \cdot \nabla u)\psi + \mathbb{D}(u) : \mathbb{D}\psi) dx\, dt = \int_{Q^T} f \psi dx\, dt,$$

for any ψ from $H^1(Q) \cap L^2(0,T; K_\sigma(\chi) \cap H^1(\Omega))$, and consequently $H^1(Q) \cap L^2(0,T; K(\chi) \cap H^1(\Omega))$ since σ was arbitrary. The space K_σ represents the set of rigid motions:

$$K_\sigma(\chi) = \{\psi \in H_0^1(\Omega) \mid \text{div } \psi = 0, \mathbb{D}\psi = 0 \quad \text{for} \quad x \in \Omega_{S_\sigma}\}.$$

From energy estimates (4.2.1) we obtain that

$$\mathbb{D}(u)(x,t) = 0 \quad \text{and} \quad u = v, \text{ for } x \in \Omega_S(t),\ t \in [0,T].$$

□

4.3 Tangential regularity

Motivated by works [7, 8], we prove regularity of the solution in direction of a non-degerate family of vector fields, consider $d = 2$. Due to assumed regularity of Ω and $\Omega_S(t)$, there exists a vector field $X \in C^2$, such that

$$X \cdot \tau = 1,\ X \cdot n = 0 \quad \text{on } \partial\Omega_S(t) \cup \partial\Omega. \tag{4.3.1}$$

We are interested in the regularity of a function f along X:

$$\partial_X f := \sum_{j=1}^{d} X^j \partial_{x_j} f.$$

We have the identity

$$\partial_X f = \text{div}\,(fX) - f \,\text{div}\, X. \tag{4.3.2}$$

Let $X(t)$ be the vector field X_0 transported by the flow associated to v, i.e.

$$X_t(\psi_t(x)) = \partial_{X_0}\psi_t(x) = \nabla_x \psi_t(x) X_0,$$

where

$$\psi_t(x) = x + \int_0^t v(s, \psi_s(x)) ds. \tag{4.3.3}$$

As observed in [8], the family X satisfies certain transport equation. For the sake of completeness we present a proof of this property, which is a first point in proving regularity of X. First we differentiate (4.3.3) along X_0

$$\partial_{X_0}\psi_t(x) = \partial_{X_0} x + \int_0^t \partial_{X_0} v(s, \psi_s(x)) \partial_{x_i} \psi_s(x) ds. \tag{4.3.4}$$

Next, we take the material derivative of both sides, which corresponds to time derivative in Lagrangian coordinate x:

$$\frac{d}{dt}\partial_{X_0}\psi_t(x) = \partial_{X_0} v(s, \psi_s(x)) \partial_{x_i}\psi_s(x) = X_0 \partial_\psi v(s, \psi_s(x)) \partial_{x_i}\psi_s(x) ds. \tag{4.3.5}$$

By the definition of X_t we get

$$\frac{d}{dt}X_t(\psi_t(x)) = X_t^i(\psi_t(x))\partial_{\psi_t}v(t,\psi_t(x)), \tag{4.3.6}$$

that is

$$\partial_t X_t(\psi_t(x)) + v\cdot\nabla X_t(\psi_t(x)) = X_t^i(\psi_t(x))\partial_{\psi_t}v(t,\psi_t(x)). \tag{4.3.7}$$

Coming back to Euler system of coordinates with $\psi_t^{-1}(x)$ we get

$$\begin{cases}(\partial_t + v\cdot\nabla)X = \partial_X v\\ X_{|t=0} = X_0.\end{cases} \tag{4.3.8}$$

We assume that $X_0 \equiv 0$ and $v \equiv 0$ in some neighbourhood of $\partial\Omega$. Then (4.3.8) implies that $X \equiv 0$ in certain neighbourhood of $\partial\Omega$.

4.3.1 Regularity of the tangential vector field

Lemma 4.3.1. *Assume $v \in L^\infty(W^{2,\infty}) \cap L^1(W^{3,\infty})$ and X is defined by equation (4.3.8). Then*

$$\|\nabla X\|_{L^\infty(Q^T)} \le C(1+t), \qquad \|\nabla^2 X\|_{L^\infty(Q^T)} \leqslant C(1+t^2), \tag{4.3.9}$$

where $C = C(v, X_0)$.

Proof. Let us define

$$V(t) := \int_0^t \|\nabla v(s)\|_{L^\infty} ds. \tag{4.3.10}$$

The standard L^∞ estimate for equation (4.3.8) (see (4.11) in [8]) gives

$$\|X(t)\|_{L^\infty} \leqslant \|X_0\|_{L^\infty} e^{V(t)}. \tag{4.3.11}$$

Next we proceed similarly to the proof of Lemma 3.2 in [16]. We first derive the estimates on L^p for finite p, which does not depend on p, therefore we can pass to the limit $p = \infty$. Differentiating equation (4.3.8) with respect to space variable x_j, we get

$$\partial_t\partial_{x_j}X + v\cdot\nabla\partial_{x_j}X + \partial_{x_j}v\cdot\nabla X = \partial_{x_j}X\cdot\nabla v + \partial_X\partial_{x_j}v. \tag{4.3.12}$$

We multiply the above equality by $|\nabla X|^{p-2}\partial_{x_j}X$ and integrate over the spatial domain. Then dividing both sides of the resulting equation by $\|\nabla X\|_{L^p}^{p-1}$ and applying Gronwall's inequality, we obtain

$$\|\nabla X(t)\|_{L^p} \leqslant e^{CV(t)}\big(\|\nabla X_0\|_{L^p} + \int_0^t e^{-CV(s)}\|\partial_X \nabla v\|_{L^p} ds\big). \tag{4.3.13}$$

We use the inequality (4.3.11) to obtain

$$\|\partial_X\nabla v\|_{L^p} \leqslant \|X\|_{L^\infty}\|\nabla^2 v\|_{L^p} \leqslant Ce^V\|X_0\|_{L^\infty}, \tag{4.3.14}$$

and implement in (4.3.13) to get bound on the gradient of the vector field

$$\|\nabla X(t)\|_{L^p} \leqslant e^{CV(t)}\big(\|\nabla X_0\|_{L^p} + C\int_0^t e^{-CV(s)}e^{V(s)}\|X_0\|_{L^\infty} ds\big) \leqslant C_v(\bar{C}_0 + CC_0 t). \tag{4.3.15}$$

The inequality in (4.3.15) establishes a bound on the norm of the gradient of the vector field $X(t)$ in the L^p space, where $1 \leqslant p < \infty$. We note that the bound does not explicitly depend on p (we precisely justify this independence in the second part of the proof where we estimate the second gradient), which means the inequality holds for $p=\infty$, which proves the first inequality in (4.3.9).

In order to show the estimate for $\nabla^2 X$ we differentiate equation (4.3.12) with respect to the spatial variable x_i. We obtain the following expression for the second-order spatial derivatives of the X:

$$\begin{aligned}\partial_t\partial_{x_i}\partial_{x_j}X+\partial_{x_i}v\cdot\nabla\partial_{x_j}X+v\cdot\nabla\partial_{x_i}\partial_{x_j}X+\partial_{x_i}\partial_{x_j}v\cdot\nabla X+\partial_{x_j}v\cdot\nabla\partial_{x_i}X\\ =\partial_{x_i}\partial_{x_j}\partial_X v.\end{aligned}\tag{4.3.16}$$

In line with the approach used in (4.3.15), we establish bounds on the second derivatives of X. For this puporpose we multiply (4.3.16) by $|\nabla^2 X|^{p-2}\nabla^2_{x_i,x_j}X$. We begin by addressing the most challenging term in the resulting expression:

$$|\int \partial_{x_i}\partial_{x_j}v\cdot\nabla X|\nabla^2X|^{p-2}\nabla^2_{x_i,x_j}X\,dx|\leqslant C\|\nabla^2 v\|_{L^\infty}\|\nabla X\|_{L^p}\|\nabla^2 X\|_{L^p}^{p-1}.$$

By estimating the remaining terms in (4.3.16) in a similar manner, we arrive at the following result

$$\frac{d}{dt}\|\nabla^2X\|^p_{L^p}\leqslant p\left[\|\nabla v\|_{L^\infty}\|\nabla^2X\|^p_{L^p}+(\|\nabla^2 v\|_{L^\infty}\|\nabla X\|_{L^p}+\|\nabla^2\partial_X v\|_{L^p})\|\nabla^2X\|^{p-1}_{L^p}\right].\tag{4.3.17}$$

Now, applying the bound from (4.3.15) to the above inequality and dividing both sides by $\|\nabla^2X\|^{p-1}_{L^p}$, we obtain

$$\frac{d}{dt}\|\nabla^2X\|_{L^p}\le C\left(\|\nabla v\|_{L^\infty}\|\nabla^2X\|_{L_p}+(1+t)\|\nabla^2 v\|_{L^\infty}+\|\nabla^2\partial_X v\|_{L^p}\right).\tag{4.3.18}$$

For the last term on the RHS of the above inequality we have the relation

$$\partial_{x_i}\partial_{x_j}\partial_X v=\partial_{x_i}\partial_{x_j}X\cdot\nabla v+\partial_{x_j}X\cdot\nabla\partial_{x_i}v+\partial_{x_i}X\cdot\nabla\partial_{x_j}v+\partial_X\partial_{x_i}\partial_{x_j}v,\tag{4.3.19}$$

for all $1\leqslant i,j\leqslant d$, which allows to estimate

$$\|\nabla^2\partial_X v\|_{L^p}\leqslant\|\nabla^2X\|_{L^p}\|\nabla v\|_{L^\infty}+C\|\nabla X\|_{L^p}\|\nabla^2 v\|_{L^\infty}+\|X\|_{L^\infty}\|\nabla^3 v\|_{L^p}.\tag{4.3.20}$$

Therefore we can rewrite (4.3.18) as

$$\frac{d}{dt}\|\nabla^2X\|_{L^p}\le C\left(\|\nabla v\|_{L^\infty}\|\nabla^2X\|_{L_p}+(1+t)\|\nabla^2 v\|_{L^\infty}+\|\nabla^3 v\|_{L^p}\right),\tag{4.3.21}$$

which by Gronwall inequality gives

$$\|\nabla^2X(t)\|_{L^p}\leqslant e^{CV(t)}\left(\|\nabla^2X(0)\|_{L^p}+\int_0^t((1+t)\|\nabla^2v\|_{L^\infty}+\|\nabla^3v\|_{L^p})\,ds\right).\tag{4.3.22}$$

Therefore the assumed regularity of v implies

$$\|\nabla^2X(t)\|_{L^p}\leqslant C(1+t^2)\tag{4.3.23}$$

with some universal constant C, for fixed time with $1\leqslant p<\infty$. Again, the constant does not depend on p, which completes the proof of (4.3.9). □

4.3.2 Estimate of first order tangential derivative

To simplify the notation, we write u instead u_m. Let us differentiate (4.1.2) along the vector field X, we get

$$\begin{aligned}\partial_t(\partial_X u - \partial_X v) - \text{div }\left(\mu_m \mathbb{D}(\partial_X u - \partial_X v)\right) + \nabla \partial_X p + \eta_m(\partial_X u - \partial_X v) \\ = \partial_X f + \text{div }\mathbb{D}(\partial_X v) - \partial_t \partial_X v + \mathcal{R}_1,\end{aligned} \tag{4.3.24}$$

$$\text{div }(\partial_X u) = \sum_{k,j=1}^{d} \partial_{x_k} X^j \partial_{x_j} u^k + \sum_{j=1}^{d} X^j \partial_{x_j}(\text{div }u) = \sum_{k,j=1}^{d} \partial_{x_k} X^j \partial_{x_j} u^k, \tag{4.3.25}$$

where $\mathcal{R}_1$ is a reminder

$$\begin{aligned}\mathcal{R}_1 =& \partial_t X^i \partial_{x_i} u - \frac{1}{2}\text{div }\left[\mu_m\left(\nabla(u-v)\,\nabla X + \nabla^T X \nabla^T(u-v)\right)\right] \\ &- X^i_{,k}\partial_{x_i}(\mu_m \mathbb{D}(u-v)) + X^i_{,k}\partial_{x_i} p - X^i_{,k}\partial_{x_i}(\mathbb{D}v) - \frac{1}{2}\text{div }(\nabla v \nabla X + \nabla^T X \nabla^T v) \\ &- \partial_X\left[(u\cdot\nabla)u\right].\end{aligned} \tag{4.3.26}$$

The last term in the second line of the above expression corresponds to the nonlinear propagated convection term of Navier-Stokes equations, that is

$$\partial_X\left[(u\cdot\nabla)u\right] := (\partial_X u \cdot \nabla)u + (u\cdot\nabla)\partial_X u - u^l \partial_{x_l} X^i \partial_{x_i} u^k.$$

By using (4.3.8) we can rewrite $\mathcal{R}_1$ as follows

$$\begin{aligned}\mathcal{R}_1 =& (\partial_X v - v\cdot\nabla X)\cdot\nabla u - \text{div }(\mu_m(\nabla(u-v)\,\nabla X + \nabla^T X \nabla^T(u-v))) \\ &- X^i_{,k}\partial_{x_i}(\mu_m \mathbb{D}(u-v)) + X^i_{,k}\partial_{x_i} p - X^i_{,k}\partial_{x_i}(\mathbb{D}v) - \frac{1}{2}\text{div }(\nabla v \nabla X + \nabla^T X \nabla^T v) \\ &+ (u\cdot\nabla X)\cdot\nabla u - (\partial_X u\cdot\nabla)u - (u\cdot\nabla)(\partial_X u - \partial_X v) - (u\cdot\nabla)\partial_X v\big).\end{aligned} \tag{4.3.27}$$

Proof of Theorem 4.1.1. Recall that by (4.3.8) we have $X \equiv 0$ close to the boundary of Ω. Therefore we can test equation (4.3.24) by $\partial_X(u-v) \in H^1_0(\Omega)$, getting

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_X u - \partial_X v|^2 dx + \int \mu_m \mathbb{D}(\partial_X u - \partial_X v)^2 - \int_\Omega \partial_X p \text{ div }\partial_X(u-v) \\ &+ \int_\Omega \eta_m(\partial_X u - \partial_X v)^2 dx = \int_\Omega (\partial_X f + \text{div }(\mathbb{D}\partial_X v) - \partial_t\partial_X v + \mathcal{R}_1)(\partial_X u - \partial_X v)\,dx.\end{aligned} \tag{4.3.28}$$

For the nonlinear component of the term $\mathcal{R}_1$, we will apply the standard approach to nonlinear forms and utilize a Ladyzhenskaya-type inequality (B.0.10). Additionally, we will use the bounds from (4.3.11) and (4.3.15)

$$\begin{aligned}
\Big| \int_\Omega \Big((u\cdot\nabla X)\cdot\nabla u - (u\cdot\nabla)\partial_X v\Big)(\partial_X u - \partial_X v)\,dx \Big| \\
&\leqslant \|u\|_{L^4}\|\nabla u\|_{L^2}\|\nabla X\|_{L^\infty}\|\partial_X(u-v)\|_{L^4} + \|u\|_{L^4}\|\nabla\partial_X v\|_{L^2}\|\partial_X(u-v)\|_{L^4} \\
&\leqslant \big[C_X\|\nabla u\|_{L^2} + \|\partial_X v\|^2_{L^2}\big]\|u\|^{1/2}_{L^2}\|\nabla u\|^{1/2}_{L^2}\|\partial_X(u-v)\|^{1/2}_{L^2}\|\nabla\partial_X(u-v)\|^{1/2}_{L^2} \\
&\quad \leqslant C(\epsilon_1)\big[C(X)\|\nabla u\|^2_{L^2} + \|\nabla\partial_X v\|^2_{L^2}\big] \\
&\qquad + \epsilon_1\|u\|_{L^2}\|\nabla u\|_{L^2}\|\partial_X(u-v)\|_{L^2}\|\nabla\partial_X(u-v)\|_{L^2}.
\end{aligned} \tag{4.3.29}$$

We use Ladyzhenskya inequalities for the nonlinear term of the reminder $\mathcal{R}_1$

$$\begin{aligned}
\Big|\int_\Omega -(\partial_X u\cdot\nabla)u(\partial_X(u-v))\,dx\Big| &\leqslant \|\partial_X(u-v)\|^2_{L^4}\|\nabla u\|_{L^2} \\
&+ \|\partial_X(u-v)\|_{L^4}\|\partial_X v\|_{L^4}\|\nabla u\|_{L^2} \\
&\leqslant \|\partial_X(u-v)\|_{L^2}\|\nabla\partial_X(u-v)\|_{L^2}\|\nabla u\|_{L^2} + \|\partial_X(u-v)\|_{L^4}\|\partial_X v\|_{L^4}\|\nabla u\|_{L^2} \\
&\leqslant (\|\partial_X(u-v)\|_{L^2}\|\nabla u\|_{L^2} + \|\partial_X v\|_{L^4}\|\nabla u\|_{L^2})\|\nabla\partial_X(u-v)\|_{L^2}.
\end{aligned} \tag{4.3.30}$$

Combining estimates (4.3.29) and (4.3.30), we obtain the following estimate

$$\begin{aligned}
\Big|\int_\Omega\Big((u\cdot\nabla X)\cdot\nabla u - (\partial_X u\cdot\nabla)u - (u\cdot\nabla)\partial_X v\Big)(\partial_X u - \partial_X v)\,dx\Big| \\
\leqslant \epsilon_2\|\nabla\partial_X(u-v)\|^2_{L^2} + (C\|u\|^2_{L^2}\|\nabla u\|^2_{L^2} + \|\nabla u\|^2_{L^2})\|\partial_X(u-v)\|^2_{L^2} + C_b,
\end{aligned} \tag{4.3.31}$$

we denote by C_b a term that is well-defined due to the assumptions on v and the estimate in (4.2.1), and which is integrable in time:

$$C_b := C\|\nabla u\|^2_{L^2} + C\|\partial_X v\|^2_{H^1}\|u\|^2_{H^1}.$$

Next, we apply integration by parts to the terms involving discontinuities and the pressure term in $\mathcal{R}_1$. Then, by applying Hölder's inequality and using the estimates (4.3.30)-(4.3.31), we get

$$\begin{aligned}
\Big|\int_\Omega \mathcal{R}_1(\partial_X u - \partial_X v)dx\Big| \leqslant & C_1\|\mu_m^{1/2}\nabla(u-v)\|_{L^2}\|\mu_m^{1/2}\nabla\partial_X(u-v)\|_{L^2} \\
&+ C_2\|p\|_{L^2}\|\nabla\partial_X(u-v)\|_{L^2} \\
&+ \epsilon_2\|\nabla\partial_X(u-v)\|^2_{L^2} + (C\|u\|^2_{L^2}\|\nabla u\|^2_{L^2} \\
&+ \|\nabla u\|^2_{L^2})\|\partial_X(u-v)\|^2_{L^2} + C_b.
\end{aligned} \tag{4.3.32}$$

We use Bogovskii approach (see Appendix (B.0.1)), where the functional (B.0.4) in our case will take a form

$$\begin{aligned}
\mathcal{F}(\psi) =& \frac{d}{dt}(\partial_X u - \partial_X v, \psi) + (\mu_m\mathbb{D}(\partial_X u - \partial_X v), \mathbb{D}\psi) + (\eta_m(\partial_X u - \partial_X v), \psi) \\
&+ (\partial_X f, \psi) + (\mathbb{D}\partial_X v, \mathbb{D}\psi) + \frac{d}{dt}(\partial_X v, \psi) + (\mathcal{R}_1, \psi),
\end{aligned} \tag{4.3.33}$$

for any $\psi \in H_0^1(D)$. The pressure $\partial_X p$ satisfies the following inequality ([23])

$$\left|\int_\Omega \partial_X p\, dx\right| = \left|-\int_\Omega p \operatorname{div} X\, dx + \int_{\partial\Omega} X^i n^i p\, ds\right| \leqslant \|\operatorname{div} X\|_{L^2}\|p\|_{L^2}.$$

On the other hand,

$$\mathcal{F}(\psi) = (\partial_X p, \operatorname{div}\psi). \tag{4.3.34}$$

According to [11], there exists a function ψ that satisfies the following problem

$$\begin{gathered}\operatorname{div}\psi = \partial_X p - \frac{1}{|\Omega|}\int_\Omega \partial_X p,\\ \psi \in H_0^1, (\Omega)\\ \|\psi\|_{H^1} \leqslant C\|\partial_X p\|_{L^2}.\end{gathered} \tag{4.3.35}$$

We estimate separately $\mathcal{R}_1$, recall

$$\begin{aligned}\mathcal{R}_1 = {}& (\partial_X v - v\cdot\nabla X)\cdot\nabla u - \frac{1}{2}\operatorname{div}\left(\mu_m(\nabla(u-v)\,\nabla X + \nabla^T X\nabla^T(u-v))\right)\\ &- X^i_{,k}\partial_{x_i}(\mu_m\mathbb{D}(u-v)) + X^i_{,k}\partial_{x_i}p - X^i_{,k}\partial_{x_i}(\mathbb{D}v) - \frac{1}{2}\operatorname{div}\left(\nabla v\nabla X + \nabla^T X\nabla^T v\right)\\ &- (\partial_X u\cdot\nabla)u - (u\cdot\nabla)\partial_X u + u^l\partial_{x_l}X^i\partial_{x_i}u^k.\end{aligned} \tag{4.3.36}$$

We begin by addressing the challenging part of the nonlinear term in the final line of the above expression. By applying integration by parts, we obtain the following result:

$$\begin{aligned}-\int_\Omega[(\partial_X u\cdot\nabla)u\psi - u^j\partial_{x_j}X^i\partial_{x_i}u^k\,\psi^k]dx &= -\int_\Omega[X^i\partial_{x_i}u^j\partial_{x_j}u^k\psi^k - u^j\partial_{x_j}X^i\partial_{x_i}u^k\,\psi^k]dx\\ = -\int_\Omega[-\partial_{x_j}X^i\partial_{x_i}u^j u^k\psi^k - u(\partial_X u\cdot\nabla)\psi + \partial_{x_j}X^i\partial_{x_i}u^j u^k\psi^k\\ + u^j\partial_{x_i}\partial_{x_j}(X^i)u^k\,\psi^k + u^j\partial_{x_j}X\partial_{x_i}\psi^k u^k]\,dx.\end{aligned} \tag{4.3.37}$$

The first and third terms of the above expression cancel out. For the remaining nonlinear term, applying integration by parts once again, we obtain:

$$\int_\Omega -(u\cdot\nabla)\partial_X u\,\psi\,dx = \int_\Omega (u\cdot\nabla)\psi\partial_X u\,dx. \tag{4.3.38}$$

Combining (4.3.37)-(4.3.38) we get

$$\begin{aligned}&\left|\int_\Omega(\partial_X u\cdot\nabla)\psi u - u^j\cdot\partial_{x_i}\partial_{x_j}(X^i)\cdot u\,\psi - u^j\partial_{x_j}X\partial_{x_i}\psi^k u^k + (u\cdot\nabla)\psi\partial_X u)\,dx\right|\\ &\quad\leqslant \|\partial_X u\|_{L^4}\|\nabla\psi\|_{L^2}\|u\|_{L^4} + C(X)\|u\|^2_{L^4}(\|\psi\|_{L^2} + \|\nabla\psi\|_{L^2})\\ &\qquad + \|u\|_{L^4}\|\partial_X u\|_{L^4}\|\nabla\psi\|_{L^2}\\ &\quad\leqslant \left(2\|u\|^{1/2}_{L^2}\|\nabla u\|^{1/2}_{L^2}\|\partial_X u\|^{1/2}_{L^2}\|\nabla\partial_X u\|^{1/2}_{L^2} + C(X)\|u\|_{L^2}\|\nabla u\|_{L^2}\right)\|\nabla\psi\|_{L^2}\\ &\quad\leqslant \left(2c(\epsilon)\|u\|_{L^2}\|\nabla u\|_{L^2}\|\partial_X u\|_{L^2} + \epsilon\|\nabla\partial_X u\|_{L^2} + C(X)\|u\|_{L^2}\|\nabla u\|_{L^2}\right)\|\nabla\psi\|_{L^2}.\end{aligned} \tag{4.3.39}$$

Final bound for the the reminder term $\mathcal{R}_1$ with test function ψ will take the form

$$\begin{aligned}&\Big|\int_\Omega \mathcal{R}_1\,\psi\,dx\Big| \leqslant \Big(C(X,v)\|\mu_m\nabla(u-v)\|_{L^2}+C(X,v)\|p\|_{L^2}\\&+2c(\epsilon)\|u\|_{L^2}\|\nabla u\|_{L^2}\|\partial_X u\|_{L^2}+\epsilon\|\nabla\partial_X u\|_{L^2}+C(X,v)\|u\|_{L^2}\|\nabla u\|_{L^2}\Big)\|\nabla\psi\|_{L^2}\\&:=\mathcal{R}_{press}\|\nabla\psi\|_{L^2}.\end{aligned} \tag{4.3.40}$$

Following the same steps as in the proof of Lemma B.0.1, we deduce that the pressure term can be estimated as follows:

$$\begin{aligned}\|\partial_X p\|_{L^2}\leqslant{}&\|\partial_t\partial_X u\|_{H^{-1}}+\|\mu_m\nabla(\partial_X u-\partial_X v)\|_{L^2}+\|\eta_m(\partial_X u-\partial_X v)\|_{L^2}\\&+\|\partial_X f\|_{L^2}+\|\partial_X\partial_t v\|_{L^2}+\|\nabla\partial_X v\|_{L^2}+\mathcal{R}_{press}.\end{aligned} \tag{4.3.41}$$

We have

$$\begin{aligned}&\int_0^T \mathcal{R}_{press}^2\,dt\le C(X,m,v)\|\mu_m^{1/2}\nabla(u-v)\|^2_{L^2(L^2)}+C(X,v)\|p\|^2_{L^2(L^2)}\\&+\big(C(X,v)+C(\epsilon)\|\partial_X u\|^2_{L^\infty(L^2)}\big)\|u\|^2_{L^\infty(L^2)}\|\nabla u\|^2_{L^2(L^2)}+\epsilon\|\nabla\partial_X u\|^2_{L^2(L^2)}.\end{aligned} \tag{4.3.42}$$

The estimate for the $\partial_t\partial_X u$ in $L^2(0,T;H^{-1}(\Omega))$ follows from that fact the we use in (4.3.33) test function $\phi(x)\in H^1_0(\Omega)$, which is divergence-free. As a result, we obtain $(\partial_X p,\operatorname{div}\phi):=0$. By (4.3.33) and (4.3.34), this implies

$$\begin{aligned}\|\partial_t\partial_X u\|_{H^{-1}}\leqslant{}&\|\mu_m\nabla(\partial_X u-\partial_X v)\|_{L^2}+\|\eta_m(\partial_X u-\partial_X v)\|_{L^2}\\&+\|\partial_X f\|_{L^2}+\|\nabla\partial_X v\|_{L^2}+\mathcal{R}_{press}.\end{aligned} \tag{4.3.43}$$

Plugging this estimate into (4.3.41), we get

$$\begin{aligned}\|\partial_X p\|_{L^2}\leqslant{}&C(\|\mu_m\nabla(\partial_X u-\partial_X v)\|_{L^2}+\|\eta_m(\partial_X u-\partial_X v)\|_{L^2}\\&+\|\partial_X f\|_{L^2}+\|\partial_X\partial_t v\|_{L^2}+\|\nabla\partial_X v\|_{L^2}+\mathcal{R}_{press}).\end{aligned} \tag{4.3.44}$$

By applying Hölder, Poincaré and Korn inequalities we can further simplify and bound the terms in (4.3.28):

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\partial_X u-\partial_X v|^2dx+\int_\Omega\mu_m|\nabla(\partial_X u-\partial_X v)|^2+\int_\Omega\eta_m|\partial_X u-\partial_X v|^2dx\\&\leqslant(\|\partial_X f\|_{L^2}+C(v))\|\nabla\partial_X(u-v)\|_{L^2}+|\int_\Omega\mathcal{R}_1(\partial_X(u-v))dx|\\&\quad+\|\partial_X p\|_{L^2}\|\operatorname{div}\partial_X(u-v)\|_{L^2}\\&\quad\leqslant\big(\|\partial_X f\|_{L^2}+C(v)\big)\|\nabla\partial_X(u-v)\|_{L^2}+|\int_\Omega\mathcal{R}_1(\partial_X(u-v))dx|\\&\quad\quad+\|\partial_X p\|_{L^2}\|\nabla(u-v)\|_{L^2},\end{aligned} \tag{4.3.45}$$

where in the second inequality we used (4.3.25). By using the estimates (4.3.44) and (4.3.32) to the above inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_X u - \partial_X v|^2 dx + \int_\Omega \mu_m |\nabla(\partial_X u - \partial_X v)|^2 + \int_\Omega \eta_m |\partial_X u - \partial_X v|^2 dx \\
&\quad \leqslant \big(\|\partial_X f\|_{L^2} + C(v)\big)\|\nabla \partial_X (u-v)\|_{L^2} \\
&\qquad + C_1 \|\mu_m^{1/2}\nabla(u-v)\|_{L^2}\|\mu_m^{1/2}\nabla\partial_X(u-v)\|_{L^2} + C_2\|p\|_{L^2}\|\nabla\partial_X(u-v)\|_{L^2} \\
&\qquad + \epsilon_2 \|\nabla\partial_X(u-v)\|_{L^2}^2 + (C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\partial_X(u-v)\|_{L^2}^2 + C_b \\
&\qquad + C_3\big[\|\mu_m \nabla(\partial_X u - \partial_X v)\|_{L^2} + \|\eta_m(\partial_X u - \partial_X v)\|_{L^2} + \|\partial_X f\|_{L^2} \\
&\qquad + C(v) + \mathcal{R}_{press}\big]\|\nabla(u-v)\|_{L^2}.
\end{aligned} \tag{4.3.46}
$$

Applying Young's inequality with a small parameter ϵ to the above expression, we obtain:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\partial_X u - \partial_X v|^2 dx + \int \mu_m |\nabla(\partial_X u - \partial_X v)|^2 + \int_\Omega \eta_m |\partial_X u - \partial_X v|^2 dx \\
&\quad \leqslant C(\epsilon_3)\big(\|\partial_X f\|_{L^2}^2 + C(v)\big) + \epsilon_3\|\nabla\partial_X(u-v)\|_{L^2}^2 \\
&\quad + C_4\|\mu_m^{1/2}\nabla(u-v)\|_{L^2}^2 + \epsilon_3\|\mu_m^{1/2}\nabla\partial_X(u-v)\|_{L^2}^2 + C_5\|p\|_{L^2}^2 + \epsilon_3\|\nabla\partial_X(u-v)\|_{L^2}^2 \\
&\quad + \epsilon_2\|\nabla\partial_X(u-v)\|_{L^2}^2 + (C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\partial_X(u-v)\|_{L^2}^2 + C_b \\
&\quad + C_3\varepsilon\big[\|\mu_m\nabla\partial_X(u-v)\|_{L^2}^2 + \|\eta_m\partial_X(u-v)\|_{L^2}^2 + \|\partial_X f\|_{L^2}^2 \\
&\quad + C(v) + C(X)\|\mu_m\nabla(u-v)\|_{L^2}^2 + C(X)\|p\|_{L^2}^2 \\
&\quad + 2c(\epsilon_4)\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\partial_X u\|_{L^2}^2 + \epsilon_4\|\nabla\partial_X u\|_{L^2}^2 + C(X)\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\big] \\
&\quad + C_m(\varepsilon)\|\nabla(u-v)\|_{L^2}^2.
\end{aligned} \tag{4.3.47}
$$

The terms involving ϵ_i for $i = 1, ..4$ in equation (4.3.47) are absorbed by the left-hand side. We select a specific ε that depends on m in such a way that the term $\|\mu_m\nabla\partial_X(u-v)\|_{L^2}^2$ is also absorbed by the left-hand side. As a result, we derive:

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_X u - \partial_X v\|_{L^2}^2 + \|\mu_m^{1/2}\nabla(\partial_X u - \partial_X v)\|_{L^2}^2 + \|\eta_m^{1/2}(\partial_X u - \partial_X v)\|_{L^2}^2 \\
&\quad \leqslant C(\epsilon_3)\big(\|\partial_X f\|_{L^2}^2 + C(v)\big) + +(m^{1/2}) \\
&\quad + (C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\partial_X(u-v)\|_{L^2}^2 + C_b \\
&\quad + \|\partial_X f\|_{L^2}^2 + C\|f\|_{L^2}^2 \\
&\quad + 2c(\epsilon_4)\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\partial_X u\|_{L^2}^2 + C(X)\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 \\
&\quad + C_m(\varepsilon)\|\nabla(u-v)\|_{L^2}^2.
\end{aligned} \tag{4.3.48}
$$

Applying Gronwall's inequality to the above equation, we get

$$
\begin{aligned}
&\|\partial_X u - \partial_X v\|_{L^\infty(L^2)} + \|\mu_m^{1/2}\nabla(\partial_X u - \partial_X v)\|_{L^2(L^2)} + \|\eta_m^{1/2}(\partial_X u - \partial_X v)\|_{L^2(L^2)} \\
&\quad \leqslant C\big(\|\partial_X u_0\|_{L^2} + \|\partial_X f\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} + M\big)\Big[\exp\big(\int_0^T \psi(t)dt\big)\Big]^{1/2},
\end{aligned} \tag{4.3.49}
$$

where $\psi(t) := (1 + \|u(t)\|_{L^2}^2)\|\nabla u(t)\|_{L^2}^2$, and

$$
M := C(v) + \|u\|_{L^\infty(L^2)}\|\nabla u\|_{L^2(L^2)} + C_m\|\nabla(u-v)\|_{L^2(L^2)},
$$

which gives the estimate for the first three terms on the RHS of (4.1.11).

By squaring both sides of (4.3.43), integrating over time, and applying the energy estimate (4.2.1) we obtain that $\partial_t \partial_X u \in L^2(0,T;H^{-1}(\Omega))$ with the estimate

$$\begin{aligned}&\|\partial_t\partial_X u\|_{L^2(0,T;H^{-1}(\Omega))} \leq C(m)\big(\|\mu_m^{1/2}\nabla\partial_X(u-v)\|_{L^2(L^2)} + \|\eta_m^{1/2}\partial_X(u-v)\|_{L^2(L^2)}\big)\\&+\|\partial_X f\|_{L^2(L^2)} + C(X,m,\Omega,u_0,f,v) + \big(C(X)+C(\epsilon)\|\partial_X u\|_{L^\infty(L^2)}\big)C(\Omega,u_0,f,v)\\&+\epsilon\|\nabla\partial_X u\|_{L^2(L^2)},\end{aligned} \tag{4.3.50}$$

where we have used (4.2.1) to estimate the RHS of (4.3.42). This completes the proof of (4.1.11).

□

Appendix A

Supplementary proofs for Chapter 2

A.0.1 Preliminaries

We recall the particular case of Sobolev embedding theorem [30], which will be used throughout Chapter 2. Let Ω be a Lipschitz open bounded set in $\mathbb{R}^d$, then for $u \in H_0^1(\Omega)$ we have

$$\|u\|_{L^q(\Omega)} \leqslant c(q,\Omega)\|u\|_{H_0^1(\Omega)} \tag{A.0.1}$$

$\forall q, \quad 1 \leqslant q < \infty$, in case $d = 2$, and $1 \leqslant q \leqslant 6$ in case $d = 3$.

Let us define a trilinear and continuous form $b : V \times V \times V \to \mathbb{R}$, where V is defined in (2.2.2)

$$b(u,v,w) = \sum_{i,j=1}^{d} \int_\Omega u_i(\partial_{x_i} v) w_j \, dx. \tag{A.0.2}$$

Then there holds

Lemma A.0.1. *For any open set $\Omega \subset \mathbb{R}^d$,*

$$b(u,v,v) = 0, \, \forall u \in V, \, v \in H_0^1(\Omega) \cap L^d(\Omega) \tag{A.0.3}$$

$$b(u,v,w) = -b(u,w,v), \, \forall u \in V, \, v, \, w \in H_0^1(\Omega) \cap L^d(\Omega). \tag{A.0.4}$$

Proof. For $u \in \mathcal{V}$ (see (2.2.1)-(2.2.2)) and $v \in \mathcal{D}(\Omega)$, we have

$$\begin{aligned} \int_\Omega u_i \partial_{x_i} v_j v_j \, dx &= \int_\Omega u_i \partial_{x_i} \frac{(v_j)^2}{2} \, dx \\ &= -\frac{1}{2} \int_\Omega \partial_{x_i} u_i (v_j)^2 \, dx. \end{aligned} \tag{A.0.5}$$

The second equality is obtained by integration by parts and the fact that v vanished at the boundary of the domain, which proves (A.0.3) since

$$b(u,v,v) = -\frac{1}{2} \sum_{j=1}^{d} \int_\Omega \text{div } u_i (v_j)^2 \, dx = 0. \tag{A.0.6}$$

Since $\mathcal{V}$ is dense in V conclusion follows. Property (A.0.4) is a consequence of (A.0.3) when we replace v by $v + w$, and we use the multilinear properties of b. □

A.0.2 Proof of Theorems 2.2.1 and 2.2.2

Proof of Theorem 2.2.1. The existence of u_m is proved by Galerkin method (Temam [30]): we construct the approximate solution of (2.1.5) and then pass to the limit. Since m is fixed, we denote u_m and v_m by u and v, respectively.

Since the space V is separable and $\mathcal{V}$ is dense in V, there exists a sequence $w_1, w_2, ..., w_N...$ of linearly independent elements of $\mathcal{V}$ which is total in V. For each n we define an approximate solution u^N of (2.1.5) by

$$u^N = \sum_{j=1}^{N} c_{jN} w_j \tag{A.0.7}$$

with unknown coefficients $c_{jN} \in \mathbb{R}$, satisfying

$$(\nu \mathbb{D}u^N, \mathbb{D}w_j) + b(u^N, u^N, w_j) = \langle f(t), w_j \rangle \tag{A.0.8}$$

for $j = 1, 2..., N$. The equations (A.0.7) and (A.0.8) are a system of nonlinear equations for $c_{1N}, ..., c_{NN}$, the existence of a solution of this system follows from the Lemma 1.4 ([30]) Ch.II, Lemma 1.4.), that is consequence of the Brouwer Fixed Point Theorem:

Lemma A.0.2. *Let X be finite a dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $[\cdot]$ and let P be a continuous mapping from X into itself such that for some $k > 0$*

$$[P(\xi), \xi] > 0 \ for \ \ [\xi] = k.$$

Then there exists $\xi \in X$, $[\xi] \leqslant k$, such that

$$P(\xi) = 0.$$

We apply this lemma for proving the existence of u^N as follows:

Let X be the space spanned by $w_1, w_2, ..., w_N$; the scalar product on X is the scalar product induced by V, and $P = P_N$, where $P_N : X \to X$, is defined by

$$[P_N(u), v] = (\nu \mathbb{D}u, \mathbb{D}v) + b(u, u, v) - (f, v), \ \forall \, u, v \in X.$$

Let us check that scalar product $[\cdot,\cdot]$ is positive

$$[P_N(u), u] = \|\nu \mathbb{D}u\|^2 + b(u, u, u) - (f, u) = \|\nu \mathbb{D}u\|^2 - (f, u) \geqslant c(\nu)\|u\|^2 - \|f\|\|u\|.$$

In the last inequality were used the Korn and Cauchy-Schwartz inequalities. Therefore

$$[P_N(u), u] \geqslant \|u\|(c(\nu)\|u\| - \|f\|). \tag{A.0.9}$$

It follows that $[P_N u, u] > 0$ for $\|u\| = k$, and $k > \frac{1}{c(\nu)}\|f\|$. It follows that, there exists a solution u^N of (A.0.7)-(A.0.8). We multiply (A.0.8) by c_{jN}, this gives

$$\|\nu \mathbb{D}u^N\|_{L^2}^2 + b(u^N, u^N, u^N) = (f, u^N).$$

We know that, trilinear form $b(u^N, u^N, u^N) = 0$ by (A.0.3), and get

$$\|\nu \mathbb{D}u^N\|_{L^2}^2 = (f, u^N) \leqslant \|f\|_{L^2}\|u^N\|_{L^2}. \tag{A.0.10}$$

By Korn inequality we get a priori estimate

$$\|u^N\|_V \leqslant \|f\|_{L^2}. \tag{A.0.11}$$

Hence the sequence remains bounded in V, there exists a subsequence $k \longrightarrow \infty$ such that $u^{N_k} \rightharpoonup u$ in V. From compact embedding of $V \hookrightarrow L^2$, so we have also $u^{N_k} \longrightarrow u$ in $L^2(\Omega)$.
If u^N converges to u in $W^{1,2}$ weakly and in L^2 strongly, then we need to show that

$$b(u^N, u^N, v) \longrightarrow b(u, u, v), \ \forall v \in V.$$

Then we can pass to the limit in (A.0.8) with the subsequence $k \longrightarrow \infty$, we find that

$$(\nu \mathbb{D}u, \mathbb{D}v) + b(u, u, v) = \langle f(t), v \rangle$$

for any $v = w_1, ..., w_N ....$ The above equation is also true for any v which is the linear combination of $w_1, ..., w_N ....$ Since this combination are dense in V, a continuity argument shows that the above equation holds for each $v \in V$ and that u is a solution of (2.1.5).

From the properties of trilinear form we have

$$b(u^N, u^N, v) = -b(u^N, v, u^N) = -\sum_{i,j=1}^{d} \int_\Omega u_i^N u_j^N \partial_{x_i} v_j \, dx.$$

We know that $u_i^N \rightarrow u_i$ converges strongly in $L^2(\Omega)$, since $\partial_{x_i} v \in L^\infty(\Omega)$, so we have

$$\int_\Omega u_i^N u_j^N \partial_{x_i} v^j \rightarrow \int_\Omega u_i u_j \partial_{x_i} v_j \, dx.$$

Hence $b(u^N, v, u^N)$ converges to $b(u, v, u) = -b(u, u, v)$.

□

Proof of Theorem 2.2.2. Using energy estimates from (A.0.15) we deduce that for the domain Ω_S we get

$$m \int_S |\mathbb{D}(u_m)|^2 \, dx \leqslant \|f\|_{L^2} \|u_m\|_{L^2}. \tag{A.0.12}$$

Specifically, as $m \rightarrow \infty$, we have $\mathbb{D}u_m \rightarrow 0$ in Ω_S, and u_m converges strongly to u in $L^2(\Omega)$, where u is a weak solution of (1.2.3). Thus, in the limit as $m \rightarrow \infty$, we obtain rigid motion.

□

A.0.3 Energy and Bogovskii type estimates

We need to derive some energy estimates for Stokes system of equations for the (2.3.1) case, to show that the velocity vector field is in Hilbert space.

From Korn inequality (A.0.22) for Stokes system (2.3.1) we have:

$$\int_\Omega \nu(x) |\mathbb{D}u|^2 dx \geqslant c \int_\Omega |\nabla u|^2 dx. \tag{A.0.13}$$

Using Hölder and Poincaré inequalities we get:

$$\int_\Omega \nu(x) |\mathbb{D}u|^2 dx \leqslant \|f\|_{L^2} \|u\|_{L^2} \leqslant c_p \|f\|_{L^2} \|\nabla u\|_{L^2}. \tag{A.0.14}$$

We get energy estimate:

$$\|\nabla u\|_{L^2} \leqslant C_1 \|f\|_{L^2}. \tag{A.0.15}$$

The Definition 2.3.1 has no information about the pressure field. Since u is a weak solution, we know from [[30], Lemma 2.1] that there exists $p \in L^2(\Omega)$ such that

$$(\nu \mathbb{D}u, \mathbb{D}\psi) = -(f, \psi) + (p, \operatorname{div} \psi) \tag{A.0.16}$$

holds for $\psi \in C_0^\infty(\Omega)$. So, to every weak solution we are able to associate a pressure p in such a way that equation (A.0.16) holds. We formulate the following result for our case.

Lemma A.0.3. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, and let $f \in L^2(\Omega)$. A vector field $u \in W^{1,2}(\Omega)$ satisfies equation (2.3.4) for all $\phi \in \mathcal{V}(\Omega)$ if and only if there exists a pressure $p \in L^2(\Omega)$ such that equation (A.0.16) holds for every $\psi \in C_0^\infty(\Omega)$. Then*

$$p \in L^2(\Omega).$$

Finally, if we normalize p by the condition

$$\int_\Omega p = 0, \tag{A.0.17}$$

then the following estimate holds

$$\|p\|_{L^2} \leqslant c_b \left(\|\nu \mathbb{D}u\|_{L^2} + \|f\|_{L^2}\right). \tag{A.0.18}$$

Proof. The existence of pressure p follows from Temam ([30], Lemma 2.1). Let us consider the functional

$$\mathcal{F}(\psi) = (\nu \mathbb{D}u, \mathbb{D}\psi) + (f, \psi)$$

for $\psi \in H_0^1(\Omega)$. By assumption, $\mathcal{F}$ is bounded in $H_0^1(\Omega)$ and is identically zero in $\mathcal{V}(\Omega)$. Consider the problem

$$\begin{aligned} \operatorname{div} \psi &= p \\ \psi &\in H_0^1(\Omega) \\ \|\psi\|_{H^1} &\leqslant c_1 \|p\|_{L^2} \end{aligned} \tag{A.0.19}$$

with Ω bounded and satisfying the cone condition. Since

$$\int_\Omega p = 0,$$

from Theorem III.3.1 (in [[11]]) we deduce the existence of ψ solving equation (A.0.16). If we replace such a ψ into equation (A.0.16) and use equation (A.0.17) together with the Hölder inequality and Poincaré inequality we have

$$\begin{aligned} \|p\|_{L^2}^2 &\leqslant \|\nu \mathbb{D}u\|_{L^2} \|\mathbb{D}\psi\|_{L^2} + \|f\|_{L^2} \|\psi\|_{L^2} \\ \leqslant \|\nu \mathbb{D}u\|_{L^2} \|\nabla \psi\|_{L^2} + \|f\|_{L^2} \|\psi\|_{L^2} &\leqslant c_1 \|\nu \mathbb{D}u\|_{L^2} \|p\|_{L^2} + c_2 \|f\|_{L^2} \|p\|_{L^2}, \end{aligned} \tag{A.0.20}$$

$$\|p\|_{L^2} \leqslant c \|\nu \mathbb{D}u\|_{L^2} + c \|f\|_{L^2}$$

The proof is therefore completed. □

A.0.4 Korn inequality

Lemma A.0.4 (Korn inequality)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be an open bounded domain of class* C^2*. Then there exists constant* $c > 0$ *such that*

$$\int_\Omega \nu(x)|\mathbb{D}u|^2 dx \geqslant c_1(\nu_0)\|u\|^2_{H^1(\Omega)} \tag{A.0.21}$$

for all $u \in V$*.*

Proof. The proof is based on the results from [25], Lemma 2.1. We will prove it in general case where viscosity is bounded $\nu(x) \geq \nu_0$, where ν_0 is a fixed constant. Without loss of generality we could assume that $u \in C^2_c(\Omega)$. Consider,

$$\int_\Omega (\mathbb{D}u)^2 \, dx = \frac{1}{4}\int_\Omega \sum_{i,j=1}^{2} (u^i_{,j} + u^j_{,i})^2 dx = \frac{1}{2}\Big(\|\nabla u\|^2_{L^2} + \int_\Omega \sum_{i,j=1}^{2} u^i_{,j} u^j_{,i} dx \Big). \tag{A.0.22}$$

Integration by parts of the last term on the RHS of (A.0.22) gives us

$$\int_\Omega \sum_{i,j=1}^{2} u^i_{,j} u^j_{,i} = \int_\Omega \sum_{i,j=1}^{2} u^i_{,i} u^j_{,j} + \int_{\partial\Omega} \sum_{i,j=1}^{2} \left(u^i_{,j} u^j n_i - u^i_{,i} u^j n_j \right). \tag{A.0.23}$$

As u is compactly supported in Ω, it follows that the second term of the RHS of (A.0.23) is zero, so we get (A.0.22).

□

Appendix B

Supplementary proofs for Chapter 4

B.0.1 Bogovskii type estimates

Introduction of the pressure. The Definition 4.1.1 has no information about the pressure field. Since m is fixed, we use the notation u instead u_m. The motivation is to introduce the pressure using basic theory [30], let us set

$$U(t) = \int_0^t u(s)ds,\ \beta(t) = \int_0^t u^i(s)\nabla_i u(s)ds,\ F(t) = \int_0^t f(s)ds,\ V(t) = \int_0^t v(s)ds$$

u is a weak solution of 4.1.1 then, for any $d = 2, 3$,

$$U,\ \beta,\ F \in \mathcal{C}([0,T]; V').$$

Integrating, we see

$$(\mu_m \mathbb{D}(U(t) - V(t)), \mathbb{D}\phi) + (\eta_m(U(t) - V(t)), \phi) = \langle g(t), \phi \rangle, \quad \phi \in V,\ t \in [0;T]$$

with

$$g(t) = F(t) - \beta(t) - u(t) + u_0 + \operatorname{div} \mathbb{D}V(t).$$

from Proposition I.1.1 and Proposition I.1.2.[30], we get for each $t \in [0;T]$ there exists $P(t) \in L^2$ such that

$$-\operatorname{div}(\mu_m \mathbb{D}(U(t) - V(t))) + \eta_m(U(t) - V(t)) + \nabla P = g(t)$$

The gradient operator is an isomorphism from $L^2/\mathbb{R}$ into $H^{-1}(D)$. Observing that

$$\nabla P = g(t) + \operatorname{div}(\mu_m \mathbb{D}(U(t) - V(t))) - \eta_m(U(t) - V(t))$$

we have $\nabla P \in \mathcal{C}(0,T; H^{-1}(\Omega))$, and $P \in \mathcal{C}([0,T]; L^2)$. So, we can differentiate the above equation in the distribution sense in $Q^T = [0,T] \times \Omega$, setting

$$p = \frac{\partial P}{\partial t},$$

we obtain (4.1.2). We want to obtain some regularity for p. Let us test (4.2.4) in such a way that

$$\begin{aligned} \langle u_t, \psi \rangle + (\mu_m \mathbb{D}(u - v), \mathbb{D}\psi) + ((u \cdot \nabla)u, \psi) + (\eta_m(u - v), \psi) \\ = (f, \psi) + (p, \operatorname{div} \psi) + (\mathbb{D}v, \mathbb{D}\psi) \end{aligned} \tag{B.0.1}$$

holds for $\psi \in C_0^\infty(\Omega)$. So, to every weak solution we are able to associate a pressure p in such a way that equation (B.0.1) holds. We formulate the following result for our case.

Lemma B.0.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^2, and let $f \in L^2(Q^T)$. A vector field $u \in W^{1,2}(Q^T)$ satisfies equation (4.1.1) for all $\phi \in \mathcal{V}(Q^T)$ if and only if there exists a pressure $p \in L^2(Q^T)$ such that equation (B.0.1) holds for every $\psi \in C_0^\infty(Q^T)$. Then*

$$p \in L^2(Q^T).$$

Finally, if we normalize p by the condition

$$\int_\Omega p = 0 \quad \textit{for a.e.}. \tag{B.0.2}$$

The following estimate holds

$$\begin{aligned} \|p\|_{L^2(L^2)} \leqslant & C(\|\mu_m \nabla(u-v)\|_{L^2(L^2)} + \|\eta_m(u-v)\|_{L^2(L^2)} \\ & + \|u\|_{L^\infty(L^2)}\|u\|_{L^2(H^1)} + \|f\|_{L^2(L^2)} + \tilde{C}_v) \end{aligned} \tag{B.0.3}$$

Proof. Let us consider the functional

$$\begin{aligned} \mathcal{F}(\psi) = & \frac{d}{dt}(u-v,\psi) + (\mu_m \mathbb{D}(u-v), \mathbb{D}\psi) + (\eta_m(u-v),\psi) + ((u\cdot\nabla)u,\psi) - (f,\psi) \\ & - (v_t,\psi) + (\nabla v, \nabla\psi)) \end{aligned} \tag{B.0.4}$$

for $\psi \in H_0^1(\Omega)$. By assumption, $\mathcal{F}$ is bounded in $H_0^1(\Omega)$ and is identically zero in $\mathcal{V}(\Omega)$. Consider the problem for fixed t

$$\begin{aligned} \operatorname{div}\psi &= p \\ \psi &\in H_0^1(\Omega) \\ \|\psi\|_{H^1} &\leqslant c_1\|p\|_{L^2} \end{aligned} \tag{B.0.5}$$

with Ω bounded and satisfying the cone condition. Since

$$\int_\Omega p = 0, \tag{B.0.6}$$

from Theorem III.3.1 (in [[11]) we deduce the existence of ψ solving equation (A.0.16). If we replace such a ψ into equation (A.0.16) and use equation (B.0.6) together with the Hölder inequality and Poincaré inequality we have

$$\begin{aligned} \|p\|_{L^2}^2 \leqslant & c\big(\|\partial_t(u-v)\|_{H^{-1}} + \|\mu_m\nabla(u-v)\|_{L^2} + \|\eta_m(u-v)\|_{L^2} + \|u\|_{L^4}^2 \\ & + \|\partial_t v\|_{L^2} + \|\nabla v\|_{L^2} + \|f\|_{L^2}\big)\|\psi\|_{H^1} \\ \leqslant & c_2\Big(\|\partial_t(u-v)\|_{H^{-1}} + \|\mu_m\nabla(u-v)\|_{L^2} + \|\eta_m(u-v)\|_{L^2} + \|u\|_{L^2}\|\nabla u\|_{L^2} \\ & + \|\partial_t v\|_{L^2} + \|\nabla v\|_{L^2} + \|f\|_{L^2}\Big)\|p\|_{L^2}. \end{aligned} \tag{B.0.7}$$

After dividing both sides of the above inequality by $\|p\|_{L^2}$, we get

$$\|p\|_{L^2} \leqslant c_2(\|\partial_t(u-v)\|_{H^{-1}} + \|\mu_m \nabla(u-v)\|_{L^2} + \|\eta_m(u-v)\|_{L^2} + \|u\|_{L^2}\|u\|_{H^1} \\ + \|f\|_{L^2} + C_v). \tag{B.0.8}$$

We apply the energy estimate from (4.2.2), square both sides of the expression, and then integrate over time:

$$\begin{aligned} \|p\|_{L^2(L^2)} &\leqslant c_2(\|\partial_t(u-v)\|_{L^2(H^{-1})} + \|\mu_m \nabla(u-v)\|_{L^2(L^2)} \\ &+ \|u\|_{L^\infty(L^2)}\|u\|_{L^2(H^1)} + \|\eta_m(u-v)\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} + \tilde{C}_v) \\ &\leqslant C(\|\mu_m \nabla(u-v)\|_{L^2(L^2)} + \|u\|_{L^\infty(L^2)}\|u\|_{L^2(H^1)} \\ &+ \|\eta_m(u-v)\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} + \tilde{C}_v). \end{aligned} \tag{B.0.9}$$

We obtained Bogovskii type estimate the proof is therefore completed. □

B.0.2 Generalization of Lions-Aubin compactness result

In this subsection, we revisit the Lions-Aubin argument applied to an appropriate projection of the velocity field onto the "space of rigid velocities," as established in [14](see Lemma 3.4).

Let X be a Hilbert space with an inner product $(\cdot,\cdot)$, and Y be its closed subspace. Y is also a Hilbert space with the inner product $(\cdot,\cdot)$. We denote by P the orthogonal projector in X on Y, that is $PX := Y$. Let X_0 and X_1 are Banach spaces such that $X_0 \subset X$, with the compact embedding, and $Y \subset X_1$, with the embedding being dense.

Lemma B.0.2. *Let $\{v_k\}$ be a sequence of functions such that*

$$\|v_k\|_{L^p(0,T;X_0)}, \quad \|\partial_t P v_k\|_{L^p(0,T;X_1)} \leqslant C,$$

for $1 < p < \infty$. Then the sequence Pv_k is compact in $L^2(0,T;X)$.

B.0.3 Inequalities

Ladyzhenskaya inequalities: For any $u \in W_0^{1,2}(\Omega)$, and open bounded domain $\Omega \subset \mathbb{R}^d$ we have [18],

in the case $d = 2$

$$\|u\|_{L^4(\Omega)} \leqslant \|\nabla u\|^{1/2}_{L^2(\Omega)}\|u\|^{1/2}_{L^2(\Omega)}, \tag{B.0.10}$$

and in the case $d = 3$

$$\|u\|_{L^4(\Omega)} \leqslant \|\nabla u\|^{3/4}_{L^2(\Omega)}\|u\|^{1/4}_{L^2(\Omega)}. \tag{B.0.11}$$

Young's inequality

$$ab \leqslant \frac{1}{\lambda}\epsilon^{\lambda}a^{\lambda} + \frac{1}{\lambda'}\epsilon^{-\lambda'}b^{\lambda'} \tag{B.0.12}$$

where a, b, ϵ any positive numbers with $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$.

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