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p-adic local Langlands correspondence and geometry

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I dedicate this thesis to my parents and my brother.

"Na moje wiersze, napisane tak wcześnie,
że nie wiedziałam, że jestem poetą [...]
Na moje wiersze, jak na drogocenne wina,
przyjdzie kiedyś pora!"
Marina Cwietajewa "Na moje wiersze"

"idź bo tylko tak będziesz przyjęty do grona
zimnych czaszek
do grona twoich przodków: Gilgamesza
Hektora Rolanda
obrońców królestwa bez kresu i miasta
popiołów"
Zbigniew Herbert "Przesłanie Pana
Cogito"

"Wam wszystkie szczyty były małe,
a miękkim czerstwy chleb!"
Marina Cwietajewa "Generałom 1812
roku"

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Last, but not least, I thank my parents and my brother. This thesis is dedicated to them.

Résumé

Résumé

Cette these concerne la geometrie de la correspondence de Langlands p-adique. On donne la formalisation des methodes de Emerton, qui permettrait d'établir la conjecture de Fontaine-Mazur dans le cas general des groupes unitaires. Puis, on verifie que ce formalism est satisfait dans la cas de $U(3)$ oú on utilise la construction de Breuil-Herzig pour la correspondence p-adique.

De point de vue local, on commence l'étude de cohomologie modulo p et p-adiques de tour de Lubin-Tate pour $\mathrm{GL}_2(\mathbb{Q}_p)$. En particulier, on demontre que on peut retrouver la correspondence de Langlands p-adique dans la cohomologie completée de tour de Lubin-Tate.

Mots-clefs

program de Langlands, variétés de Shimura, espaces de Rapoport-Zink, représentations galoisiennes

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Abstract

This thesis concerns the geometry behind the p-adic local Langlands correspondence. We give a formalism of methods of Emerton, which would permit to establish the Fontaine-Mazur conjecture in the general case for unitary groups. Then, we verify that our formalism works well in the case of $U(3)$ where we use the construction of Breuil-Herzig as the input for the p-adic correspondence.

From the local viewpoint, we start a study of the modulo p and p-adic cohomology of the Lubin-Tate tower for $\mathrm{GL}_2(\mathbb{Q}_p)$. In particular, we show that we can find the local p-adic Langlands correspondence in the completed cohomology of the Lubin-Tate tower.

Keywords

Langlands program, Shimura varieties, Rapoport-Zink spaces, Galois representations

Contents

Introduction

The p-adic Langlands program in recent years burst out with activity. The goal is to construct a bijection between p-adic local Galois representations $\rho : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_n(E)$ (where F and E are finite extensions of $\mathbb{Q}_p$) and certain admissible Banach representations of $\mathrm{GL}_n(F)$ over E. Up to now, we have only a complete picture for $\mathrm{GL}_2(\mathbb{Q}_p)$ (by the work of Berger, Breuil, Colmez, Emerton, Kisin, Paskunas). In this case we have a local construction of Colmez as well as Emerton's proof that the p-adic local Langlands correspondence appears in the completed cohomology of modular curves. Let us explain this last result as it is the starting point of this thesis.

For any compact open subgroup $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ let us define open modular curves as complex varieties

$$Y(K) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C}\backslash\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K$$

Each $Y(K)$ has a natural model over $\mathbb{Q}$ which we denote also by $Y(K)$. Let us fix a compact open subgroup K^p of $\mathrm{GL}_2(\mathbb{A}_f^p)$. The p-adic completed cohomology of Emerton is defined by

$$\widehat{H}^i(K^p)_E = \left(\varprojlim_s \varinjlim_{K_p} H^i_{et}(Y(K_pK^p), \mathbb{Z}/p^s\mathbb{Z})\right) \otimes_{\mathbb{Z}_p} E$$

where K_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ and H^i_{et} denotes the étale cohomology groups. Hence there is a natural action of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{H}^i(K^p)_E$.

Let Σ be a set of places which contains p and all the primes at which K^p is not hyperspecial. We let $\mathbb{T}_E = E[T_l, S_l]_{l\notin\Sigma}$ be the abstract Hecke algebra generated by standard Hecke operators T_l and S_l. The Hecke algebra $\mathbb{T}_E$ acts on $\widehat{H}^i(K^p)_E$. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be a continuous Galois representation. We say that ρ is *pro-modular* if the Hecke system λ of $\mathbb{T}_E$ associated to ρ is such that the λ-isotypic part $\widehat{H}^1(K^p)_E[\lambda]$ of the completed cohomology is non-zero. Let $\rho_p = \rho_{|\,\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)}$ be the restriction of a pro-modular representation ρ and let $B(\rho_p)$ be the admissible Banach E-representation corresponding to ρ_p by the p-adic local Langlands correspondence. One of the main results of Emerton says that we have a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p)$-injection

$$\rho \otimes_E B(\rho_p) \hookrightarrow \widehat{H}^1(K^p)_E$$

Actually Emerton proves even more. Namely, he describes almost completely $\widehat{H}^1(K^p)_E$, but we shall need only the above weak local-global compatibility result.

One of the upshots of the above injection is the proof of the Fontaine-Mazur conjecture for GL_2 over $\mathbb{Q}$. This conjecture states that if a continuous Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ is unramified almost everywhere and ρ_p is de Rham, then ρ is modular (arises as a Galois representation associated to a classical modular form). Using the above weak local-global compatibility, Emerton proves that if ρ is pro-modular and ρ_p is de Rham with distinct Hodge-Tate weights, then ρ is in fact modular. The idea is that modularity is connected with certain locally algebraic vectors being non-zero, and this statement follows from the inclusion above and the fact that the locally algebraic vectors of $B(\rho_p)$ are non-zero, when ρ_p is de Rham with distinct Hodge-Tate weights. We remark, that for GL_2 over $\mathbb{Q}$ Emerton is able to deduce from this apparently weak statement of the Fontaine-Mazur conjecture, the full version by appealing to modularity lifting theorems and Serre's conjecture. In this work we shall deal only with deducing that pro-modular Galois representations which are regular de Rham above places dividing p are modular. We will refer to this statement as the pro-modular Fontaine-Mazur conjecture.

Our point of departure is the weak local-global compatibility of Emerton. In Chapter I we give a general formalism which allows us to prove the pro-modular Fontaine-Mazur conjecture for unitary groups $U(n)$ compact at infinity assuming the existence of a certain weak approximation of the p-adic local Langlands correspondence satisfying some natural hypotheses. A natural question is to ask for examples. The construction of Breuil-Herzig is the first succesful construction in the p-adic Langlands program, which goes beyond GL_2. They have associated to upper-triangular representations $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ admissible Banach E-representations $\Pi(\rho_p)^{ord}$, which are built out of principal series. Conjecturally, $\Pi(\rho_p)^{ord}$ should account for the ordinary part of the full p-adic local Langlands correspondence $\Pi(\rho_p)$ if it exists. We will show that their construction satisfies our formalism. At the end of Chapter I we will prove the pro-modular Fontaine-Mazur conjecture in the ordinary totally indecomposable setting for $U(n)$ (see Corollary I.4.19):

Theorem .0.1. *Let $z \in X_{K^p}(E)$, where X_{K^p} is the eigenvariety of some tame level K^p associated to $U(n)$ and let ρ be the Galois representation associated to z. For each $v \mid p$ we assume that*

1. *ρ_v is ordinary, de Rham and regular;*
2. *the reduction $\bar{\rho}_v$ is generic and totally indecomposable.*

Then z is modular (i.e. ρ is isomorphic to the Galois representation associated to a classical automorphic representation of $U(n)$).

In Chapter II (joint work with John Bergdall) we show that the construction of Breuil-Herzig appears in the completed cohomology group of $U(3)$. Let F/F^+ be a CM extension of number fields in which p is totally split and denote $G = U(3)$ a definite unitary group in three variables attached to F/F^+. Let us fix a compact open subgroup $K^p \subset G(\mathbb{A}_F^{p\infty})$. With this data in hand, we can define the completed cohomology group of Emerton

$$\widehat{H}^0(K^p)_E = \left(\varprojlim_s \varinjlim_{K_p} H^0(G(\mathbb{Q})\backslash G(\mathbb{A}_F^\infty)/K_pK^p, \mathbb{Z}/p^s\mathbb{Z}) \right) \otimes_{\mathbb{Z}_p} E$$

where K_p runs over open compact subgroups of $G(\mathbb{Q}_p)$. This space can be seen as a model for p-adic automorphic representations on $U(3)$.

If π is an automorphic representation on $U(3)$ then it has an associated (in the usual sense) global Galois representation $\rho = \rho_\pi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_3(E)$ (extending E if neccessary; work of Blasius and Rogawski in this case). If π has tame level K^p then ρ_π is unramified away from a finite set depending on K^p.

For each place $v \mid p$ of F^+ we write $v = \tilde{v}\tilde{v}^c$ and consider the local Galois representation $\rho_v := \rho_{\tilde{v}} : \mathrm{Gal}(\overline{F}_{\tilde{v}}/F_{\tilde{v}}) \simeq \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_3(E)$. If $\rho_{\tilde{v}}$ is generic and ordinary (these notions are defined in Chapter II) then the same is true for $\rho_{\tilde{v}^c}$ and $\Pi(\rho_{\tilde{v}})^{\mathrm{ord}}$ only depends on $v \mid p$ in F^+, where $\Pi(\rho_{\tilde{v}})^{\mathrm{ord}}$ is the representation of $\mathrm{GL}_3(F_{\tilde{v}})$ associated to $\rho_{\tilde{v}}$ by [BH]. Hence we will denote it by $\Pi(\rho_v)^{\mathrm{ord}}$. The following theorem (Theorem II.3.24) is our main result. It is a weak form, in the case of $U(3)$, of Conjecture 4.2.2 in [BH].

Theorem .0.2. *Suppose that for all $v \mid p$, ρ_v is generic ordinary and totally indecomposable. Then there is a closed embedding*

$$\widehat{\bigotimes}_{v|p}\Pi(\rho_v)^{\mathrm{ord}} \hookrightarrow \widehat{H}^0(K^p)_E.$$

We emphasize that ρ is assumed to be modular in the above theorem

The techniques that we use in the proof of this theorem may be important in their own right. Namely, we establish a connection between refinements of classical points on the $U(3)$-eigenvariety and certain principal series. This allows us to conclude by using an adjunction formula for the Jacquet functor of Emerton. We remark that these methods should generalize to $U(n)$, as well as to $U(2,1)$ for example, and this is the subject of our current work in progress. This approach is a natural generalization of [BE10].

Searching for a way to generalize the p-adic local Langlands correspondence, lead us to think about the geometric methods. The classical local Langlands correspondence was proved only after the use of geometric methods by Harris-Taylor. They have used global (Shimura varieties) as well as local objects (Rapoport-Zink spaces) in order to deduce the correspondence. It is tempting to follow their approach also in the p-adic case. Even though some of the methods are no longer available (harmonic analysis), there are also new purely p-adic phenomena. We have started this introduction by recalling Emerton results which constitute the global geometric part of the p-adic Langlands program. In Chapters III and IV we have investigated local geometric methods and we have obtained partial results in this direction.

Chapter III focuses on the mod p cohomology of the Lubin-Tate tower. Our main results are
(1) In the first cohomology group $H^1_{LT,\bar{\mathbb{F}}_p}$ of the Lubin-Tate tower for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears the mod p local Langlands correspondence and the naive mod p Jacquet-Langlands correspondence, meaning that there is an injection of representations

$$\pi \otimes \bar{\rho} \hookrightarrow H^1_{LT,\bar{\mathbb{F}}_p}$$

and $\sigma \otimes \pi \otimes \bar{\rho}$ appears as a subquotient in $H^1_{LT,\bar{\mathbb{F}}_p}$, where π is a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\bar{\rho}$ is its associated local mod p Galois representation and σ is the naive mod p Jacquet-Langlands correspondence (definition is given in Chapter III).
(2) The first cohomology group $H^1_{LT,c,\bar{\mathbb{F}}_p}$ with compact support of the Lubin-Tate tower does not contain any supersingular representations. This suprising result shows that the mod p situation is much different from its mod l analogue. It also permits us to show that the mod p local Langlands correspondence appears in H^1 of the ordinary locus. Again this fact is different from the l-adic setting for supercuspidal representations.
To obtain those results, especially (1), we compare modular curves with their supersingular locus, which contains multiple copies of the Lubin-Tate tower. We work at the rigid-analytic level with Berkovich spaces.

In Chapter IV we study the p-adic completed cohomology of the Lubin-Tate tower. Our main results are analogous to those mentioned above, though we take a slightly different approach, by working with adic spaces. This allows us to work with modular curves of infinite level, which are perfectoid spaces by the recent work of Scholze. This seems to be a conceptually clearer approach. One of our main results is the following

Theorem .0.3. *Let $\rho : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$ be a pro-modular representation (i.e. associated to some p-adic Hecke eigensystem on the Hecke algebra). Assume that $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible. Then we have a $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection*

$$B(\rho_p) \otimes_E \rho_p \hookrightarrow H^1(\mathcal{M}_{LT,\infty}, E)$$

where $B(\rho_p)$ is the p-adic Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to ρ_p by the p-adic local Langlands correspondence and $\mathcal{M}_{LT,\infty}$ is the Lubin-Tate space at infinity associated to $\mathrm{GL}_2(\mathbb{Q}_p)$.

Convention: In each chapter we refer to the results from this chapter by simply using the arabic numbers. For example, in Chapter III, Corollary 8.4 will refer to Corollary III.8.4.

Chapter I

Weak local-global compatibility and ordinary representations

I.1 Introduction

In [Eme11a], Emerton has shown that the completed cohomology of modular curves realises the p-adic local Langlands correspondence and used this result to prove the Fontaine-Mazur conjecture for $\mathrm{GL}_2(\mathbb{Q})$. We start from the observation that Emerton's methods can be well formalized to work for other groups, at least if we assume certain hypotheses, for example the existence of the p-adic Langlands correspondence. Fortunately, only a part of properties of the conjectural p-adic local Langlands correspondence are needed for applications to the pro-modular Fontaine-Mazur conjecture. We list them under hypothesis (H1) in the body of this chapter. After introducing this local definition, we move to the global setting. We work on the unitary Shimura varieties of type $U(n)$. After establishing certain basic results on the completed cohomology of these objects, we introduce the notion of an allowable set, which is a dense set of points on the eigenvariety, such that the specialisation at its points of a certain universal deformation of $\bar{\rho}$ lies in the completed cohomology of our Shimura varieties. This gives a necessary global condition to link the local hypothesis (H1) with the completed cohomology. Having to deal only with allowable sets is easier, as we may hope that the description of the p-adic Langlands correspondence for certain representations (regular and crystalline) will be explicit.

We remark that eventually we use two deformation arguments: one at the local level and the other at the global level (the existence of allowable points). They are related to two hypotheses ((H1) and (H2) respectively) on our global Galois representation $\bar{\rho}$. Assuming also a mild hypothesis (H3), we are able to prove the pro-modular Fontaine-Mazur conjecture for $U(n)$ in the following form (actually, we develop even more general formalism):

Proposition I.1.1. *Let F be a CM field and let E be a finite extension of $\mathbb{Q}_p$. Let $\rho : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_n(E)$ be a continuous Galois representation such that*

(1) ρ is pro-modular.

(2) ρ_v is de Rham and regular for every $v|p$.

(3) $\bar{\rho}$ satisfies hypotheses (H1)-(H3).

Then ρ is a twist of a Galois representation associated to an automorphic form on $U(n)$.

The pro-modularity condition is explained in Section 3. It should not be very restrictive, as it is believed that any representation ρ for which $\bar{\rho}$ is modular, is pro-modular (this is proved by Emerton for GL_2 over $\mathbb{Q}$).

As a corollary to this proposition, we obtain a version of the Fontaine-Mazur conjecture on the respective eigenvariety.

Corollary I.1.2. *Let $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_n(k)$ be a continuous Galois representation which satisfies hypotheses (H1)-(H3). Let $\mathcal{X}[\bar{\rho}]$ be the $\bar{\rho}$-part of the eigenvariety $\mathcal{X}$ associated to $U(n)$ by the construction of Emerton from [Eme06c]. Let $x \in \mathcal{X}[\bar{\rho}]$ be an E-point such that its associated representation $\rho_x : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_n(E)$ is de Rham and regular at every place of F above p. Then x is modular.*

There is one principal example (besides $\mathrm{GL}_2(\mathbb{Q}_p)$) when our formalism is satisfied and it was the motivation behind writing this chapter - namely, the recent construction of the ordinary representations of Breuil-Herzig ([BH]). We review this setting in the second part of this chapter and then we prove unconditionally the pro-modular Fontaine-Mazur conjecture for $U(n)$ in the ordinary totally indecomposable setting at the end of this chapter. Interestingly, the proof is relatively simple and we do not use in it the full construction of Breuil-Herzig. Our main unconditional result is

Theorem I.1.3. *Let $z \in X^{\mathrm{ord}}_{K^p}(E)$, where $X^{\mathrm{ord}}_{K^p}$ is the ordinary part of the eigenvariety of some tame level K^p associated to $U(n)$ and let ρ be the Galois representation associated to z. For each $v \mid p$ we assume that*

1. *ρ_v is de Rham and regular;*

2. *the reduction $\bar{\rho}_v$ is generic and totally indecomposable.*

Then z is classical (i.e. z arises from a classical automorphic representation of $U(n)$).

This result is also implied by a well-known classicality theorem of Hida. Nevertheless, our proof is completely different.

I.2 Definitions and basic facts

Let L denote an imaginary quadratic field in which p splits and let c be the complex conjugation. Choose a prime u above p. Let us denote by F^+ a totally real field of degree d. Set $F = LF^+$. We will assume that p totally decomposes in F. Let D/F be a central simple algebra of dimension n^2 such that F is the centre of D, the opposite algebra D^{op} is isomorphic to $D \otimes_{L,c} L$ and D is split at all primes above u. Choose an involution of the second kind $*$ on D and assume that there exists a homomorphism $h : \mathbb{C} \to D_{\mathbb{R}}$ for which $b \mapsto h(i)^{-1} b^* h(i)$ is a positive involution on $D_{\mathbb{R}}$.

Define the reductive group

$$G(R) = \{(\lambda, g) \in R^\times \times D^{op} \otimes_{\mathbb{Q}} R \mid gg^* = \lambda\}.$$

We assume that G is a unitary group of signature $(0, n)$ at all infinite places.

We choose a p-adic field E with ring of integers $\mathcal{O}$ and residue field k. These will be our coefficient rings.

We will fix an integral model of G over $\mathcal{O}_{F^+}[1/N]$ (see for example 4.1 in [BH] for details). We consider 0-dimensional Shimura varieties $S_K = G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ for G, where K is a compact open subgroup of $G(\mathbb{A}_f)$.

Let W be a finite-dimensional representation of G over E. By the construction described in Chapter 2 of [Eme06c], we can associate to W a local system $\mathcal{V}_W$ on S_K.

Let us fix a finite set Σ of primes w of F, such that $w_{|F^+}$ splits and w does not divide pN. We can now define the abstract Hecke algebra

$$\mathbb{T}_\Sigma^{abs} = \mathcal{O}[T_w^{(i)}]_{w\notin\Sigma}$$

where $T_w^{(i)}$ are the Hecke operators for $1 \leq i \leq n$. The operator $T_w^{(i)}$ acts on the Shimura variety S_K by a double coset $\mathrm{GL}_n(\mathcal{O}_{F_w})\left(\begin{smallmatrix} 1_{n-j} & 0 \\ 0 & \varpi_w 1_j \end{smallmatrix}\right)\mathrm{GL}_n(\mathcal{O}_{F_w})$, where ϖ_w is a uniformiser of $\mathcal{O}_{F_w}$.

We define the completed cohomology of Emerton by

$$\widehat{H}^0(K^p) = (\varprojlim_s \varinjlim_{K_p} H^0(S_{K_pK^p}, \mathcal{O}/\varpi^s\mathcal{O})) \otimes_{\mathcal{O}} E$$

where $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$ are open compact subgroups. We also define its $\mathcal{O}$-submodule

$$\widehat{H}^0(K^p)_{\mathcal{O}} = \varprojlim_s \varinjlim_{K_p} H^0(S_{K_pK^p}, \mathcal{O}/\varpi^s\mathcal{O})$$

We will fix the tame level K^p for the rest of the text. Let $K^\Sigma = \prod_{l\notin\Sigma} G(\mathbb{Z}_l)$. We assume that $(K^p)_l = G(\mathbb{Z}_l)$ at each $l \notin \Sigma$.

We write $\mathbb{T}(K_pK^p)$ for the image of $\mathbb{T}_\Sigma^{abs}$ in $\mathrm{End}_{\mathcal{O}}(H^0(S_{K_pK^p}, \mathcal{O}))$. Then we define

$$\mathbb{T} = \mathbb{T}(K^p) := \varprojlim_{K_p} \mathbb{T}(K_pK^p)$$

where the limit runs over open compact subgroups K_p of $G(\mathbb{Q}_p)$. We remark that $\mathbb{T}$ has a finite number of maximal ideals and is a product of its localisation at those maximal ideals. We refer the reader to p. 28 of [Sor12] for details. In particular, if $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}$, then $\mathbb{T}_{\mathfrak{m}}$ is a direct factor of $\mathbb{T}$.

We define also

$$H^0(K^p, \mathcal{V}_W) = \varinjlim_{K_p} H^0(S_{K_pK^p}, \mathcal{V}_W)$$

where W is an irreducible algebraic representation of G and $\mathcal{V}_W$ is the E-local system on $(S_K)_K$ associated to W.

We recall the definition of locally algebraic vectors from [Eme11b].

Definition I.2.1. *Let G be the group of $\mathbb{Q}_p$-points in some connected linear algebraic group $\mathbb{G}$ over $\mathbb{Q}_p$ and let V be a representation of G over E. Let W be a finite-dimensional algebraic representation W of $\mathbb{G}$ over E. A vector v in V is locally W-algebraic if there exists an open subgroup H of G, a natural number n, and an H-equivariant homomorphism $W^n \to V$ whose image contains the vector v. We write V_{W-la} for the set of locally W-algebraic vectors of V.*

Emerton proved in Proposition 4.2.2 of [Eme11b] that V_{W-la} is a G-invariant subspace of V.

Definition I.2.2. *A vector v in V is locally algebraic, if it is locally W-algebraic for some finite-dimensional algebraic representation W of $\mathbb{G}$. We denote the set of locally algebraic vectors of V by $V_{l.alg}$.*

It is a G-invariant subspace of V by Proposition 4.2.6 of [Eme11b]. We have the following proposition

Proposition I.2.3. *We have a $G(\mathbb{A}_{\Sigma_0})$-equivariant isomorphism*

$$\widehat{H}^0(K^p)_{l.alg} \simeq \bigoplus_W H^0(K^p, \mathcal{V}_W) \otimes W^\vee$$

where the sum is taken over all isomorphism classes of irreducible algebraic representations of G.

Proof. This follows from the Emerton spectral sequence. See Corollary 2.2.18 of [Eme06c]. □

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ which we fix and let $\bar{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(k)$ be the continuous Galois representation which is unramified outside Σ and whose characteristic polynomial satisfies

$$\mathrm{char}\bar{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w) = \sum_{i=0}^{n} (-1)^{n-i}\,\mathrm{Nm}(w)^{i(i-1)/2} T_w^{(i)} X^i \mod \mathfrak{m}$$

for all places w which do not belong to Σ and which split when restricted to F^+. This is the Galois representation associated to $\mathfrak{m}$. We refer the reader to Proposition 3.4.2 in [CHT08] for the construction. We remark that we can suppose that $\bar{\rho}_{\mathfrak{m}}$ is valued in $\mathrm{GL}_n(k)$ after possibly extending E (which we allow).

We assume that the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ is non-Eisenstein, that is $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible. We let $\rho_{\mathfrak{m}}$ be the universal automorphic deformation of $\bar{\rho}_{\mathfrak{m}}$ over $\mathbb{T}_{\mathfrak{m}}$ (its construction is standard and we do not recall it here; precise references may be found in Section 4.3 of [CS13]). It is an n-dimensional Galois representation over $\mathbb{T}_{\mathfrak{m}}$ which satisfies

$$\mathrm{char}\rho_{\mathfrak{m}}(\mathrm{Frob}_w) = \sum_{i=0}^{n} (-1)^{n-i}\,\mathrm{Nm}(w)^{i(i-1)/2} T_w^{(i)} X^i$$

for all places w which do not belong to Σ and which split when restricted to F^+.

I.3 General formalism

We now explain the general formalism for proving the pro-modular Fontaine-Mazur conjecture which we specialize at the end to the ordinary setting.

Let $\mathbb{T}'_{\mathfrak{m}}$ be a local complete reduced $\mathcal{O}$-algebra finite over $\mathbb{T}_{\mathfrak{m}}$ and let $\rho'_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\mathbb{T}'_{\mathfrak{m}})$ be the pushout of the universal representation $\rho_{\mathfrak{m}}$ to $\mathbb{T}'_{\mathfrak{m}}$. In what follows, we will always write $\mathfrak{p}'$ for an ideal of $\mathbb{T}'_{\mathfrak{m}}$ and $\mathfrak{p}$ for its inverse image in $\mathbb{T}_{\mathfrak{m}}$. In particular, we will write $\mathfrak{m}'$ for the maximal ideal of $\mathbb{T}'_{\mathfrak{m}}$.

We will make certain hypotheses (the last one depending on an ideal $\mathfrak{p}' \in \mathrm{Spec}\,\mathbb{T}'_{\mathfrak{m}}$):

- (H1) There exists an admissible representation $\Pi(\rho'_{\mathfrak{m},v})$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathbb{T}'_{\mathfrak{m}}$ associated to each local representation $\rho'_{\mathfrak{m},v}$ for $v|p$. This representation is such that for each prime ideal $\mathfrak{p}'$ of $\mathbb{T}'_{\mathfrak{m}}$ which comes from $\mathrm{Spm}(\mathbb{T}'_{\mathfrak{m}}[1/p])$ (where Spm is the maximal spectrum, i.e. the set of maximal ideals) for which $\rho'_{\mathfrak{m},v}/\mathfrak{p}'[1/p]$ is regular and de Rham at all places v dividing p, the locally algebraic vectors of $\Pi(\rho'_{\mathfrak{m},v})/\mathfrak{p}'[1/p]$ are non-zero for all $v|p$. Moreover we assume that the k-representation $\pi_{\mathfrak{m},v} := \Pi(\rho'_{\mathfrak{m},v})/\mathfrak{m}'$ is of finite length.

- (H2): There exists an allowable set of points for $\Pi(\rho'_{\mathfrak{m},v})$ (for each $v|p$), that is, there exists a dense set of points $\mathcal{C}$ in $\mathrm{Spec}(\mathbb{T}'_{\mathfrak{m}})$ which is contained in $\mathrm{Spm}(\mathbb{T}'_{\mathfrak{m}}[1/p])$ and such that for each $\mathfrak{p}' \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathbb{T}_{\mathfrak{m}}[G(\mathbb{Q}_p)]}(\widehat{\otimes}_{v|p}\Pi(\rho'_{\mathfrak{m},v})/\mathfrak{p}', \widehat{H}^0(K^p)) \neq 0$$

- (H3)[$\mathfrak{p}'$]: Every non-zero $\mathbb{T}_{\mathfrak{m}}[G(\mathbb{Q}_p)]$-linear map

$$\widehat{\otimes}_{v|p}\Pi(\rho'_{\mathfrak{m},v})/\mathfrak{p}' \to \widehat{H}^0(K^p)$$

  is an embedding.

Let us make some comments before showing how these hypotheses imply the promodular Fontaine-Mazur conjecture.

The hypothesis (H1) gives an existence of a representation which shall be viewed as an approximation of the p-adic local Langlands correspondence applied to $\rho'_{\mathfrak{m}}$. In what follows (H1) will be satisfied by using the construction of Breuil-Herzig of the ordinary part of the p-adic local Langlands correspondence.

Regarding the hypothesis (H3)[$\mathfrak{p}'$] we will not say anything here. It is needed to deduce that certain locally algebraic vectors are non-zero.

We are left with discussing (H2). Let us define

$$\Pi_p = \widehat{\otimes}_{v|p}\Pi(\rho'_{\mathfrak{m},v})$$

We define $\mathbb{T}'_{\mathfrak{m}}$-module

$$X = \mathrm{Hom}_{\mathbb{T}_{\mathfrak{m}}[G(\mathbb{Q}_p)]}(\Pi_p, \widehat{H}^0(K^p)_{\mathcal{O}})$$

of $\mathbb{T}_{\mathfrak{m}}[G(\mathbb{Q}_p)]$-linear homomorphisms which are $G(\mathbb{Q}_p)$-equivariant and continuous, where Π_p is given the $\mathfrak{m}$-adic topology.

The hypothesis (H2) is equivalent to demanding the existence of an allowable set for $\bar{\rho}$ that is a dense subset $\mathcal{C}$ on $\mathrm{Spec}\,\mathbb{T}'_{\mathfrak{m}}$, such that for all $\mathfrak{p}' \in \mathcal{C}$ we have

$$X[\mathfrak{p}'] = \mathrm{Hom}_{\mathbb{T}_{\mathfrak{m}}[G(\mathbb{Q}_p)]}(\Pi_p/\mathfrak{p}', \widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}) \neq 0$$

Let us prove a preliminary lemma:

Lemma I.3.1. *$\mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O}) \otimes_{\mathcal{O}} E$ is a finitely generated $\mathbb{T}'_{\mathfrak{m}}[1/p]$-module.*

Proof. By Proposition C.5 of [Eme11a] we have to show that X is cofinitely generated. By Definition C.1, because $\widehat{H}^0(K^p)_{\mathcal{O},\mathfrak{m}}$ is ϖ-adically complete, separated and $\mathcal{O}$-torsion free, we are left to show that $(X/\varpi X)[\mathfrak{m}']$ is finite-dimensional over k. But we have

$$(X/\varpi X)[\mathfrak{m}'] \hookrightarrow \mathrm{Hom}_{k[G(\mathbb{Q}_p)]}(\Pi_p/\mathfrak{m}', \widehat{H}^0(K^p)_{k,\mathfrak{m}})$$

and we show that Hom is finite-dimensional. Because $\Pi_p/\mathfrak{m}' = \otimes_{v|p}\pi_{\mathfrak{m},v}$ and each $\pi_{\mathfrak{m},v}$ is of finite length, for each v we can choose a finite-dimensional k-subspace W_v of $\pi_{\mathfrak{m},v}$ which generates $\pi_{\mathfrak{m},v}$ as a $\mathrm{GL}_n(\mathbb{Q}_p)$-representation. Let $W = \otimes_{v|p}W_v$. Since W_v is smooth and finite-dimensional we can choose a compact open subgroup K_v fixing W_v point-wise. Let $K_p = \prod_{v|p} K_v$. By restriction we have

$$\mathrm{Hom}_{k[G(\mathbb{Q}_p)]}(\Pi_p/\mathfrak{m}', \widehat{H}^0(K^p)_{k,\mathfrak{m}}) \hookrightarrow \mathrm{Hom}_{k[K_p]}(W, \widehat{H}^0(K^p)_{k,\mathfrak{m}})$$

Since K_p acts trivially on W we moreover have

$$\mathrm{Hom}_{k[K_p]}(W, \widehat{H}^0(K^p)_{k,\mathfrak{m}}) \simeq W^\vee \otimes_k H^0(S_{K_pK^p}, k)_\mathfrak{m}$$

which is of finite dimention over k. □

Lemma I.3.2. *Assume (H2). Then $X[\mathfrak{p}'] \neq 0$ for all $\mathfrak{p}' \in \operatorname{Spec} \mathbb{T}'_\mathfrak{m}$.*

Proof. By Lemma C.14 of [Eme11a], we have

$$(\mathbb{T}'_\mathfrak{m}/\mathfrak{p}') \otimes_{\mathbb{T}'_\mathfrak{m}} \mathrm{Hom}_\mathcal{O}(X, \mathcal{O}) \otimes_\mathcal{O} E \simeq \mathrm{Hom}_\mathcal{O}(X[\mathfrak{p}'], \mathcal{O}) \otimes_\mathcal{O} E$$

and so it suffices to show that the elements on the right are non-zero for all $\mathfrak{p}'$ if and only if they are non-zero for all $\mathfrak{p}'$ in $\mathcal{C}$. Consider things in more generality. Let M be a finitely generated $\mathbb{T}'_\mathfrak{m}[1/p]$-module such that $M/\mathfrak{p}'M \neq 0$ for all $\mathfrak{p}' \in \mathcal{C}$. Because $\mathbb{T}'_\mathfrak{m}[1/p]/\mathfrak{p}'$ is a field, it follows that $M/\mathfrak{p}'M$ is a faithful $\mathbb{T}'_\mathfrak{m}[1/p]/\mathfrak{p}'$-module. If $t \in \mathbb{T}'_\mathfrak{m}[1/p]$ acts by 0 on M then it acts by 0 on $M/\mathfrak{p}'M$ for all $\mathfrak{p}'$, and as $M/\mathfrak{p}'M \neq 0$ if $\mathfrak{p}' \in \mathcal{C}$, we have $t \in \mathfrak{p}'$ for all $\mathfrak{p}' \in \mathcal{C}$, that is $t \in \cap_{\mathfrak{p}' \in \mathcal{C}} \mathfrak{p}' = 0$. So $\mathbb{T}'_\mathfrak{m}[1/p]$ acts faithfully on M. Now, let $\mathfrak{p}'$ be any maximal ideal of $\mathbb{T}'_\mathfrak{m}[1/p]$ and suppose that $M/\mathfrak{p}'M = 0$, that is $M = \mathfrak{p}'M$. As M is finitely generated $\mathbb{T}'_\mathfrak{m}[1/p]$-module, Nakayama's lemma gives us a non-zero element t of $\mathbb{T}'_\mathfrak{m}[1/p]$ such that $tM = 0$, which is impossible as we have shown above. We deduce that $M/\mathfrak{p}'M \neq 0$ for all $\mathfrak{p}'$. Applying this reasoning to $M = \mathrm{Hom}_\mathcal{O}(X, \mathcal{O}) \otimes_\mathcal{O} E$ which is finitely generated by Lemma I.3.1, we conclude. □

Definition I.3.3. *We say that a representation $\rho : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_n(E)$ is pro-modular with respect to $\mathbb{T}'_\mathfrak{m}$ if there exists a prime ideal $\mathfrak{p}'$ of $\mathbb{T}'_\mathfrak{m}$ such that $\rho \simeq \rho_\mathfrak{m}/\mathfrak{p}[1/p]$ and $\widehat{H}^0(K^p)[\mathfrak{p}] \neq 0$, where $\mathfrak{p}$ is the inverse image of $\mathfrak{p}'$ in $\mathbb{T}_\mathfrak{m}$.*

One natural source of pro-modular representations are representations attached to points on the eigenvariety for G. We shall review this notion later on.

We say that ρ is modular if it is the Galois representation associated to some automorphic representation of G of tame level K^p. This is equivalent to $\widehat{H}^0(K^p)_{l.alg}[\mathfrak{p}] \neq 0$ by Proposition I.2.3. Our three hypotheses imply the pro-modular Fontaine-Mazur conjecture in the following form.

Theorem I.3.4. *Let $\rho : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_n(E)$ be a pro-modular Galois representation with respect to $\mathbb{T}'_\mathfrak{m}$ with the associated prime ideal $\mathfrak{p}'$ of $\mathbb{T}'_\mathfrak{m}$. Assume that ρ is de Rham and regular at all places dividing p. Assume also that hypotheses (H1),(H2) and (H3)[$\mathfrak{p}$'] hold. Then ρ is modular.*

Proof. As ρ_v is de Rham and regular for every $v|p$, by (H1) we have that $\Pi(\rho_v)_{l.alg} \neq 0$ for every $v|p$. By Lemma I.3.2 and the hypothesis (H3)[$\mathfrak{p}'$] we conclude that also

$$\widehat{H}^0(K^p)_{l.alg}[\mathfrak{p}] \neq 0$$

which is what we wanted. □

In the rest of this text we will explain the ordinary setting.

I.4 Ordinary case

In this section, we show that the ordinary part of Breuil-Herzig ([BH]) fulfills the formalism presented in the previous section.

I.4.1 Preliminaries on reductive groups

We recall certain results on reductive groups used in [BH]. Let G be a split connected reductive $\mathbb{Z}_p$-group with a Borel subgroup B and a torus $T \subset B$. We let $(X(T), R, X^\vee(T), R^\vee)$ be the root datum of G, where $R \subset X(T)$ (respectively $R^\vee \subset X^\vee(T)$) is the set of roots (resp. coroots). For each $\alpha \in R$, let s_α be the reflection on $X(T)$ associated to α. Let W be the Weyl group, the subgroup of automorphisms of $X(T)$ generated by s_α for $\alpha \in R$.

We fix a subset of simple roots $S \subset R$ and we let $R^+ \subset R$ be the set of positive roots (roots in $\oplus_{\alpha\in S}\mathbb{Z}_{\geq 0}\alpha$). Let G^{der} be the derived group of G and let $\hat{G}$ be the dual group scheme of G (which we get by taking the dual root datum). We have also dual groups $\hat{B}$ and $\hat{T}$.

To $\alpha \in R$ one can associate a root subgroup $U_\alpha \subset G$. We have $\alpha \in R^+$ if and only if $U_\alpha \subset B$. We let $\mathfrak{g}_\alpha$ be the Lie algebra of U_α. We call a subset $C \subset R$ closed if for each $\alpha \in C, \beta \in C$ such that $\alpha + \beta \in R$, we have $\alpha + \beta \in C$. If $C \subset R^+$ is a closed subset, we let $U_C \subset U$ be the Zariski closed subgroup of B generated by the root subgroups U_α for $\alpha \in C$. We let $B_C = TU_C$ be the Zariski closed subgroup of B determined by C.

Let us spell out all the assumptions that we put on G and its dual group $\hat{G}$. We suppose throughout this text that both G and $\hat{G}$ have connected centers. Moreover we suppose that G^{der} is simply connected (some of these conditions are equivalent, see Proposition 2.1.1 in [BH]). This condition implies that there exist fundamental weights λ_α for $\alpha \in S$. They satisfy for any $\beta \in S$

$$\langle \lambda_\alpha, \beta^\vee \rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

We define as in Section 3.1 of [BH] a twisting element θ for G by setting $\theta = \sum_{\alpha\in S} \lambda_\alpha$. For any $\alpha \in S$ we have $\langle \theta, \alpha^\vee \rangle = 1$.

If $C \subset R$ is a closed subset, write G_C for the Zariski closed subgroup scheme of G generated by T, U_α and $U_{-\alpha}$ for $\alpha \in C$. For $C = \{\alpha\}$ we write simply G_α for G_C. A subset $J \subset S$ of pairwise orthogonal roots is closed (see the proof of Lemma 2.3.7 in [BH]) and hence we can define G_J as above.

Lemma I.4.1 (Lemma 3.1.4, [BH]). *Let $J \subset S$ be a subset of pairwise orthogonal roots. Then there is a subtorus $T'_J \subset T$ which is central in G_J such that*

$$G_J \simeq T'_J \times \mathrm{GL}_2^J$$

We use this lemma in the construction of $\Pi(\rho)^{ord}$ which we define as a sum over certain induced representations of $G_J(\mathbb{Q}_p)$. We construct representations of $G_J(\mathbb{Q}_p)$ by using the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

I.4.2 Ordinary part of the p-adic local Langlands correspondence

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and let k be its residue field. We fix also a uniformiser ϖ. Let A be a complete local Noetherian $\mathcal{O}$-algebra with residue field k.

We have

$$T(\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{Spec}(\mathbb{Q}_p)}(\mathrm{Spec}(\mathbb{Q}_p), \mathrm{Spec}(\mathbb{Q}_p[X(T)])) = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Q}_p^\times) =$$

$$= \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times = X(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times$$

To a continuous character

$$\hat{\chi} : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab} \to \hat{T}(A)$$

we can associate a continuous character $\chi : T(\mathbb{Q}_p) \to A^\times$ by taking the composite of the maps

$$T(\mathbb{Q}_p) \simeq X(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Q}_p^\times \hookrightarrow X(\hat{T}) \otimes_{\mathbb{Z}} \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab} \to X(\hat{T}) \otimes_{\mathbb{Z}} \hat{T}(A) \to A^\times$$

where the first injection comes from the local class field theory.

We define the p-adic cyclotomic character $\epsilon : G_{\mathbb{Q}_p} \to A^\times$ by composing the standard p-adic cyclotomic character which takes values in $\mathcal{O}^\times$ with the inclusion $\mathcal{O}^\times \hookrightarrow A^\times$. By the local class field theory we can also consider it as a character of $\mathbb{Q}_p^\times$ which we tacitly do in what follows.

Let us consider a continuous homomorphism

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \hat{G}(A)$$

Definition I.4.2. *We say that ρ is **triangular** when it takes values in our fixed Borel $\hat{B}(A)$ of $\hat{G}(A)$.*

We let $C_\rho \subset R^{+\vee}$ be the smallest closed subset such that $\hat{B}_{C_\rho}(A)$ contains all the $\rho(g)$ for $g \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ (compare with Lemma 2.3.1 of [BH]). Thus ρ factorises via $\hat{B}_{C_\rho}(A)$

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \hat{B}_{C_\rho}(A) \subset \hat{B}(A) \subset \hat{G}(A)$$

We associate a character $\hat{\chi}_\rho$ to ρ by composing ρ with the natural surjection

$$\hat{\chi}_\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \hat{B}_{C_\rho}(A) \twoheadrightarrow \hat{T}(A)$$

We attach to $\hat{\chi}_\rho$ a continuous character $\chi_\rho : T(\mathbb{Q}_p) \to A^\times$ by the local class field theory as above.

Definition I.4.3. *We say that a triangular ρ is **generic** if $\alpha^\vee \circ \hat{\chi}_\rho \notin \{1, \epsilon, \epsilon^{-1}\}$ for all $\alpha \in R^+$ (or equivalently all $\alpha \in R$). The same definition applies to the reduction $\bar{\rho}$ of ρ.*

In what follows we will consider only triangular representations ρ. We assume that $\bar{\rho}$ is generic.

We now construct several representations of $G(\mathbb{Q}_p)$ over A attached to ρ. Let $I \subset S^\vee$ be a subset of pairwise orthogonal roots. We shall firstly construct an admissible continuous representation $\tilde{\Pi}(\rho)_I$ of $G_{I^\vee}(\mathbb{Q}_p)$ over A. We imitate the proof of Proposition 3.3.3 in [BH], though we present a simplified construction, because we do not need to show unicity of $\tilde{\Pi}(\rho)_I$. Only later on and under additional assumptions we will show that we retrieve the construction of Breuil and Herzig over fields. Then we obtain a representation $\Pi(\rho)^{ord}$ of $G(\mathbb{Q}_p)$ over A, which generalizes the construction of Breuil and Herzig over fields, and which we define as a direct limit of $\Pi(\rho)_I$ over different I (where $\Pi(\rho)_I$ is simply $\tilde{\Pi}(\rho)_I$ induced to $G(\mathbb{Q}_p)$). In particular, we shall consider a representation

$\Pi(\rho)_\emptyset = (\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_\rho \cdot (\epsilon^{-1}\circ\theta))^{\mathcal{C}^0}$ which we use for the proof of the pro-modular Fontaine-Mazur conjecture. All these representations are functorial in A and hence behave well with respect to reduction modulo prime ideals.

If $\beta \in I^\vee$ and $\chi_\beta : T_\beta(\mathbb{Q}_p) \to A^\times$ is a continuous character, we define

$$\Pi_\beta(\chi_\beta) = \left(\mathrm{Ind}_{\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \chi_\beta \cdot (\epsilon^{-1}\circ\theta)_{|T_\beta(\mathbb{Q}_p)}\right)^{\mathcal{C}^0}$$

This is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ which we use as a building block. We let $\rho_\beta : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_{2,\beta^\vee}(A)$ be the representation which we get by composing $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \hat{B}(A)$ with $\hat{B}(A) \to \hat{B}_\beta(A) \to \mathrm{GL}_{2,\beta^\vee}(A)$. We define $\mathcal{E}_\beta$ as the representation attached to the 2-dimensional Galois representation ρ_β by the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In order to have a functorial construction we fix a quasi-inverse MF^{-1} to the Colmez functor MF for $\mathrm{GL}_2(\mathbb{Q}_p)$ (we use the notation of Emerton from [Eme11a]). For any β it sends a lifting of $\bar{\rho}_\beta$ to a lifting of $\bar{\pi}_\beta$ with a central character (where $\bar{\pi}_\beta$ is the smooth representation over k corresponding to $\bar{\rho}_\beta$ by the mod p local Langlands correspondence). Then we define

$$\mathcal{E}_\beta = \mathrm{MF}^{-1}(\rho_\beta)$$

We remark that over k this is an extension of $\Pi_\beta(s_\beta(\chi_{\rho|T_\beta(\mathbb{Q}_p)}))$ by $\Pi_\beta(\chi_{\rho|T_\beta(\mathbb{Q}_p)})$ because ρ_β is lower-triangular with the appropiate character on the diagonal (see Proposition 3.4.2 in [Eme11a]).

Let $\chi'_{\rho,I^\vee} = \chi_{\rho|T'_{I^\vee}(\mathbb{Q}_p)}$. We define an admissible continuous representation of $T'_{I^\vee}(\mathbb{Q}_p)\times \mathrm{GL}_2(\mathbb{Q}_p)^{I^\vee}$

$$\tilde{\Pi}(\rho)_I = \chi'_{\rho,I^\vee} \cdot (\epsilon^{-1}\circ\theta_{|T'_{I^\vee}(\mathbb{Q}_p)}) \otimes_A (\widehat{\otimes}_{\beta\in I^\vee}\mathcal{E}_\beta)$$

This is exactly the representation we look for.

We set

$$\Pi(\rho)_I = \left(\mathrm{Ind}_{B^-(\mathbb{Q}_p)G_{I^\vee}(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \tilde{\Pi}(\rho)_I\right)^{\mathcal{C}^0}$$

where we view $\tilde{\Pi}(\rho)_I$ as a continuous representation of $B^-(\mathbb{Q}_p)G_{I^\vee}(\mathbb{Q}_p)$ by inflation. By the proposition above and by Theorem 3.1.1 in [BH] (which holds in our setting verbatim), the representation $\Pi(\rho)_I$ of $G(\mathbb{Q}_p)$ is admissible and continuous.

We now use an argument similar to the one of Breuil-Herzig appearing before Lemma 3.3.5 in [BH] to construct a direct limit. Following the proof of Proposition 3.4.2 of [Eme11a] we have natural injections of $\Pi_\beta(\chi_{\rho|T_\beta(\mathbb{Q}_p)})$ into $\mathcal{E}_\beta$. Indeed, Proposition 3.2.4 of [Eme11a] gives us a natural embedding $\chi_{\rho|T_\beta(\mathbb{Q}_p)} \hookrightarrow \mathrm{Ord}(\mathcal{E}_\beta)$, where we have denoted by Ord the ordinary part functor of Emerton. By adjointness property of Ord this gives us a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant injection $\Pi_\beta(\chi_{\rho|T_\beta(\mathbb{Q}_p)}) \hookrightarrow \mathcal{E}_\beta$. We remark that those injections will be functorial because of Proposition 3.2.4 of [Eme11a] and because we have fixed a quasi-inverse MF^{-1}.

By Theorem 4.4.6 and Corollary 4.3.5 of [Eme10a], we have for $I' \subset I$

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\Pi(\rho)_{I'}, \Pi(\rho)_I) \simeq \mathrm{Hom}_{G_{I^\vee}(\mathbb{Q}_p)}\left((\mathrm{Ind}_{(B^-(\mathbb{Q}_p)\cap G_{I^\vee}(\mathbb{Q}_p))G_{I'^\vee}(\mathbb{Q}_p)}^{G_{I^\vee}(\mathbb{Q}_p)} \tilde{\Pi}(\rho)_{I'})^{C^0}, \tilde{\Pi}(\rho)_I\right)$$

Observe that our injections

$$\Pi_\beta(\chi_{\rho|T_\beta(\mathbb{Q}_p)}) \hookrightarrow \mathcal{E}_\beta$$

invoked above induce an injection

$$\mathrm{Ind}_{(B^-(\mathbb{Q}_p)\cap G_{I^\vee}(\mathbb{Q}_p))G_{I'^\vee}(\mathbb{Q}_p)}^{G_{I^\vee}(\mathbb{Q}_p)}(\tilde{\Pi}(\rho)_{I'})^{C^0} \hookrightarrow \tilde{\Pi}(\rho)_I$$

and hence also a $G(\mathbb{Q}_p)$-equivariant injection

$$\Pi(\rho)_{I'} \hookrightarrow \Pi(\rho)_I$$

This actually gives a compatible system of injections, by which we mean that for any $I'' \subset I' \subset I$, the corresponding diagram of injections is commutative. We then define an admissible continuous representation of $G(\mathbb{Q}_p)$ over A by

$$\Pi(\rho)^{ord} = \varinjlim_{I} \Pi(\rho)_I$$

where I runs over subsets of $S^\vee$ of pairwise orthogonal roots.

I.4.3 Compatibility with the construction of Breuil-Herzig

We study in this section how $\Pi(\rho)^{ord}$ behaves with respect to reduction modulo prime ideals in A. Recall that G and its dual are split, hence we can canonically identify $R^\vee(A)$ and $R^\vee(A/\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ of A.

Lemma I.4.4. *Let $A \to A'$ be a morphism of complete local $\mathcal{O}$-algebras and let ρ be triangular over A with $\bar{\rho}$ generic. Then*

$$\Pi(\rho \otimes_A A')_I \simeq \Pi(\rho)_I \otimes_A A'$$

for any subset $I \subset S^\vee$ of pairwise orthogonal roots and

$$\Pi(\rho \otimes_A A')^{\mathrm{ord}} \simeq \Pi(\rho)^{\mathrm{ord}} \otimes_A A'$$

Proof. Observe that $\rho \otimes_A A'$ is triangular because ρ is. By the definition of $\Pi(\rho)_I$ it is enough to check, that the construction of $\tilde{\Pi}(\rho)_I$ we have given above is compatible with the base change $A \to A'$. This follows from the fact that the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is compatible with the base change $A \to A'$.

□

To put more content into this lemma let us specialize to the totally indecomposable case.

Definition I.4.5. *We say that ρ is **totally indecomposable** if $C_\rho = R^{+\vee}$ is minimal among all conjugates of ρ by B (equivalently, $C_{b\rho b^{-1}} = R^{+\vee}$ for all $b \in B$).*

We prove now that for GL_n we retrieve the construction of Breuil-Herzig after reducing modulo $\mathfrak{p}$. Before continuing, we shall give another characterisation of totally indecomposable representations valable for $G = \mathrm{GL}_n$.

Lemma I.4.6. *Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(A)$ be a triangular representation and A be a field. The following conditions are equivalent:*

1. *All semi-simple subquotients of ρ are simple (equivalently, the graded pieces of the filtration by the socle are irreducible).*

2. *B is the unique Borel that contains the image of ρ (equivalently, the image of ρ fixes a unique Borel B (flag)). Here B is the Borel we have fixed before in the definition of being triangular.*

3. *ρ is totally indecomposable.*

Proof. (1. $\Leftrightarrow$ 2.) If there exists $\mathrm{soc}_{j+1} / \mathrm{soc}_j$ which is not irreducible then we can construct two distinct flags which are stable by the image of ρ. On the other hand, if there exists two distinct flags fixed by the image of ρ, say

$$V_1 \subset V_2 \subset ... \subset V_n$$

and

$$V'_1 \subset V'_2 \subset ... \subset V'_n$$

and let j be the smallest index such that $V_j \neq V'_j$. Then $(V_j + V'_j)/V_{j-1}$ is of dimension 2 and semi-simple, hence ρ is not totally indecomposable.

(2. $\Leftrightarrow$ 3.) Suppose that ρ stabilizes another Borel B' (apart from B). Let $b \in B$ be an element which conjugates B' into a Borel containing the maximal torus T. This Borel $bB'b^{-1}$ is of the form $w(B)$ for some w in the Weyl group. Hence we see that $C_{b\rho b^{-1}}$ is contained in the intersection of $R^{+\vee}$ and $w(R^{+\vee})$, and in particular is different from $R^{+\vee}$.

If C_ρ is different from $R^{+\vee}$, then there exists a positive simple root α which does not belong to C_ρ. It follows that $s_\alpha(C_\rho)$ is contained in $R^{+\vee}$ and hence the image of ρ is contained in $s_\alpha(B)$. □

Lemma I.4.7. *Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(\mathcal{O})$ be a triangular representation such that $\bar{\rho}$ is triangular, generic and totally indecomposable. Then $\rho_E = \rho \otimes_{\mathcal{O}} E$ is also totally indecomposable and generic.*

Proof. The statement about genericity of ρ_E is clear. Let us prove that it is totally indecomposable. Let us denote by $\bar{\chi}_j$ characters appearing on the diagonal of $\bar{\rho}$ which we have supposed to be pairwise distinct hence linearly independent. Let B be a Borel in $\mathrm{GL}_n(E)$ containing the image of ρ. It corresponds to a flag

$$V_1 \subset V_2 \subset ... \subset V_n = E^n$$

By intersection with $\mathcal{O}^n$ we obtain a flag

$$\omega_1 \subset \omega_2 \subset ... \subset \omega_n = \mathcal{O}^n$$

of $\mathcal{O}^n$ stable by the image of ρ which reduces to the standard flag modulo $\mathfrak{m}$ by the hypothesis that $\bar{\rho}$ is totally indecomposable. In particular, we see that G acts on V_i/V_{i-1} by a character χ_i with values in $\mathcal{O}^\times$ which lifts the character $\bar{\chi}_i$. By genericity of $\bar{\rho}$, the characters χ_i are mutually distinct and each appears in the semi-simplification of ρ with multiplicity 1.

Suppose now that we have another Borel B' different from B and stable by the image of ρ with the associated flag

$$V'_1 \subset V'_2 \subset ... \subset V'_n$$

Let i be the smallest index i such that $V'_i \neq V_i$. Then G acts on the 2-dimensional subquotient $(V_i + V'_i)/V_{i-1}$ by the character χ_i, which contradicts the fact that χ_i appears with multiplicity 1. Hence $B' = B$ and we see that ρ is totally indecomposable by Lemma I.4.6. □

Proposition I.4.8. *Suppose that $\bar{\rho}$ is generic, triangular and totally indecomposable and ρ is triangular. Then for any morphism $A \to E'$ (where E' is a finite extension of E), the E'-Banach representation $\Pi(\rho)^{\mathrm{ord}} \otimes_A E'$ is the representation $\Pi(\rho \otimes_A E')^{\mathrm{ord}}$ of Breuil and Herzig.*

Proof. By Lemma I.4.4 we can suppose that $A = \mathcal{O}_{E'}$. Observe that $\rho_{E'} = \rho \otimes_{\mathcal{O}_{E'}} E'$ is generic and totally indecomposable by Lemma I.4.7. To finish the proof we have to show that $\rho_{E'}$ is a good conjugate of itself (Definition 3.2.4 in [BH]). This follows from (3) of Lemma I.4.6 and we conclude by Lemma 3.3.5 of [BH]. ☐

I.4.4 Universal ordinary modular representation

In this subsection we will apply the formalism developed above to a particular example. We consider triangular deformations of modular representations and our goal is to define $\Pi(\rho_{\mathfrak{m},w})^{ord}$, where $\rho_{\mathfrak{m},w}$ is a certain universal modular Galois representation at a place $w|p$.

We take up the setting of Section 2. For each place $w|p$ of F^+ we choose a place $\tilde{w}$ of F, so as to get an identification

$$G(\mathbb{Q}_p) \simeq \prod_{w|p} \mathrm{GL}_n(F_{\tilde{w}}) = \mathrm{GL}_n(\mathbb{Q}_p)^f$$

where $f = [F^+ : \mathbb{Q}]$. We denote by B the upper triangular Borel subgroup and we have $B \simeq B_n(\mathbb{Q}_p)^f$.

We will now define a certain quotient $\mathbb{T}(K^p)^{\mathrm{ord}}$ of $\mathbb{T}(K^p)$. There are two equivalent approaches for this.

Firstly, we may follow Geraghty who introduced in 2.4 [Ger10] a certain direct factor $\mathbb{T}(K^pK_p(n))^{\mathrm{ord}}$ of $\mathbb{T}(K^pK_p(n))$ where $K_p(n)$ denotes the group of matrices in $\mathrm{GL}_n(\mathbb{Z}_p)^f$ that reduce to a unipotent upper-triangular matrix mod p^n. More precisely, in loc. cit is defined the algebra $\mathbb{T}_\lambda^{T,\mathrm{ord}}(U(l^{n,n}), \mathcal{O})$. There, λ is a dominant weight for G (but we take $\lambda = 0$ in this case), $U(l^{n,n})$ is our $K^pK_p(n)$ (our p is denoted by l), T is our Σ. Beware that Geraghty's algebra contains diamond operators at places above p (his l), in contrast with ours. So our $\mathbb{T}(K^pK_p(n))^{\mathrm{ord}}$ is the image of $\mathbb{T}(K^p)$ in Geraghty's $\mathbb{T}_0^{T,\mathrm{ord}}(U(l^{n,n}), \mathcal{O})$. When n varies, these constructions are compatible and we may take the projective limit $\mathbb{T}_0^{T,\mathrm{ord}}(U(l^\infty), \mathcal{O})$. We get a quotient $\mathbb{T}(K^p)^{\mathrm{ord}}$ as the image of the natural map $\mathbb{T}(K^p) \to \mathbb{T}_0^{T,\mathrm{ord}}(U(l^\infty), \mathcal{O})$ in Geraghty's notation on p.14 of loc. cit.

Alternatively, we may use Emerton's ordinary part functor and define

$$\mathbb{T}(K^p)^{\mathrm{ord}} := \text{image of } \mathbb{T}(K^p) \text{ in } \mathrm{End}_{\mathcal{O}}(\mathrm{Ord}_B(\widehat{H}^0(K^p)))$$

Note that $\mathrm{Ord}_B(\widehat{H}^0(K^p))$ is a continuous representation of $T(\mathbb{Q}_p)$ over $\mathbb{T}(K^p)$ and in particular is a $\mathbb{T}(K^p)[[T(\mathbb{Z}_p)]]$-module. Then Geraghty's algebra can be identified with the image of $\mathbb{T}(K^p)[[T(\mathbb{Z}_p)]]$ in $\mathrm{End}_{\mathcal{O}}(\mathrm{Ord}_B(\widehat{H}^0(K^p)))$ (compare with 5.6 of [Eme11a]).

Let now $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}(K^p)$ as in Section 2. We say that $\mathfrak{m}$ is **ordinary** if it comes from a maximal ideal of $\mathbb{T}(K^p)^{\mathrm{ord}}$. The quotient map $\mathbb{T}(K^p)_{\mathfrak{m}} \to \mathbb{T}(K^p)_{\mathfrak{m}}^{\mathrm{ord}}$ is a surjection.

Let us fix an ordinary non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbb{T}(K^p)$. Recall that we have defined $\bar{\rho}_{\mathfrak{m}}$ and $\rho_{\mathfrak{m}}$ in Section 2. For any prime ideal $\mathfrak{p}$ in $\mathbb{T}(K^p)_{\mathfrak{m}}^{\mathrm{ord}}$ coming from the maximal spectrum $\mathrm{Spm}(\mathbb{T}(K^p)_{\mathfrak{m}}^{\mathrm{ord}}[1/p])$, we will write

$$\rho_{\mathfrak{p}} := \rho_{\mathfrak{m}} \otimes_{\mathbb{T}(K^p)_{\mathfrak{m}}^{\mathrm{ord}}} \mathbb{T}(K^p)_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{p}[1/p]$$

which is a continuous Galois representation over a finite extension of $\mathbb{Q}_p$. We will need the following result of Geraghty:

Proposition I.4.9. *Consider the set P^{cris}_{autom} of maximal ideals in $\mathbb{T}(K^p)^{\mathrm{ord}}_{\mathfrak{m}}[1/p]$ such that $H^0(K^pK_p(0), \mathcal{V}_W)[\mathfrak{p}]$ is non-zero for some irreducible algebraic representation W of G. Then:*

- *This set is Zariski dense in* $\mathrm{Spec}(\mathbb{T}(K^p)^{\mathrm{ord}}_{\mathfrak{m}}[1/p])$.
- *For any $\mathfrak{p}$ in P^{cris}_{autom}, the representation $\rho_{\mathfrak{p}}$ is triangularisable (and crystalline) at each place dividing p.*

Proof. The point 2. follows from Corollary 2.7.8 of [Ger10]. The point 1. is the density of cristalline points which is proved in Corollary 4 of [Sor12], or can be deduced from the density result of Hida used by Geraghty in the proof of Corollary 3.1.4 in [Ger10]. □

As a consequence of this proposition, the residual representation $\bar{\rho}_{\mathfrak{m},w}$ is triangularisable for each $w|p$.

We now assume further that $\bar{\rho}_{\mathfrak{m},w}$ is totally indecomposable and generic for each $w|p$. Note that generic was only defined for triangular representations. However the definition extends unambiguously to triangularisable representations, provided they are totally indecomposable, because such representations factor through a unique Borel subgroup.

In what follows we will write $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ for $\mathbb{T}(K^p)^{\mathrm{ord}}_{\mathfrak{m}}$. This should cause no confusion.

Our goal is to define $\Pi(\rho_{\mathfrak{m},w})^{ord}$ where $\rho_{\mathfrak{m},w}$ is the restriction of $\rho_{\mathfrak{m}}$ to the decomposition group $G_{F_w} = \mathrm{Gal}(\bar{F}_w/F_w)$ for any place $w|p$ of F. In order to do so, we need to prove that $\rho_{\mathfrak{m},w}$ is triangularisable. We basically do so, but not over $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ but rather over a bigger $\mathcal{O}$-algebra $\mathbb{T}'_{\mathfrak{m}}$. This is sufficient for our applications.

Following Geraghty (Section 3.1 of [Ger10]) we introduce a subfunctor G of $\mathrm{Spec}\,\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}} \times \mathcal{F}$ defined on A-points as the set of $\mathcal{O}$-homomorphisms $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}} \to A$ and filtrations $\mathrm{Fil} \in \mathcal{F}(A)$ ($\mathcal{F}$ is the flag variety) preserved by the induced representation $\rho_{A,w}$. In fact, Geraghty defined this functor over a universal ring R, but we shall need it only over the Hecke algebra.

This functor is representable by a closed subscheme $\mathcal{G}$ of $\mathrm{Spec}\,\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}} \times \mathcal{F}$ (Lemma 3.1.2 in [Ger10]). We consider the resulting morphism $f : \mathcal{G} \to \mathrm{Spec}\,\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$.

Proposition I.4.10. *The morphism $f : \mathcal{G} \to \mathrm{Spec}\,\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ is proper with geometric fibres of cardinal one.*

Proof. The properness of f follows from that of the flag variety (cf. the proof of Lemma 3.1.3 in [Ger10]). Let us now prove that each geometric fibre is of cardinal one. Let us denote by $\bar{\chi}_{1,w}, \bar{\chi}_{2,w}, ..., \bar{\chi}_{n,w}$ characters of G_{F_w} appearing on the diagonal of $\bar{\rho}_{\mathfrak{m},w}$. Firstly, we remark that geometric fibres are non-empty. Indeed, f is dominant by Proposition I.4.9, hence surjective since it is proper. On the other hand, there is at most one filtration Fil over each geometric point, because $\bar{\rho}_{\mathfrak{m},w}$ is generic and totally indecomposable hence each j-th graded piece $gr_j = \mathrm{Fil}_j \,/\, \mathrm{Fil}_{j-1}$ has to be a lifting of $\bar{\chi}_{j,w}$ (see proofs of Lemma I.4.6 and Lemma I.4.7). This allows us to conclude. □

By Proposition above and Zariski Main Theorem we conclude that f is finite and hence $\mathcal{G} = \mathrm{Spec}\,\mathbb{T}'_{\mathfrak{m},w}$ for some $\mathcal{O}$-algebra $\mathbb{T}'_{\mathfrak{m},w}$ finite over $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$.

Corollary I.4.11. *The morphism $f : \operatorname{Spec} \mathbb{T}'_{\mathfrak{m},w} \to \operatorname{Spec} \mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ is a homeomorphism which induces an isomorphism of residual fields at each prime $\mathfrak{p}' \in \operatorname{Spec} \mathbb{T}'_{\mathfrak{m},w}$ with perfect residual field.*

Proof. It follows from the fact that geometric fibres of f are of cardinal one. □

We define $\mathbb{T}'_{\mathfrak{m}}$ to be the tensor product over $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ of $\mathbb{T}'_{\mathfrak{m},w}$ for all $w|p$. This is still an $\mathcal{O}$-algebra finite over $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ with $\operatorname{Spec} \mathbb{T}'_{\mathfrak{m}}$ homeomorphic to $\operatorname{Spec} \mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$.

Consider the base-change of $\rho_{\mathfrak{m}}$ to $\mathbb{T}'_{\mathfrak{m}}$, that is $\rho'_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\mathbb{T}'_{\mathfrak{m}})$. By what we have said above, $\rho'_{\mathfrak{m},w}$ can be conjugated to a triangular representation $\rho''_{\mathfrak{m},w}$ for each $w|p$, which is also generic and totally indecomposable at each $w|p$, because $\bar{\rho}_{\mathfrak{m},w}$ is by our assumption. By Corollary above, for each prime ideal $\mathfrak{p}$ associated to an automorphic representation π on $G(\mathbb{A})$, there exists a unique $\mathfrak{p}'$ in $\mathbb{T}'_{\mathfrak{m}}$ such that $\rho'_{\mathfrak{m}}/\mathfrak{p}'\rho'_{\mathfrak{m}}[1/p] \simeq \rho_{\mathfrak{m}}/\mathfrak{p}\rho_{\mathfrak{m}}[1/p]$.

The above discussion leads us to the following definition

$$\Pi(\rho_{\mathfrak{m},w})^{ord} := \Pi(\rho''_{\mathfrak{m},w})^{ord}$$

and similarly

$$\Pi(\rho_{\mathfrak{m},w})_I := \Pi(\rho''_{\mathfrak{m},w})_I$$

for any I, in particular for $I = \emptyset$ which we shall use below. These are representations over $\mathbb{T}'_{\mathfrak{m}}$. To conclude using our precedent results that the reduction modulo prime ideals of $\Pi(\rho_{\mathfrak{m},w})^{ord}$ is well-behaved and compatible with the construction of Breuil-Herzig we need the following fact.

Lemma I.4.12. *For each prime ideal $\mathfrak{p}$ of $\mathbb{T}'_{\mathfrak{m}}$ which comes from a maximal ideal of $\mathbb{T}'_{\mathfrak{m}}[1/p]$, the representation $\Pi(\rho''_{\mathfrak{m},w})^{ord}/\mathfrak{p}[1/p]$ does not depend on the chosen triangulation $\rho''_{\mathfrak{m},w}$ of $\rho'_{\mathfrak{m},w}$ (where by triangulation of $\rho'_{\mathfrak{m},w}$ we mean a triangular representation which can be conjugated to $\rho'_{\mathfrak{m},w}$).*

Proof. By Proposition I.4.8 we deal with the construction of Breuil-Herzig and hence we can use facts from [BH]. We have to prove that for any triangulation $\rho''_{\mathfrak{m},w}$ the reduction $\rho''_{\mathfrak{m},w}/\mathfrak{p}$ is a good conjugate of $\rho_{\mathfrak{m},w}/\mathfrak{p}$ (Definition 3.2.4 in [BH]). This would give our claim by Lemma 3.3.5 of [BH]. By our assumption that $\bar{\rho}_{\mathfrak{m},w}$ is generic triangular and totally indecomposable, any triangular lift ρ of $\bar{\rho}_{\mathfrak{m},w}$ is totally indecomposable and generic by Lemma I.4.7. Then we conclude by (3) of Lemma I.4.6 that each triangulation of $\rho'_{\mathfrak{m},w}/\mathfrak{p}$ (in particular $\rho''_{\mathfrak{m},w}/\mathfrak{p}$) is a good conjugate of $\rho'_{\mathfrak{m},w}/\mathfrak{p}$.

□

We summarize our efforts so far in the following theorem.

Theorem I.4.13. *Let $\mathfrak{m}$ be an ordinary non-Eisenstein ideal of $\mathbb{T}$ such that $\bar{\rho}_{\mathfrak{m},w}$ is totally indecomposable and generic for each $w|p$ in F. Then we have for any prime ideal $\mathfrak{p}'$ of $\mathbb{T}'_{\mathfrak{m}}$ (with the inverse image $\mathfrak{p}$ in $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$) which comes from a maximal ideal of $\mathbb{T}'_{\mathfrak{m}}[1/p]$:*

$$\Pi(\rho'_{\mathfrak{m},w})^{ord}/\mathfrak{p}'\Pi(\rho'_{\mathfrak{m},w})^{ord}[1/p] \simeq \Pi(\rho_{\mathfrak{m},w}/\mathfrak{p}\rho_{\mathfrak{m},w}[1/p])^{ord}$$

Similar compatibilities with reduction modulo prime ideals hold for $\Pi(\rho_{\mathfrak{m},w})_I$.

I.4.5 On the pro-modular Fontaine-Mazur conjecture

We come back to our general formalism which we will apply to $\Pi(\rho_{\mathfrak{m},w})_{\emptyset}$. We assume that $\mathfrak{m}$ is a non-Eisenstein ordinary ideal of $\mathbb{T}$ such that $\bar{\rho}_{\mathfrak{m},w}$ is triangular, generic and totally indecomposable for each $w|p$. We take $\mathbb{T}'_{\mathfrak{m}}$ to be $\mathbb{T}'_{\mathfrak{m}}$ from preceding sections. We start with two lemmas:

Lemma I.4.14. *Let $\psi_1, \psi_2 : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to E$ be two de Rham characters such that $\psi_1\psi_2^{-1} \notin \{1, \varepsilon, \varepsilon^{-1}\}$ and let $0 \to \psi_1 \to V \to \psi_2 \to 0$ be the non-split extension (there is a unique one; see below). Suppose that V is de Rham. Then $\mathrm{HT}(\psi_1) < \mathrm{HT}(\psi_2)$ (normalizing $\mathrm{HT}(\varepsilon) = -1$).*

Proof. The fact that there is a unique extension of ψ_1 by ψ_2 follows from the fact that $H^1 = H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \psi_1\psi_2^{-1})$ is of dimension 1 because $\psi_1\psi_2^{-1}$ is generic. Observe that $V \in H^1$. One can define the Selmer group $H^1_g = H^1_g(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p), \psi_1\psi_2^{-1})$ which measures whether V is de Rham (we refer the reader to Chapter II of [Ber13]; Definition is given before Proposition 2.17). By Corollary 2.18 of [Ber13] we see that $V \in H^1_g$. Hence H^1_g is of dimension one. But Proposition 2.19 of [Ber13] gives us a formula for the dimension of H^1_g, by which we infer in our case that $dim H^1_g = 1$ is equal to the number of negative Hodge-Tate numbers (compare with the discussion after Proposition 2.19 in [Ber13]). Hence $\mathrm{HT}(\psi_1) < \mathrm{HT}(\psi_2)$. ☐

Recall that we have defined the character θ in Section 4.1. For GL_n this character is simply $diag(z_1, ..., z_n) \mapsto \prod_i z_i^{1-i}$.

Lemma I.4.15. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(E)$ be a de Rham, triangular, totally indecomposable, generic Galois representation. Then the character $\chi_\rho \cdot (\varepsilon^{-1} \circ \theta)$ is locally algebraic dominant.*

Proof. Triangularity permits us to define χ_ρ. It is clear that the character is locally algebraic because ρ is de Rham. We conclude that $\chi_\rho \cdot (\varepsilon^{-1} \circ \theta)$ is dominant by applying the lemma above to each pair of consecutive characters on the diagonal of ρ (which we can do because ρ is totally indecomposable and generic). ☐

We can now check that for representations $\Pi(\rho_{\mathfrak{m},w})_{\emptyset}$ hypothesis (H1) holds:

(H1): We have to check that if $\mathfrak{p}$ is a prime ideal of $\mathbb{T}'_{\mathfrak{m}}$ corresponding to the Galois representation $\rho_{\mathfrak{p}}$ which is de Rham and regular at all places $w|p$, then locally algebraic vectors in $\Pi(\rho_{\mathfrak{m},w})_{\emptyset}/\mathfrak{p}[1/p] = \Pi(\rho_{\mathfrak{p},w})_{\emptyset}[1/p]$ are non-zero. Indeed, the locally algebraic vectors in $\Pi(\rho_{\mathfrak{p},w})_{\emptyset}[1/p]$ are non-zero because it is the representation induced from the locally algebraic dominant character $\chi = \chi_\rho \otimes (\varepsilon^{-1} \circ \theta)$ (by Lemma I.4.15); to see it we write $\chi = \chi_{sm}\delta_W$ for this character, where χ_{sm} is smooth and δ_W is algebraic corresponding to an irreducible algebraic representation W of $G(\mathbb{Q}_p)$. We have $W = (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \delta_W)^{alg}$. Then the universal completion of the locally algebraic representation $(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{sm})^{sm} \otimes W$ is equal to $(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi)^{\mathcal{C}^0} = \Pi(\rho_{\mathfrak{p},w})_{\emptyset}[1/p]$ because χ is unitary (we inject the locally algebraic induction into the continuous induction by sending $f_{sm} \otimes f_{alg}$ to $f_{sm} \cdot f_{alg}$, where f_{sm},f_{alg} are functions on smooth, respectively algebraic part). In particular, the set of locally algebraic vectors in $\Pi(\rho_{\mathfrak{p},w})_{\emptyset}[1/p]$ is non-empty. The fact that $\Pi(\rho_{\mathfrak{m},w})_{\emptyset}/\mathfrak{m}'$ is of finite length is clear from the definition.

For $\mathfrak{p} \in P^{cris}_{autom}$ as in Proposition I.4.9, we know that each $\rho_{\mathfrak{p},w}$ is crystalline triangularisable. Our hypothesis on $\bar{\rho}_{\mathfrak{m},w}$ implies that $\rho_{\mathfrak{p},w}$ is also totally indecomposable (Lemma I.4.7) and generic. So we may unambigously associate to it a character $\chi_{\rho_{\mathfrak{p},w}}$ of $T_n(\mathbb{Q}_p)$.

Let us recall a classical local-global compatibility result.

Lemma I.4.16. *Fix $\mathfrak{p} \in P^{cris}_{autom}$. Let W be the irreducible algebraic representation of $G(\mathbb{Q}_p)$ such that $H^0(K^p, \mathcal{V}_W)[\mathfrak{p}] \neq 0$. Let π be an automorphic representation such that $\pi_f^{K^p} \subset H^0(K^p, \mathcal{V}_W)[\mathfrak{p}]$. Then*

$$W^\vee = \otimes_{w|p} (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p))} (\chi_{\rho_\pi, w} \cdot (\varepsilon^{-1} \circ \theta))_{alg})^{alg}$$

$$\pi_p = \otimes_{w|p} (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} (\chi_{\rho_\pi, w} \cdot (\varepsilon^{-1} \circ \theta))_{sm})^{sm}$$

where we have denoted by $(.)_{sm}$ (respectively, $(.)_{alg}$) the smooth (resp. algebraic) part of the character.

Proof. The first isomorphism follows from Corollary 2.7.8(i) of [Ger10] with the following dictionary:

- our W is Geraghty's M_λ, therefore $W^\vee = \mathrm{Ind}_{B^-}^{\mathrm{GL}_n} (w_0 \lambda)^{-1}$
- his $\lambda = (\lambda_\tau)_{\tau: F^+ \hookrightarrow E} = (\lambda_w)_{w|p}$ since p was assumed to be totally split in F.
- for each $w|p$, loc. cit. tells us that $(\chi_{\rho_{\pi,w}})_{alg} = (w_0 \lambda_w)^{-1} \cdot \theta$

The second isomorphism follows from Corollary 2.7.8(ii) of [Ger10] and the first formula on p. 27 of [Ger10] (proof of Lemma 2.7.5). Namely, loc. cit. tells us that π_w is the unramified subquotient of $(n - \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} (\chi_{\rho_\pi})_{sm}) \otimes |\det|^{(n-1)/2}$ (normalized induction). But the genericity of $\rho_{\mathfrak{p},w}$ implies that

$$\pi_w = (n - \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} (\chi_{\rho_\pi})_{sm}) \otimes |\det|^{(n-1)/2}$$

and smooth representation theory tells us that this is also

$$(n - \mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} (\chi_{\rho_\pi})_{sm}) \otimes |\det|^{(n-1)/2} = (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)} (\chi_{\rho_\pi})_{sm} \delta_B^{-1/2}) \otimes |\det|^{(n-1)/2}$$

where δ_B is the modulus character. We conclude by observing that

$$\delta_B^{-1/2} \cdot |\det|^{(n-1)/2} = (\varepsilon^{-1} \circ \theta)_{sm} : (z_1, ..., z_n) \mapsto \prod |z_i|^{i-1}$$

□

Using $\Pi(\rho_{\mathfrak{m},w})_\emptyset$ we can make use of our formalism (Theorem I.3.4) to get the pro-modular Fontaine-Mazur conjecture in the following form.

Theorem I.4.17. *Let $\mathfrak{m}$ be an ordinary non-Eisentein ideal of $\mathbb{T}$ such that $\bar{\rho}_{\mathfrak{m},w}$ is totally indecomposable and generic for all $w|p$. Let ρ be pro-modular with respect to $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$ and de Rham regular. Then ρ is modular.*

Proof. To conclude by Theorem I.3.4 we have to check that hypotheses (H2) and (H3) hold for $\Pi(\rho_{\mathfrak{m},w})_\emptyset$. The hypothesis (H2) says that there exists an allowable set for $\Pi(\rho_{\mathfrak{m},w})_\emptyset$, which means that there exists a dense set of prime ideals $\mathfrak{p}$ in the Hecke algebra $\mathbb{T}'_{\mathfrak{m}}$ for which the associated Galois representation $\rho_{\mathfrak{p},w}$ ($w|p$ is a split place) gives a Banach representation $\Pi(\rho_{\mathfrak{p},w})_\emptyset$ with an injection

$$\otimes_{w|p} \Pi(\rho_{\mathfrak{p},w})_\emptyset \hookrightarrow \widehat{H}^0(K^p)_E$$

We can prove it for the set P^{cris}_{autom} which is dense by Proposition I.4.9. Indeed, if $\mathfrak{p}$ corresponds to a classical automorphic representation π with the Galois representation ρ_π. Following Lemma I.4.16 we put $\chi_w = \chi_{\rho_\pi,w} \cdot (\varepsilon^{-1} \circ \theta)$. Take $\chi = \otimes_{w|p}\chi_w$ and write $\chi = \chi_{sm}\delta_W$ as above in the verification of (H1). Then by the description of locally algebraic vectors of $\widehat{H}^0(K^p)_{E,l.alg}$ from Proposition I.2.3 we see that $W^\vee \otimes (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{sm})^{sm}$ injects into $\widehat{H}^0(K^p)_{E,l.alg}$, (we use here Lemma I.4.16). Hence taking the completion we see that the universal completion $W^\vee \otimes \widehat{(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}} \chi_{sm})^{sm}$ of $W^\vee \otimes (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{sm})^{sm}$ sits in $\widehat{H}^0(K^p)_E$. But because χ is unitary, we have

$$W^\vee \otimes \widehat{(\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}} \chi_{sm})^{sm} = (\mathrm{Ind}_{B^-(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi)^{\mathcal{C}^0} = \otimes_{w|p}\Pi(\rho_{\mathfrak{p},w})_\emptyset$$

by which we conclude.

The hypothesis (H3)[$\mathfrak{p}'$] ($\mathfrak{p}'$ is the prime ideal of $\mathbb{T}'_{\mathfrak{m}}$ corresponding to ρ by our pro-modularity assumption and Corollary I.4.11) says that we have a closed injection

$$\Pi(\rho_{\mathfrak{p}',w})_\emptyset \hookrightarrow \widehat{H}^0(K^p)_E[\mathfrak{p}']$$

This is clear in our context because $\rho_{\mathfrak{p}',w}$ is generic and hence $\Pi(\rho_{\mathfrak{p}',w})_\emptyset$ is irreducible (see Theorem 3.1.1(ii) in [BH]).

This allows us to conclude.

$\square$

We can get a more explicit result by using eigenvarieties. In [Eme06c] Emerton has constructed the eigenvariety $\mathcal{X}$ associated to the group G using completed cohomology. We do not recall here this construction explicitly, but let us mention that $\mathcal{X}$ parametrises (certain) pro-modular representations. Let $\mathcal{X}[\bar{\rho}_{\mathfrak{m}}]^{\mathrm{ord}}$ be the $\bar{\rho}_{\mathfrak{m}}$-part of the eigenvariety associated to $U(n)$ parametrising ordinary points (i.e. points associated to ordinary p-adic automorphic forms). In particular every point $x \in \mathcal{X}[\bar{\rho}_{\mathfrak{m}}]^{\mathrm{ord}}$ is pro-modular with respect to $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}}$. We denote by λ_x its corresponding Hecke character and by ρ_x its associated Galois representation.

The above theorem implies the following result

Corollary I.4.18. *Let $\mathfrak{m}$ be an ordinary non-Eisenstein ideal of $\mathbb{T}$ such that $\bar{\rho}_{\mathfrak{m},w}$ is totally indecomposable and generic for all $w|p$. Let x be an E-point on the eigenvariety $\mathcal{X}[\bar{\rho}_{\mathfrak{m}}]^{\mathrm{ord}}(E)$ such that for each place $w|p$ the representation $\rho_{x,w}$ is regular and de Rham. Then x is classical.*

Proof. Modularity is clear so we need only to comment upon the classicality of x. It follows under our assumptions by Lemma I.4.15 that $\rho_{x,w}$ has dominant weights which is enough to conclude that x is classical as it was modular. $\square$

Chapter II

Ordinary representations for $U(3)$ (joint with J. Bergdall)

II.1 Introduction

Breuil and Herzig have recently pursued the construction of interesting p-adic Banach space representations associated to local Galois representations by concentrating on the ordinary case [BH]. In fact, if L is a finite extension of $\mathbf{Q}_p$ and representation $\rho_p : G(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_n(L)$ is generic and ordinary then Breuil and Herzig constructed an admissible continuous unitary representation $\Pi(\rho_p)^{\mathrm{ord}}$ on an L-Banach space by taking successive extensions of unitary principal series. Their recipe takes as key input the splitting behavior of ρ_p and thus forsees compatibility between the Galois and automorphic sides of a p-adic Langlands program for $\mathrm{GL}_n(\mathbf{Q}_p)$.

It is further conjectured that if ρ_p gives rise to $\Pi(\rho_p)$ under a conjectural local Langlands correspondence for $\mathrm{GL}_n(\mathbf{Q}_p)$ then $\Pi(\rho_p)^{\mathrm{ord}}$ should account for the maximal piece of $\Pi(\rho_p)$ which can be built out of principal series alone. That there is, or may be, a discrepancy between $\Pi(\rho_p)$ and $\Pi(\rho_p)^{\mathrm{ord}}$ is an interesting feature of the situation beyond $\mathrm{GL}_2(\mathbf{Q}_p)$. Nevertheless, here we are concerned with the representation $\Pi(\rho)^{\mathrm{ord}}$ and the insights it can bring concerning the p-adic local Langlands program.

In this work we explore the global aspect of [BH]. Let F/F^+ be a CM extension of number fields in which p is totally split and denote $G = U(3)$ a definite unitary group in three variables attached to F/F^+. Let us fix a compact open subgroup $K^p \subset G(\mathbf{A}_F^{p\infty})$. With this data in hand, we can define the completed cohomology group of Emerton

$$\widehat{H}^0(K^p)_L = \left(\varprojlim_n \varinjlim_{K_p} H^0(G(\mathbf{Q})\backslash G(\mathbf{A}_F^\infty)/K_pK^p, \mathbf{Z}_p/p^n\mathbf{Z}_p) \right) \otimes_{\mathbf{Z}_p} L$$

where K_p runs over open compact subgroups of $G(\mathbf{Q}_p)$. This space can be seen as a model for p-adic automorphic representations on $U(3)$.

If π is an automorphic representation on $U(3)$ then it has an associated (in the usual sense) global Galois representation $\rho = \rho_\pi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_3(L)$ (extending L if neccessary). If π has tame level K^p then ρ_π is unramified away from a finite set depending on K^p.

For each place $v \mid p$ we write $v = \tilde{v}\tilde{v}^c$ and consider the local Galois representation $\rho_v := \rho_{\tilde{v}} : \mathrm{Gal}(\overline{F}_{\tilde{v}}/F_{\tilde{v}}) \simeq \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \to \mathrm{GL}_3(L)$. If $\rho_{\tilde{v}}$ is generic and ordinary then the same is true for $\rho_{\tilde{v}^c}$ and $\Pi(\rho_{\tilde{v}})^{\mathrm{ord}}$ only depends on $v \mid p$ in F^+. The following (see Theorem II.3.24) is our main result. It is a weaker form of the Conjecture 4.2.2 in [BH].

Theorem A. *Suppose that for all $v \mid p$, ρ_v is generic ordinary and totally indecomposable. Then there is a closed embedding*

$$\widehat{\bigotimes}_{v|p} \Pi(\rho_v)^{\mathrm{ord}} \hookrightarrow \widehat{H}^0(K^p)_L.$$

This result, or the conjectures of Breuil and Herzig, can be seen as a generalization, from GL_2 to $U(3)$, of the work of Breuil and Emerton [BE10]. There are similarities and differences between our work and [BE10]. Our representation-theoretic results naturally overlap with [BE10]. Most notably we give an adjunction formula (Theorem II.3.12) between certain principal series and completed cohomology. The theorem is then reduced, via the adjunction formula, to exhibiting the existence, or non-existence, of certain points on an eigenvariety X_{K^p} for $U(3)$ (see Section II.2 for information on eigenvarieties). The same reduction is done in [BE10] but our argument works intrinsically on the eigenvariety by studying the variation of Galois representations and making use of Kisin's famous result on the analytic continuation of crystalline periods over p-adic families. We remark also that a similar result is proven in [BH] using completely different techniques. However, they put stronger hypotheses on ρ, in particular they assume that each $\bar{\rho}_v$ is totally indecomposable (see Theorem 4.4.8 in [BH]).

To end this introduction, let us remark briefly on the setting we have restricted ourselves to at the present. As indicated following Theorem A, the adjunction formula reduces our weak form of Breuil-Herzig to the existence or not of certain Hecke eigensystems in spaces of p-adic automorphic forms. In the totally indecomposable case, this amounts to showing certain Hecke eigensystems do not exist in spaces of p-adic automorphic forms. The converse, constructing "overconvergent companion forms", is a more serious matter. In our work in progress that construction will allow us to prove Theorem A with only the hypothesis that ρ_v be indecomposable above p. This is done by the computations of this text along with generalizing the Galois-theoretic construction of companion forms in [Ber]. For general conjectures in this line, see [Bre] (which also contains partial results).

We remark that our methods are general enough to be applicable in other contexts. First, the restriction to $n = 3$ is only used for brevity at the moment and we plan to have a sequel dealing with general $n \geq 3$. Moreover, we can also prove similar results in the non-compact case for unitary groups $U(2,1)$. Details for that will be provided elsewhere.

II.1.1 Notations

We introduce notations which we will use constantly throughout the text. Let $G = \mathrm{GL}_3(\mathbf{Q}_p)$ with the Borel B being the upper triangular matrices and its opposite Borel B^- the lower triangular matrices. The diagonal torus is T. The modulus character is $\delta_B : T \to \mathbf{Q}_p^\times$ given by $|\cdot|^2 \otimes 1 \otimes |\cdot|^{-2}$. For an algebraic weight k we denote by δ_k the corresponding highest weight character of T.

Let S_3 be the symmetric group.

We denote the cyclotomic character by ε and we normalize the local class field theory so that $\varepsilon = z|z|$ (as a character of $\mathbf{Q}_p^\times$) with the Hodge-Tate weight -1.

II.2 Eigenvarieties

Before discussing eigenvarieties, we begin with local preliminaries on two separate notions of refinements.

II.2.1 Refinements

Here we recall two notions of refinement, one on the Galois side and one on the automorphic side. In the case of Galois representations, we explicitly highlight the case of crystalline representations which are ordinary.

Suppose that $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_3(L)$ is a crystalline representation all of whose crystalline eigenvalues are distinct and lie in $L^\times$. We further assume that ρ is regular, i.e. the Hodge-Tate weights are distinct, with Hodge-Tate weights $h_1 < h_2 < h_3$.

Definition II.2.1. *A refinement R of ρ is an ordering $R = (\phi_1, \phi_2, \phi_3)$ for the crystalline eigenvalues appearing in $D_{\mathrm{cris}}(\rho)$.*

If $\{\phi_1, \ldots, \phi_i\}$ is a list of crystalline eigenvalues then we denote by $\mathrm{wt}(\phi_1, \ldots, \phi_i)$ the Hodge-Tate weight of the line $D_{\mathrm{cris}}(\wedge^i \rho)^{\varphi = \phi_1 \cdots \phi_i} \subset D_{\mathrm{cris}}(\wedge^i \rho)$. It must be a sum of distinct Hodge-Tate weights for ρ. Thus a refinement R defines an ordering (s_1, s_2, s_3) of the Hodge-Tate weights by declaring $\mathrm{wt}(\phi_1) = s_1$, $\mathrm{wt}(\phi_1, \phi_2) = s_1 + s_2$ and s_3 is the unique weight not equal to s_1 or s_2.

Definition II.2.2. *If R is a refinement then we define its weight type as the permutation $\tau \in S_3$ such that $s_i = h_{\tau(i)}$. If $\tau = 1$ we say that R is non-critical. Otherwise, we say that R is critical of type τ.*

For example if $\rho = \psi_1 \oplus \psi_2$ for two crystalline characters ψ_i then there are two refinements, one non-critical and one critical of type $\tau = (12)$.

We now specialize to the case that ρ is upper triangularizable (and still regular with distinct crystalline eigenvalues). Thus we assume that

$$\rho \sim \begin{pmatrix} \psi_1 & * & * \\ 0 & \psi_2 & * \\ 0 & 0 & \psi_3 \end{pmatrix} \tag{II.1}$$

with the ψ_i crystalline characters. Without loss of generality (this uses the crystalline property) we also assume that ψ_i has Hodge-Tate weight h_i, i.e. that ρ is ordinary. If we denote by ϕ_{ψ_i} the crystalline eigenvalue of ψ_i then $v_p(\phi_{\psi_i}) = h_i$. Since $D_{\mathrm{cris}}(\rho)$ is weakly admissible we have that

$$\begin{aligned} \mathrm{wt}(\phi_{\psi_i}) &\leq v_p(\phi_{\psi_i}) = h_i && \text{(for } i = 1, 2, 3) \\ \mathrm{wt}(\phi_{\psi_i}, \phi_{\psi_j}) &\leq v_p(\phi_{\psi_i}) + v_p(\phi_{\psi_j}) = h_i + h_j && \text{(for each pair } (i, j)) \end{aligned}$$

The representation (II.1) fixes one particular refinement $R_{\mathrm{nc}} = (\phi_{\psi_1}, \phi_{\psi_2}, \phi_{\psi_3})$ of ρ. Any other refinement R must be of the form $R = R_{\mathrm{nc}}^\sigma := (\phi_{\psi_{\sigma(i)}})_i$ with $\sigma \in S_3$.

Lemma II.2.3. *R_{nc} is always non-critical. If ρ is totally indecomposable then $R_{\mathrm{nc}}^{(12)}$ and $R_{\mathrm{nc}}^{(23)}$ are non-critical as well.*

Proof. Since $\mathrm{wt}(\phi_{\psi_1}) \leq h_1$ and h_1 is the least Hodge-Tate weight we have $\mathrm{wt}(\phi_{\psi_1}) = h_1$. Similarly, $\mathrm{wt}(\phi_{\psi_1}, \phi_{\psi_2}) = h_1 + h_2$. Thus R_{nc} is non-critical.

Now assume that ρ is totally indecomposable and consider $R_{\mathrm{nc}}^{(12)} = (\phi_{\psi_2}, \phi_{\psi_1}, \phi_{\psi_3})$ with weight ordering (s_1, s_2, s_3). By what we just said, $s_1 + s_2 = \mathrm{wt}(\phi_{\psi_2}, \phi_{\psi_1}) = h_1 + h_2$ and so $R_{\mathrm{nc}}^{(12)}$ is critical if and only if $\mathrm{wt}(\phi_{\psi_2}) = h_2$. In that case, however, $D_{\mathrm{cris}}(\rho)^{\varphi = \phi_{\psi_2}} \subset D_{\mathrm{cris}}(\rho)$ is a weakly-admissible filtered φ-module and thus ψ_2 must define a subrepresentation of ρ, contradicting that ρ is totally indecomposable.

The case of $R_{\mathrm{nc}}^{(23)}$ is similar. If it were critical then the quotient ρ/ψ_1 would split into $\psi_2 \oplus \psi_3$. ∎

Remark II.2.4. *If ρ is totally indecomposable then it is possible that a R^σ is critical if $\sigma \notin \{1, (12), (23)\}$.*

We end this section with a short definition.

Definition II.2.5. *Suppose that ρ is ordinary and write ρ as in* (II.1)*. We say that ρ is generic ordinary if $p\phi_{\psi_i} \neq \phi_{\psi_{i+1}}$ for $i = 1, 2$.*

We suppose that ρ is ordinary as in (II.1). Using local class field theory we write each character ψ_i as

$$\psi_i = z^{-h_i} \operatorname{nr}(\phi_{\psi_i}).$$

It follows from the definition and this expression that ρ is generic ordinary if and only if $\psi_i\psi_j \notin \{1, \varepsilon^{\pm 1}\}$ for each $i \neq j$. Thus ρ is generic ordinary if and only if ρ is generic ordinary in the sense of [BH, Section 3.3].

We now consider the automorphic side. The analog of a crystalline $G_{\mathbf{Q}_p}$-representation is an unramified principal representations of $\mathrm{GL}_3(\mathbf{Q}_p)$. Denote by $B(\mathbf{Q}_p)$ the upper triangular Borel in $\mathrm{GL}_3(\mathbf{Q}_p)$ and $T(\mathbf{Q}_p)$ the diagonal torus inside $B(\mathbf{Q}_p)$. Suppose that θ is a smooth character of $T(\mathbf{Q}_p)$. In that case we can form the smooth induction $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta)^{\mathrm{sm}}$. If π is smooth admissible representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ which is unramified then π necessarily appears as the unique unramified Jordan-Holder factor of $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta)^{\mathrm{sm}}$ for some smooth θ. The character θ is well-defined up to an action of S_3: if $\sigma \in S_3$ then we denote

$$\theta^{(\sigma)} = \theta^\sigma(\delta_{B(\mathbf{Q}_p)}^{-1/2})^\sigma \delta_{B(\mathbf{Q}_p)}^{1/2}.$$

Then, the unique characters θ' such that π appears in the Jordan-Holder series for $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta')^{\mathrm{sm}}$ are those of the form $\theta' = \theta^{(\sigma)}$.

Definition II.2.6. *Let π be an unramified smooth admissible representation of* $\mathrm{GL}_3(\mathbf{Q}_p)$*. A refinement of π is the choice of a smooth character θ such that $\pi \subset \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta)^{\mathrm{sm}}$.*

In the terminology of [BC09], a refinement is the choice of θ such that π appears as a Jordan-Holder factor of $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta)^{\mathrm{sm}}$ and our refinement is their accessible refinement. Thus every $\sigma \in S_3$ defines a refinement $\theta^{(\sigma)}$ but only some σ define an accessible refinement. To that point, however, there is an equivalence

$$\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta)^{\mathrm{sm}} \text{ is unramified} \iff p^{j-i}\frac{\theta_i(p)}{\theta_j(p)} \neq p^{\pm 1} \text{ for } i \neq j. \tag{II.2}$$

If that is the case then $\pi = \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta)^{\mathrm{sm}}$, every refinement is accessible and thus $\pi \subset \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\theta^{(\sigma)})^{\mathrm{sm}}$ for all $\sigma \in S_3$. Thus except in the exceptional case (II.2) , one refinement θ for π gives all the refinements $\theta^{(\sigma)}$ for π.

II.2.2 Definite eigenvarieties

Here we will outline definite eigenvarieties. Our approach is to first describe the eigenvariety and its properties, and second to refer to an explicit construction of Emerton. The explicit construction will be used in Section II.3 to generalize certain results of [BE10].

We fix F^+ a totally real field extension of the rational number $\mathbf{Q}$ and F/F^+ a CM extension. We assume that p is totally split in F and we let Σ_p be the set of places $v \mid p$ in F^+. For each $v \in \Sigma_p$ we fix now the choice of a place $\tilde{v}$ above v.

Let $G = \mathrm{U}(3, F/F^+)$ be a definite unitary group in three variables. If w is a place of F^+ split in F then each choice $\tilde{w}$ of place over w defines an isomorphism

$$G(F_w^+) \overset{\tilde{w}}{\simeq} \mathrm{GL}_3(F_{\tilde{w}})$$

In particular, for each $v \in \Sigma_p$ we have a fixed isomorphism $G(F_v^+) \simeq \mathrm{GL}_3(\mathbf{Q}_p)$. Denote now $G_{\Sigma_p} = G(F^+ \otimes_{\mathbf{Q}} \mathbf{Q}_p)$. Under these identifications we define T_{Σ_p} to be the diagonal torus, B_{Σ_p} the upper triangular Borel and $B_{\Sigma_p}^-$ its lower triangular Borel. We denote as well $N_{\Sigma_p}^0 = N(\mathcal{O}_{F^+} \otimes_{\mathbf{Z}} \mathbf{Z}_p)$ integral points of the unipotent part N_{Σ_p} of B_{Σ_p}. Finally, we let $T_{\Sigma_p}^+ = \{t \in T_{\Sigma_p} \mid tN_{\Sigma_p}^0 t^{-1} \subset N_{\Sigma_p}^0\}$.

We fix a compact open subgroup $K^p \subset G(\mathbf{A}_{F^+}^{p\infty})$. We factor K^p into a product $K^p = \prod_{v \nmid p} K_v^p$. Choose a finite set of places Σ^p of F^+ such that if $v \notin \Sigma^p$ then K_v^p is maximal hyperspecial compact in $G(F_v^+)$. We write the above factorization as

$$K^p = \prod_{v \notin \Sigma} K_v^p \times \prod_{v \in \Sigma} K_v^p =: K^{p\Sigma^p} K_{\Sigma^p}^p.$$

We define the unramified Hecke algebra

$$\mathcal{H}(K^p)^{\mathrm{nr}} := \mathcal{H}(G(\mathbf{A}_{F^+}^{p\Sigma^p})//K^{p\Sigma^p}).$$

The places $\tilde{v}$ above $v \in \Sigma_p$ define isomorphisms $G(F_v^+) \simeq \mathrm{GL}_n(F_{\tilde{v}}) = \mathrm{GL}_3(\mathbf{Q}_p)$. If we denote T_v the diagonal torus in $G(F_v^+)$ under this identification then we can define $\mathcal{T}_v := \mathrm{Hom}_{\mathrm{loc\,.\,an}}(T_v, \mathbf{G}_m^{\mathrm{rig}})$ and $\mathcal{T}_{\Sigma_p} = \prod_{v \in \Sigma_p} \mathcal{T}_v = \mathrm{Hom}_{\mathrm{loc\,.\,an}}(T_{\Sigma_p}, \mathbf{G}_m^{\mathrm{rig}})$.

In what follows we fix an isomorphism $\bar{\mathbf{Q}}_p \simeq \mathbf{C}$. Suppose that $k \in \mathcal{T}_{\Sigma_p}$ is a dominant weight for G_{Σ_p} and let W_k be the irreducible algebraic representation of G_{Σ_p} with highest weight k. The space $\mathcal{A}_k(G, K^p)$ of automorphic forms of weight k and tame level K^p decomposes as a $\mathcal{H}(K^p)^{\mathrm{nr}}$-module

$$\mathcal{A}_k(G, K^p) \simeq \bigoplus_{\pi_\infty \simeq W_k} (\pi_f^{K^{p\Sigma_p}})^{m(\pi)} \tag{II.3}$$

with π running over irreducible automorphic representations for $G(\mathbf{A}_{F^+})$ and $m(\pi)$ the multiplicity of π appearing in $L^2(G(F^+)\backslash G(\mathbf{A}_{F^+}))$. If π is an automorphic representation for $G(\mathbf{A}_{F^+})$ of tame level K^p we denote by $\lambda_\pi : \mathcal{H}(K^p)^{\mathrm{nr}} \to \bar{\mathbf{Q}}_p$ the canonical character.

At p we consider a place $v \in \Sigma_p$ and denote by π_v the local component of π, a smooth admissible representation of $G(F_v^+) \simeq \mathrm{GL}_3(\mathbf{Q}_p)$. Write

$$\pi_{\Sigma_p} = \bigotimes_{v \in \Sigma_p} \pi_v.$$

We say that π is unramified at p if π_{Σ_p} is unramified, or equivalently, each π_v is unramified. If π_{Σ_p} is unramified then a refinement θ is the choice of a tuple $\theta = (\theta_v)_{v \in \Sigma_p} \in \mathcal{T}_{\Sigma_p}$ where θ_v is a refinement of π_v for each $v \in \Sigma_v$. Equivalently a refinement θ is the choice of a smooth character of T_{Σ_p} such that $\pi_{\Sigma_p} \subset \mathrm{Ind}_{B_{\Sigma_p}}^{G_{\Sigma_p}} \theta$.

We now consider the infinite component π_{∞_v} (identified using our choice $\tilde{v} \mid v$ for $v \in \Sigma_p$) of π. It is an irreducible algebraic representation of the compact group $G(F_{\infty_v}^+) \simeq \mathrm{U}(3, \mathbf{R})$ and thus has an associated dominant weight $k_v = (k_{1,v} \geq k_{2,v} \geq k_{3,v})$. We denote by $k = (k_v)_{v \in \Sigma_p}$ the corresponding highest weight for G_{Σ_p} and the highest weight character

$$\delta_k := (z_1^{k_{1,v}} \otimes z_2^{k_{2,v}} \otimes z_3^{k_{3,v}})_v \in \mathcal{T}_{\Sigma_p}.$$

Thus, for a given π unramified at p, the choice of a refinement θ defines a locally algebraic character $\chi := \theta\delta_k \in \mathcal{T}_{\Sigma_p}$.

We use $X = X_{K^p}$ to denote the eigenvariety of tame level K^p. It is a reduced rigid analytic space equipped as well with the following data

1. a finite map $\chi : X \to \mathcal{T}_{\Sigma_p}$
2. a character $\lambda : \mathcal{H}(K^p)^{\mathrm{nr}} \to \Gamma(X, \mathcal{O}_X^{\mathrm{rig}})$, and
3. a Zariski dense set of points $X_{\mathrm{cl}} \subset X(\overline{\mathbf{Q}}_p)$.

If $x \in X(L)$ we denote by $\chi_x = \chi(x) \in \mathcal{T}_{\Sigma_p}(\overline{\mathbf{Q}}_p)$ and by λ_x the induced character

$$\mathcal{H}(K^p)^{\mathrm{nr}} \xrightarrow{\lambda} \Gamma(X, \mathcal{O}_X^{\mathrm{rig}}) \xrightarrow{\mathrm{eval}_x} L.$$

Then the natural map

$$\begin{aligned} X(\overline{\mathbf{Q}}_p) &\to \mathcal{T}_{\Sigma_p}(\overline{\mathbf{Q}}_p) \times \mathrm{Hom}(\mathcal{H}(K^p)^{\mathrm{nr}}, \overline{\mathbf{Q}}_p) \\ x &\mapsto (\chi_x, \lambda_x). \end{aligned}$$

defines a bijection between X_{cl} and pairs (χ, λ_π) attached to classical automorphic representations π and for $G(\mathbf{A}_{F^+})$ which are unramified at p and the choice of a refinement described above.

If $x \in X_{\mathrm{cl}}$ then χ_x knows both the weight of x and information at p. We separate this out as follows. If $v \in \Sigma_p$ we denote $\mathcal{W}_v = \mathrm{Hom}(T(\mathcal{O}_{F_v^+}), \mathbf{G}_m^{\mathrm{rig}})$ and $\mathcal{W}_{\Sigma_p} = \prod_{v \in \Sigma_p} \mathcal{W}_v$. Thus there is a natural projection $\mathcal{T}_{\Sigma_p} \to \mathcal{W}_{\Sigma_p}$ and we let the weight map κ be the composition

$$X \xrightarrow{\chi} \mathcal{T}_{\Sigma_p} \xrightarrow{\mathrm{proj}} \mathcal{W}_{\Sigma_p}$$

This deserves to be called the weight map for the following reason. If $z = (\chi, \lambda)$ is a classical point associated to an automorphic representation π of weight $k = (k_v)$ then $\chi = \theta\delta_k$ with θ smooth, so that $\delta(z) = \delta_k$.

The succinct description of the eigenvariety is enough for some purposes. But the conjectures of Breuil and Herzig [BH] deal with certain subrepresentations of spaces of p-adic automorphic forms and to understand this, it will be convenient for us to describe one source of an eigenvariety construction, due to Emerton.

We let $L/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_L$ and uniformizer ϖ_L. The p-adically completed cohomology of tame level K^p and with coefficients in L is by definition

$$\widehat{H}^0(K^p)_L = \left(\varprojlim_n \varinjlim_{K_p} H^0(G(\mathbf{Q})\backslash G(\mathbf{A}_F^\infty)/K_pK^p, \mathbf{Z}_p/p^n\mathbf{Z}_p) \right) \otimes_{\mathbf{Z}_p} L$$

where K_p runs over compact open subgroups of $G(\mathbf{Q}_p)$. This is an L-Banach space equipped with a continuous representation of $G_{\Sigma_p} \times \mathcal{H}(K^p)^{\mathrm{nr}}$. If $\lambda : \mathcal{H}(K^p)^{\mathrm{nr}} \to \overline{\mathbf{Q}}_p$ is a character we denote by $\widehat{H}^0(K^p)_L^\lambda$ the corresponding G_{Σ_p}-representation of the eigenspace with respect to the character λ.

Within the space $\widehat{H}^0(K^p)_L$ we can take the locally analytic vectors $\widehat{H}^0(K^p)_{L,\mathrm{an}}$ and then this supplies us with a G_{Σ_p}-representation to which we can apply Emerton's Jacquet functor (see [Eme06a]). Thus we have a representation $J_{B_{\Sigma_p}}(\widehat{H}^0(K^p)_{L,\mathrm{an}})$ of the torus T_{Σ_p}. For $\chi \in \mathcal{T}_{\Sigma_p}$, we denote by $J^\chi_{B_{\Sigma_p}}(-)$ the χ-eigenspace. We can read off the χ-eigenspace by the isomorphism

$$J^\chi_{B_{\Sigma_p}}(\widehat{H}^0(K^p)_{L,\mathrm{an}}) \simeq \widehat{H}^0(K^p)_{L,\mathrm{an}}^{N^0_{\Sigma_p}, T^+_{\Sigma_p} = \chi}.$$

The above isomorphism is Hecke equivariant and thus we can put the Hecke eigensystem on both sides. We refer the reader to Proposition 2.3.3 in [Eme06c] for it and for the following proposition.

Proposition II.2.7 (Emerton)**.** *Suppose that L'/L is a finite extension. A pair $(\chi, \lambda) \in \mathcal{T}_{\Sigma_p}(L') \times \mathrm{Hom}(\mathcal{H}(K^p)^{\mathrm{nr}}, L')$ defines a point $(\chi, \lambda) \in X(L')$ if and only if*

$$J^{\chi}_{B_{\Sigma_p}}(\widehat{H}^0(K^p)^{\lambda}_{L,\mathrm{an}}) = \widehat{H}^0(K^p)^{N^0_{\Sigma_p}, T^+_{\Sigma_p}=\chi,\lambda}_{L,\mathrm{an}} \neq 0.$$

II.2.3 The refined family of Galois representations

The eigenvariety carries as well a p-adic family of Galois representations. Suppose that π is a classical automorphic representation on $G(\mathbf{A}_{F^+})$ of tame level K^p. We denote $\Sigma = \Sigma^p \cup \Sigma_p$ and by the same symbol the set of places of F above Σ. The work of Blasius and Rogawski (and many others in a more general situation) associates to π a p-adic Galois representation

$$\rho_\pi : G_{F,\Sigma} \to \mathrm{GL}_3(\overline{\mathbf{Q}}_p).$$

It satisfies local-global compatibility at each finite place. If $\tilde{w}$ is a place of F and ρ is a representation of G_F, we denote by $\rho_{\tilde{w}}$ the restriction $\rho|_{G_{F_{\tilde{w}}}}$. If $v \in \Sigma_p$ we further denote by ρ_v the representation $\rho_v := \rho_{\tilde{v}}$. The representations ρ_π satisfy the following

1. If $\tilde{w} \notin \Sigma$ is a split place of F then the Frobenius semi-simple Weil-Deligne representation associated to $\rho_{\tilde{w}}$ corresponds to the $\mathrm{GL}_3(F_{\tilde{w}}) \overset{\tilde{w}}{\simeq} \mathrm{GL}_3(F^+_w)$-representation $\pi_w|\det^{-1}|$ under the local Langlands correspondence for $\mathrm{GL}_3(F_{\tilde{w}})$.

2. If $v \in \Sigma_p$ then $\rho_{\pi,v} := \rho_{\pi,\tilde{v}}$ is de Rham with Hodge-Tate weights $h_{i,v} = -k_{i,v} + i - 1$.

3. If π_{Σ_p} is unramified then $\rho_{\pi,v}$ is crystalline for each $v \in \Sigma_p$. Moreover, if we make the choice of a refinement θ for π_{Σ_p} then the set of crystalline eigenvalues of $\rho_{\pi,v}$ are given by $\{p^2\theta_{1,v}(p), p\theta_{2,v}(p), \theta_{3,v}(p)\}$.

By Chebotarev's theorem, the first condition classifies ρ_π up to the semi-simplification.

Consider the eigenvariety $X = X_{K^p}$ of tame level K^p again. In that case, the above associates to each classical point $z \in X_{\mathrm{cl}}$ a Galois representation $\rho_z := \rho_\pi$ independent of the refinement.

The choice of a refinement for π_{Σ_p}, however has the following interpretation. Suppose that $z \in X_{\mathrm{cl}}$ corresponds to the refinement θ of π_{Σ_p}. Then, for all v we can define a refinement

$$R_{z,v} := (\phi_{1,v}, \phi_{2,v}, \phi_{3,v}) = (p^2\theta_{1,v}(p), p\theta_{2,v}(p), \theta_{3,v}(p))$$

of the local representation $\rho_{z,v}$. Recall that in Section II.2.1 we defined the weight type of a refinement and what it means for a refinement to be non-critical.

Definition II.2.8. *For a point $z \in X_{\mathrm{cl}}$ we define its weight type to be the Σ_p-tuple $\tau = (\tau_v)_{v \in \Sigma_p}$ where τ_v is the weight type of $R_{z,v}$. We say that z is non-critical if it is of weight type $(1_v)_v$ and critical of type τ otherwise.*

If we denote $t_z : G_{F,\Sigma} \to \overline{\mathbf{Q}}_p$ the function $g \mapsto \mathrm{tr}(\rho_z(g))$ then t_z is a three dimensional pseudocharacter of $G_{F,\Sigma}$ and extends on the eigenvariety to a global pseudocharacter

$$t : G_{F,\Sigma} \to \Gamma(X, \mathcal{O}^{\mathrm{rig}}_X).$$

In particular, for each closed point $x \in X(\overline{\mathbf{Q}}_p)$, specializing the pseudocharacter at x we get $t_x : G_{F,\Sigma} \to \overline{\mathbf{Q}}_p$ and thus by Taylor's theorem [Tay91, Theorem 1(2)] there is a unique semi-simple representation $\rho_x : G_{F,\Sigma} \to \mathrm{GL}_3(\overline{\mathbf{Q}}_p)$ such that $\mathrm{tr}(\rho_x) = t_x$.

Denote now

$$\mathcal{W}_{\Sigma_p} = \prod_{v \in \Sigma_p} \mathcal{W}_v \qquad \mathcal{W}_v = \mathrm{Hom}(T(\mathcal{O}_{F_v^+}), \mathbf{G}_m^{\mathrm{rig}}).$$

Finally denote by η the composition

$$X \xrightarrow{\kappa} \mathcal{W}_{\Sigma_p} \xrightarrow{\log} \prod_{v \in \Sigma_p} \left(\mathbf{G}_a^{\mathrm{rig}}\right)^3 \xrightarrow{s} \prod_{v \in \Sigma_p} \left(\mathbf{G}_a^{\mathrm{rig}}\right)^3 \tag{II.4}$$

where s is the affine change of coordinates

$$\begin{aligned} \left(\mathbf{G}_a^{\mathrm{rig}}\right)^3 &\xrightarrow{s} \left(\mathbf{G}_a^{\mathrm{rig}}\right)^3 \\ (u_1, u_2, u_3) &\mapsto (-u_1, -u_2 + 1, -u_3 + 2). \end{aligned}$$

If $x \in X(\overline{\mathbf{Q}}_p)$ then we write $\eta_{\Sigma_p}(x) = (\eta_{i,v}(x))_v \in \prod_{v \in \Sigma_p} \left(\overline{\mathbf{Q}}_p\right)^3$. In this notation, if $z \in X_{\mathrm{cl}}$ then $\eta_{i,v}(z) = h_{i,v}$ is the ith Hodge-Tate weight at the place $\tilde{v}$. Thus by [BC09, Lemma 7.5.12] we have that for a general $x \in X(\overline{\mathbf{Q}}_p)$, $\eta(x)$ is recording the Hodge-Tate-Sen weights of $\rho_{x,v}$.

If $x \in X(L)$ and $v \in \Sigma_p$ we denote by $\chi_{v,x} = \chi_{1,v,x} \otimes \chi_{2,v,x} \otimes \chi_{3,v,x}$ the vth coordinate of the $\chi_x \in \mathcal{T}_{\Sigma_p}(L)$. Let $n = 3$ for clarity. We then define analytic functions $F_{i,v} \in \Gamma(X, \mathcal{O}_X^{\mathrm{rig}})$ by

$$F_{i,v}(x) = p^{n+1-2i} \chi_{i,v,x}(p). \tag{II.5}$$

If $z = (\theta\delta_k, \lambda) \in X_{\mathrm{cl}}$ is a classical point then

$$\begin{aligned} p^{\eta_{i,v}(z)} F_{i,v}(z) &= p^{h_{i,v}} p^{n+1-2i} \theta_{i,v}(p) p^{k_{i,v}} \\ &= p^{n-i} \theta_{i,v}(p). \end{aligned}$$

Thus for $z \in X_{\mathrm{cl}}$ the collection $R_z = (R_{z,v})$ of refinements is given by

$$R_{z,v} = (p^{\eta_{1,v}(z)} F_{1,v}(z), p^{\eta_{2,v}(z)} F_{2,v}(z), p^{\eta_{3,v}(z)} F_{3,v}(z)).$$

In particular, X together with the Galois pseudorepresentation t of $G_{F,\Sigma}$ forms a refined family of Galois representations in the sense of [BC09, Ch. 4]. This implies that we have the analytic continuouity of crystalline periods: for each $x \in X(\overline{\mathbf{Q}}_p)$, each $v \in \Sigma_p$ and each i we have

$$\mathrm{Fil}^0 D_{\mathrm{cris}}(\wedge^i \rho_{x,v}(\eta_{1,v}(x) + \cdots + \eta_{i,v}(x)))^{\varphi = F_{1,v}(x) \cdots F_{i,v}(x)} \neq 0.$$

In particular, we note the following result

Proposition II.2.9. *Suppose that $x \in X(\overline{\mathbf{Q}}_p)$ and $\eta_{i,v}(x) \in \mathbf{Z}$ for all i, v. Then*

$$\mathrm{Fil}^{\eta_{1,v}(x) + \cdots + \eta_{i,v}(x)} D_{\mathrm{cris}}(\wedge^i \rho_{x,v})^{\varphi = p^{\eta_{1,v}(x)} F_{1,v}(x) \cdots p^{\eta_{i,v}(x)} F_{i,v}(x)} \neq 0.$$

The proof of this has been given in varying levels of generality in many different articles now: [Kis03], [BC09], [Liu13], [KPX].

II.3 Ordinary representations and the conjecture of Breuil-Herzig

II.3.1 Generic ordinary points on eigenvarieties

We now further elucidate the previous sections in the case of ordinary points on X. We use the notation of the previous section.

Recall that a classical point $z = (\chi, \lambda)$ in $X(\overline{\mathbf{Q}}_p)$ is associated to the choice of a classical automorphic representation π of $G(\mathbf{A}_{F^+})$ with tame level K^p, spherical above p, and the choice of a smooth character θ such that

$$\pi_{\Sigma_p} \subset \mathrm{Ind}_{B_{\Sigma_p}}^{G_{\Sigma_p}} \theta.$$

The character χ is then specified as $\chi = \theta\delta_k$ where $k = (k_v)$ is the list of dominant weights of π_∞.

Fix for the moment an automorphic representation π such that π has tame level K^p, π_{Σ_p} is unramified, π has weight $k = (k_v)_{v\in\Sigma_p}$.

Definition II.3.1. *We say that π is generic ordinary if $\rho_{\pi,v}$ is generic ordinary for all $v \in \Sigma_p$.*

Suppose that π is generic ordinary and write

$$\rho_{\pi,v} \sim \begin{pmatrix} \psi_{1,v} & * & * \\ & \psi_{2,v} & * \\ & & \psi_{3,v} \end{pmatrix} \tag{II.6}$$

with the characters $\psi_{i,v}$ crystalline of Hodge-Tate weights $h_{1,v} < h_{2,v} < h_{3,v}$ for each $v \in \Sigma_p$. If we denote $\phi_{\psi_{i,v}}$ the crystalline eigenvalue of $\psi_{i,v}$ then

$$\psi_{i,v} = z^{-h_{i,v}}\, \mathrm{nr}(\phi_{\psi_{i,v}}), \tag{II.7}$$

seen as a character of $\mathbf{Q}_p^\times$ using local class field theory. Now consider the smooth character of $\mathbf{Q}_p^\times$ given by

$$\theta_v^{\mathrm{nc}} = |\cdot|^2\, \mathrm{nr}(\phi_{\psi_{1,v}}) \otimes |\cdot|\, \mathrm{nr}(\phi_{\psi_{2,v}}) \otimes \mathrm{nr}(\phi_{\psi_{3,v}}).$$

The character θ^{nc} is a refinement of π_{Σ_p} and defines a point $z_{\mathrm{nc}} = (\theta^{\mathrm{nc}}\delta_k, \lambda) \in X_{\mathrm{cl}}$.

Since $\rho_{\pi,v}$ is assumed generic ordinary, it is easy to see θ_v^{nc} satisfies the condition (II.2) for all $v \in \Sigma_p$. Thus the smooth admissible representation $\mathrm{Ind}_{B(F_v^+)}^{G(F_v^+)}(\theta_v^{\mathrm{nc}})^{\mathrm{sm}}$ is unramified and we can explicitly list the other refinements of π_{Σ_p} as follows. If $\sigma = (\sigma_v)_v \in (S_3)_{\Sigma_p}$ then we denote by $\theta^{\mathrm{nc},(\sigma)} = (\theta_v^{\mathrm{nc},(\sigma_v)})_v$ the Σ_p-tuple of characters given by

$$\theta_v^{\mathrm{nc},(\sigma_v)} := (\theta_v^{\mathrm{nc}})^{\sigma_v} (\delta_{B(F_v^+)}^{-1/2})^{\sigma_v} \delta_{B(F_v^+)}^{1/2}.$$

By the criterion (II.2) we have that $\theta^{\mathrm{nc},(\sigma)}$ defines a refinement of π_{Σ_p} and all the refinements are of this form. Thus each of the points $z_{\mathrm{nc}}^{(\sigma)} = (\theta^{\mathrm{nc},(\sigma)}\delta_k, \lambda) \in X_{\mathrm{cl}}$ is a point such that $\rho_{z_{\mathrm{nc}}^{(\sigma)}} = \rho_\pi$.

Definition II.3.2. *A point $z \in X_{\mathrm{cl}}$ is called generic ordinary if there exists a generic ordinary π such that $z = z_{\mathrm{nc}}^{(\sigma)}$ as above.*

We could equivalently phrase the definition as follows. Suppose that $z \in X_{\mathrm{cl}}$ is a point such that $\rho_{z,v}$ is generic ordinary for all $v \in \Sigma_p$. We then write $\rho_{z,v}$ as in (II.6) and this necessarily defines a point $z_{\mathrm{nc}} = (\theta^{\mathrm{nc}}\delta_k, \lambda) \in X_{\mathrm{cl}}$; the point z is then one of the points $z_{\mathrm{nc}}^{(\sigma)} = (\theta^{\mathrm{nc},(\sigma)}\delta_k, \lambda)$. The tuple σ is then well-defined.

It will be convenient for us to realize that in terms of the Galois characters $\psi_{i,v}$ we have (use (II.7)) that

$$\theta_v^{\mathrm{nc},(\sigma_v)} = (\psi_{\sigma_v(1)} z^{h_{\sigma_v(1)}} \otimes \psi_{\sigma_v(2)} z^{h_{\sigma_v(2)}} \otimes \psi_{\sigma_v(3)} z^{h_{\sigma_v(3)}})(|\cdot|^2 \otimes |\cdot| \otimes 1).$$

Evaluating at p we get that

$$p^{2-i}\theta_{i,v}^{\mathrm{nc},(\sigma_v)}(p) = \phi_{\psi_{\sigma_v(i),v}}.$$

Thus in the generic ordinary case, to give a refinement of π_{Σ_p} is the same as to give a collection $R = (R_v)_{v\in\Sigma_p}$ of refinements of $\rho_{\pi,v}$.

Definition II.3.3. *An element $\sigma \in (S_3)_{\Sigma_p}$ is called simple if $\sigma_v \in \{1, (12), (23)\}$ for all $v \in \Sigma_p$. A generic ordinary point $z \in X_{\mathrm{cl}}$ is called simple if $z = z_{\mathrm{nc}}^{(\sigma)}$ with σ simple.*

Recall that in Section II.2.3 we defined what it means for a point $z \in X_{\mathrm{cl}}$ to be critical or non-critical.

Proposition II.3.4. *Suppose that z is generic ordinary and write $z = z_{\mathrm{nc}}^{(\sigma)}$.*

- *If $\sigma = 1$ then z is non-critical.*
- *If z is simple and $\rho_{z,v}$ is totally indecomposable for all $v \in \Sigma_p$ then $z_{nc}^{(\sigma)}$ is non-critical.*

Proof. We write $\rho_{z,v}$ as in (II.6) and define the refinement $R_{\mathrm{nc},v} = (\phi_{\psi_{1,v}}, \phi_{\psi_{2,v}}, \phi_{\psi_{3,v}})$ of $\rho_{z,v}$. The corresponding refinement at $z_{\mathrm{nc}}^{(\sigma)}$ is given by $R_{\mathrm{nc},v}^{\sigma}$. Thus both statements follow from Lemma II.2.3. □

II.3.2 Bad points on the eigenvariety

We now introduce the notion of badness for points $z = (\chi, \lambda)$ on the eigenvariety X. This notion will turn out to be crucial when relating principal series to the completed cohomology group. It is related to the notion of z being critical but we will not make this completely precise in this work. It can be defined in general for any connected reductive algebraic group (G, B) to which we can associate an eigenvariety and we will take this approach in a sequel.

Here though we consider the group G_{Σ_p} and its upper triangular Borel B_{Σ_p} and torus T_{Σ_p}. It will be important for us to use the infinitesimal actions of the Lie algebras of these groups. Thus we use $\mathfrak{g}_{\Sigma_p}$, $\mathfrak{b}_{\Sigma_p}$, $\mathfrak{t}_{\Sigma_p}$, $\mathfrak{n}_{\Sigma_p}^-$ (for the opposite unipotent to $\mathfrak{n}_{\Sigma_p} = \mathfrak{b}_{\Sigma_p}/\mathfrak{t}_{\Sigma_p}$) etc. to denote the corresponding Lie algebras. We use $\mathfrak{t}_{\Sigma_p}^* = \mathrm{Hom}(\mathfrak{t}_{\Sigma_p}, \overline{\mathbf{Q}}_p)$ to denote the dual space to $\mathfrak{t}_{\Sigma_p}$ (i.e. the linear dual space of $\mathfrak{t}_{\Sigma_p} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$). If $\chi \in \mathcal{T}_{\Sigma_p}(\overline{\mathbf{Q}}_p)$ then its differential $d\chi$ is an element of $\mathfrak{t}_{\Sigma_p}^*$.

If we drop the subscript Σ_p it is because we are talking about the group $\mathrm{GL}_3(\mathbf{Q}_p)$. We use the notation $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$ to denote the usual two simple positive roots of $\mathfrak{gl}_3$. They generate $\mathfrak{b}$; the third positive root of $\mathfrak{gl}_3$ is $\alpha_3 = \alpha_1 + \alpha_2$. Since $\mathfrak{g}_{\Sigma_p} = (\mathfrak{gl}_3)_{\Sigma_p}$, the roots of $\mathfrak{g}_{\Sigma_p}$ are of the form $\alpha = (\alpha_v)_{v\in\Sigma_p}$ where α_v is a root of $\mathfrak{gl}_3$ and $\alpha_v = 0$ except possibly at one place $v \in \Sigma_p$. A root $\alpha \neq 0$ is positive if and only if $\alpha_v > 0$ for the unique place $v \in \Sigma_p$ at which $\alpha_v \neq 0$. Since the roots of $\mathfrak{g}_{\Sigma_p}$ are all algebraic, we naturally confuse notation and speak about roots $\alpha \in \mathfrak{t}^*_{\Sigma_p}$ and $\alpha \in \mathcal{T}_{\Sigma_p}$.

If $\delta \in \mathcal{T}_{\Sigma_p}$ and α is a root, denote by $\alpha^\vee$ the corresponding co-root. We then consider the locally analytic character $\langle \delta, \alpha^\vee \rangle := \delta \circ \alpha^\vee : \mathbf{G}_m^{\mathrm{rig}} \to \mathbf{G}_m^{\mathrm{rig}}$. Let ρ_0 denote the half-sum of the positive roots.

Definition II.3.5. *A character $\delta \in \mathcal{T}_{\Sigma_p}$ is:*

- *α-integral if $\langle \delta, \alpha^\vee \rangle$ is of the form $z \mapsto z^k$ for some $k \in \mathbf{Z}$, in which case we use $\langle \delta, \alpha^\vee \rangle$ to also denote this integer, and*
- *α-dominant if δ is α-integral and $\langle \delta + \rho_0, \alpha^\vee \rangle > 0$.*

A character $\chi \in \mathcal{T}_{\Sigma_p}$ is locally α-dominant if it is of the form $\chi = \theta\delta$ where θ is smooth and δ is α-dominant.

Recall that the Weyl group associated to our choice of root system has an associated "dot action" on elements of $\mathfrak{t}^*_{\Sigma_p}$. The Weyl group $(S_3)_{\Sigma_p}$ acts on weights $\mu \in \mathfrak{t}^*_{\Sigma_p}$ by the formula

$$\sigma \cdot \mu = \sigma(\mu + \rho_0) - \rho_0,$$

where the σ on the right hand side is the usual permutation action.

For each simple positive root α we have the associate Weyl element $s_\alpha \in (S_3)_{\Sigma_p}$ and the dot action on $\mathfrak{t}_{\Sigma_p}$ extends to elements $\chi \in \mathcal{T}_{\Sigma_p}$ which are locally α-integral in an apparent way. Indeed, when χ is α-integral the element $s_\alpha \cdot d\chi \in \mathfrak{t}^*_{\Sigma_p}$ differs from $d\chi$ only by integers. And thus there is a natural character $s_\alpha \cdot \chi \in \mathcal{T}_{\Sigma_p}$ such that $d(s_\alpha \cdot \chi) = s_\alpha \cdot \chi$ and the smooth parts of χ and $s_\alpha \cdot \chi$ are the same. As an example, consider an element $\delta \in \mathcal{T}^{\mathrm{alg}}_{\Sigma_p}$ of the form $\delta_v = z^{k_{1,v}} \otimes z^{k_{2,v}} \otimes z^{k_{3,v}}$. Then

$$(s_\alpha \cdot \delta)_v = \begin{cases} z^{k_{1,v}} \otimes z^{k_{2,v}} \otimes z^{k_{3,v}} & \text{if } \alpha_v = 0 \\ z^{k_{2,v}-1} \otimes z^{k_{1,v}+1} \otimes z^{k_{3,v}} & \text{if } \alpha_v = \alpha_1 \\ z^{k_{1,v}} \otimes z^{k_{3,v}-1} \otimes z^{k_{2,v}+1} & \text{if } \alpha_v = \alpha_2. \end{cases} \tag{II.8}$$

This construction can be iterated to define a dot action of Weyl group elements $s_{\alpha_1} \cdots s_{\alpha_r}$ on locally integral elements of $\mathcal{T}_{\Sigma_p}$. For the reader who is familiar with Verma modules (see below), the next definition will look familiar.

Definition II.3.6. *Suppose that $\chi, \chi' \in \mathcal{T}_{\Sigma_p}$. We write $\chi' \uparrow \chi$ if $\chi = \chi'$, or there exists a simple positive root α such that χ is locally α-dominant and $s_\alpha \cdot \chi = \chi'$. We say that χ' is strongly linked to χ if there exists a sequence of simple positive roots $s_{\alpha_1}, \cdots, s_{\alpha_r}$ such that*

$$\chi' = (s_{\alpha_1} \cdots s_{\alpha_r}) \cdot \chi \uparrow (s_{\alpha_2} \cdots s_{\alpha_r}) \cdot \chi \uparrow \cdots \uparrow s_{\alpha_r} \cdot \chi \uparrow \chi.$$

Before defining bad, let us point out what is happening in the most interesting case where $\chi = \theta\delta_k$ and δ_k is a dominant weight (that is $k_1 > k_2 > k_3$). Since this is just an illustration, let us assume for simplicity that there is only one place and thus two simple positive roots α_1 and α_2. In that case, $s_\alpha \cdot \delta$ is defined for all simple positive roots α. Moreover, if k is regular then it is easy to see that $w \cdot \chi$ is defined for all elements $w \in S_3$ and that $w \cdot \chi$ is strongly linked to χ by a chain of length equal to the length of w in the Bruhat order on S_3.

We now return to the eigenvariety setting of the previous sections. Thus we have our eigenvariety $X_{K^p}(\overline{\mathbf{Q}}_p)$ of tame level K^p parameterizing eigensystems $(\chi, \lambda) \in \mathcal{T}_{\Sigma_p} \times \mathcal{H}(K^p)^{\mathrm{nr}}$ acting on spaces of p-adic automorphic forms.

Definition II.3.7. *Suppose that $z = (\chi, \lambda) \in X_{K^p}(\overline{\mathbf{Q}}_p)$. We say that the point z is bad if there exists a pair $(\chi', \lambda) \in X_{K^p}(\overline{\mathbf{Q}}_p)$ such that $\chi' \neq \chi$ and χ' is strongly linked to χ. If z is bad, we denote by $w \cdot z := (w \cdot \chi, \lambda) \in X_{K^p}(\overline{\mathbf{Q}}_p)$ (where $w \cdot \chi$ is strongly linked to χ) the corresponding companion points on $X_{K^p}(\overline{\mathbf{Q}}_p)$.*

Recall that in Section II.2.3 we defined the function η interpolating the Hodge-Tate-Sen weights and the functions $F_i \in \Gamma(X, \mathcal{O}_X^{\mathrm{rig}})$ which, after normalization, interpolate crystalline eigenvalues. For each simple positive root α of G_{Σ_p} we simultaneously view its associated reflexion s_α as an element of the Weyl groups $(S_3)_{\Sigma_p}$. Thus for each $v \in \Sigma_p$ we have a natural action of α_v on $\{1, 2, 3\}$.

Lemma II.3.8. *Suppose that z is bad and let $x(z) = w \cdot z$ be a point strongly linked to z. Then $\rho_z \simeq \rho_{x(z)}$. Furthermore, for all $v \in \Sigma_p$ and $i = 1, 2, 3$ we have*

1. *$\eta_{i,v}(x(z)) = \eta_{w_v(i),v}(z)$ and*
2. *$p^{\eta_{i,v}(x(z))} F_{i,v}(x(z)) = p^{\eta_{i,v}(z)} F_{i,v}(z)$.*

Proof. By definition $\lambda_z = \lambda_{x(z)}$ and thus $\rho_z \simeq \rho_{x(z)}$ up to semi-simplifcation by Chebotarev. However, since each is assumed semi-simple we get equality on the nose. The other two identities follow from definition of bad points and the description (II.4) (respectively (II.5)) of the map η (respectively the F_i). □

Proposition II.3.9. *If $z \in X_{\mathrm{cl}}$ is bad then z is critical.*

Proof. Suppose that z is bad and let x be a point strongly linked to z. Without loss of generality we can assume that $x = s_\alpha \cdot z$ for some simple positive root α. By Lemma II.3.8, the point x has integral Hodge-Tate weights at each place $v \in \Sigma_p$ and $\rho_{x,v} \simeq \rho_{z,v}$. Thus Proposition II.2.9 and Lemma II.3.8 implies that

$$0 \neq \mathrm{Fil}^{\eta_{\alpha_v(i),v}(z)}\, D_{\mathrm{cris}}(\rho_{z,v})^{\varphi = p^{\eta_{1,v}(z)} F_{1,v}(z)}$$
$$0 \neq \mathrm{Fil}^{\eta_{\alpha_v(1),v}(z) + \eta_{\alpha_v(2),v}(z)}\, D_{\mathrm{cris}}(\wedge^2 \rho_{z,v})^{\varphi = p^{\eta_{1,v}(z) + \eta_{2,v}(z)} F_{1,v}(z) F_{2,v}(z)}.$$

If $\alpha_v = e_1 - e_2$ for some $v \in \Sigma_p$ then the first equation implies that z is critical; if $\alpha_v = e_2 - e_3$ for some $v \in \Sigma_p$ then the second equation implies the same. □

Remark II.3.10. *The converse of Proposition II.3.9 is also true but requires slightly more work. Since we do not need it here, we save it for a sequel. Both these statements can be also deduced from the work of Christophe Breuil. The statement of the proposition follows from Lemma 6.4 and Proposition 8.1 in [Bre]. The converse follows from Section 9 of [Bre].*

Corollary II.3.11. *If $z \in X_{\mathrm{cl}}$ is a simple, generic ordinary point and $\rho_{z,v}$ is totally indecomposable then z is not bad.*

Proof. Combine Proposition II.3.4 and Proposition II.3.9. □

II.3.3 Principal series and completed cohomology

We now use results on bad points to prove that certain principal series appear in the completed cohomology. Our main result in this subsection is an adjunction formula analogous to Theorem 5.5.1 and Proposition 5.2.1 in [BE10]. We apply this result to points $z_{\mathrm{nc}}^{(\sigma)}$ defined above. Our adjunction formula follows from the more general adjunction formula given by the main result of Emerton [Eme10a]. In order to apply Emerton's result, we first need to verify two separate lemmas.

We introduce some notations from [Eme06c]. Let $\mathcal{E}$ be the coherent sheaf on $\mathcal{T}_{\Sigma_p}$ that corresponds to the strong dual of $J_{B_{\Sigma_p}}(\widehat{H}^0(K^p)_{L,\mathrm{an}})$ and let $\mathcal{A}$ be the commutative subring of $\mathrm{End}(\mathcal{E})$ generated by $\mathcal{H}(K^p)^{nr}$. Proposition 2.3.8 of [Eme06c] shows that $\mathrm{Spec}\,\mathcal{A}$ contains the eigenvariety $X = X_{K^p}$ as a closed subspace. We will prove our adjunction formula for points of $\mathrm{Spec}\,\mathcal{A}$.

Let (χ, λ) be a point of $\mathrm{Spec}\,\mathcal{A}$. We say that (χ, λ) is *bad* if there exists a pair $(\chi', \lambda) \in X_{K^p}(\overline{\mathbf{Q}}_p)$ such that $\chi' \neq \chi$ and χ' is strongly linked to χ. This extends Definition II.3.7.

We write L_χ for the one-dimensional space underlying χ. Let $\delta_{B_{\Sigma_p}} = (\delta_{B_v})_{v\in\Sigma_p}$ be the Σ_p-tuple of modulus characters. We now prove our main theorem of this subsection, which is the following adjunction formula.

Theorem II.3.12. *Let $z = (\chi, \lambda) \in \mathrm{Spec}\,\mathcal{A}$ be not bad. Then there exists an isomorphism*

$$\mathrm{Hom}_{G_{\Sigma_p}}((\mathrm{Ind}_{B^-_{\Sigma_p}}^{G_{\Sigma_p}} \chi\delta^{-1}_{B_{\Sigma_p}})^{\mathrm{an}}, \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \simeq \widehat{H}^0(K^p)_{L,\mathrm{an}}^{N^0_{\Sigma_p}, T^+_{\Sigma_p}=\chi,\lambda} \simeq J^\chi_{B_{\Sigma_p}}(\widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}})$$

where we have denoted by $(\mathrm{Ind}_{B^-_{\Sigma_p}}^{G_{\Sigma_p}} \chi\delta^{-1}_{B_{\Sigma_p}})^{\mathrm{an}}$ the analytic induction of $\chi\delta^{-1}_{B_{\Sigma_p}}$ seen as a $B^-_{\Sigma_p}$-representation.

Before giving the proof we recall necessary material on Verma modules together with some constructions of Emerton. We will ignore some of subscripts Σ_p to make our writing clearer and write simply $\mathfrak{g}, \mathfrak{b}, G, T, T^+, B, B^-, N, N^0, \delta$ for $\mathfrak{g}_{\Sigma_p}, \mathfrak{b}_{\Sigma_p}, G_{\Sigma_p}, T_{\Sigma_p}, T^+_{\Sigma_p}, B_{\Sigma_p}, B^-_{\Sigma_p}, N_{\Sigma_p}, N^0_{\Sigma_p}, \delta_{B_{\Sigma_p}}$.

Recall that if H is any subgroup of G, then a $(\mathfrak{g}, H)$-module is a $\mathfrak{g}$-module V with a linear action of H such that for any $X \in \mathfrak{g}, h \in H, v \in V$ we have $h \cdot X \cdot v = \mathrm{Ad}_h(X)hv$.

If a $\mathfrak{g}$-module V is locally $\mathfrak{n}$-nilpotent, then it carries a canonical $(\mathfrak{g}, N)$-structure obtained by integrating the $\mathfrak{n}$-action. Moreover, if V has a structure of a $(\mathfrak{g}, T)$-module, then the latter extends to a $(\mathfrak{g}, B)$-structure by integration of the $\mathfrak{n}$-action.

For a locally analytic character χ of T we define

$$M(\chi) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L_\chi$$

and

$$M(\chi)^\vee = \mathrm{Hom}_{U(\mathfrak{b}^-)}(U(\mathfrak{g}), L_\chi)^{\mathfrak{n}^\infty}$$

We endow $M(\chi)$ (respectively, $M(\chi)^\vee$) with a $\mathfrak{g}$-action by left translations (respectively, right translations) on $U(\mathfrak{g})$. This action is locally nilpotent. We let T act by the adjoint action on $U(\mathfrak{g})$ and by the character χ on L_χ. This gives a structure of a $(\mathfrak{g}, T)$-module on both $M(\chi)$ and $M(\chi)^\vee$, which extends canonically to a structure of a $(\mathfrak{g}, B)$-module.

Recall the decompostion $U(\mathfrak{g}) = U(\mathfrak{b}^-) \oplus U(\mathfrak{g})\mathfrak{n}$. We consider the map

$$U(\mathfrak{g}) \to U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n} \to U(\mathfrak{b}^-) \xrightarrow{\chi} L$$

as an element $v^\vee$ of $M(\chi)^\vee$. It is killed by $\mathfrak{n}$ and is a χ-eigenvector for T, therefore there is a unique $(\mathfrak{g}, B)$-equivariant map

$$\alpha_\chi : M(\chi) \to M(\chi)^\vee$$

which takes $v_\chi := 1 \otimes 1$ to $v^\vee$.

Lemma II.3.13.

(1) The map α_χ is the unique (up to a scalar) $(\mathfrak{g}, B)$-equivariant map $M(\chi) \to M(\chi)^\vee$. Its image $L(\chi)$ is an irreducible $(\mathfrak{g}, B)$-module.

(2) Modules $M(\chi)$ and $M(\chi)^\vee$ have finite length and their simple subquotients are of the form $L(\chi')$ where χ' is strongly linked to χ.

Proof. (1) As $\mathfrak{g}$-modules, $M(\chi)$ is the Verma module $\mathrm{Ver}(d\chi)$, $M(\chi)^\vee$ is the dual Verma module $\mathrm{Ver}(d\chi)^\vee$ and α_χ is the usual (unique up to a scalar) map $\mathrm{Ver}(d\chi) \to \mathrm{Ver}(d\chi)^\vee$ whose image is known to be the simple $\mathfrak{g}$-module $L(d\chi)$ of highest weight $d\chi$. Since α_χ is also B-equivariant, we see that its image is a simple $(\mathfrak{g}, B)$-module which extends $L(d\chi)$.

(2) The corresponding assertion for Verma modules is well-known. From the way it is proved for Verma modules, it suffices to check that if μ' is strongly linked to $d\chi$ then

(a) There is a unique χ' strongly linked to χ with $d\chi' = \mu'$.

(b) Any $\mathfrak{g}$-map $\mathrm{Ver}(d\chi') \to \mathrm{Ver}(d\chi)$ is in fact T-equivariant, that is, is a map $M(\chi') \to M(\chi)$.

(b)' Any $\mathfrak{g}$-map $\mathrm{Ver}(d\chi)^\vee \to \mathrm{Ver}(d\chi')^\vee$ is in fact T-equivariant, that is, is a map $M(\chi)^\vee \to M(\chi')^\vee$.

To prove (a), observe that if we write $d\gamma := \mu' - d\chi$, then we know that $d\gamma$ is integral hence is the derivative of an algebraic character γ. Thus $\chi' := \gamma \cdot \chi$ is the desired character.

The point (b) will follow if we prove that T acts via χ' on any highest weight vector v of $M(\chi)$ with weight $d\chi'$. But we know that v is of the form $X \cdot v_\chi$ for some $X \in U(\mathfrak{n}^-)$ of weight $d\gamma = d\chi' - d\chi$ for the adjoint action of $\mathfrak{t}$. Such an X is also a γ-eigenvector for the adjoint action of T on $U(\mathfrak{n}^-)$. It follows that $X \cdot v_\chi$ is an eigenvector for T with weight $\gamma \cdot \chi = \chi'$.

A dual argument proves (b)'. □

We now derive another description of $M(\chi)^\vee$. It is well-known but we could not find a good reference. We define $\mathcal{C}^{pol}(N, L_\chi)$ to be the space of L_χ-valued polynomial functions on N. It carries a natural structure of B-module as explained after Lemma 2.5.3 in [Eme13]. It carries also a natural $\mathfrak{g}$-action defined as follows: any $f \in \mathcal{C}^{pol}(N, L_\chi)$ may be extended to a locally analytic function on the big cell B^-N by putting $\tilde{f}(b^-n) := \chi(b^-)f(n)$. Since B^-N is open in G we can make $X \in \mathfrak{g}$ act by left invariant derivation on f, that is $Xf := \partial_X \tilde{f}_{|N}$. We then have a unique $(\mathfrak{g}, B)$-equivariant map $\beta_\chi : M(\chi) \to \mathcal{C}^{pol}(N, L_\chi)$ which takes $1 \otimes 1$ to the constant function with value 1.

Lemma II.3.14. *There is a $(\mathfrak{g}, B)$-equivariant isomorphism $\mathcal{C}^{pol}(N, L_\chi) \simeq M(\chi)^\vee$ that carries the map β_χ to α_χ up to a scalar.*

Proof. Emerton constructs an isomorphism of vector spaces in (2.5.7) of [Eme13]. The T-equivariance of this map is clear from the definitions. We note that Emerton defines the $\mathfrak{g}$-action on $\mathcal{C}^{pol}(N, L_\chi)$ via this isomorphism. It follows from Lemma 2.5.24 of [Eme13] that his action coincides with the one we have defined above. The remaining assertion follows from the unicity of α_χ. □

Proof of the adjunction formula. Let us now start the proof of the adjunction formula. We consider the following commutative diagram:

$$
\begin{tikzcd}
\mathrm{Hom}_G(\mathrm{Ind}_{B^-}^G(\chi\delta^{-1}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \arrow[r, "(1)"] \arrow[d, "\simeq (a)"] & \mathrm{Hom}_T(\chi, J_B(\widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}})) \arrow[d, "\simeq (b)"] \\
\mathrm{Hom}_{(\mathfrak{g},B)}(\mathcal{C}^{pol}(N, L_\chi) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \arrow[r, "(2)"] \arrow[d, "= (c)"] & \mathrm{Hom}_{(\mathfrak{g},B)}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \chi \otimes C_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \arrow[d, "= (d)"] \\
\mathrm{Hom}_{(\mathfrak{g},B)}(M(\chi)^\vee \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \arrow[r, "(3)"] \arrow[d, "(3a)"] & \mathrm{Hom}_{(\mathfrak{g},B)}(M(\chi) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \\
\mathrm{Hom}_{(\mathfrak{g},B)}(L(\chi) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \arrow[ur, "(3b)"] &
\end{tikzcd}
$$

Let us now explain all the identifications and maps. We want to prove that (1) is an isomorphism.

We denote by $\mathcal{C}_c^{\mathrm{lp}}(N_{\Sigma_p}, L_\chi)$ the space of compactly supported locally L_χ-valued polynomial functions on N_{Σ_p}. Recall that $\mathcal{C}_c^{lp}(N, L_{\chi\delta^{-1}}) = \mathcal{C}^{pol}(N, L_\chi) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}})$. Because of the natural open immersion $N_{\Sigma_p} \hookrightarrow G_{\Sigma_p}/B^-_{\Sigma_p}$, we can regard $\mathcal{C}_c^{\mathrm{lp}}(N_{\Sigma_p}, L_\chi)$ as a $(\mathfrak{g}, B_{\Sigma_p})$-invariant subspace of $\mathrm{Ind}_{B^-_{\Sigma_p}}^{G_{\Sigma_p}}(\chi)^{\mathrm{an}}$. The inclusion of $\mathcal{C}_c^{\mathrm{sm}}(N_{\Sigma_p}, L_\chi)$ in $\mathcal{C}_c^{\mathrm{lp}}(N_{\Sigma_p}, L_\chi)$ thus induces a $(\mathfrak{g}, B_{\Sigma_p})$-equivariant map

$$
U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathcal{C}_c^{\mathrm{sm}}(N_{\Sigma_p}, L_\chi) \to \mathcal{C}_c^{\mathrm{lp}}(N_{\Sigma_p}, L_\chi) \tag{II.9}
$$

For (a) let $\mathrm{Ind}_{B^-}^G(\chi\delta^{-1})(N)$ denote the subspace of $\mathrm{Ind}_{B^-}^G(\chi\delta^{-1})$ of functions supported on N. It generates $\mathrm{Ind}_{B^-}^G(\chi\delta^{-1})$ as a G-representation (Lemma 2.4.13 of [Eme13]). Moreover, $\mathcal{C}_c^{lp}(N, L_{\chi\delta^{-1}})$ is dense in $\mathrm{Ind}_{B^-}^G(\chi\delta^{-1})(N)$ as a $(\mathfrak{g}, B)$-representation (Proposition 2.7.9 in [Eme13]). Those are basic ingredients to prove the following isomorphisms:

$$
\mathrm{Hom}_G(\mathrm{Ind}_{B^-}^G(\chi\delta^{-1}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \simeq \mathrm{Hom}_{(\mathfrak{g},B)}(\mathrm{Ind}_{B^-}^G(\chi\delta^{-1})(N), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \simeq
$$

$$
\simeq \mathrm{Hom}_{(\mathfrak{g},B)}(\mathcal{C}_c^{lp}(N, L_{\chi\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}})
$$

which are proved as Theorems 4.1.5 and 4.2.18 in [Eme13].

The isomorphism (b) results from Theorem 3.5.6 in [Eme06a] (see also (0.17) in [Eme13]). The identification (c) is Lemma II.3.14.

The map (2) is induced by II.9. It comes from a natural map of $(\mathfrak{g}, B)$-modules:

$$
U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L_\chi \to \mathcal{C}^{pol}(N, L_\chi)
$$

which arises from the map

$$
\alpha_\chi : M(\chi) \to M(\chi)^\vee
$$

which we have introduced before the proof. Hence it factors through

$$
M(\chi) \twoheadrightarrow L(\chi) \hookrightarrow M(\chi)^\vee
$$

which gives maps (3a) and (3b).

Lemma II.3.15. *If $L(\mu) \neq L(\chi)$ is a constituent of $M(\chi)$ or $M(\chi)^\vee$ then*

$$\mathrm{Hom}_{(\mathfrak{g},B)}(L(\mu) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) = 0$$

Proof. We have

$$\mathrm{Hom}_{(\mathfrak{g},B)}(L(\mu) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \subset$$

$$\subset \mathrm{Hom}_{(\mathfrak{g},B)}(M(\mu) \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) =$$

$$= \mathrm{Hom}_B(\mathcal{C}_c^{sm}(N, L_{\mu\delta^{-1}}), \widehat{H}^0(K^p)^\lambda_{L,\mathrm{an}}) \simeq \widehat{H}^0(K^p)^{N^0, T^+=\mu,\lambda}_{L,\mathrm{an}}$$

where the last isomorphism is given by evaluating homomorphisms on the characteristic function $1_{N^0} \in \mathcal{C}_c^{sm}(N, L_{\mu\delta^{-1}})$ (we remark that we normalize the action of T^+ as Emerton in Definition 3.4.2 in [Eme06a] and hence we get a twist by δ). We conclude by remarking that

$$\widehat{H}^0(K^p)^{N^0, T^+=\mu,\lambda}_{L,\mathrm{an}} = 0$$

by the assumption that (χ, λ) is not bad. □

Corollary II.3.16. *The map (3b) is an isomorphism and the map (3a) is injective.*

Proof. It follows from Lemma II.3.15 and the structure of Verma modules which we have recalled above. □

To finish the proof of the adjunction formula we need to prove that (3a) is surjective.

Let us denote by $\mathfrak{p}_\lambda$ the ideal of the Hecke algebra $\mathcal{H}(K^p)^{nr}$ corresponding to λ and by $\mathfrak{p}^+_{id}$ the ideal of $L[T^+]$ corresponding to the character id. We start with an auxilary lemma.

Lemma II.3.17. *Let V, W be $(\mathfrak{g}, B)$-modules. We have*

$$\mathrm{Hom}_{(\mathfrak{g},B)}(V \otimes \mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), W) = \mathrm{Hom}_{\mathfrak{g}}(V, W)^{N^0, T^+=id}$$

We remark again that the twist by δ in the T^+-action appears because we use the normalization of Emerton from Definition 3.4.2 in [Eme06a] in defining the action of the monoid T^+ on the N^0-invariants. If we were to make T^+ act by correspondences $[N^0 t^+ N^0]$, then we would have $T^+ = \delta^{-1}$ on the right.

Proof. Both sides are isomorphic to $\mathrm{Hom}_B(\mathcal{C}_c^{sm}(N, L_{\delta^{-1}}), \mathrm{Hom}_{\mathfrak{g}}(V, W))$, where we pass to the right-hand side by evaluation at the characteristic function 1_{N^0}. □

We consider the exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(M(\chi)^\vee / L(\chi), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0} \to$$

$$\to \mathrm{Hom}_{\mathfrak{g}}(M(\chi)^\vee, \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0} \to$$

$$\to \mathrm{Hom}_{\mathfrak{g}}(L(\chi), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0} \to$$

$$\to \mathrm{Ext}^1_{\mathfrak{g}}(M(\chi)^\vee / L(\chi), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}$$

We want to prove that

$$\mathrm{Hom}_{\mathfrak{g}}(M(\chi)^\vee, \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}[\mathfrak{p}_\lambda, \mathfrak{p}^+_{id}] \simeq \mathrm{Hom}_{\mathfrak{g}}(L(\chi), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}[\mathfrak{p}_\lambda, \mathfrak{p}^+_{id}]$$

Because of the exact sequence above it is enough to show

$$\mathrm{Hom}_{\mathfrak{g}}(M(\chi)^{\vee}/L(\chi), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} = 0$$

$$\mathrm{Ext}^1_{\mathfrak{g}}(M(\chi)^{\vee}/L(\chi), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} = 0$$

where the subscript $\mathfrak{p}_\lambda, \mathfrak{p}^+_{id}$ denotes the localisation at respective ideals. By devissage, it suffices to show that

Lemma II.3.18. *For μ strongly linked to χ we have*

$$(a)\, \mathrm{Hom}_{\mathfrak{g}}(L(\mu), \widehat{H}^0(K^p)^{\lambda}_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} = 0$$

$$(b)\, \mathrm{Ext}^1_{\mathfrak{g}}(L(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} = 0$$

Proof. (a) We can replace $L(\mu)$ by $M(\mu)$. Thus it is enough to prove that

$$\mathrm{Hom}_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} = 0$$

But $\mathrm{Hom}_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0} = \widehat{H}^0(K^p)^{N^0,\mathfrak{t}=\mu}_{L,\mathrm{an}}$. It follows from Lemma 2.3.4(ii) of [Eme06a] that $\mathcal{H}(K^p)^{\mathrm{nr}}$ and T^+ act by compact operators on $\widehat{H}^0(K^p)^{N^0,\mathfrak{t}=\mu}_{L,\mathrm{an}}$ because this space is of compact type. Hence the localisation $(\widehat{H}^0(K^p)^{N^0,\mathfrak{t}=\mu}_{L,\mathrm{an}})_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}}$ is finite dimensional. Therefore

$$\mathrm{Hom}_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}}$$

is of $\mathfrak{p}_\lambda$-torsion and of $\mathfrak{p}^+_{id}$-torsion. Hence if this space is non-zero then also

$$\mathrm{Hom}_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}[\mathfrak{p}_\lambda, \mathfrak{p}^+_{id}] = \widehat{H}^0(K^p)^{N^0,T^+=\mu,\lambda}_{L,\mathrm{an}}$$

is non-zero which would contradict our assumption that (χ, λ) is not bad. Hence we conclude.

(b) Let us prove firstly that $\mathrm{Ext}^1_{\mathfrak{g}}(L(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}}$ is of $\mathfrak{p}_\lambda$-torsion and of $\mathfrak{p}^+_{id}$-torsion. By the exact sequence

$$\mathrm{Hom}_{\mathfrak{g}}(\ker(M(\mu) \to L(\mu)), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0} \to \mathrm{Ext}^1_{\mathfrak{g}}(L(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0} \to$$

$$\to \mathrm{Ext}^1_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}$$

and (a) it suffices to prove that $\mathrm{Ext}^1_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} = 0$. We have

$$\mathrm{Ext}^1_{\mathfrak{g}}(M(\mu), \widehat{H}^0(K^p)_{L,\mathrm{an}}) = \mathrm{Ext}^1_{\mathfrak{b}}(\mu, \widehat{H}^0(K^p)_{L,\mathrm{an}}) = H^1(\mathfrak{b}, \widehat{H}^0(K^p)_{L,\mathrm{an}}(-\mu))$$

We know that $\widehat{H}^0(K^p)_{L,\mathrm{an}}$ is injective as a $G(\mathbf{Z}_p)$-module. The standard argument is given for example in the proof of Proposition 4.9 in [Cho13], from which we infer that in fact $\widehat{H}^0(K^p)_{L,\mathrm{an}} \simeq \mathcal{C}^{la}(G(\mathbf{Z}_p), L)^{\oplus r}$ for some integer $r > 0$ as $G(\mathbf{Z}_p)$-modules. Assume we know that $\mathcal{C}^{la}(G(\mathbf{Z}_p), L)$ is $\mathfrak{b}$-acyclic and invariant under μ-torsion (the second one is clear), then

$$H^1(\mathfrak{b}, \widehat{H}^0(K^p)_{L,\mathrm{an}}(-\mu)) = 0$$

This proves the claim. Now to prove (b) completely, remark that we get the surjection

$$\mathrm{Hom}_{\mathfrak{g}}(\ker(M(\mu)\to L(\mu)),\widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}} \twoheadrightarrow \mathrm{Ext}^1_{\mathfrak{g}}(L(\mu),\widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}}$$

hence it suffices to show that $\mathrm{Hom}_{\mathfrak{g}}(\ker(M(\mu)\to L(\mu)),\widehat{H}^0(K^p)_{L,\mathrm{an}})^{N^0}_{\mathfrak{p}_\lambda,\mathfrak{p}^+_{id}}$ vanishes. But this follows from (a) by devissage.

Let us now prove that $\mathcal{C}^{la}(G(\mathbf{Z}_p),L)$ is indeed $\mathfrak{b}$-acyclic (it is stated in the proof of Proposition 5.1.2 in [BE10]; the following proof was communicated to us by Christophe Breuil). Recall that we have a topological decomposition $G(\mathbf{Z}_p)=G(\mathbf{Z}_p)/B(\mathbf{Z}_p)\times B(\mathbf{Z}_p)$ which gives us a $B(\mathbf{Z}_p)$-equivariant isomorphism

$$\mathcal{C}^{la}(G(\mathbf{Z}_p),L)\simeq \mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),L)\otimes_L \mathcal{C}^{la}(B(\mathbf{Z}_p),L)$$

See Proposition 2.1.11 in [Eme10b]. The cohomological complex for the $\mathfrak{b}$-cohomology is made of

$$\mathrm{Hom}_L(\bigwedge^q \mathfrak{b},\mathcal{C}^{la}(G(\mathbf{Z}_p),L))$$

and by the above fact this is isomorphic to

$$\mathrm{Hom}_L\left(\bigwedge^q \mathfrak{b},\mathcal{C}^{la}(B(\mathbf{Z}_p),L)\right)\widehat{\otimes}_L\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),L)$$

Observe that tensoring by $\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),L)$ is exact in the category of $\mathfrak{b}$-modules over L. Indeed, for any $\mathfrak{b}$-module M, we have a $\mathfrak{b}$-equivariant isomorphism

$$M\widehat{\otimes}_L\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),L)\simeq \mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),M)$$

Now if $0\to H\to M\to N\to 0$ is exact then

$$0\to\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),H)\to\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),M)\to\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),N)\to 0$$

is exact by the proof of Proposition 2.1.23 in [Eme11b]. Hence $-\widehat{\otimes}_L\mathcal{C}^{la}(G(\mathbf{Z}_p)/B(\mathbf{Z}_p),L)$ is an exact functor.

It suffices now to prove that $\mathcal{C}^{la}(B(\mathbf{Z}_p),L)$ is $\mathfrak{b}$-acyclic. This is proved along the proof of Proposition 3.1 in [ST05].

□

This finishes the proof of the adjunction formula. □

Due to various convenient normalizations we will now write points $z=(\chi,\lambda)\in\operatorname{Spec}\mathcal{A}$ as pairs $(\xi^{(13)}\delta_{B_{\Sigma_p}},\lambda)$ where $\xi\in\mathcal{T}_{\Sigma_p}$ and $\xi^{(13)}$ is the usual twisting by the longest Weyl-element (13):

$$(\xi_1\otimes\xi_2\otimes\xi_3)^{(13)}=\xi_3\otimes\xi_2\otimes\xi_1$$

and $\delta_{B_{\Sigma_p}}=(\delta_{B_v})_{v\in\Sigma_p}$ is the Σ_p-tuple of modulus characters. This allows us to pass between Borel B_{Σ_p} and its opposite $B^-_{\Sigma_p}$:

$$(\mathrm{Ind}^{G_{\Sigma_p}}_{B_{\Sigma_p}}\xi)^{\mathrm{an}}=(\mathrm{Ind}^{G_{\Sigma_p}}_{B^-_{\Sigma_p}}\chi\delta^{-1}_{B_{\Sigma_p}})^{\mathrm{an}}$$

We will use that tacitly in the rest of the text.

We now come back to the situation from Section 3.1. Recall that for $v \in \Sigma_p$, we have associated to an automorphic representation π of weight $k = (k_v)_v$ an upper-triangular crystalline Galois representation $\rho_{\pi,v}$ with crystalline characters $\psi_{1,v}, \psi_{2,v}, \psi_{3,v}$ on the diagonal and Hodge-Tate weights $h_{1,v} < h_{2,v} < h_{3,v}$. We have denoted by $\phi_{\psi_{i,v}}$ the crystalline eigenvalue of $\psi_{i,v}$ and we have written $\psi_{i,v} = z^{-h_{i,v}} \operatorname{nr}(\phi_{\psi_{i,v}})$. For each $\sigma_v \in S_3$ we have defined a smooth character

$$\theta_v^{\mathrm{nc},(\sigma_v)} = (\psi_{\sigma_v(1)} z^{h_{\sigma_v(1)}} \otimes \psi_{\sigma_v(2)} z^{h_{\sigma_v(2)}} \otimes \psi_{\sigma_v(3)} z^{h_{\sigma_v(3)}})(|\cdot|^2 \otimes |\cdot| \otimes 1)$$

and a classical point on the eigenvariety for each $\sigma = (\sigma_v)_{v\in\Sigma_p} \in (S_3)_{\Sigma_p}$ given by

$$z_{\mathrm{nc}}^{(\sigma)} = (\theta^{\mathrm{nc},(\sigma)}\delta_k, \lambda)$$

In order to apply the theorem to the points $z_{\mathrm{nc}}^{(\sigma)}$ we write them in the form $z_{\mathrm{nc}}^{(\sigma)} = (\chi^{(13)}\delta_{B_{\Sigma_p}}, \lambda)$. We look at the refinements one place $v \in \Sigma_p$ at a time. We regard three cases as $\sigma_v \in \{(1)_v, (12)_v, (23)_v\}$:

$$\begin{aligned}
(1)_v &: (\psi_{1,v}|\cdot|^2 \otimes \psi_{2,v}|\cdot|z \otimes \psi_{3,v}z^2)_v \\
&= \Big(\psi_{3,v}z^2|\cdot|^2 \otimes \psi_{2,v}|\cdot|z \otimes \psi_{1,v}\Big)_v^{(13)} \delta_{B_{\Sigma_p}} \\
&= \Big(\psi_{3,v}\varepsilon^2 \otimes \psi_{2,v}\varepsilon \otimes \psi_{1,v}\Big)_v^{(13)} \delta_{B_{\Sigma_p}} \\
(12)_v &: (\psi_{2,v}|\cdot|^2 z^{h_{2,v}-h_{1,v}} \otimes \psi_{1,v}|\cdot|z^{h_{1,v}-h_{2,v}+1} \otimes \psi_{3,v}z^2)_v \\
&= \Big(\psi_{3,v}|\cdot|^2 z^2 \otimes \psi_{1,v}|\cdot|z^{h_{1,v}-h_{2,v}+1} \otimes \psi_{2,v}z^{h_{2,v}-h_{1,v}}\Big)_v^{(13)} \delta_{B_{\Sigma_p}} \\
&= \Big(\psi_{3,v}\varepsilon^2 \otimes \psi_{1,v}\varepsilon z^{h_{1,v}-h_{2,v}} \otimes \psi_{2,v}z^{h_{2,v}-h_{1,v}}\Big)_v^{(13)} \delta_{B_{\Sigma_p}} \\
(23)_v &: (\psi_{1,v}|\cdot|^2 \otimes \psi_{3,v}|\cdot|z^{h_{3,v}-h_{2,v}+1} \otimes \psi_{2,v}z^{h_{2,v}-h_{3,v}+2})_v \\
&= \Big(\psi_{2,v}|\cdot|^2 z^{h_{2,v}-h_{3,v}+2} \otimes \psi_{3,v}|\cdot|z^{h_{3,v}-h_{2,v}+1} \otimes \psi_{1,v}\Big)_v^{(13)} \delta_{B_{\Sigma_p}} \\
&= \Big(\psi_{2,v}\varepsilon^2 z^{h_{2,v}-h_{3,v}} \otimes \psi_{3,v}\varepsilon z^{h_{3,v}-h_{2,v}} \otimes \psi_{1,v}\Big)_v^{(13)} \delta_{B_{\Sigma_p}}
\end{aligned} \tag{II.10}$$

We denote

$$\begin{aligned}
\chi_v^{(1)} &= \psi_{3,v}\varepsilon^2 \otimes \psi_{2,v}\varepsilon \otimes \psi_{1,v} \\
\chi_v^{(12)} &= \psi_{3,v}\varepsilon^2 \otimes \psi_{1,v}\varepsilon z^{h_{1,v}-h_{2,v}} \otimes \psi_{2,v}z^{h_{2,v}-h_{1,v}} \\
\chi_v^{(23)} &= \psi_{2,v}\varepsilon^2 z^{h_{2,v}-h_{3,v}} \otimes \psi_{3,v}\varepsilon z^{h_{3,v}-h_{2,v}} \otimes \psi_{1,v}
\end{aligned}$$

We could make a similar easy computation for any other $\sigma_v \in S_3$, but explicitly we will need only $(1), (12), (23)$. Hence we could also define $\chi_v^{(\sigma_v)}$ for any $\sigma_v \in S_3$. We use it just for the sake of a better exposition. For every $\sigma = (\sigma_v)_v \in (S_3)_{\Sigma_p}$ we put $\chi^{(\sigma)} = (\chi_v^{(\sigma_v)})_{v\in\Sigma_p}$ and we define a locally analytic representation

$$E_{\mathrm{an}}(z_{\mathrm{nc}}^{(\sigma)}) = (\mathrm{Ind}_{B_{\Sigma_p}}^{G_{\Sigma_p}} \chi^{(\sigma)})^{\mathrm{an}} \tag{II.11}$$

associated to a point $z_{\mathrm{nc}}^{(\sigma)}$. Similarly we define a continuous representation

$$E_{C^0}(z_{\mathrm{nc}}^{(\sigma)}) = (\mathrm{Ind}_{B_{\Sigma_p}}^{G_{\Sigma_p}} \chi^{(\sigma)})^{C^0}$$

associated to a point $z_{\mathrm{nc}}^{(\sigma)}$. Notice that if $\chi^{(\sigma)}$ is a unitary character of G_{Σ_p} then the universal unitary completion of $E_{\mathrm{an}}(z_{\mathrm{nc}}^{(\sigma)})$ is $E_{C^0}(z_{\mathrm{nc}}^{(\sigma)})$. Summarizing our results so far, we get:

Corollary II.3.19. *If $z_{\mathrm{nc}}^{(\sigma)}$ is a simple, generic ordinary classical point such that $\rho_{z_{\mathrm{nc}}^{(\sigma)},v}$ is totally indecomposable for all $v \in \Sigma_p$, then there exists a non-zero map*

$$E_{\mathrm{an}}(z_{\mathrm{nc}}^{(\sigma)}) \to \widehat{H}^0(K^p)_L^\lambda$$

Proof. By Corollary II.3.11 we know that $z_{\mathrm{nc}}^{(\sigma)}$ is not bad, hence we get the desired map by Theorem II.3.12. □

Let us also define for each $v \in \Sigma_p$ two characters

$$\chi_{\mathrm{comp},v}^{(12)} = \psi_{3,v}\varepsilon^2 \otimes \psi_{1,v}\varepsilon \otimes \psi_{2,v}$$

$$\chi_{\mathrm{comp},v}^{(23)} = \psi_{2,v}\varepsilon^2 \otimes \psi_{3,v}\varepsilon \otimes \psi_{1,v}$$

and then, for any $\sigma = (\sigma_v)_{v\in\Sigma_p} \in \{(12),(23)\}_{v\in\Sigma_p}$ a character $\chi_{\mathrm{comp}}^{(\sigma)} = (\chi_{\mathrm{comp},v}^{(\sigma_v)})_{v\in\Sigma_p}$. Here the subscript "comp" stands for "companion", which comes from the proof of the following result.

Corollary II.3.20. *If $z_{\mathrm{nc}}^{(\sigma)}$ is a simple, generic ordinary classical point such that $\rho_{z_{\mathrm{nc}}^{(\sigma)},v}$ is totally indecomposable for all $v \in \Sigma_p$ then*

$$\mathrm{Hom}_{G_{\Sigma_p}}(\mathrm{Ind}_{B_{\Sigma_p}}^{G_{\Sigma_p}}(\chi_{\mathrm{comp}}^{(\sigma)})^{C^0}, \widehat{H}^0(K^p)_L^\lambda) = 0$$

Proof. We remark that $\chi_{\mathrm{comp}}^{(\sigma)}$ is a refinement of a point $x^{(\sigma)} = (\chi_{\mathrm{comp}}^{(\sigma)}, \lambda) \in \mathrm{Spec}\,\mathcal{A}$ which is strongly linked to $z_{\mathrm{nc}}^{(\sigma)}$. As $z_{\mathrm{nc}}^{(\sigma)}$ is not bad, $x^{(\sigma)}$ does not appear on the eigenvariety and thus is also not bad. We conclude by the adjunction formula applied to $x^{(\sigma)}$. □

II.3.4 Unitary completions of locally analytic representations

We now relate our work to certain representations which arise in the work of Breuil-Herzig. We will first describe a certain locally analytic representation of $\mathrm{GL}_3(\mathbf{Q}_p)$ associated to a totally indecomposable generic ordinary representation ρ_p by Breuil and Herzig. Then we show that it arises as a $\mathrm{GL}_3(\mathbf{Q}_p)$-subrepresentation of a completed cohomology space.

For each simple root r for $\mathrm{GL}_3(\mathbf{Q}_p)$, positive with respect to the upper triangular Borel $B(\mathbf{Q}_p)$, we get a Levi component G_r of a parabolic P_r such that r is the unique simple positive root appearing in $B(\mathbf{Q}_p) \cap G_r$. For example,

$$B(\mathbf{Q}_p) \cap G_{e_1-e_2} = \begin{pmatrix} * & * & \\ & * & \\ & & * \end{pmatrix} \subset G_{e_1-e_2} = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix} \subset P_{e_1-e_2} = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}.$$

Suppose that ρ_p is a generic ordinary, crystalline representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights $h_1 < h_2 < h_3$ and write

$$\rho_p \sim \begin{pmatrix} \psi_1 & * & * \\ & \psi_2 & * \\ & & \psi_3 \end{pmatrix}$$

with ψ_i having weight h_i.

We begin by describing the representation $\Pi(\rho_p)^{\mathrm{ord}}$ constructed by Breuil and Herzig in the totally indecomposable case. By [BH, Proposition B.2], each simple root $r = e_i - e_j$ gives rise to a unique non-split extension

$$0 \to \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_2\varepsilon \otimes \psi_1)^{C^0} \to E_{e_1-e_2} \to \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_1\varepsilon \otimes \psi_2)^{C^0} \to 0, \tag{II.12}$$

$$0 \to \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_3\varepsilon^2 \otimes \psi_2\varepsilon)^{C^0} \to E_{e_2-e_3} \to \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_2\varepsilon^2 \otimes \psi_3\varepsilon)^{C^0} \to 0. \tag{II.13}$$

First define

$$\Pi_{\mathrm{nc}}(\rho_p) = \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_3\varepsilon^2 \otimes \psi_2\varepsilon \otimes \psi_1)^{C^0}.$$

and then define

$$\Pi(\rho_p)_{e_1-e_2} = \mathrm{Ind}_{P_{e_1-e_2}}^{\mathrm{GL}_3(\mathbf{Q}_p)}(E_{e_1-e_2} \otimes \psi_3\varepsilon^2)^{C^0}, \text{ and} \tag{II.14}$$
$$\Pi(\rho_p)_{e_2-e_3} = \mathrm{Ind}_{P_{e_2-e_3}}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_1 \otimes E_{e_2-e_3})^{C^0}.$$

By (II.12) and (II.13) we have two non-split extensions of $\mathrm{GL}_3(\mathbf{Q}_p)$-representations

$$0 \to \Pi_{\mathrm{nc}}(\rho_p) \to \Pi(\rho_p)_{e_1-e_2} \to \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_3\varepsilon^2 \otimes \psi_1\varepsilon \otimes \psi_2)^{C^0} \to 0, \text{ and} \tag{II.15}$$
$$0 \to \Pi_{\mathrm{nc}}(\rho_p) \to \Pi(\rho_p)_{e_2-e_3} \to \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_2\varepsilon^2 \otimes \psi_3\varepsilon \otimes \psi_1)^{C^0} \to 0.$$

Finally we define the amalgamated sum $\Pi(\rho_p)^{\mathrm{ord}} = \Pi(\rho_p)_{e_1-e_2} \oplus_{\Pi_{\mathrm{nc}}(\rho_p)} \Pi(\rho_p)_{e_2-e_3}$. This is the representation $\Pi(\rho_p)^{\mathrm{ord}}$ of [BH, §3] up to normalizations (see the remarks following Conjecture II.3.23 below).

In order to proceed we have to obtain another description of $\Pi(\rho_p)^{\mathrm{ord}}$. We start with a simple general lemma.

Lemma II.3.21. *Let $P \subset G$ be a parabolic subgroup of a p-adic reductive group G. Let $(V_{\mathrm{an}}, \pi_{\mathrm{an}})$ be a locally analytic L-representation of P and let (V, π) be its universal unitary completion, which we assume to be non-zero. Suppose also that we have a P-equivariant injection $V_{\mathrm{an}} \hookrightarrow V$ (which is neccessarily dense). Then we have a dense P-equivariant injection*

$$\mathrm{Ind}_P^G(V_{\mathrm{an}})^{\mathrm{an}} \hookrightarrow \mathrm{Ind}_P^G(V)^{C^0}$$

Proof. Let $f \in \mathrm{Ind}_P^G(V)^{C^0}$. We want to approximate it by functions in $\mathrm{Ind}_P^G(V_{\mathrm{an}})^{\mathrm{an}}$. So let us take any $\epsilon > 0$. Remark that we have $P\backslash G \simeq P_0\backslash G_0$ where $P_0 \subset G_0$ are respectively maximal compact subgroups of P, G. Let $\bar{U}$ be the opposite unipotent and let H be any sufficiently small compact open subgroup of G. We have a decomposition $G_0 = \coprod_{g_0} P_0 \times (H \cap \bar{U}) \times g_0$ into finitely many pieces. Let us choose an element $v_0 \in V_{\mathrm{an}}$ such that for any $h \in H \cap \bar{U}$ we have $|f(hg_0) - v_0| < \epsilon$. This is possible by density of V_{an} in V and choosing H sufficiently small (H will depend on f). Then we define $f_{\mathrm{an}}(phg_0) = \pi_{\mathrm{an}}(p)v_0$ for $p \in P$ and $h \in H \cap \bar{U}$. We make a similar definition for other pieces in the decomposition hence obtaining a function $f_{\mathrm{an}} : G \to V_{\mathrm{an}}$. Observe that because π is unitary, π is bounded on P and hence there exists $\sup_{p\in P} |\pi(p)|$ which is finite. We have

$$|f(phg_0) - f_{\mathrm{an}}(phg_0)| = |\pi(p)| \cdot |f(hg_0) - v_0| < \epsilon \cdot \sup_{p\in P} |\pi(p)|$$

which allow us to conclude as ϵ was arbitrary. $\square$

We return to the extension classes of (II.12) and (II.13). By [BE10, Théorème 2.2.2] (see also [Eme06b, §6.3] but note that the Hodge-Tate weights of the reference(s) are the negatives of ours) says each extension may be written explicitly as a universal unitary completion

$$\begin{aligned} E_{e_1-e_2} &\simeq \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_1\varepsilon z^{h_1-h_2} \otimes \psi_2 z^{h_2-h_1})^{\mathrm{an},\wedge} \text{ and} \\ E_{e_2-e_3} &\simeq \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_2\varepsilon^2 z^{h_2-h_3} \otimes \psi_3\varepsilon z^{h_3-h_2})^{\mathrm{an},\wedge}. \end{aligned} \tag{II.16}$$

Thus we obtain the description of the extensions (II.15).

Proposition II.3.22. *We have*

$$\begin{aligned} \Pi(\rho_p)_{e_1-e_2} &\simeq \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_3\varepsilon^2 \otimes \psi_1\varepsilon z^{h_1-h_2} \otimes \psi_2 z^{h_2-h_1})^{\mathrm{an},\wedge} \textit{ and} \\ \Pi(\rho_p)_{e_2-e_3} &\simeq \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_2\varepsilon^2 z^{h_2-h_3} \otimes \psi_3\varepsilon z^{h_3-h_2} \otimes \psi_1)^{\mathrm{an},\wedge}. \end{aligned}$$

Proof. The proofs are symmetric so we only cover the first isomorphism. We write

$$\begin{aligned} I &= \mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_3(\mathbf{Q}_p)}(\psi_3\varepsilon^2 \otimes \psi_1\varepsilon z^{h_1-h_2} \otimes \psi_2 z^{h_2-h_1})^{\mathrm{an}} \\ &\simeq \mathrm{Ind}_{P_{e_1-e_2}}^{\mathrm{GL}_3(\mathbf{Q}_p)}\left(\psi_3\varepsilon^2 \otimes \mathrm{Ind}_{\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\psi_1\varepsilon z^{h_1-h_2} \otimes \psi_2 z^{h_2-h_1})^{\mathrm{an}}\right)^{\mathrm{an}}, \end{aligned}$$

the tensor representation in the second line being seen as a representation of $P_{e_1-e_2}$ via inflation $P_{e_1-e_2} \to G_{e_1-e_2}$. By (II.14), (II.16) and Lemma II.3.21 we see that there is a dense inclusion $I \hookrightarrow \Pi(\rho_p)_{e_1-e_2}$. This induces a non-zero map $I^\wedge \to \Pi(\rho_p)_{e_1-e_2}$. Since both representations are admissible, to show that it is an isomorphism we need only show it is injective.

Since I is dense in $\Pi(\rho_p)_{e_1-e_2}$ we can pull back the unit ball and obtain an open, separated, $\mathrm{GL}_3(\mathbf{Q}_p)$-stable lattice $\Lambda_1 \subset I$ such that $\Pi(\rho_p)_{e_1-e_2} \simeq \varprojlim_n \Lambda_1/p^n\Lambda_1 \otimes \mathbf{Q}_p$. Since $I^\wedge \neq 0$ we also obtain a minimal $\mathrm{GL}_3(\mathbf{Q}_p)$-stable lattice Λ_0 such that $I^\wedge \simeq \varprojlim_n \Lambda_0/p^n\Lambda_0 \otimes \mathbf{Q}_p$. By minimiality $\Lambda_0 \subset \Lambda_1$; since each is separated we have

$$I^\wedge \simeq \varprojlim_n \Lambda_0/p^n\Lambda_0 \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \hookrightarrow \varprojlim_n \Lambda_1/p^n\Lambda_1 \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \simeq \Pi(\rho_p)_{e_1-e_2}.$$

This completes the proof. □

II.3.5 On a conjecture of Breuil and Herzig

We now go back to our global setting. We suppose that $z = (\chi_z, \lambda_z) \in X(L)$ is a generic ordinary point such that $\rho_{z,v}$ is totally indecomposable for each place $v \in \Sigma_p$. We define a representation

$$\Pi(\rho_{z,p})^{\mathrm{ord}} := \widehat{\bigotimes}_{v\in\Sigma_p} \Pi(\rho_{z,v})^{\mathrm{ord}}$$

Note that this only depends on ρ_z, not on z itself (i.e. on λ_z, not χ_z). We use $\widehat{H}^0(K^p)_L^{\lambda_z,\mathrm{ord}}$ to denote the ordinary part of the G_{Σ_p}-representation $\widehat{H}^0(K^p)_L^{\lambda_z}$. Let us now recall a special case of the conjecture of Breuil and Herzig which describes this subspace.

Conjecture II.3.23 ([BH, Conjecture 4.2.2])**.** *Suppose that $z \in X(L)$ is a generic ordinary point and $\rho_{z,v}$ is totally indecomposable at each place $v \in \Sigma_p$. Then there exists an integer $d \geq 1$ and a G_{Σ_p}-equivariant isomorphism*

$$\left(\Pi(\rho_{z,p})^{\mathrm{ord}}\right)^{\oplus d} \simeq \widehat{H}^0(K^p)_L^{\lambda_z,\mathrm{ord}}.$$

Note that there are the following differences between our normalizations of those of [BH]. First, we have used the upper triangular Borels throughout. Second, the representation defined in [BH, §3], temporarily denoted by $\Pi(\rho_p)^{\mathrm{ord,BH}}$, differs from ours by the equation

$$\Pi(\rho_p)^{\mathrm{ord,BH}} \otimes \varepsilon^2 \circ \det = \Pi(\rho_p)^{\mathrm{ord}}.$$

Thus the conjecture we have written is the same as [BH, Conjecture 4.2.2]. We further remark that the integer d should only depend on K^p and λ_z (e.g. because the right hand side depends only on those things).

We cannot completely prove the conjecture of Breuil-Herzig but we do give the following strong evidence.

Theorem II.3.24. *Suppose that $z \in X_{\mathrm{cl}}(L)$ is a generic ordinary classical point and $\rho_{z,v}$ is totally indecomposable at each place $v \in \Sigma_p$. Then there exists an integer $d \geq 1$, depending only on K^p and λ_z, and a G-equivariant closed embedding*

$$\left(\Pi(\rho_{z,p})^{\mathrm{ord}}\right)^{\oplus d} \hookrightarrow \widehat{H}^0(K^p)_L^{\lambda_z,\mathrm{ord}}.$$

In order to prepare the proof we need an intermediate result, due to Chenevier, on multiplicities in spaces of p-adic automorphic forms. Recall from Definition II.3.3 that an element $\sigma \in (S_3)_{\Sigma_p}$ is said to be simple if $\sigma_v \in \{1, (12), (23)\}$ for all $v \in \Sigma_p$. We recall also that by Proposition II.3.22 $E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)})$ is an extension of $E_{C^0}(\chi_{\mathrm{comp}}^{(\sigma)})$ by $E_{C^0}(z_{\mathrm{nc}}^{(1)})$.

Proposition II.3.25. *Suppose that σ is simple. Then the natural map*

$$\mathrm{Hom}(E_{C^0}(z_{\mathrm{nc}}^{(1)}), \widehat{H}_L^{0,\lambda_z}) \to \mathrm{Hom}(E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)}), \widehat{H}_L^{0,\lambda_z})$$

is an isomorphism.

Proof. We first note that we have a G_{Σ_p}-equivariant short exact sequence

$$0 \to E_{C^0}(z_{\mathrm{nc}}^{(1)}) \to E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)}) \to E_{C^0}(\chi_{\mathrm{comp}}^{(\sigma)}) \to 0$$

and thus an exact sequence

$$0 \to \mathrm{Hom}(E_{C^0}(\chi_{\mathrm{comp}}^{(\sigma)}), \widehat{H}_L^{0,\lambda_z}) \to \mathrm{Hom}(E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)}), \widehat{H}_L^{0,\lambda_z}) \to \mathrm{Hom}(E_{C^0}(z_{\mathrm{nc}}^{(1)}), \widehat{H}_L^{0,\lambda_z}).$$

By Corollary II.3.20 the first space is actually zero. Thus it suffices to check that the dimensions of the two spaces in the proposition are the same. Since $z_{\mathrm{nc}}^{(\sigma)}$ is not bad for each simple σ we can apply Theorem II.3.12. So we aim to show that

$$\dim J_{B_{\Sigma_p}}^{\chi^{(13)}_{z_{\mathrm{nc}}^{(\sigma)}}\delta_{B_{\Sigma_p}}}(\widehat{H}^0(K^p)_{L,\mathrm{an}}^{\lambda_z}) = \dim J_{B_{\Sigma_p}}^{\chi^{(13)}_{z_{\mathrm{nc}}^{(1)}}\delta_{B_{\Sigma_p}}}(\widehat{H}^0(K^p)_{L,\mathrm{an}}^{\lambda_z}) \tag{II.17}$$

is independent of simple $\sigma \in (S_3)_{\Sigma_p}$.

Here is where we apply the results of Chenevier. By [Bre, Proposition 7.2] the dimensions (II.17) are the same as the dimensions of spaces of p-adic automorphic forms defined in terms of locally analytic Iwahori principal series [Che11, §4]. Since σ is simple, the points $z_{\mathrm{nc}}^{(\sigma)}$ are all non-critical by Proposition II.3.4. Thus by [Che11, Proposition 4.2], every p-adic automorphic form with system of eigenvalues λ_z and refinement $\chi_{z_{\mathrm{nc}}^{(\sigma)}}$ is actually classical: there must be a classical automorphic representation π as in (II.3) on which $\mathcal{H}^{\mathrm{nr}}(K^p)$ acts via $\lambda_\pi = \lambda_z$ and $\pi_{\Sigma_p} \subset \mathrm{Ind}_{B_{\Sigma_p}}^{G_{\Sigma_p}} \theta_{z_{\mathrm{nc}}^{(\sigma)}}$. Since z is generic ordinary, the

Iwahori invariants $\pi_{\Sigma_p}^{\mathrm{Iw}}$ contain each $\theta_{z_{\mathrm{nc}}^{(\sigma)}}$ with multiplicity one. Thus in the notation of (II.3) we get

$$\dim J_{B_{\Sigma_p}}^{\chi_{z_{\mathrm{nc}}^{(\sigma)}}^{(13)}\delta_{B_{\Sigma_p}}}(\widehat{H}^0(K^p)_{L,\mathrm{an}}^{\lambda_z}) = \sum_{\lambda_\pi=\lambda_z} m(\pi)\dim(\pi_f^{K^p})$$

is independent of σ. $\square$

We are now in a position to prove Theorem II.3.24.

Proof of Theorem II.3.24. Observe that if σ is simple then $E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)})$ contains $E_{C^0}(z_{\mathrm{nc}}^{(1)})$. Thus we can denote by $\oplus_\sigma E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)})$ the amalgamated sum of all these representations over $E_{C^0}(z_{\mathrm{nc}}^{(1)})$. We have two observations. First, for any $\mathrm{GL}_3(\mathbf{Q}_p)$-representation M,

$$\mathrm{Hom}(\oplus_\sigma E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)}), M) = \times_{\mathrm{Hom}(E_{C^0}(z_{\mathrm{nc}}^{(1)}),M)} \mathrm{Hom}(E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)}), M). \tag{II.18}$$

Second, we see by inspection that

$$\oplus_\sigma E_{\mathrm{an}}^{\wedge}(z_{\mathrm{nc}}^{(\sigma)}) \simeq \Pi(\rho_{z,p})^{\mathrm{ord}}. \tag{II.19}$$

To prove the theorem now, we take $M = \widehat{H}^0(K^p)_{L,\mathrm{an}}^{\lambda_z}$. We plug (II.19) into the left hand side of (II.18) and we apply Proposition II.3.25 to compute the right hand side. Thus we get

$$\mathrm{Hom}(\Pi(\rho_{z,p})^{\mathrm{ord}}, \widehat{H}^0(K^p)_L^{\lambda_z}) = \mathrm{Hom}(E_{C^0}(z_{\mathrm{nc}}^{(1)}), \widehat{H}^0(K^p)_L^{\lambda_z}) \neq 0.$$

Now notice that $E_{C^0}(z_{\mathrm{nc}}^{(1)})$ is topologically irreducible and so any map $E_{C^0}(z_{\mathrm{nc}}^{(1)}) \to \widehat{H}^0(K^p)_L^{\lambda_z}$ is necessarily a closed embedding. Since every map $\Pi(\rho_{z,p})^{\mathrm{ord}} \to \widehat{H}^0(K^p)_L^{\lambda_z}$ is uniquely determined by its restriction to $E_{C^0}(z_{\mathrm{nc}}^{(1)})$ we deduce that every map $\Pi(\rho_{z,p})^{\mathrm{ord}} \to \widehat{H}^0(K^p)_L^{\lambda_z}$ is a closed embedding as well. $\square$

Chapter III

On mod p non-abelian Lubin-Tate theory

III.1 Introduction

By the recent work of Emerton (see [Eme11a]) we know that the p-adic completed (resp. mod p) cohomology of the tower of modular curves realizes the p-adic (resp. mod p) Local Langlands correspondence. In this Chapter we will obtain an analogous but weaker result for the mod p cohomology of the Lubin-Tate tower over $\mathbb{Q}_p$. In fact, we will analyse both the cohomology with compact support and the cohomology without support of the Lubin-Tate tower. Here are the two main results which we prove:

(1) In the first cohomology group $H^1_{LT,\bar{\mathbb{F}}_p}$ of the Lubin-Tate tower appears the mod p local Langlands correspondence and the naive mod p Jacquet-Langlands correspondence, meaning that there is an injection of representations

$$\pi \otimes \bar{\rho} \hookrightarrow H^1_{LT,\bar{\mathbb{F}}_p}$$

and $\sigma \otimes \pi \otimes \bar{\rho}$ appears as a subquotient in $H^1_{LT,\bar{\mathbb{F}}_p}$, where π is a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\bar{\rho}$ is its associated local mod p Galois representation and σ is the naive mod p Jacquet-Langlands correspondence (for details, see Section 8).

(2) The first cohomology group $H^1_{LT,c,\bar{\mathbb{F}}_p}$ with compact support of the Lubin-Tate tower does not contain any supersingular representations. This suprising result shows that the mod p situation is much different from its mod l analogue. It also permits us to show that the mod p local Langlands correspondence appears in H^1 of the ordinary locus - again a fact which is different from the l-adic setting for supercuspidal representations.

Before sketching how we obtain the above results, let us outline the first main difference with the non-abelian Lubin-Tate theory in the l-adic case. When $l \neq p$ the comparison between the Lubin-Tate tower and the modular curve tower is made via vanishing cycles. For that, we need to know that the stalks of vanishing cycles gives the cohomology of the Lubin-Tate tower, or in other words we need an analogue of the theorem proved by Berkovich in [Ber96]. But when $l = p$, the statement does not hold anymore (see Remark 3.8.(iv) in [Ber96]) and hence we cannot imitate directly the arguments from the l-adic theory.

To circumvent this difficulty, we work from the beginning at the rigid-analytic level and consider embeddings from the ordinary and the supersingular tubes into modular curves. This gives two long exact sequences of cohomology, depending on whether we take compact support or a support in the ordinary locus and we start our analysis by resuming facts about the geometry of modular curves. We recall a decomposition of the ordinary locus, which proves that its cohomology is induced from some proper parabolic subgroup of GL_2. We use this fact several times in order to have vanishing of the cohomology of ordinary locus after localising at a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$. We then recall standard facts about admissible representations and review the functor of localisation at π which comes out of the work of Paskunas.

We then turn to the analysis of the supersingular locus. In this context, naturally appears a quaternion algebra $D^\times/\mathbb{Q}$ which is ramified exactly at p and ∞. We define the local fundamental representation of Deligne in our setting (which appeared for the first time in the letter of Deligne [Del73]) and we show a decomposition of the cohomology of supersingular locus. At this point we will be able to show that H^1 of the tower of modular curves injects into H^1 of the Lubin-Tate tower hence proving part of (1).

Having established this result, we start analysing mod p representations of the p-adic quaternion algebra and define a candidate for the mod p Jacquet-Langlands correspondence $\sigma_{\mathfrak{m}}$ which we later show to appear in the cohomology. It will a priori depend on a global input, namely a maximal ideal $\mathfrak{m}$ of a Hecke algebra corresponding to some modular mod p Galois representation $\bar{\rho}$, but we conjecture that it is independent of $\mathfrak{m}$. This is reasonable as it would follow from the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture. After further analysis of $\sigma_{\mathfrak{m}}$ we are able to finish the proof of (1).

Using similar techniques, we start analysing the cohomology with compact support of the Lubin-Tate tower. By using the Hochschild-Serre spectral sequence, we are able to reduce (2) to the question of whether supersingular Hecke modules of the mod p Hecke algebra at the pro-p Iwahori level appear in the H^1_c of the Lubin-Tate tower at the pro-p Iwahori level. We solve this question by explicitly computing some cohomology groups.

While proving the above theorems, we will also prove that the first cohomology group of the Lubin-Tate tower and the first cohomology group of the ordinary locus are non-admissible smooth representations. In particular, they are much harder to describe than their mod l analogues. Moreover our model for the mod p Jacquet-Langlands correspondence $\sigma_{\mathfrak{m}}$ (actually we propose three candidates for the correspondence which we discuss in Section 7.4) is a representation of $D^\times(\mathbb{Q}_p)$ of infinite length. This indicates that already for $D^\times(\mathbb{Q}_p)$ the mod p Langlands correspondence is complicated (as in the work of [BP12], representations in question are not of finite length). On the other hand, the case of $D^\times(\mathbb{Q}_p)$ is much simpler than that of $\mathrm{GL}_2(F)$ for F a finite extension of $\mathbb{Q}_p$, and hence we might be able to describe $\sigma_{\mathfrak{m}}$ precisely. Natural question in this discussion is the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture (see Conjecture 4.7 in [BDJ10]) which says that we have an isomorphism

$$\mathbf{F}[\mathfrak{m}] \simeq \sigma_{\mathfrak{m}} \otimes \pi^p(\bar{\rho})$$

where $\mathbf{F}$ denotes locally constant functions on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)$ with values in $\bar{\mathbb{F}}_p$ and $\pi^p(\bar{\rho})$ is a representation of $\mathrm{GL}_2(\mathbb{A}_f^p)$ associated to $\bar{\rho}$ by the modified Langlands correspondence.

At the end, we remark that our arguments work well in the $l \neq p$ setting and omit the use of vanishing cycles. As some of our arguments are geometric, we can also get similar results in the p-adic setting. We hope to return to this issue in our future work. Also, the geometry of modular curves is very similar to the geometry of Shimura curves and hence we hope that some of the reasonings in this Chapter will give an insight into the nature of the mod p local Langlands correspondence of $\mathrm{GL}_2(F)$ for F a finite extension of $\mathbb{Q}_p$.

III.2 Geometry of modular curves

Let $X(Np^m)$ be the Katz-Mazur compactification of the modular curve associated to the moduli problem $(\Gamma(p^n), \Gamma_1(N))$ (see [KM85]) which is defined over $\mathbb{Z}[1/N, \zeta_{p^n}]$, where ζ_{p^n} is a primitive p^n-th root of unity, that is $X(Np^m)$ parametrizes (up to isomorphism) triples (E, ϕ, α), where E is an elliptic curve, $\phi : (\mathbb{Z}/p^n\mathbb{Z})^2 \to E[p^n]$ is a Drinfeld level structure and $\alpha : \mathbb{Z}/N\mathbb{Z} \to E[N]$ is a $\Gamma_1(N)$-structure. We consider the integral model of it defined over $\mathbb{Z}_p^{nr}[\zeta_{p^n}]$, where $\mathbb{Z}_p^{nr}$ is the maximal unramified extension of $\mathbb{Z}_p$, which we will denote also by $X(Np^m)$. Let us denote by $X(Np^m)^{an}$ the analytification of $X(Np^m)$ which is a Berkovich space.

Recall that there exists a reduction map $\pi : X(Np^m)^{an} \to \overline{X(Np^m)}$, where $\overline{X(Np^m)}$ is the special fiber of $X(Np^m)$. We define $\overline{X(Np^m)_{ss}}$ (resp. $\overline{X(Np^m)_{ord}}$) to be the set of supersingular (resp. ordinary) points in $\overline{X(Np^m)}$. Define the tubes $X(Np^m)_{ss} = \pi^{-1}(\overline{X(Np^m)_{ss}})$ and $X(Np^m)_{ord} = \pi^{-1}(\overline{X(Np^m)_{ord}})$ inside $X(Np^m)^{an}$ of supersingular and ordinary points respectively.

III.2.1 Two exact sequences

We know that $X(Np^m)_{ss}$ is an open analytic subspace of $X(Np^m)^{an}$ isomorphic to some copies of Lubin-Tate spaces, where the number of copies is equal to the number of points in $\overline{X(Np^m)_{ss}}$ (see section 3 of [Buz03]). We have a decomposition $X(Np^m)^{an} = X(Np^m)_{ss} \cup X(Np^m)_{ord}$ and we put $j : X(Np^m)_{ss} \hookrightarrow X(Np^m)^{an}$ and $i : X(Np^m)_{ord} \to X(Np^m)^{an}$. We remark that j is an open immersion of Berkovich spaces. Let F be a sheaf in the étale topos of $X(Np^m)^{an}$. By the general formalism of six operations (due in this setting to Berkovich, see [Ber93]) we have a short exact sequence:

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

which gives a long exact sequence of étale cohomology groups:

$$... \to H^0(X(Np^m)_{ord}, i^*F) \to H^1_c(X(Np^m)_{ss}, F) \to$$

$$\to H^1(X(Np^m)^{an}, F) \to H^1(X(Np^m)_{ord}, i^*F) \to ...$$

On the other hand, we can consider a similar exact sequence for the cohomology without compact support, but instead considering support on the ordinary locus. This results in the long exact sequence

$$... \to H^1_{X_{ord}}(X(Np^m)^{an}, F) \to H^1(X(Np^m)^{an}, F) \to$$

$$\to H^1(X(Np^m)_{ss}, i^*F) \to H^2_{X_{ord}}(X(Np^m)^{an}, F) \to ...$$

where we have denoted by $H^1_{X_{ord}}(X(Np^m)^{an}, F)$ the étale cohomology of $X(Np^m)^{an}$ with support on $X(Np^m)_{ord}$. Because of the vanishing of the cohomology with compact support of the supersingular locus localised at π (see the explanation in the next sections), this exact sequence will be of more importance to us later on. We will analyse those two exact sequences simultanously.

III.2.2 Decomposition of ordinary locus

Let us recall that we have the Weil pairing on elliptic curves

$$e_{p^m} : E[p^m] \times E[p^m] \to \mu_{p^m}$$

Denote by ζ_{p^m} a p^m-th primitive root of unity. For $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$ we define a substack $X(Np^m)_a$ of $X(Np^m)$ as the moduli problem which classifies elliptic curves E with level structures (ϕ, α) such that $e_{p^m}(\phi(\begin{smallmatrix}1\\0\end{smallmatrix}), \phi(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^a$. This moduli problem is representable by a scheme over $\mathbb{Z}_p^{nr}[\zeta_{p^m}]$ (see chapter 9 of [KM85]). Moreover the coproduct $\coprod_a X(Np^m)_a$ is a regular model of $X(Np^m)$ over $\mathbb{Z}_p^{nr}[\zeta_{p^m}]$.

Let us denote by $\overline{X(Np^m)}_{a,ord}$ the ordinary locus of the reduction of $X(Np^m)_a$. We recall (see for example chapter 13 of [KM85]) that the set of irreducible components of $\overline{X(Np^m)}_{ord}$ consists of smooth curves $C_{a,b}(Np^m)$ defined on points by:

$$C_{a,b}(Np^m)(S) = \{(E, \phi, \alpha) \in \overline{X(Np^m)}_{a,ord}(S) \mid e_{p^m}(\phi(\begin{smallmatrix}1\\0\end{smallmatrix}), \phi(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^a \text{ and } \operatorname{Ker}\phi = b\}$$

where $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$ and $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ is regarded as a line in $\mathbb{Z}/p^m\mathbb{Z} \times \mathbb{Z}/p^m\mathbb{Z}$. We observe that $\zeta_{p^m}^a = 1$ modulo p.

We are interested in lifting $C_{a,b}(Np^m)$ to characteristic zero and so we put

$$\mathbb{X}_{a,b}(Np^m) = \pi^{-1}(C_{a,b}(Np^m))$$

Hence $\{\mathbb{X}_{a,b}(Np^m)\}$ for $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$, $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ form a decomposition of the ordinary locus $X(Np^m)_{ord}$ because different $C_{a,b}(Np^m)$ intersect only at supersingular points. The spaces $\mathbb{X}_{a,b}(Np^m)$ may be regarded as analytifications of Igusa curves. For a detailed discussion, see [Col05]. We do not determine here whether $\mathbb{X}_{a,b}(Np^m)$ are precisely the connected components of $X(Np^m)_{ord}$. We remark also that one can give a moduli description of each $\mathbb{X}_{a,b}(Np^m)$.

There is an action of $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ on $X(Np^m)^{an}$ which is given on points by:

$$(E, \phi, \alpha) \cdot g = (E, \phi \circ g, \alpha)$$

for $g \in \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$. Observe that if $e_{p^m}(\phi(\begin{smallmatrix}1\\0\end{smallmatrix}), \phi(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^a$, then $e_{p^m}((\phi \circ g)(\begin{smallmatrix}1\\0\end{smallmatrix}), (\phi \circ g)(\begin{smallmatrix}0\\1\end{smallmatrix})) = \zeta_{p^m}^{a\cdot \det g}$ for $g \in \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ and so g induces an isomorphism between $\mathbb{X}_{a,b}(Np^m)$ and $\mathbb{X}_{a\cdot\det g, g^{-1}\cdot b}(Np^m)$.

For $b \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$ there is a Borel subgroup $B_m(b)$ in $\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$ which fixes b and hence the Borel subgroup $B_m(b)^+ = B_m(b) \cap \mathrm{SL}_2(\mathbb{Z}/p^m\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{Z}/p^m\mathbb{Z})$ stabilises $\mathbb{X}_{a,b}(Np^m)$.

Let $b = \infty = (\begin{smallmatrix}1\\0\end{smallmatrix}) \in \mathbb{P}^1(\mathbb{Z}/p^m\mathbb{Z})$. By the above considerations we have

$$H^i(X(Np^m)_{ord}, i^*F) = \bigoplus_{a,b} H^i(\mathbb{X}_{a,b}(Np^m), (i^*F)_{|\mathbb{X}_{a,b}(Np^m)}) \simeq$$

$$\simeq \operatorname{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})} \left(\bigoplus_a H^i(\mathbb{X}_{a,\infty}(Np^m), (i^*F)_{|\mathbb{X}_{a,\infty}})) \right)$$

and also

$$H^i_{X_{ord}}(X(Np^m)^{an}, F) = \bigoplus_{a,b} H^i_{\mathbb{X}_{a,b}(Np^m)}(X(Np^m)^{an}, F) \simeq$$

$$\simeq \operatorname{Ind}_{B_m(\infty)}^{\mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z})} \left(\bigoplus_a H^i_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, F) \right)$$

Those results will be extremely useful for us later on, when we introduce the localisation at a given supersingular representation.

III.2.3 Supersingular points

Let us denote by D the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at p and at ∞. We recall the description of supersingular points $\overline{X(Np^m)_{ss}}$ which has appeared in [Del73] and then was explained in [Car86], sections 9.4 and 10.4. Fix a supersingular elliptic curve $\overline{E}$ over $\mathbb{F}_p$ and a two-dimensional vector space V over $\mathbb{Q}_p$. Let $\det(\overline{E}) = \mathbb{Z}$ be the determinant of $\overline{E}$. Denote by $W(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ the Weil group of $\mathbb{F}_p$ and put

$$\Delta = \left(W(\bar{\mathbb{F}}_p/\mathbb{F}_p) \times \mathrm{Isom}(\det(\overline{E}) \otimes_{\mathbb{Z}} \mathbb{Q}_p(1), \wedge^2 V)\right)/\sim$$

where $\sim$ is defined by $(\sigma, \beta) \sim (\sigma\,\mathrm{Frob}^k, p^{-k}\beta)$ for $k \in \mathbb{Z}$, where $\mathrm{Frob} : x \mapsto x^p$ is a Frobenius map. We define K_m to be the kernel of $D^\times(\mathbb{Z}_p) \to D^\times(\mathbb{Z}/p^m\mathbb{Z})$ and we let $K(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) \mid a \equiv 1 \mod N \text{ and } c \equiv 0 \mod N\}$, viewed as a subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$ by the diagonal embedding. Then:

$$\overline{X(Np^m)_{ss}} = \Delta/K_m \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

Every $\delta \in \Delta/K_m$ furnishes a supersingular elliptic curve $E(\delta)$, so that for every $\delta \in \Delta$ we can consider the Lubin-Tate tower $LT_\delta = \varprojlim_m LT_\delta(p^m)$, which is the generic fiber of the deformation space of the formal group attached to $E(\delta)$ and where $LT_\delta(p^m)$ denotes the generic fiber of the deformation space of formal groups with p^m-level structure (see [Dat12] for details on the Lubin-Tate tower). Let us denote by $\mathbb{E}(\delta)$ the universal formal group deforming the formal group attached to $E(\delta)$ and let $\mathbb{E}(\Delta) = \coprod_{\delta\in\Delta} \mathbb{E}(\delta)$. By 9.4 of [Car86], the universal formal group over $\varprojlim_{N,p^m} \overline{X(Np^m)_{ss}}$ is isomorphic to $\mathbb{E}(\Delta) \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)$ and hence we conclude that

$$\varprojlim_{Np^m} X(Np^m)_{ss} \simeq LT_\Delta \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)$$

where $LT_\Delta = \coprod_{\delta\in\Delta} LT_\delta$. We also get a description at a finite level

$$X(Np^m)_{ss} \simeq LT_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

where $LT_{\Delta/K_m} = \coprod_{\delta\in\Delta/K_m} LT_\delta(p^m)$.

These results will allow us later on to define the local fundamental representation and analyze the action of the quaternion algebra $D^\times$.

III.3 Admissibility of cohomology groups

In this section we will recall the notion of admissibility in the context of mod p representations. It will be crucial in our study of cohomology.

III.3.1 General facts and definitions

We start with general facts about admissible representations. In our definitions, we will follow [Eme10a]. Let k be a field of characteristic p and let G be a connected reductive group over $\mathbb{Q}_p$.

Definition III.3.1. *Let V be a representation of G over k. A vector $v \in V$ is smooth if v is fixed by some open subgroup of G. Let V_{sm} denote the subset of smooth vectors of V. We say that a G-representation V over k is smooth if $V = V_{sm}$.*

A smooth G-representation V over k is admissible if V^H is finitely generated over k for every open compact subgroup H of G.

Proposition III.3.2. *The category of admissible k-representations is abelian.*

Proof. This category is (anti-)equivalent to the category of finitely generated augmented modules over certain completed group rings. See Proposition 2.2.13 and 2.4.11 in [Eme10a]. $\square$

Now, we will prove an analogue of Lemma 13.2.3 from [Boy99] in the $l = p$ setting. We will later apply this lemma to the cohomology of the ordinary locus to force its vanishing after localisation at a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Lemma III.3.3. *For any smooth admissible representation (π, V) of the parabolic subgroup $P \subset G$ over k, the unipotent radical U of P acts trivially on V.*

Proof. Let L be a Levi subgroup of P, so that $P = LU$. Let $v \in V$ and let $K_P = K_L K_U$ be a compact open subgroup of P such that $v \in V^{K_P}$. We choose an element z in the centre of L such that:

$$z^{-n} K_P z^n \subset ... \subset z^{-1} K_P z \subset K_P \subset z K_P z^{-1} \subset ... \subset z^n K_P z^{-n} \subset ...$$

and $\bigcup_{n \geq 0} z^n K_P z^{-n} = K_L U$. For every n and m, modules $V^{z^{-n} K_P z^n}$ and $V^{z^{-m} K_P z^m}$ are of the same length as they are isomorphic via $\pi(z^{n-m})$ and hence we have not only an isomorphism but an equality $V^{z^{-n} K_P z^n} = V^{z^{-m} K_P z^m}$. Thus for every $x \in V^{K_P}$ we have $x \in V^{K_P} = V^{z^{-n} K_P z^n} = V^{K_L U}$ which is contained in V^U. $\square$

We also record the following result of Emerton for the future use.

Lemma III.3.4. *Let $V = \mathrm{Ind}_P^G W$ be a parabolic induction. If V is a smooth admissible representation of G over k, then W is a smooth admissible representation of P over k.*

Proof. This follows from Theorem 4.4.6 in [Eme10a]. $\square$

III.3.2 Cohomology and admissibility

In [Eme06c], Emerton has introduced the completed cohomology, which plays a crucial role in the p-adic Langlands program. The most important thing for us right now is the fact that those cohomology groups for modular curves are admissible as $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. We have

Proposition III.3.5. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation*

$$\widehat{H}^1(X(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. This is Theorem 2.1.5 of [Eme06c] (see also Theorem 1.16 in [CE12]). $\square$

By formal properties of the category of admissible representations, which form a Serre subcategory of the category of smooth representations (see Proposition 2.2.13 in [Eme10a]), the above result permits us to deduce admissibility for other cohomology groups which are of interest to us. Let us remark that we can define also the cohomology of the Lubin-Tate tower with compact support:

Remark III.3.6. *A priori, cohomology with compact support is a covariant functor. But using the adjunction map*

$$\Lambda \to \pi_* \pi^* \Lambda \simeq \pi_! \pi^! \Lambda$$

where Λ *is a constant sheaf and* $\pi : X(Np^{m+1})_{ss} \to X(Np^m)_{ss}$ *is finite (hence* $\pi_* = \pi_!$*) and étale (hence* $\pi^! = \pi^*$*) by the properties of Lubin-Tate tower, we get maps* $H^i_c(X(Np^m)_{ss}, \Lambda) \to H^i_c(X(Np^{m+1})_{ss}, \Lambda)$ *compatible with* $H^i(X(Np^m)^{an}, \Lambda) \to H^i(X(Np^{m+1})_{ss}, \Lambda)$*.*

We start firstly by analysing cohomology groups which appear in the exact sequence for the cohomology with compact support. We have

Proposition III.3.7. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation*

$$\widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p) = \varinjlim_m H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. The number of connected components of $X(Np^m)_{ord}$ is finite and let $d(Np^m)$ be their number. For $s > 0$, we have

$$H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p) = (\bar{\mathbb{F}}_p)^{d(Np^m)}$$

hence $\varinjlim_m H^0(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$ is admissible. □

We deduce

Proposition III.3.8. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation*

$$\widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p) = \varinjlim_m H^1_c(X(Np^m)_{ss}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. We consider the exact sequence from 2.1:

$$... \to \widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p) \to \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)^{an}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)_{ord}, \bar{\mathbb{F}}_p) \to ...$$

and we conclude using the fact that admissible representations form a Serre subcategory of smooth representations and the propositions proved above. □

We remark that the cohomology with compact support of the Lubin-Tate tower is much easier to work with than the cohomology without the support. This is because the latter will turn out to be non-admissible.

We finish this section with the following proposition

Proposition III.3.9. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation*

$$\widehat{H}^1_{X_{ord}}(X(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1_{X_{ord}}(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

is admissible.

Proof. This follows from the exact sequence (we use the notations from the previous section)

$$H^0(X(Np^m)_{ss}, \bar{\mathbb{F}}_p) \to H^1_{X_{ord}}(X(Np^m)^{an}, \bar{\mathbb{F}}_p) \to H^1(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

and Proposition 3.5. Again we use here the fact that admissible representations form a Serre subcategory of smooth representations. □

III.4 Supersingular representations

In this section we recall results on the structure of admissible representations and we apply them to the exact sequence of cohomology groups that we have introduced before, getting the first comparison between the cohomology of the Lubin-Tate tower and the cohomology of the tower of modular curves. We will start with a reminder on the mod p local Langlands correspondence. The reader should consult [Ber11] for references to proofs of cited facts.

III.4.1 Mod p local Langlands correspondence

Let ω_n be the fundamental character of Serre of level n which is defined on inertia group I via $\sigma \mapsto \frac{\sigma(p^{1/p^{n-1}})}{p^{1/p^{n-1}}}$. Let ω be the mod p cyclotomic character. For $h \in \mathbb{N}$, we write Ind ω_n^h for the unique semisimple $\bar{\mathbb{F}}_p$-representation of $G_{\mathbb{Q}_p}$ which has determinant ω^h and whose restriction to I is isomorphic to $\omega_n^h \oplus \omega_n^{ph} \oplus ... \oplus \omega_n^{p^{n-1}h}$. If $\chi : G_{\mathbb{Q}_p} \to k^\times$ is a character, we will denote by $\rho(r, \chi)$ the representation $\mathrm{Ind}(\omega_2^{r+1}) \otimes \chi$ which is absolutely irreducible if $r \in \{0, ..., p-1\}$. In fact, any absolutely irreducible representation of $G_{\mathbb{Q}_p}$ of dimension 2 is isomorphic to some $\rho(r, \chi)$ for $r \in \{0, ..., p-1\}$. We remark that Ind ω_2^{r+1} is not isomorphic to the induced representation $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$, because of the condition which we put on the determinant. In fact, computing the determinant of $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$, one sees that

$$\mathrm{Ind}\ \omega_2^{r+1} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^{r+1} \cdot \mathrm{sgn})$$

where sgn is the $\bar{\mathbb{F}}_p$-character of $G_{\mathbb{Q}_{p^2}}$ which factors through $\mathbb{F}_{p^2}^\times \times \mathbb{Z}$ and which is trivial on $\mathbb{F}_{p^2}^\times$ and takes the Frobenius of $G_{\mathbb{Q}_{p^2}}$ to -1 in $\mathbb{Z}$ (we have to make a choice of a uniformiser to have the map $G_{\mathbb{Q}_{p^2}} \twoheadrightarrow \mathbb{F}_{p^2}^\times \times \mathbb{Z}$, but in this context it suffices to take p).

On the GL_2-side, one considers representations $\mathrm{Sym}^r k^2$ inflated to $\mathrm{GL}_2(\mathbb{Z}_p)$ and then extended to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by making p acts by identity. We then consider the induced representation

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2$$

One can show that the endomorphism ring (a Hecke algebra) $\mathrm{End}_{k[\mathrm{GL}_2(\mathbb{Q}_p)]}(\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2)$ is isomorphic to $k[T]$, where T corresponds to the double class $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. For a character $\chi : G_{\mathbb{Q}_p} \to k^\times$ and $\lambda \in k$. we introduce representations:

$$\pi(r, \lambda, \chi) = \frac{\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r k^2}{T - \lambda} \otimes (\chi \circ \det)$$

For $r \in \{0, ..., p-1\}$ such that $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$, the representation $\pi(r, \lambda, \chi)$ is irreducible. One proves that $\chi \circ \det$, $\mathrm{Sp} \otimes (\chi \circ \det)$ (Sp is the special representation which we do not define here) and $\pi(r, \lambda, \chi)$ for $r \in \{0, ..., p-1\}$ and $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$ are all the smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

This explicit description gives a mod p correspondence by associating $\rho(r, \chi)$ to $\pi(r, 0, \chi)$.

III.4.2 Supersingular representations

Let us a fix a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $\bar{\mathbb{F}}_p$-vector space with a central character ξ. Recall the following result of Paskunas:

Proposition III.4.1. *Let τ be an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ admitting a central character. If $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi,\tau) \neq 0$ then $\tau \simeq \pi$.*

Proof. See [Pas10] and [Pas11] for the case $p = 2$. □

This result permits us to conclude that the $\mathrm{GL}_2(\mathbb{Q}_p)$-block of any supersingular representation consists of one element - the supersingular representation itself. Here, by a $\mathrm{GL}_2(\mathbb{Q}_p)$-block we mean an equivalence class for a relation defined as follows. We write $\pi \sim \tau$ if there exists a sequence of irreducible smooth admissible $\bar{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$: $\pi_0 = \pi, \pi_1, ..., \pi_n = \tau$ such that for each i one of the following conditions holds:
1) $\pi_i \simeq \pi_{i+1}$,
2) $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_i, \pi_{i+1}) \neq 0$,
3) $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_{i+1}, \pi_i) \neq 0$.

One can find a description of all $\mathrm{GL}_2(\mathbb{Q}_p)$-blocks in [Pas11] or [Pas13]. The general result of Gabriel on the block decomposition of locally finite categories gives:

Proposition III.4.2. *We have a decomposition:*

$$\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p) = \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)_{(\pi)} \oplus \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)^{(\pi)}$$

where $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)$ is the (abelian) category of smooth admissible $\bar{\mathbb{F}}_p$-representations admitting a central character ξ, $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)_{(\pi)}$ (resp. $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\bar{\mathbb{F}}_p)^{(\pi)}$) is the subcategory of it consisting of representations Π whose all the irreducible subquotients are (resp. are not) isomorphic to π.

Proof. See Proposition 5.32 in [Pas13]. □

This result permit us to consider the localisation functor with respect to π

$$V \mapsto V_{(\pi)}$$

on the category of admissible representations such that all irreducible subquotients of $V_{(\pi)}$ are isomorphic to the fixed π.

Remark III.4.3. *We note that the condition on the existence of central characters is not important. Central characters always exist by the work of Berger ([Ber12]) in the mod p case.*

III.4.3 Cohomology with compact support

We apply the localisation functor to the three admissible terms in the exact sequence obtained from 2.1:

$$... \to \widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p) \to \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)^{an}, \bar{\mathbb{F}}_p) \to \widehat{H}^1(X(N)_{ord}, \bar{\mathbb{F}}_p) \to ...$$

getting the exact sequence

$$\widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p)_{(\pi)} \to \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p)_{(\pi)} \to \widehat{H}^1(X(N)^{an}, \bar{\mathbb{F}}_p)_{(\pi)}$$

For $a \in \mathbb{Z}_p^\times$ let us define in the light of 2.2

$$\widehat{H}^1(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p) = \varinjlim_m H^1(\mathbb{X}_{a,\infty}(Np^m), \bar{\mathbb{F}}_p)$$

Recall now, that after 2.2, $\widehat{H}^0(X(N)_{ord}, \bar{\mathbb{F}}_p)$ is an admissible representation isomorphic to the induced representation

$$\mathrm{Ind}_{B_\infty(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(\bigoplus_a \widehat{H}^0(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p) \right)$$

where $B_\infty(\mathbb{Q}_p)$ is the Borel subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$, a goes over $\mathbb{Z}_p^\times$ and we mean by $\bigoplus_a \widehat{H}^0(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p)$ smooth functions on $\mathbb{Z}_p^\times$ with values in $\bigoplus_a \widehat{H}^0(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p)$. On this representation unipotent group acts trivially by lemma 3.3 (which we can use thanks to lemma 3.4) and hence we see that it is induced from the tensor product of characters. This means that after localisation at π this representation vanishes

$$\mathrm{Ind}_{B_\infty(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \left(\bigoplus_a \widehat{H}^0(\mathbb{X}_{a,\infty}(N), \bar{\mathbb{F}}_p) \right)_{(\pi)} = 0$$

and we arrive at

Theorem III.4.4. *We have an injection of representations*

$$\widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p)_{(\pi)} \hookrightarrow \widehat{H}^1(X(N), \bar{\mathbb{F}}_p)_{(\pi)}$$

By taking yet another direct limit, we define

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} = \varinjlim_N \widehat{H}^1_c(X(N)_{ss}, \bar{\mathbb{F}}_p)$$

$$\widehat{H}^1_{\bar{\mathbb{F}}_p} = \varinjlim_N \widehat{H}^1(X(N), \bar{\mathbb{F}}_p)$$

Corollary III.4.5. *We have an injection of representations*

$$(\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p})_{(\pi)} \hookrightarrow (\widehat{H}^1_{\bar{\mathbb{F}}_p})_{(\pi)}$$

We define also for a future use

$$\widehat{H}^1_{ord,\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1(X(Np^m)_{ord}, \bar{\mathbb{F}}_p)$$

and for $a \in \mathbb{Z}_p^\times$

$$\widehat{H}^1_{a,\infty,\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1(\mathbb{X}_{a,\infty}(Np^m), \bar{\mathbb{F}}_p)$$

III.4.4 Cohomology without support

We can apply similar reasoning as above to the situation without compact support. The roles of the ordinary locus and the supersingular locus are interchanged. By using again the decomposition of the ordinary locus and lemmas 3.3 and 3.4, we get that the localisation of $\widehat{H}^1_{X_{ord}}$ vanishes

$$\widehat{H}^1_{X_{ord}}(X(N), \bar{\mathbb{F}}_p)_{(\pi)} = 0$$

and hence we get

Theorem III.4.6. *We have an injection of representations*

$$(\widehat{H}^1_{\bar{\mathbb{F}}_p})_{(\pi)} \hookrightarrow \widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$$

where $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ is defined similarly as above.

Later on, we will show that $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ is a non-admissible representation, and this is why we cannot localise it at π. Let us finish by giving another definition for a future use (where $a \in \mathbb{Z}_p^\times$)

$$\widehat{H}^1_{\mathbb{X}_{a,\infty},\bar{\mathbb{F}}_p} = \varinjlim_N \varinjlim_m H^1_{\mathbb{X}_{a,\infty}(Np^m)}(X(Np^m)^{an}, \bar{\mathbb{F}}_p)$$

III.5 New vectors

Because there does not exist at the moment the Colmez functor in the context of quaternion algebras, which would be similar to the one considered for example in [Pas13], we are forced to look for a global definition of the mod p Jacquet-Langlands correspondence. To do that, we prove an analogue of a classical theorem of Casselman in the context of the modified mod l Langlands correspondence of Emerton-Helm (see [EH11]), which amounts to the statement that for any prime $l \neq p$, and for any local two-dimensional Galois representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$, there exists a compact, open subgroup $K_l \subset \mathrm{GL}_2(\mathbb{Z}_l)$ such that $\pi_l(\rho)^{K_l}$ has dimension 1, where $\pi_l(\rho)$ is the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_l)$ associated to ρ by [EH11].

Let $\mathfrak{b}$ be an ideal of $\mathbb{Z}_p$ and put $\Gamma_0(\mathfrak{b}) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \mod \mathfrak{b}\}$. Let us recall the classical result of Casselman (see [Cas73]):

Theorem III.5.1. *Let π be an irreducible admissible infinite-dimensional representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\bar{\mathbb{Q}}_l$-vector space and let ϵ be the central character of π. Let $c(\pi)$ be the conductor of π which is the largest ideal of $\mathbb{Z}_p$ such that the space of vector v with $\pi(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})v = \epsilon(a)v$, for all $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(c(\pi))$ is not empty. Then this space has dimension one.*

We will prove that the result holds also modulo p for the modified mod l Langlands correspondence. For that we need to assume that our prime p is odd.

Theorem III.5.2. *Let $\pi = \pi(\rho)$ be the mod p admissible representation of $\mathrm{GL}_2(\mathbb{Q}_l)$ associated by the modified mod l Langlands correspondence to a Galois representation $\rho : G_{\mathbb{Q}_l} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$. Then there exists an open, compact subgroup K of $\mathrm{GL}_2(\mathbb{Q}_l)$ such that $\dim_{\bar{\mathbb{F}}_p} \pi^K = 1$.*

Proof. We recall the results of [EH11] concerning the construction of the modified mod l Langlands correspondence. By Proposition 5.2.1 of [EH11], the theorem is true when ρ^{ss} is not a twist of $1 \oplus |\cdot|$, by the reduction modulo p of the classical result of Casselman from [Cas73], which in the $l \neq p$ situation was proved by Vigneras in [Vig89b] (see Theorem 23 and Proposition 24). When this is not the case, we can suppose that in fact $\rho^{ss} = 1 \oplus |\cdot|$ and we go by case-by-case analysis of the possible forms of $\pi(\rho)$ as described in [EH11] after Proposition 5.2.1 and in [Hel12]. The $\pi(\rho)$'s which appear are mostly extensions of four kinds of representations (and some combinations of them): trivial representation 1, $|\cdot| \circ \det$, the Steinberg St, $\pi(1)$ of Vigneras (see [Vig89b]).

1) Suppose $0 \to \pi(1) \to \pi(\rho) \to 1 \to 0$. In this case $l \equiv -1 \mod p$. Let $\Gamma_0(p) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) | c \equiv 0 \mod p, a \equiv d \equiv 1 \mod p\}$. Then we have a long exact sequence associated with higher invariants by $\Gamma_0(p)$ (which we denote by $R^i(.)^{\Gamma_0(p)}$):

$$0 \to \pi(\rho)^{\Gamma_0(p)} \to 1 \to R^1\pi(1)^{\Gamma_0(p)}$$

as $\pi(1)^{\Gamma_0(p)} = 0$ by the Proposition 24 of [Vig89b]. We conclude by observing that $R^1\pi(1)^{\Gamma_0(p)} = 0$ because $|\Gamma_0(p)| = p^\infty$ and $l \not| |\Gamma_0(p)|$ by our assumption.

2) In the same way we deal with the situation when $\pi(\rho)$ is an extension of $|\cdot|\circ\det$ by $\pi(1)$ with the same assumption on l.

3) When $l \equiv -1 \mod p$ it is also possible to have $0 \to \pi(1) \to \pi(\rho) \to 1 \oplus |\cdot|\circ\det \to 0$. Look at $\mathrm{GL}_2(\mathbb{Z}_p)$-invariants. The associated long exact sequence is

$$0 \to \pi(\rho)^{\mathrm{GL}_2(\mathbb{Z}_p)} \to (1 \oplus |\cdot|\circ\det)^{\mathrm{GL}_2(\mathbb{Z}_p)} \to R^1\pi(1)^{\mathrm{GL}_2(\mathbb{Z}_p)} = \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \pi(1)^{I+pM_2(\mathbb{Z}_p)})$$

Let us denote by $\mathcal{E}$ the extension of 1 by $\pi(1)$ which we get from $0 \to \pi(1) \to \pi(\rho) \to 1 \oplus |\cdot|\circ\det \to 0$. We remark that $\pi(1)^{I+pM_2(\mathbb{Z}_p)}$ defines the same representation mod p as the reduction of $\pi(1)$. The last map in the above exact sequence is explicit

$$(1 \oplus |\cdot|\circ\det)^{\mathrm{GL}_2(\mathbb{Z}_p)} \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \pi(1)^{I+pM_2(\mathbb{Z}_p)})$$
$$(a, b) \mapsto (a+b)\mathcal{E}$$

and we see that it gives a line in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \pi(1)^{I+pM_2(\mathbb{Z}_p)})$ and hence the kernel, i.e. $\pi(\rho)^{\mathrm{GL}_2(\mathbb{Z}_p)}$, is one-dimensional as $(1 \oplus |\cdot|\circ\det)^{\mathrm{GL}_2(\mathbb{Z}_p)}$ has dimension two.

4) The last non-banal case with which we have to deal is the case when p is odd, $l \equiv 1 \mod p$ and we have an extension:

$$0 \to \mathrm{St} \to \pi(\rho) \to 1 \to 0$$

In this case $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(1, \mathrm{St})$ (here by St we mean in fact $\mathrm{St}^{I+pM_2(\mathbb{Z}_p)}$ but that also defines the Steinberg representation mod p hence we use the same notation) is two-dimensional - see Lemma 4.2 in [Hel12]. We look at the reduction map

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(1, \mathrm{St}) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \mathrm{St})$$

Let us denote by $\mathcal{E}$ the image of the class $[\pi(\rho)]$ of $\pi(\rho)$ in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \mathrm{St})$ under the above reduction. We have two cases to consider. Suppose firstly that $\mathcal{E} = 0$. Then we claim that $K = \mathrm{GL}_2(\mathbb{Z}_p)$ works. Indeed we have in this case

$$0 \to \pi(\rho)^K \to 1^K \to \mathrm{Ext}^1_K(1, \mathrm{St})$$

and as the image of 1^K in $\mathrm{Ext}^1_K(1, \mathrm{St})$ is $\mathcal{E}$, we conclude by assumption.

Now let us suppose that $\mathcal{E} \neq 0$. Then we claim that the Iwahori subgroup $K = I$ works. We have

$$0 \to \mathrm{St}^K \to \pi(\rho)^K \to 1^K \to \mathrm{Ext}^1_K(1, \mathrm{St})$$

The image of 1^K in $\mathrm{Ext}^1_K(1, \mathrm{St})$ is non-zero by assumption, because $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Z}_p)}(1, \mathrm{St}) \hookrightarrow \mathrm{Ext}^1_K(1, \mathrm{St})$. Hence $\pi(\rho)^K$ is isomorphic to St^K which is of dimension one.

5) We remark that there is also the so-called banal case when l is not congruent to ± 1 modulo p. In this case, there are two situations to consider. In the first one $\pi(\rho) = \mathrm{St} \otimes |\cdot|\circ\det$ and we can take $K = I$, the Iwahori subgroup. In the second one $\pi(\rho)$ is the unique non-split extension of $|\cdot|\circ\det$ by $\mathrm{St} \otimes |\cdot|\circ\det$. Because $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{F}_p)}(1, \mathrm{St}) = 0$ as we are in the banal case, we conclude as above that $K = \mathrm{GL}_2(\mathbb{Z}_p)$ works. □

III.6 The fundamental representation

Following the original Deligne's approach to the non-abelian Lubin-Tate theory, we define the local fundamental representation. Using it, we refine the Lubin-Tate side of the injections we have considered. Then we recall Emerton's results on the cohomology of the tower of modular curves, yielding by a comparison an information on the local fundamental representation. Our arguments are similar to those given in [Del73].

III.6.1 Cohomology of the supersingular tube

We have introduced in section 2.3, the set Δ, spaces $LT_{\Delta/K_m} = \coprod_{\delta\in\Delta/K_m} LT_\delta$ and we have obtained a description of the supersingular tube

$$X(Np^m)_{ss} \simeq LT_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N)$$

Definition III.6.1. *Define the fundamental representation by*

$$\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p} = \varinjlim_m H^1_c(LT_{\Delta/K_m}, \bar{\mathbb{F}}_p)$$

Similarly we introduce the fundamental representation without support denoting it by $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$.

From the description of supersingular points, we have

$$H^1_c(X(Np^m)_{ss}, \bar{\mathbb{F}}_p) = H^1_c(LT_{\Delta/K_m} \times_{D^\times(\mathbb{Q})} \mathrm{GL}_2(\mathbb{A}_f^p)/K(N), \bar{\mathbb{F}}_p) =$$

$$= \{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K(N) \to H^1_c(LT_{\Delta/K_m}, \bar{\mathbb{F}}_p)\}^{D^\times(\mathbb{Q}_p)}$$

We take the direct limit:

$$\varinjlim_m H^1_c(X(Np^m)_{ss}, \bar{\mathbb{F}}_p) \simeq \{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K(N) \to \varinjlim_m H^1_c(LT_{\Delta/K_m}, \bar{\mathbb{F}}_p)\}^{D^\times(\mathbb{Q}_p)}$$

Take the limit over N to obtain

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} \simeq \{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \to \widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}\}^{D^\times(\mathbb{Q}_p)} \simeq$$

$$\simeq \Big(\{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \to \bar{\mathbb{F}}_p\} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}\Big)^{D^\times(\mathbb{Q}_p)}$$

Let

$$\mathbf{F} = \{f : D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f) \to \bar{\mathbb{F}}_p\}$$

where f are locally constant functions, then

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} \simeq \Big(\mathbf{F} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}\Big)^{D^\times(\mathbb{Q}_p)} \tag{III.1}$$

We get a similar result for the cohomology without support

$$\widehat{H}^1_{ss,\bar{\mathbb{F}}_p} \simeq \Big(\mathbf{F} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}\Big)^{D^\times(\mathbb{Q}_p)}$$

III.6.2 Emerton's results

We recall Emerton's results on the completed cohomology of modular curves. Remark that we are using implicitly the comparison theorem for étale cohomology of a scheme and its analytification which is proved in [Ber95].

Let us fix a finite set $\Sigma = \Sigma_0 \cup \{p\}$. Let $K^\Sigma = \prod_{l\notin\Sigma} K_l$ where $K_l = \mathrm{GL}_2(\mathbb{Z}_l)$ and choose an open, compact subgroup K_{Σ_0} of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$. Let $\bar{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ be an odd, irreducible, continuous representation unramified outside Σ. Remark that by Serre's conjecture (see [Kha06]) $\bar{\rho}$ is modular. Let us denote by $\mathfrak{m}$ the maximal ideal in the Hecke algebra $\mathbb{T}(K_{\Sigma_0})$ which corresponds to $\bar{\rho}$. We write also $\bar{\rho}_{|G_{\mathbb{Q}_p}} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_\mathbb{Q}} \alpha$, where α can be considered as a character of $\mathbb{Q}_{p^2}^\times$ by the local class field theory. For the definitions, see Section 5 of [Eme11a].

Theorem III.6.2. *Assuming that $\bar{\rho}$ satisfies certain technical hypotheses (see below), we have an isomorphism*

$$\widehat{H}^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}]^{K^\Sigma} \simeq \pi \otimes_{\bar{\mathbb{F}}_p} \pi_{\Sigma_0}(\bar{\rho}) \otimes_{\bar{\mathbb{F}}_p} \bar{\rho}$$

where π is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\bar{\rho}$ by the mod p local Langlands correspondence and $\pi_{\Sigma_0}(\bar{\rho})$ is a representation $\mathrm{GL}_2(\mathbb{A}_f^{\Sigma_0})$ associated to $\bar{\rho}$ by the modified local Langlands correspondence mod l for $l \in \Sigma_0$ (see [EH11]).

For the exact assumptions, see Proposition 6.1.20 in [Eme11a]. Those assumptions are not important for our applications, as we can always find $\bar{\rho}$ which satisfies them and which at p is isomorphic to our fixed irreducible Galois representation $\bar{\rho}_p$ (see below).

III.6.3 Comparison

We will use results of Emerton to describe a part of $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$. We start by comparing mod p Hecke algebras for GL_2 and for $D^\times$. On $\mathbf{F}$, after taking K^Σ-invariants, there is an action of a Hecke algebra. For $l \notin \Sigma$, we have a Hecke operator T_l acting on functions of $D^\times(\mathbb{A}_f)$ by

$$T_l(f)(x) = f(xg) + \sum_{i=0}^{l-1} f(xg_i)$$

where $g = \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $g_i = \left(\begin{smallmatrix} 1 & 0 \\ i & l \end{smallmatrix}\right)$ are both considered as elements of $D^\times(\mathbb{A}_f)$ having 1 at places different from l. Let us denote by $\mathbb{T}^D(K_{\Sigma_0})$ the Hecke algebra, which is the free $\mathcal{O}$-algebra spanned by the operators T_l and S_l for all $l \notin \Sigma$, where $S_l = [K_{\Sigma_0}K^\Sigma \left(\begin{smallmatrix} l & 0 \\ 0 & l \end{smallmatrix}\right) K_{\Sigma_0}K^\Sigma]$. By the results of Serre (see letter to Tate from [Ser96]), systems of eigenvalues for (T_l) of $\mathbb{T}^D(K_{\Sigma_0})$ on $\mathbf{F}$ are in bijection with systems of eigenvalues for (T_l) of $\mathbb{T}(K_{\Sigma_0})$ coming from mod p modular forms. This allows us to identify maximal ideals of $\mathbb{T}^D(K_{\Sigma_0})$ with those of $\mathbb{T}(K_{\Sigma_0})$ and in what follows we will make no distinction between them.

Let $\bar{\rho}_p$ be the local Galois representation associated to a supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the mod p Langlands correspondence. We assume that there exists a representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ which is odd, irreducible, continuous, unramified outside a finite set $\Sigma = \Sigma_0 \cup \{p\}$, and such that $\bar{\rho}_{|G_{\mathbb{Q}_p}} = \bar{\rho}_p$. This is always the case by the main result of [BG13]. See also the introduction to [Bre03] for a discussion (especially Conjecture 1.5) of the reductions of Galois representations associated to modular forms.

Let us denote by $\mathfrak{m}$ the maximal ideal in the Hecke algebra $\mathbb{T}(K_{\Sigma_0})$ corresponding to $\bar{\rho}$. Results of Emerton apply to $\bar{\rho}$ because we have assumed that $\bar{\rho}_p$ is irreducible. We denote by $K_{\mathfrak{m},\Sigma_0}$ an open compact subgroup of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$ for which $\pi_{\Sigma_0}(\bar{\rho})^{K_{\mathfrak{m},\Sigma_0}}$ is a one-dimensional vector space (see section III.5). We put $K_{\mathfrak{m}} = K_{\mathfrak{m},\Sigma_0}K^\Sigma$ and we define:

$$\sigma_{\mathfrak{m}} = \mathbf{F}[\mathfrak{m}]^{K_{\mathfrak{m}}}$$

This is a representation of $D^\times(\mathbb{Q}_p)$. We remark that Breuil and Diamond in [BD12] also define a representation of $D^\times(\mathbb{Q}_p)$ which serves as a model for a local representation which should appear conjecturally at the place p in the local-global compatibility of the Buzzard-Diamond-Jarvis conjecture (see the next section for a discussion). Their construction is different from our and uses "‘types"’ instead of new vectors.

Let us look again at our cohomology groups. Taking $K_{\mathfrak{m}}$-invariants, which commute with $D^\times(\mathbb{Q}_p)$-invariants, we get

$$\left(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}\right)^{K_{\mathfrak{m}}} \simeq \left(\mathbf{F}^{K_{\mathfrak{m}}} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}\right)^{D^\times(\mathbb{Q}_p)}$$

Let us define the dual $\sigma_{\mathfrak{m}}^{\vee} = \mathrm{Hom}_{\bar{\mathbb{F}}_p}(\sigma_{\mathfrak{m}}, \bar{\mathbb{F}}_p)$. It is not necesarily a smooth representation. Taking $[\mathfrak{m}]$-part we get:

$$\left(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}[\mathfrak{m}]\right)^{K_{\mathfrak{m}}} \simeq \left(\sigma_{\mathfrak{m}} \otimes_{\bar{\mathbb{F}}_p} \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}\right)^{D^\times(\mathbb{Q}_p)} =: \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^{\vee}]$$

Thus, by the results proven earlier, we have

$$\pi \otimes_{\bar{\mathbb{F}}_p} \bar{\rho} \simeq \left(\widehat{H}^1_{\bar{\mathbb{F}}_p}[\mathfrak{m}]\right)^{K_{\mathfrak{m}}}_{(\pi)} \hookrightarrow \left(\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}[\mathfrak{m}]\right)^{K_{\mathfrak{m}}} \simeq \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^{\vee}]$$

and we arrive at

Theorem III.6.3. *We have a $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection:*

$$\pi \otimes_{\bar{\mathbb{F}}_p} \bar{\rho} \hookrightarrow \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^{\vee}]$$

We will strengthen this result after proving additional facts about $\sigma_{\mathfrak{m}}$. It is also possible to obtain the analogous result in the p-adic setting. Details will appear elsewhere.

III.6.4 The mod p Jacquet-Langlands correspondence

We have defined above

$$\sigma_{\mathfrak{m}} = \mathbf{F}[\mathfrak{m}]^{K_{\mathfrak{m}}}$$

This is a mod p representation of $D^\times(\mathbb{Q}_p)$ which is one of our candidates for the mod p Jacquet-Langlands correspondence we search for. We will analyse this representation more carefully in the next section, getting a result about its socle. The question we do not answer here is whether this local representation is independent of the Hecke ideal $\mathfrak{m}$ and if yes, how to construct it by local means. We make a natural conjecture

Conjecture III.6.4. *Let $\mathfrak{m}$ and $\mathfrak{m}'$ be two maximal ideals of the Hecke algebra, which correspond to Galois representations $\bar{\rho}$ and $\bar{\rho}'$ such that $\bar{\rho}_p \simeq \bar{\rho}'_p$. Then we have a $D^\times(\mathbb{Q}_p)$-equivariant isomorphism*

$$\sigma_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}'}$$

This conjecture is natural in view of the fact that $\sigma_{\mathfrak{m}}$ should play a role of the mod p Jacquet-Langlands correspondence and it should depend only on a local data. In fact, this conjecture follows from the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture (see Conjecture 4.7 in [BDJ10])

Conjecture III.6.5. *We have a $D^\times(\mathbb{A})$-equivariant isomorphism*

$$\mathbf{F}[\mathfrak{m}] \simeq \sigma \otimes \pi^p(\bar{\rho})$$

where σ is a $D^\times(\mathbb{Q}_p)$-representation which depends only on $\bar{\rho}_p$, where $\bar{\rho}$ is the Galois representation associated to $\mathfrak{m}$.

The conjecture of Buzzard-Diamond-Jarvis would be proved if one could show the existence of an analogue of the Colmez functor in the context of quaternion algebras. Then, the methods of Emerton from [Eme11a] could be applied to give a proof. We come back to the discussion of the mod p Jacquet-Langlands correspondence at the end of the next section.

III.7 Representations of quaternion algebras: mod p theory

In this section we analyse more carefully mod p representations of quaternion algebras, especially representations $\sigma_{\mathfrak{m}}$ defined in the preceding section. We also define a naive mod p Jacquet-Langlands correspondence.

III.7.1 Naive mod p Jacquet-Langlands correspondence

By the work of Vigneras (see [Vig89a]), we know that all irreducible representations of $D^\times$ are of dimension 1 or 2 and are either

1) a character of $D^\times(\mathbb{Q}_p)$, or

2) are of the form $\mathrm{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}^\times}^{D^\times} \alpha$ where α is a character of $\mathbb{Q}_{p^2}^\times$.

Let $\bar{\rho}_p$ be the mod p 2-dimensional irreducible Galois representation which corresponds to the supersingular representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the mod p Local Langlands correspondence. As we have mentioned earlier, it is of the form $\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^r \cdot \mathrm{sgn}) \otimes \chi$ where χ is a character and $r \in \{1, ..., p\}$.

Definition III.7.1. *The naive mod p Jacquet-Langlands correspondence is*

$$\mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} (\omega_2^r \cdot \mathrm{sgn}) \otimes \chi \mapsto \mathrm{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}}^{D^\times} (\omega_2^r) \otimes \chi$$

where ω_2^r is treated as a character of $\mathbb{Q}_{p^2}$ by the local class field theory and χ is considered both as a character of $G_{\mathbb{Q}_p}$ and $D^\times(\mathbb{Q}_p)$. This gives a bijection between two-dimensional representations of $G_{\mathbb{Q}_p}$ and two-dimensional representations of $D^\times(\mathbb{Q}_p)$. Similar correspondence holds for characters.

We remark that one may also would like to call this the naive mod p Langlands correspondence for $D^\times(\mathbb{Q}_p)$. We get the Jacquet-Langlands correspondence in the usual sense, when we compose it with the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\alpha : \mathbb{Q}_{p^2} \to \bar{\mathbb{F}}_p^\times$ be a character. We denote by $\rho(\alpha)$ the representation of $G_{\mathbb{Q}_p}$ obtained by the local class field theory and an induction. We denote by $\sigma(\alpha)$ the $D^\times(\mathbb{Q}_p)$-representation $\mathrm{Ind}_{\mathcal{O}_D^\times \mathbb{Q}_{p^2}}^{D^\times}(\alpha)$. We remark that we also could define the naive mod p Jacquet-Langlands correspondence as

$$\rho(\alpha) \mapsto \sigma(\alpha)$$

but we have chosen our normalisation with a twist by sgn to have the same condition on determinants as for the classical l-adic Jacquet-Langlands correspondence.

III.7.2 Quaternionic forms

Let D be the quaternion algebra over $\mathbb{Q}$, ramified at p and at ∞. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and a uniformiser ϖ. Define

$$\mathbf{F} = \varinjlim_K H^0(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K, \bar{\mathbb{F}}_p)$$

$$\mathbf{F}_{\mathcal{O}} = \varinjlim_K H^0(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A}_f)/K, \mathcal{O})$$

Define also $\mathbf{F}_L = \mathbf{F}_{\mathcal{O}} \otimes_{\mathcal{O}} L$. We can make similar definitions for other $\mathbb{F}_p$ or $\mathbb{Z}_p$-algebras (for example for finite extensions of $\mathbb{F}_p$ or for $\bar{\mathbb{Z}}_p$ in $\mathbf{F}_{\bar{\mathbb{Z}}_p}$ which we will use in the text).

Recall that we have fixed a finite set $\Sigma = \Sigma_0 \cup \{p\}$ and chosen an open, compact subgroup K_{Σ_0} of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$. On each of the above spaces, after taking K^Σ-invariants, there is an action of the Hecke algebra $\mathbb{T}^D(K_{\Sigma_0})$. Recall also that we have defined $\bar\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ an odd, irreducible, continuous representation unramified outside Σ and we have denoted by $\mathfrak{m}$ the maximal ideal in $\mathbb{T}(K_{\Sigma_0})$ (or in $\mathbb{T}^D(K_{\Sigma_0})$) which corresponds to $\bar\rho$. We write

$$\bar\rho_{|G_{\mathbb{Q}_p}} = \rho(\alpha)$$

where α can be considered as a character of $\mathbb{Q}_{p^2}^\times$ by the local class field theory.

Proposition III.7.2. *Take an open, compact subgroup K_p of $D^\times(\mathbb{Q}_p)$ and choose K_{Σ_0} to be an open, compact subgroup of $\prod_{l\in\Sigma_0} \mathrm{GL}_2(\mathbb{Z}_l)$ such that $K_pK_{\Sigma_0}K^\Sigma$ is neat. Then $\mathbf{F}_\mathfrak{m}^{K_{\Sigma_0}K^\Sigma}$ is injective as a smooth representation of K_p.*

We do not define the notion of neatness for which we refer to section 0.6 in [Pin90]. We only need this condition to ensure that K_p acts freely as in the proof below. Any sufficiently small open compact subgroup is neat.

Proof. Let M be any smooth finitely generated representation of K_p. Hence M is of finite dimension and its dual is also smooth. We have

$$\mathbf{F}^{K_{\Sigma_0}K^\Sigma} = \varinjlim_{K'_p} \mathbf{F}^{K'_pK_{\Sigma_0}K^\Sigma}$$

where $K'_p \subset K_p$ runs over sufficiently small, normal open subgroups of $\mathcal{O}_D^\times$, so that K'_p acts trivially on M. We can associate to M a local system $\mathcal{M}$ on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_{\Sigma_0}K^\Sigma$. Because K_p acts freely on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K_{\Sigma_0}K^\Sigma$ by the assumption of neatness, we can descend this system to each $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K'_pK_{\Sigma_0}K^\Sigma$, where K'_p is as above. Moreover on each $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f)/K'_pK_{\Sigma_0}K^\Sigma$, $\mathcal{M}$ is a constant local system and hence:

$$\mathrm{Hom}_{K_p}(M, \mathbf{F}^{K_{\Sigma_0}K^\Sigma}) \simeq \varinjlim_{K'_p} \mathrm{Hom}_{K_p}(M, \mathbf{F}^{K'_pK_{\Sigma_0}K^\Sigma}) \simeq$$

$$\simeq \varinjlim_{K'_p} (\mathbf{F}^{K'_pK_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee))^{K_p} \simeq \mathbf{F}^{K_pK_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee)$$

where $\mathbf{F}(\mathcal{M}^\vee) = H^0(D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_f), \mathcal{M}^\vee)$. Because $\mathbf{F}^{K_pK_{\Sigma_0}K^\Sigma}(\mathcal{M}^\vee)$ is an exact functor (there is no H^1), we get the result. □

We will now start to analyse socles of quaternionic forms $\mathbf{F}_\mathfrak{m}^{K_{\Sigma_0}K^\Sigma}$. Let us start with the following lemma:

Lemma III.7.3. *Let β be a finite dimensional $\bar{\mathbb{F}}_p$-representation of $\mathcal{O}_D^\times$. We have*

$$\mathrm{Hom}_{\mathcal{O}_D^\times}(\beta^\vee, \mathbf{F}_\mathfrak{m}^{K^p}) \simeq \mathbf{F}_\mathfrak{m}^{K^p}\{\beta\}$$

where $\mathbf{F}_\mathfrak{m}^{K^p}\{\beta\}$ is the space of automorphic functions $D(\mathbb{Q})\backslash D(\mathbb{A}_f)/K^p \to \beta$.

Proof. The isomorphism is given by an explicit map. See Lemma 7.4.3 in [EGH13]. □

Proposition III.7.4. *The only irreducible $\bar{\mathbb{F}}_p$-representations of $D^\times(\mathbb{Q}_p)$ which appear as submodules in $\mathbf{F}_\mathfrak{m}^{K_{\Sigma_0}K^\Sigma}$ are isomorphic to $\sigma^\vee = \sigma(\alpha)^\vee$.*

Proof. Observe that the only irreducible $\bar{\mathbb{F}}_p$-representations of $\mathcal{O}_D^\times$ which can appear in the $\mathcal{O}_D^\times$-socle of $\mathbf{F}_{\mathfrak{m}}^{K^p}$ are duals of the Serre weights of $\bar{\rho}$. This follows from the lemma above and the definition of being modular, i.e. $\bar{\rho}$ is modular of weight β (where β is a representation of $\mathcal{O}_D^\times$) if and only if there exists an open compact subset U of $D^\times(\mathbb{A}_f)$ such that $\mathbf{F}_{\mathfrak{m}}^U\{\beta\} \neq 0$. By the lemma, this is equivalent to $\mathrm{Hom}_{\mathcal{O}_D^\times}(\beta^\vee, \mathbf{F}_{\mathfrak{m}}^U) \neq 0$ which holds if and only if $\beta^\vee \in soc_{\mathcal{O}_D^\times} \mathbf{F}_{\mathfrak{m}}^U$. Now the result follows from Theorem 7 in [Kha01], as the only possible weights which can appear in the socle are $\alpha^\vee$ and $(\alpha^p)^\vee$. Hence the $D^\times(\mathbb{Q}_p)$-socle contains only $\sigma(\alpha)^\vee$. □

As a corollary we also get the $[\mathfrak{m}]$-isotypic analogue of the above

Corollary III.7.5. *The only irreducible representations which appear as submodules in* $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ *are isomorphic to* $\sigma^\vee = \sigma(\alpha)^\vee$.

We are now ready to strengthen the theorem which has appeared before

Theorem III.7.6. *The representation* $\sigma \otimes \pi \otimes \bar{\rho}$ *appears as a subquotient in* $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$

Proof. This follows from

$$\pi \otimes \bar{\rho} \hookrightarrow \widehat{H}^1_{LT,\bar{\mathbb{F}}_p}[\sigma_{\mathfrak{m}}^\vee]$$

and the fact that the only irreducible $D^\times(\mathbb{Q}_p)$-representation which appears as a quotient of $\sigma_{\mathfrak{m}}^\vee$ is σ. □

We remark that if

$$n = \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}_{D^\times(\mathbb{Q}_p)}(\sigma(\alpha)^\vee, \mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}])$$

then one conjectures that $n = 1$ (even in the more general setting, see Section 8 of [Bre14]).

Before moving further, let us recall a structure theorem of Breuil and Diamond for our $D^\times(\mathbb{Q}_p)$-representations, which shows that our candidate for the mod p Jacquet-Langlands correspondence defined above is of entirely different nature than the one with complex coefficients.

Proposition III.7.7. *The* $D^\times(\mathbb{Q}_p)$*-representation* $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ *is of infinite length.*

Proof. We give a sketch of the proof, which is contained in [BD12] as Corollary 3.2.4 (it is conditional on the local-global compatibility part of the Buzzard-Diamond-Jarvis conjecture). Firstly observe that it is enough to prove that $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ is of infinite dimension over $\bar{\mathbb{F}}_p$, because a representation of finite length will be also of finite dimension as $D^\times$ is compact modulo center. Suppose now that we have an automorphic form π such that the reduction of its associated Galois representation $\bar{\rho}_\pi$ is isomorphic to $\bar{\rho}$ and $\pi^{K_{\Sigma_0}K^\Sigma} \neq 0$. Then there is a lattice $\Lambda_\pi = \mathbf{F}_{\bar{\mathbb{Z}}_p}^{K_{\Sigma_0}K^\Sigma} \cap \pi^{K_{\Sigma_0}K^\Sigma}$ inside $\pi^{K_{\Sigma_0}K^\Sigma}$. Its reduction $\bar{\Lambda}_\pi = \Lambda_\pi \otimes_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p$ lies in $\mathbf{F}^{K_{\Sigma_0}K^\Sigma}[\mathfrak{m}]$ so it is enough to prove that we can find automorphic representations π as above with $\pi^{K_{\Sigma_0}K^\Sigma}$ of arbitrarily high dimension. This is done by explicit computations of possible lifts in [BD12]. □

This proposition indicates that $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$ is a non-admissible smooth representation.

III.7.3 Non-admissibility

We have

Proposition III.7.8. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations* $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ *and* $\widehat{H}^2_{X_{ord},\bar{\mathbb{F}}_p}$ *are non-admissible smooth* $\bar{\mathbb{F}}_p$*-representations.*

Proof. If one of them would be admissible, then also the second would because of the exact sequence

$$\widehat{H}^1_{X_{ord},\bar{\mathbb{F}}_p} \to \widehat{H}^1_{\bar{\mathbb{F}}_p} \to \widehat{H}^1_{ss,\bar{\mathbb{F}}_p} \to \widehat{H}^2_{X_{ord},\bar{\mathbb{F}}_p} \to \widehat{H}^2_{\bar{\mathbb{F}}_p}$$

It is enough to prove that $\widehat{H}^1_{ss,\bar{\mathbb{F}}_p}$ is non-admissible, or even that $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$ is non-admissible. Let us look at the Hochschild-Serre spectral sequence for the Iwahori level I of the Lubin-Tate tower

$$H^i(I, \widehat{H}^j_{LT,\bar{\mathbb{F}}_p}) \Rightarrow H^{i+j}_{LT,I,\bar{\mathbb{F}}_p}$$

where we have denoted by $H^{i+j}_{LT,I,\bar{\mathbb{F}}_p}$ the fundamental representation at I-level. Now observe that if $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$ were admissible, then $H^0(I, \widehat{H}^1_{LT,\bar{\mathbb{F}}_p})$ would be of finite dimension. Because $H^1(I, \widehat{H}^0_{LT,\bar{\mathbb{F}}_p})$ is of finite dimension (as $\widehat{H}^0_{LT,\bar{\mathbb{F}}_p}$ is), this would mean that $H^1_{LT,I,\bar{\mathbb{F}}_p}$ is finite-dimensional. But geometrically Lubin-Tate tower at level I is an annulus (this is a standard fact, one can prove it by methods of section 8.1) and hence $H^1_{LT,I,\bar{\mathbb{F}}_p}$ has to be of infinite dimension (see remark 6.4.2 in [Ber93]). This contradiction finishes the proof. □

Corollary III.7.9. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\widehat{H}^1_{LT,\bar{\mathbb{F}}_p}$ *is a non-admissible smooth* $\bar{\mathbb{F}}_p$*-representation.*

Proof. Follows from the proposition above. □

III.7.4 On mod p Jacquet-Langlands correspondence

We come once again to the discussion of the mod p Jacquet-Langlands correspondence. Remark that there are three possible candidates for the correspondence which appear in our work:

1) The 2-dimensional irreducible representation σ of $D^\times(\mathbb{Q}_p)$ defined by the naive mod p Jacquet-Langlands correspondence.

2) The representation $\sigma_\mathfrak{m}$ defined by global means and depending a priori on a maximal Hecke ideal $\mathfrak{m}$. It is of infinite length as a representation of $D^\times(\mathbb{Q}_p)$ and contains $\sigma^\vee$ in its socle.

3) The representation defined via the cohomology

$$\sigma_{LT} = \mathrm{Hom}_{G_{\mathbb{Q}_p}\times \mathrm{GL}_2(\mathbb{Q}_p)}(\bar{\rho} \otimes_{\bar{\mathbb{F}}_p} \pi, \widehat{H}^1_{LT,\bar{\mathbb{F}}_p})$$

By the results above, it contains σ as a subquotient.

In the l-adic setting, we can define representations of $D^\times(\mathbb{Q}_p)$ in the similar way and it is known that $\sigma_{LT} \simeq \sigma_\mathfrak{m}^\vee$. Moreover $\sigma_\mathfrak{m}$ in the l-adic setting is 2-dimensional (at least in the moderately ramified case). This is not the case in the mod p setting as we have showed that representations of 1) and 2) are different (one is 2-dimensional, the other is infinite-dimensional). The natural definition of the mod p correspondence seems to be σ_{LT} and it is also natural to ask what is the relation between σ_{LT} and $\sigma_{\mathfrak{m}}^{\vee}$ for appropiate $\mathfrak{m}$ as considered before.

III.8 Cohomology with compact support

In this section we will discuss what happens when we consider the cohomology with compact support. Our basic result is negative and it states that the first cohomology group with compact support of the fundamental representation $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}$ does not contain any supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as a subrepresentation. This suprising result, which is very different from the situation known in the l-adic setting where $l \neq p$, leads to a similar exact sequence as we have considered for cohomology without support, but this time, we get that $\pi \otimes \bar{\rho}$ is contained in the H^1 of the ordinary locus.

III.8.1 Geometry at pro-p Iwahori level

Let $K(1) = \left(\begin{smallmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{smallmatrix}\right)$ and let $I(1) = \left(\begin{smallmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{smallmatrix}\right)$ be the pro-p Iwahori subgroup. We let

$$\mathcal{M}_{LT,K(1)} = \operatorname{Spf} R_{K(1)}$$

$$\mathcal{M}_{LT,I(1)} = \operatorname{Spf} R_{I(1)}$$

be the formal models for the Lubin-Tate space at levels $K(1)$ and $I(1)$ respectively. We will compute $R_{I(1)}$ explicitly. This is also done in a more general setting in the work of Haines-Rapoport (see Corollary 3.4.3 in [HR12]) but here we give a short and elementary argument.

We know that $R_{I(1)} = R_{K(1)}^{I(1)}$ and hence we can use the explicit description of $R_{K(1)}$ by Yoshida to get the result (see Proposition 3.5 in [Yos10]). Let $W = W(\bar{\mathbb{F}}_p)$ be the Witt vectors of $\bar{\mathbb{F}}_p$. There is a surjection $W[[\tilde{X}_1, \tilde{X}_2]] \twoheadrightarrow R_{K(1)}$ which maps $\tilde{X}_i$ to X_i where X_i $(i = 1, 2)$ are local parameters for $R_{K(1)}$ which form a $\mathbb{F}_p$-basis of $\mathfrak{m}_{R_{K(1)}}[p] = \{x \in \mathfrak{m}_{R_{K(1)}} | [p](x) = 0\}$, where $[p]$ is explained below. We will find parameters for $R_{I(1)} = R_{K(1)}^{I(1)}$. Observe that for $b \in \mathbb{F}_p$ we have (see chapter 3 of [Yos10])

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} X_1 = X_1$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} X_2 = [b]X_1 +_{\Sigma} X_2$$

where $+_{\Sigma}$ is the addition on the universal deformation of the unique formal group over $\bar{\mathbb{F}}_p$ of height 2 and $[.]$ gives the structure of multiplication by elements of $\mathbb{Z}_p$ on the same universal deformation Σ. See Chapter 3 of [Yos10] for details. We see that X_2 is not invariant under $I(1)$ and hence we define $X_2' = \prod_{b \in \mathbb{F}_p}([b]X_1 +_{\Sigma} X_2)$ which is. We claim that (X_1, X_2') are local parameters for $R_{I(1)}$. Indeed if z belongs to $R_{I(1)} = R_{K(1)}^{I(1)}$ then we may write it as $z = P(X_1) + X_2 Q(X_1, X_2)$, where $P \in W[[X_1]]$ and $Q \in W[[X_1, X_2]]$. As $P(X_1)$ is invariant under $I(1)$, we see that also $X_2 Q(X_1, X_2)$ has to be invariant under $I(1)$. Because of the action of $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ on X_2 described above and the fact that $R_{K(1)}$ is a regular local ring hence factorial, we see that X_2' divides $X_2 Q(X_1, X_2)$ (we use here the fact that $[b]X_1 +_{\Sigma} X_2$ and $[b']X_1 +_{\Sigma} X_2$ are not associated for $b \neq b'$; this follows from

Proposition 4.2 in [Str08]). This leads to $z = P(X_1) + X_2' Q'(X_1, X_2)$ for some Q' which is $I(1)$-invariant and hence we conclude by successive approximations (since $R_{K(1)}$ is X_2'-adically complete) that there is a power series f such that $z = f(X_1, X_2')$ (we use the fact that polynomials are dense in formal series).

Let us observe that for $a \in \mathbb{F}_p^\times$ we have for $i = 1, 2$: $[a]X_i = uX_i$, where u is a unit in $R_{K(1)}$. Let us now look at the relation defining $R_{K(1)}$ inside $W[[\tilde{X}_1, \tilde{X}_2]]$ which appears in Proposition 3.5 of [Yos10]. We have

$$p = u \prod_{(a_1,a_2)\in\mathbb{F}_p^2\backslash\{0,0\}} ([a_1]X_1 +_\Sigma [a_2]X_2)$$

where u is some unit in $R_{K(1)}$. Let us write $a \sim b$ whenever $a = ub$ for some unit u in $R_{K(1)}$. Thus we have

$$p \sim \prod_{(a_1,a_2)\in\mathbb{F}_p^2\backslash\{0,0\}} ([a_1]X_1 +_\Sigma [a_2]X_2) \sim \left(\prod_{a_1\in\mathbb{F}_p^\times} [a_1]X_1\right)\left(\prod_{a_1\in\mathbb{F}_p}\prod_{a_2\in\mathbb{F}_p^\times} [a_2]([a_1/a_2]X_1 +_\Sigma X_2)\right) \sim$$

$$\sim \left(\prod_{a_1\in\mathbb{F}_p^\times} [a_1]X_1\right)\left(\prod_{a_2\in\mathbb{F}_p^\times} X_2'\right) \sim (X_1 X_2')^{p-1}$$

Hence we have $p = u'(X_1X_2')^{p-1}$ for some unit u' in $R_{K(1)}$ a priori, but we can see that u' is in fact a unit in $R_{I(1)}$. Because $W[[X,Y]]$ is a complete local ring with an algebraically closed residue field there exists a $(p-1)$-th root of u', and hence we can write $p = (X_1'X_2'')^{p-1}$. We want to conclude that this is the only relation in $R_{I(1)}$ which means that there exists a surjection

$$B = W[[\tilde{X}_1', \tilde{X}_2'']] \twoheadrightarrow R_{I(1)}$$

with kernel $f = (\tilde{X}_1'\tilde{X}_2'')^{p-1} - p$. First of all, observe that $R_{I(1)}$ and B/fB are regular local rings of dimension 2 with a surjection $B/fB \twoheadrightarrow R_{I(1)}$. We claim that this map has to be necessarily an injection also. Indeed, this holds for any surjective morphism $A \twoheadrightarrow R$ of regular local rings of the same dimension by using the fact that that for a regular local ring we have $\mathrm{gr}^\bullet_{\mathfrak{m}_A} A \simeq \mathrm{Sym}\, \mathfrak{m}_A/\mathfrak{m}_A^2$. This yields an isomorphism at the graded level which lifts to the level of rings. All in all, we conclude that

Proposition III.8.1. *We have*

$$R_{I(1)} \simeq W[[X,Y]]/((XY)^{p-1} - p)$$

This means that $\mathcal{M}_{LT,I(1)}$ is made of $p-1$ copies of an open annulus in $\mathbb{P}^1$ after a base change to $W[\sqrt[p-1]{p}]$:

$$R_{I(1)} \otimes_W W[\sqrt[p-1]{p}] \simeq \prod_{i=1}^{p-1} W[[X,Y]]/(XY - \sqrt[p-1]{p} \cdot \zeta_{p-1}^i)$$

III.8.2 Cohomology at pro-p Iwahori level

We compute $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$ (we will omit $\bar{\mathbb{F}}_p$ from the notation in what follows). Let $\mathcal{A}$ be an open annulus in $\mathbb{P}^1$. We can write a long exact sequence

$$0 \to H^0_c(\mathcal{A}) \to H^0(\mathbb{P}^1) \to H^0(\mathbb{P}^1\backslash\mathcal{A}) \to H^1_c(\mathcal{A}) \to H^1(\mathbb{P}^1)$$

We know that

$$H^1(\mathbb{P}^1) = H^0_c(\mathcal{A}) = 0$$

$$\dim_{\bar{\mathbb{F}}_p} H^0(\mathbb{P}^1) = 1$$

$$\dim_{\bar{\mathbb{F}}_p} H^0(\mathbb{P}^1 \backslash \mathcal{A}) = 2$$

and hence it follows that

$$\dim_{\bar{\mathbb{F}}_p} H^1_c(\mathcal{A}) = 1$$

Because geometrically $\mathcal{M}_{LT,I(1)}$ is made of $p-1$ copies of $\mathcal{A}$, we have

$$\dim_{\bar{\mathbb{F}}_p} H^1_c(\mathcal{M}_{LT,I(1)}) = p-1$$

Let $\mathcal{H} = \mathcal{H}_{\mathrm{GL}_2}(I(1)) = \bar{\mathbb{F}}_p[I(1)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/I(1)]$ be the mod p Hecke algebra at the pro-p Iwahori level. Let I be the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. We look at the action of $\bar{\mathbb{F}}_p[I/I(1)] \simeq \bar{\mathbb{F}}_p[(\mathbb{F}_p^\times)^2]$ on the cohomology. We know by [Str08] that it acts by determinant on connected components of $\mathcal{M}_{LT,K(1)}$ and hence on connected components of $\mathcal{M}_{LT,I(1)}$ so we have a decomposition of $H^1_c(\mathcal{M}_{LT,I(1)})$ into $p-1$ pieces of dimension 1:

$$H^1_c(\mathcal{M}_{LT,I(1)}) = \bigoplus_{\chi:\mathbb{F}_p^\times \to \bar{\mathbb{F}}_p^\times} H^1_c(\mathcal{M}_{LT,I(1)})_\chi$$

where $H^1_c(\mathcal{M}_{LT,I(1)})_\chi$ is the part of $H^1_c(\mathcal{M}_{LT,I(1)})$ on which $\bar{\mathbb{F}}_p[(\mathbb{F}_p^\times)^2]$ acts through $\chi \circ \det$.

III.8.3 Vanishing result

We will now prove that the supersingular representation π does not appear in $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}$. First of all, remark that it is enough to prove that the $\mathcal{H}$-module $\pi^{I(1)}$ does not appear in $(\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$, because the functor $\pi \mapsto \pi^{I(1)}$ induces a bijection between supersingular representations and supersingular Hecke modules (see [Vig04]). We have the Hochschild-Serre spectral sequence

$$H^i(I(1), \widehat{H}^j_{LT,c,\bar{\mathbb{F}}_p}) \Rightarrow H^{i+j}_{LT,c,I(1),\bar{\mathbb{F}}_p}$$

where we have denoted by $H^{i+j}_{LT,c,I(1),\bar{\mathbb{F}}_p}$ the fundamental representation at $I(1)$-level. This gives a long exact sequence

$$0 \to H^1(I(1), \widehat{H}^0_{LT,c,\bar{\mathbb{F}}_p}) \to H^1_{LT,c,I(1),\bar{\mathbb{F}}_p} \to (\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)} \to H^2(I(1), \widehat{H}^0_{LT,c,\bar{\mathbb{F}}_p})$$

Because $\widehat{H}^0_{LT,c,\bar{\mathbb{F}}_p} = 0$ as $H^0_c(\mathcal{M}_{LT}, \bar{\mathbb{F}}_p) = 0$ we have an $\mathcal{H}$-equivariant isomorphism

$$H^1_{LT,c,I(1),\bar{\mathbb{F}}_p} \simeq (\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$$

This means that if $\pi^{I(1)}$ appears in $(\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$ then it appears also in $H^1_{LT,c,I(1),\bar{\mathbb{F}}_p}$. But because $H^1_{LT,c,I(1),\bar{\mathbb{F}}_p}$ consists as a $\mathrm{GL}_2(\mathbb{Q}_p)^\circ = (\ker(\det))$-representation of multiple copies of $H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$, it is enough to show that $\pi^{I(1)}$ does not appear in

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Q}_p)^\circ \mathbb{Q}_p}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p) \simeq H^1_c(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)^{\oplus 2}$$

To prove it, it suffices to show that no supersingular $\mathcal{H}$-module appears in it. Let M be any supersingular $\mathcal{H}$-module. Then we know that it is 2-dimensional and of the form

$M_2(0, z, \omega)$ as in Section 3.2 of [Vig04], where ω is a character of $I/I(1)$. If we write $I/I(1) = \mathbb{F}_p^\times \times \mathbb{F}_p^\times$ and $\omega = \eta_1 \otimes \eta_2$ then $M = (\eta_1 \otimes \eta_2) \oplus (\eta_2 \otimes \eta_1)$ as a $I/I(1)$-module. If M appears in $H_c^1(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$, then $I/I(1)$ acts on M by determinant and hence $\eta_1 = \eta_2$. This would mean that $H_c^1(\mathcal{M}_{LT,I(1)})_{\eta_1}$ is at least 2-dimensional, which is a contradiction. Hence the only possibility is in this case that two products $\eta_1 \otimes \eta_1$ appear in different copies of $H_c^1(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)^{\oplus 2}$. But this is also not possible. Indeed, let us consider the operator $S = I(1)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)I(1)$. It acts on both copies of $H_c^1(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$ in the same way, as can be seen by looking at the decomposition $LT = \coprod LT^{(i)}$ (here LT means the Rapoport-Zink space for $\mathrm{GL}_2(\mathbb{Q}_p)$) which is $\mathrm{GL}_2(\mathbb{Q}_p)^\circ \times D(\mathbb{Q}_p)^\circ$-equivariant and hence we have $\mathrm{GL}_2(\mathbb{Q}_p)^\circ$-equivariantly that $LT^{(i)} \simeq LT^{(0)}$. But on the other hand, Vigneras in [Vig04] proves that S acts on M by two different scalars (one being zero, other non-zero) and hence M cannot appear in $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Q}_p)^\circ \mathbb{Q}_p}^{\mathrm{GL}_2(\mathbb{Q}_p)} H_c^1(\mathcal{M}_{LT,I(1)}, \bar{\mathbb{F}}_p)$. All in all, we conclude that $\pi^{I(1)}$ does not appear in $(\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p})^{I(1)}$ and hence

Theorem III.8.2. *The supersingular representation π does not appear in $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p}$.*

We could rephrase it also as

$$\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p,(\pi)} = 0$$

Remark III.8.3. *Observe that the above proof does not use in any particular form the fact that we are working with $\mathrm{GL}_2(\mathbb{Q}_p)$, besides the fact that the functor $\pi \mapsto \pi^{I(1)}$ induces a bijection between supersingular representations and supersingular $\mathcal{H}$-modules. Apart from that, the results of Vigneras and Yoshida holds for $\mathrm{GL}_2(F)$ as well, where F is a finite extension $\mathbb{Q}_p$ and show that there are no supersingular modules in the cohomology with compact support of the Lubin-Tate tower at the pro-p Iwahori level. This leads to the conclusion that supersingular representations of $\mathrm{GL}_2(F)$ attached to these supersingular modules by the construction of Paskunas (see [Pas04]) do not appear in the cohomology with compact support of the Lubin-Tate tower at infinite level. We remark that, contrary to $F = \mathbb{Q}_p$ case, those supersingular representations constructed by Paskunas do not conjecturally give all the supersingular representations of $\mathrm{GL}_2(F)$.*

The above theorem gives us, when combined with the exact sequence for the supersingular locus, an appearance of the mod p local Langlands correspondence in the cohomology of the ordinary locus (in contrast with the mod l situation).

Corollary III.8.4. *We have an $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$-equivariant injection*

$$\pi \otimes \bar{\rho} \hookrightarrow \widehat{H}^1_{ord,\bar{\mathbb{F}}_p}$$

Moreover, this vanishing result can be used in the study of non-admissibility and in the description of the cohomology of certain Shimura curves.

III.8.4 Non-admissibility

We will now show that our cohomology groups are non-admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We start with:

Proposition III.8.5. *The $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\widehat{H}^2_{ss,c,\bar{\mathbb{F}}_p}$ and $\widehat{H}^1_{ord,\bar{\mathbb{F}}_p}$ are non-admissible smooth $\bar{\mathbb{F}}_p$-representations.*

Proof. If one of them would be admissible, then also the second would because of the exact sequence

$$\widehat{H}^1_{ss,c,\bar{\mathbb{F}}_p} \to \widehat{H}^1_{\bar{\mathbb{F}}_p} \to \widehat{H}^1_{ord,\bar{\mathbb{F}}_p} \to \widehat{H}^2_{ss,c,\bar{\mathbb{F}}_p} \to \widehat{H}^2_{\bar{\mathbb{F}}_p}$$

But we know that $\widehat{H}^1_{ord,\bar{\mathbb{F}}_p}$ is an induced representation

$$\mathrm{Ind}^{\mathrm{GL}_2}_{B(\infty)}\left(\bigoplus_a \widehat{H}^1_{a,\infty,\bar{\mathbb{F}}_p}\right)$$

so if it were admissible, then the localisation of it at π would have to vanish. This is not possible by the corollary above. □

Corollary III.8.6. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\widehat{H}^2_{LT,c,\bar{\mathbb{F}}_p}$ *is a non-admissible smooth* $\bar{\mathbb{F}}_p$*-representation.*

Proof. Follows from the proposition above. □

III.8.5 Cohomology of Shimura curves

We will briefly sketch another consequence of vanishing of $\widehat{H}^1_{LT,c,\bar{\mathbb{F}}_p,(\pi)}$ and $H^1_c(\mathcal{M}_{LT}, \bar{\mathbb{F}}_p)_{(\pi)}$. Now recall the Faltings isomorphism (see [Far08]) which gives us

$$H^1_c(\mathcal{M}_{LT}, \bar{\mathbb{F}}_p)_{(\pi)} = H^1_c(\mathcal{M}_{Dr}, \bar{\mathbb{F}}_p)_{(\pi)} = 0$$

where $\mathcal{M}_{Dr}$ is the Drinfeld tower at infinity (see [Dat12] for details). We have a spectral sequence coming from the p-adic uniformisation of the Shimura curve Sh associated to the algebraic group G'' arising from the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at p and some other prime q:

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^{2-q}_c(\mathcal{M}_{Dr,K_p}, \bar{\mathbb{F}}_p), C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}), \bar{\mathbb{F}}_p)^{K^p}) \Rightarrow H^{p+q}_c(Sh^{an}_{K_pK^p}, \bar{\mathbb{F}}_p)$$

where we have denoted by G' the algebraic group arising from the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at q and ∞. For this, see [Far04] where it is proven for $\bar{\mathbb{Q}}_l$ but the proof works also for $\bar{\mathbb{F}}_p$ (the proof is also contained in the appendix B of [Dat06]).

Choose any non-Eisenstein maximal ideal $\mathfrak{n}$ in the Hecke algebra of G'' whose associated Galois representation corresponds at p to the supersingular representation π we have chosen before. Take the direct limit over K_p and localise the above spectral sequence at $\mathfrak{n}$ to get

$$\mathrm{Ext}^p_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^{2-q}_c(\mathcal{M}_{Dr}, \bar{\mathbb{F}}_p)_{(\pi)}, C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}), \bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}) \Rightarrow H^{p+q}_c(Sh^{an}_{K^p}, \bar{\mathbb{F}}_p)_{\mathfrak{n}}$$

The localisation of $H^{2-q}_c(\mathcal{M}_{Dr}, \bar{\mathbb{F}}_p)$ at π appears because $C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}), \bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}$ is π-isotypic. We remark here that spaces $H^{2-q}_c(\mathcal{M}_{Dr}, \bar{\mathbb{F}}_p)$ are admissible as can be seen from the spectral sequence and the fact that other appearing spaces are admissible. Using our vanishing result we get an interesting isomorphism

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}(H^2_c(\mathcal{M}_{Dr}, \bar{\mathbb{F}}_p)_{(\pi)}, C^\infty(G'(\mathbb{Q})\backslash G'(\mathbb{A}), \bar{\mathbb{F}}_p)^{K^p}_{\mathfrak{n}}) \simeq H^1_c(Sh^{an}_{K^p}, \bar{\mathbb{F}}_p)_{\mathfrak{n}}$$

This can be possibly used to study the mod p cohomology of the Shimura curve Sh. We shall treat this issue elsewhere.

III.9 Concluding remarks

Let us finish by giving some remarks and stating natural questions.

III.9.1 l-adic case

Observe that our arguments work well also in the mod $l \neq p$ setting and circumvent the use of vanishing cycles. The idea of localisation at a supersingular (supercuspidal) representation appears also in the work of Dat. See especially [Dat12] where the author discusses localisations both for GL_n and quaternion algebras and then uses it to describe the supercuspidal part of the cohomology.

One might want also to see [Shi], which bears some resemblance to certain arguments we use. Shin describes the mod l cohomology of Shimura varieties by using results of Dat about the mod l cohomology of the Lubin-Tate tower. In our work, we start from global results of Emerton to deduce from them statements about local objects.

III.9.2 Beyond modular curves

The geometric arguments we have given also applies to Shimura curves considered by Carayol in [Car86] and we can consider similar exact sequences relating the ordinary locus and the supersingular locus in this setting. Nevertheless, in this case we cannot go on with arguments as we do not have a definition of the mod p local Langlands correspondence for extensions of $\mathbb{Q}_p$. In fact, such a construction seems a little bit problematic as might be seen from the work of Breuil-Paskunas ([BP12]), where the authors show that there are much more automorphic representations than Galois representations. The hope is that by looking at the cohomology of the Lubin-Tate tower, one should be able to tell how the correspondence should look like. We will pursue this subject in our subsequent work.

III.9.3 Adic spaces

We have chosen to work with Berkovich spaces, but one might as well wonder how the things translate into the setting of adic spaces of R. Huber ([Hub96]). In fact, everything that we have considered can be rewritten in the language of adic spaces and we might consider the same long exact sequences as above (though these exact sequences will be inversed due to the fact that adic spaces behave like formal schemes). The main difference between those two contexts lies in the ordinary locus which in the case of adic spaces will contain additional points which lie in the closure of the ordinary locus from the setting of Berkovich spaces. Nevertheless, the cohomology groups in both settings will be similar and we refer a reader to Chapter IV for details. Let us remark also, that the comparison between mod p étale cohomology of a formal scheme and its (adic) analytification is proved in Theorem 3.7.2 of [Hub96].

III.9.4 Serre's letters

Though it does not appear explicitly in our work (besides the comparison of Hecke algebras), we were influenced by two letters written by Jean-Pierre Serre (see [Ser96]). It is there that in some sense appears for the first time the modified mod l Local Langlands correspondence which goes under the name of the universal unramified representation (see the letter to Kazhdan). Indeed, if we were to suppose that our global lift $\bar{\rho}$ which we have used is actually unramified everywhere outside p, then there is no need to recall either the modified mod l Local Langlands correspondence or new vectors, and we could formulate everything in the language of Serre.

Chapter IV

On p-adic non-abelian Lubin-Tate theory

IV.1 Introduction

This Chapter is a sequel to Chapter III, where we have studied the mod p étale cohomology of the Lubin-Tate tower. Here we turn to the study of the p-adic completed and analytic cohomologies. There are two goals which we want to accomplish. The first one is to show a result analogous to the one obtained in Chapter III, namely to show that the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in the étale cohomology of the Lubin-Tate tower at infinity. The methods we use are partly those of Chapter III (localisation at a supersingular representation; use of the local-global compatibility of Emerton), though we approach them differently by working in the setting of adic spaces (we have worked with Berkovich spaces in Chapter III). This gives us more freedom as we can work directly at the infinite level (modular curves at the infinite level; Lubin-Tate tower at the infinite level) thanks to the work of Scholze on perfectoid spaces ([Sch12a], [Sch13], [SW13]). In this way, we do not need anymore to pass to the limit in the cohomology, as working at the infinite level is the same as working with the completed cohomology (see Chapter IV of [Sch13] for torsion coefficients). We prove our main result (Theorem 4.3) for local Galois representations ρ_p which are restrictions of some global pro-modular (a notion from [Eme11a]) representations ρ and such that the mod p reduction $\bar{\rho}_p$ is absolutely irreducible. We need these assumptions in order to be able to use the main result of [Eme11a].

The second goal of this Chapter is to discuss the folklore conjecture which roughly states that the p-adic local Langlands correspondence appears in the de Rham cohomology of the Drinfeld tower. As far as we know, this conjecture is not stated anywhere explicitly in the literature, though there was some work done towards it. The reader should consult [Sch10] for some partial progress at the 0-th level of the tower. Thanks to the work of Scholze-Weinstein ([SW13]) we can work directly at the infinite level which we do. Moreover, because of the duality of Rapoport-Zink spaces at the infinite level (which goes back to Faltings; see Section 7 of [SW13]), we know that the Drinfeld space at infinity $\mathcal{M}_{Dr,\infty}$ is isomorphic to the Lubin-Tate space at infinity $\mathcal{M}_{LT,\infty}$ and hence we can consider only the Lubin-Tate tower which is easier to relate to modular curves.

As to the folklore conjecture, we give a short argument at the beginning of Section 4, which explains why the de Rham cohomology of $\mathcal{M}_{LT,\infty}$ simplifies greatly. The reason is that for any perfectoid space X (hence for $\mathcal{M}_{LT,\infty}$ after [SW13]) the cohomology groups of j-th differentials $H^i(X, \Omega^j_X)$ vanishes for $j > 0$ and any i. This reduces the study of the de Rham cohomology to the study of the cohomology of the structure sheaf (which we refer to as the analytic cohomology - with topology defined by open subsets) which should be a good substitute for the de Rham cohomology in the setting of perfectoid spaces. We state the folklore conjecture for the analytic cohomology in the last section.

At the end we remark that one problem with the de Rham cohomology for perfectoid spaces, if one would like to define it in some meaningful way, is the lack of finiteness result. We should mention the work of Cais ([Cai09]), where the author consider integral structures on the de Rham cohomology of curves. The aim is to p-adically complete the de Rham cohomology of the tower of modular curves, as was done with the étale cohomology by Emerton ([Eme06c]). It seems interesting to determine what one would get by applying his construction at each finite level and then passing to the limit and how it would relate to the de Rham cohomology of the modular curve at the infinite level.

IV.2 Modular curves at infinity

In this section we review the geometric background which we use. We describe modular curves (and their compactifications) at the infinite level and we deal with the ordinary locus and the supersingular locus. We will use the language of adic spaces for which the reader should consult [Hub96] and [Sch12a].

We let E be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}$ and the residue field $k = \mathcal{O}/\varpi$ where ϖ is a uniformiser. This is our coefficient field.

IV.2.1 Geometry of modular curves

We denote open modular curves over $\mathbb{C}$ for an open compact subset $K \subset \mathrm{GL}_2(\mathbb{A}_f)$ by

$$Y(K) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C}\backslash\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K$$

There is a canonical algebraic model of it over $\mathbb{Q}$. We fix some complete and algebraically closed extension C of $\mathbb{Q}_p$. Let $\mathcal{O}_C$ be the ring of integers of C. We consider modular curves as adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$ which we may do after base-changing each $Y(K)$.

We let $X(K)$ be the compactification of $Y(K)$, which we also consider as an adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$. We will work with modular curves at the infinite level. We recall Scholze's results. We use $\sim$ in the sense of Definition 2.4.1 in [SW13].

Theorem IV.2.1. *For any sufficiently small level $K^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ there exist adic spaces $Y(K^p)$ and $X(K^p)$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ such that*

$$Y(K^p) \sim \varprojlim_{K_p} Y(K_p K^p)$$

$$X(K^p) \sim \varprojlim_{K_p} X(K_p K^p)$$

where K_p runs over open compact subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. See Theorem III.1.2 in [Sch13]. □

In what follows we will write $Y = Y(K^p)$ and $X = X(K^p)$, having fixed one tame level K^p throughout the text.

For the maximal compact open subgroup $\mathrm{GL}_2(\mathbb{Z}_p)$ we can define the supersingular locus $Y(\mathrm{GL}_2(\mathbb{Z}_p)K^p)_{\mathrm{ss}}$ (respectively, the ordinary locus $Y(\mathrm{GL}_2(\mathbb{Z}_p)K^p)_{\mathrm{ord}}$) as the inverse image under the reduction of the set of supersingular points (resp. closure of the inverse image of the ordinary locus) in the special fiber of $Y(\mathrm{GL}_2(\mathbb{Z}_p)K^p)$. Then for any compact open subgroup $K_p \subset \mathrm{GL}_2(\mathbb{Z}_p)$, we define $Y(K_pK^p)_{\mathrm{ss}}$ (resp. $Y(K_pK^p)_{\mathrm{ord}}$) as the pullback of $Y(\mathrm{GL}_2(\mathbb{Z}_p)K^p)_{\mathrm{ss}}$ (resp. $Y(\mathrm{GL}_2(\mathbb{Z}_p)K^p)_{\mathrm{ord}}$). Hence $Y(K_pK^p)_{\mathrm{ord}}$ is the complement of $Y(K_pK^p)_{\mathrm{ss}}$ and hence a closed subspace of $Y(K_pK^p)$. We define similarly the supersingular locus $X(K_pK^p)_{\mathrm{ss}}$ and the ordinary locus $X(K_pK^p)_{\mathrm{ord}}$ of $X(K_pK^p)$. Using the pullback from the finite level, we define also $X_{\mathrm{ss}}, Y_{\mathrm{ss}}, X_{\mathrm{ord}}, Y_{\mathrm{ord}}$ at the infinite level. The reader may consult the discussion in [Sch13] which appears after Theorem III.1.2.

Theorem IV.2.2. *There exist adic spaces Y_{ss}, Y_{ord} and X_{ss}, X_{ord} over $\mathrm{Spa}(C, \mathcal{O}_C)$ such that*

$$Y_{\mathrm{ss}} \sim \varprojlim_{K_p} Y(K_pK^p)_{\mathrm{ss}}$$

$$Y_{\mathrm{ord}} \sim \varprojlim_{K_p} Y(K_pK^p)_{\mathrm{ord}}$$

and similarly for X_{ss} and X_{ord}. Here K_p runs over open compact subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. Follows from Proposition 2.4.3 in [SW13]. □

One of the main results of [Sch13] (Theorem III.1.2), is the construction of the Hodge-Tate period map π_{HT} which is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant morphism

$$\pi_{\mathrm{HT}} : X \to (\mathbb{P}^1)^{ad}$$

where $(\mathbb{P}^1)^{ad}$ is the adic projective line over $\mathrm{Spa}(C, \mathcal{O}_C)$. This morphism commutes with Hecke operators away from p for the trivial action of these Hecke operators on $(\mathbb{P}^1)^{ad}$. Moreover, the decomposition of X into the supersingular and the ordinary locus can be seen at the flag variety level. Namely, we have (see the discussion after Theorem III.1.2 in [Sch13])

$$X_{\mathrm{ord}} = \pi_{\mathrm{HT}}^{-1}(\mathbb{P}^1(\mathbb{Q}_p))$$
$$X_{\mathrm{ss}} = \pi_{\mathrm{HT}}^{-1}((\mathbb{P}^1)^{ad} \backslash \mathbb{P}^1(\mathbb{Q}_p))$$

We let

$$j : X_{\mathrm{ss}} \hookrightarrow X$$

denote the open immersion and we put

$$i : X_{\mathrm{ord}} \to X$$

For any injective étale sheaf I on X we have an exact sequence of global sections

$$0 \to \Gamma_{X_{\mathrm{ord}}}(X, I) \to \Gamma(X, I) \to \Gamma(X_{\mathrm{ss}}, j^*I) \to 0$$

which gives rise to the exact sequence of étale cohomology for any étale sheaf F on X (take an injective resolution $I^\bullet$ of F and apply the above exact sequence to it)

$$\ldots \to H^0(X_{\mathrm{ss}}, j^*F) \to H^1_{X_{\mathrm{ord}}}(X, F) \to H^1(X, F) \to H^1(X_{\mathrm{ss}}, j^*F) \to \ldots$$

By specialising F to a constant sheaf $\mathcal{O}/\varpi^s\mathcal{O}$ ($s > 0$) we get an exact sequence

$$\ldots \to H^0(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1_{X_{\mathrm{ord}}}(X, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1(X, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O}) \to \ldots$$

We can obtain an analogous exact sequence for analytic cohomology which we review later. In what follows we will be interested in the p-adic completed cohomology, introduced by Emerton in [Eme06c]. We define

$$H^i(X, E) = \left(\varprojlim_s H^i_{et}(X, \mathcal{O}/\varpi^s\mathcal{O}) \right) \otimes_{\mathcal{O}} E$$

Using the fact that $X \sim \varprojlim_{K_p} X(K_pK^p)$ and Theorem 7.17 in [Sch12a], we have

$$H^i(X, E) = \left(\varprojlim_s \varinjlim_{K_p} H^i_{et}(X(K_pK^p), \mathcal{O}/\varpi^s\mathcal{O}) \right) \otimes_{\mathcal{O}} E$$

which is precisely the p-adic completed cohomology of Emerton. We use similar definitions for X_{ss} and X_{ord}.

IV.2.2 Ordinary locus

We recall the decomposition of the ordinary locus, which implies that representations arising from the cohomology are induced from a Borel subgroup. This is a classical and well-known result, but we shall give it a short proof using recent results of Scholze and the fact that we are working at the infinite level. We have given a different proof in Section 2.2 of Chapter III.

Proposition IV.2.3. *The étale (and also analytic) cohomology of X_{ord} is induced from a Borel subgroup $B(\mathbb{Q}_p)$ of upper-triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$*

$$H^i_{X_{\mathrm{ord}}}(X, F) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} W(F)$$

where $F = \mathcal{O}/\varpi^s\mathcal{O}$ is an étale constant sheaf on X_{ord} and $W(F)$ is a certain cohomology space defined below in the proof which depends on F and admits an action of $B(\mathbb{Q}_p)$.

Proof. Recall that $X_{\mathrm{ord}} = \pi_{\mathrm{HT}}^{-1}(\mathbb{P}^1(\mathbb{Q}_p))$, where π_{HT} is the Hodge-Tate period map. Let $\infty = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \in \mathbb{P}^1(\mathbb{Q}_p)$. The stabilizer of ∞ is equal to the Borel subgroup $B(\mathbb{Q}_p)$ of upper-triangular matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$. We have

$$H^i_{X_{\mathrm{ord}}}(X, F) = H^i(X_{\mathrm{ord}}, i^!F) = H^0(\mathbb{P}^1(\mathbb{Q}_p), R^i\pi_{\mathrm{HT},*}(i^!F)) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\{\infty\}, R^i\pi_{\mathrm{HT},*}(i^!F))$$

where the second isomorphism follows from the continuity of π_{HT}. Those are all smooth spaces, because $H^i(X_{\mathrm{ord}}, i^!F)$ is smooth (by Theorem IV.2.2 and recalling that $F = \mathcal{O}/\varpi^s\mathcal{O}$).

□

IV.2.3 Supersingular locus

Let us denote by $\mathcal{M}_{LT,K_p}$ the Lubin-Tate space for $\mathrm{GL}_2(\mathbb{Q}_p)$ at the level K_p, where K_p is a compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$. See Section 6 of [SW13] for a definition. We just recall that this is a deformation space for p-divisible groups with an additional data and it is a local analogue of modular curves. We view it as an adic space over $\mathrm{Spa}(C, \mathcal{O}_C)$.

Once again, we would like to pass to the limit and work with the space at infinity.

Theorem IV.2.4. *There exists a perfectoid space $\mathcal{M}_{LT,\infty}$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ such that*

$$\mathcal{M}_{LT,\infty} \sim \varprojlim_{K_p} \mathcal{M}_{LT,K_p}$$

where K_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. This is Theorem 6.3.4 from [SW13]. One defines $\mathcal{M}_{LT,\infty}$ as a deformation functor of p-divisible groups with a trivialization of Tate modules. □

To compare X and $\mathcal{M}_{LT,\infty}$ (hence their cohomology groups) we use the p-adic uniformisation of Rapoport-Zink at the infinite level. Let us denote by D the quaternion algebra over $\mathbb{Q}$ which is ramified exactly at p and ∞. The p-adic uniformisation of Rapoport-Zink states

Proposition IV.2.5. *We have an isomorphism of adic spaces*

$$X_{\mathrm{ss}} \sim \varprojlim_{K_p} D^\times(\mathbb{Q}) \backslash \left(\mathcal{M}_{LT,\infty} \times \mathrm{GL}_2(\mathbb{A}_f^p) \right) / K_p K^p$$

This isomorphism is equivariant with respect to the action of the Hecke algebra of level K^p.

Proof. The uniformisation at finite level is proved in [RZ96]. We adify their construction and pass to the limit using Theorem IV.2.2. □

IV.3 On admissible representations

Having recalled the geometric results, we now pass to the results about representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We review and prove some facts about Banach admissible representations. Then we recall recent results of Paskunas which allow us to consider the localisation functor.

IV.3.1 General facts and definitions

We start with general facts about admissible representations. In our definitions, we will follow [Eme10a]. As before, let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, a uniformiser ϖ and the residue field k. Let $C(\mathcal{O})$ denote the category of complete Noetherian local $\mathcal{O}$-algebras having finite residue fields. Let us consider $A \in C(\mathcal{O})$. We let G be any connected reductive group over $\mathbb{Q}_p$.

Definition IV.3.1. *Let V be a representation of G over A. A vector $v \in V$ is smooth if v is fixed by some open subgroup of G and v is annihilated by some power $\mathfrak{m}^i$ of the maximal ideal of A. Let V_{sm} denote the subset of smooth vectors of V. We say that a G-representation V over A is smooth if $V = V_{sm}$.*

A smooth G-representation V over A is admissible if $V^H[\mathfrak{m}^i]$ (the $\mathfrak{m}^i$-torsion part of the subspace of H-fixed vectors in V) is finitely generated over A for every open compact subgroup H of G and every $i \geq 0$.

Definition IV.3.2. *We say that a G-representation V over A is ϖ-adically continuous if V is ϖ-adically separated and complete, $V[\varpi^\infty]$ is of bounded exponent, $V/\varpi^i V$ is a smooth G-representation for any $i \geq 0$.*

Definition IV.3.3. *A ϖ-adically admissible representation of G over A is a ϖ-adically continuous representation V of G over A such that the induced G-representation on $(V/\varpi V)[\mathfrak{m}]$ is admissible smooth over $A/\mathfrak{m}$.*

This definition implies that for every $i \geq 0$, the G-representation $V/\varpi^i V$ is smooth admissible. See Remark 2.4.8 in [Eme10a].

Definition IV.3.4. *We call a G-representation V over E Banach admissible if there exists a G-invariant lattice $V^\circ \subset V$ over $\mathcal{O}$ such that V° is ϖ-adically admissible as a representation of G over $\mathcal{O}$.*

Proposition IV.3.5. *The category of ϖ-adically admissible representations of G over A is abelian and moreover, a Serre subcategory of the category of ϖ-adically continuous representations.*

Proof. The category is anti-equivalent to the category of finitely generated augmented modules over certain completed group rings. See Proposition 2.4.11 in [Eme10a]. □

Now, we will prove an analogue of Lemma 13.2.3 from [Boy99] in the $l = p$ setting. We will later apply this lemma to the cohomology of the ordinary locus to force its vanishing after localisation at a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We have proved it already in the mod p setting as Lemma 3.3 in Chapter III.

Lemma IV.3.6. *For any smooth admissible representation (π, V) of the parabolic subgroup $P \subset G$ over A, the unipotent radical U of P acts trivially on V.*

Proof. Let L be a Levi subgroup of P, so that $P = LU$. Let $v \in V$ and let $K_P = K_L K_U$ be a compact open subgroup of P such that $v \in V^{K_P}$. We choose an element z in the centre of L such that:

$$z^{-n} K_P z^n \subset ... \subset z^{-1} K_P z \subset K_P \subset z K_P z^{-1} \subset ... \subset z^n K_P z^{-n} \subset ...$$

and $\bigcup_{n \geq 0} z^n K_P z^{-n} = K_L U$. For every n and m, modules $V^{z^{-n} K_P z^n}[\mathfrak{m}^i]$ and $V^{z^{-m} K_P z^m}[\mathfrak{m}^i]$ are of the same length for every $i \geq 0$, as they are isomorphic via $\pi(z^{n-m})$. We naturally have an inclusion $V^{z^{-n} K_P z^n}[\mathfrak{m}^i] \subset V^{z^{-m} K_P z^m}[\mathfrak{m}^i]$ and hence we get an equality $V^{z^{-n} K_P z^n}[\mathfrak{m}^i] = V^{z^{-m} K_P z^m}[\mathfrak{m}^i]$. By smoothness, there exists i such that $v \in V[\mathfrak{m}^i]$. Thus we have $v \in V^{K_P}[\mathfrak{m}^i] = V^{z^{-n} K_P z^n}[\mathfrak{m}^i] = V^{K_L U}[\mathfrak{m}^i]$ which is contained in $V^U[\mathfrak{m}^i]$. □

Lemma IV.3.7. *For any ϖ-adically admissible representation (π, V) of the parabolic subgroup $P \subset G$ over A, the unipotent radical U of P acts trivially on V.*

Proof. By the remark above, each $V/\varpi^i V$ is admissible, and hence the preceding lemma applies, so that U acts trivially on each $V/\varpi^i V$. But $V = \varprojlim_i V/\varpi^i V$, hence U acts trivially on V. □

Later on, we will need the following result.

Lemma IV.3.8. *Let $V = \mathrm{Ind}_P^G W$ be a parabolic induction. If V is a ϖ-adically admissible representation of G over A, then W is a ϖ-adically admissible representation of P over A.*

Proof. This follows from Theorem 4.4.6 in [Eme10a]. □

IV.3.2 Localisation functor

Let π be a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k. Recall that supersingular representations correspond to irreducible two-dimensional Galois representations under the local Langlands correspondence modulo p. See [Ber11].

In [Pas13], Paskunas has proved the following result (Proposition 5.32)

Proposition IV.3.9. *We have a decomposition:*

$$\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O}) = \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O})_{(\pi)} \oplus \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O})^{(\pi)}$$

where $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O})$ *is the (abelian) category of smooth admissible* $\mathcal{O}/\varpi^s\mathcal{O}$*-representations admitting a central character* ξ*,* $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O})_{(\pi)}$ *(resp.* $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O})^{(\pi)}$*) is the subcategory of it consisting of representations* Π *such that all irreducible subquotients of* Π *are (resp. are not) isomorphic to* π*.*

We denote the projection

$$\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O}) \mapsto \mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p),\xi}(\mathcal{O}/\varpi^s\mathcal{O})_{(\pi)}$$

by

$$V \mapsto V_{(\pi)}$$

and we refer to it as the localisation functor with respect to π. The existence of a central character follows from the work [DS13] for irreducible representations. In what follows, we will ignore the central character ξ in our notations, though whenever we localise we mean that we firstly localise the representation at ξ and then we project as above.

IV.4 p-adic Langlands correspondence and analytic cohomology

In this section we show that the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ appears in the étale cohomology of the Lubin-Tate tower at infinity. We also state a conjecture about the analytic cohomology of the Lubin-Tate perfectoid.

IV.4.1 p-adic Langlands correspondence

For this section we refer the reader to [Ber11] (for the Colmez functor) and [Pas13] (for equivalence of categories). We recall that Colmez has constructed a covariant exact functor $\mathbb{V}$

$$\mathbb{V} : \mathrm{Rep}_{\mathcal{O}}(\mathrm{GL}_2(\mathbb{Q}_p)) \to \mathrm{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$$

which sends $\mathcal{O}$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\mathcal{O}$-representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Moreover this functor is compatible with deformations and induces an equivalence of categories when restricted to appropiate sub-representations. We call the inverse of this functor the p-adic local Langlands correspondence and we denote it by $B(\cdot)$. For our applications we will only need the fact that for p-adic continuous representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$, $B(\rho)$ is a Banach admissible E-representation. Furthermore, when ρ is irreducible, then $B(\rho)$ is topologically irreducible.

Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be the reduction of ρ which we assume to be irreductible. Let π be the supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k which corresponds to $\bar{\rho}$ by the mod p local Langlands correspondence, that is $\mathbb{V}(\pi) = \bar{\rho}$. Then one knows that $B(\rho)$ is an object of the category $\mathrm{Rep}^{adm}_{\mathrm{GL}_2(\mathbb{Q}_p)}(E)_{(\pi)}$ defined above.

IV.4.2 Étale cohomology

We recall the results of Emerton on the p-adic completed cohomology and then we prove that certain p-adic Banach representations appear in the étale cohomology of $\mathcal{M}_{LT,\infty}$. From now on we work in the global setting. Let $\rho : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$ be a continuous Galois representation. We assume that it is unramified outside some finite set $\Sigma = \Sigma_0 \cup \{p\}$. Moreover we assume that its reduction $\bar{\rho}$ is modular (that is, isomorphic to the reduction of a Galois representation associated to some automorphic representation on $\mathrm{GL}_2(\mathbb{Q})$) and $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible.

Let us recall that we have introduced spaces X, Y depending on the tame level K^p. We now assume that K^p is unramified outside Σ. We shall factor K^p as $K^p = K_{\Sigma_0} K^{\Sigma_0}$. Let $\mathbb{T}_\Sigma = \mathcal{O}[T_l, S_l]_{l \notin \Sigma}$ be the commutative $\mathcal{O}$-algebra with T_l, S_l formal variables indexed by $l \notin \Sigma$. This is a standard Hecke algebra which acts on modular curves by correspondences.

To the modular Galois representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ we can associate the maximal Hecke ideal $\mathfrak{m}$ of $\mathbb{T}_\Sigma$ which is generated by ϖ (uniformiser of $\mathcal{O}$) and elements $T_l + a_l$ and $lS_l - b_l$, where l is a place of $\mathbb{Q}$ which does not belong to Σ, $X^2 + \bar{a}_l X^1 + \bar{b}_l$ is the characteristic polynomial of $\bar{\rho}(\mathrm{Frob}_l)$ and a_l, b_l are any lifts of $\bar{a}_l, \bar{b}_l$ to $\mathcal{O}$.

We let $\pi_{\Sigma_0}(\rho) = \otimes_{l \in \Sigma_0} \pi_l(\rho_l)$ be the tensor product of E-representations of $\mathrm{GL}_2(\mathbb{Q}_l)$ ($l \in \Sigma_0$) associated to $\rho_l = \rho_{|G_{\mathbb{Q}_l}}$ by the generic version of the l-adic local Langlands correspondence (see [EH11]).

We assume that ρ is pro-modular in the sense of Emerton (see [Eme11a]). Let $\mathfrak{p}$ be the prime ideal of $\mathbb{T}_\Sigma$ associated to ρ (similarly as we have associated $\mathfrak{m}$ to $\bar{\rho}$). We have an obvious inclusion $\mathfrak{p} \subset \mathfrak{m}$. We remark that pro-modularity is a weaker condition than modularity and it can be seen as saying that ρ is a Galois representation associated to some p-adic Hecke eigensystem coming from the completed Hecke algebra (the projective limit over finite level Hecke algebras). Recall that we have assumed that $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ is absolutely irreducible. This permits us to state the main result of [Eme11a] as

Theorem IV.4.1. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ be a continuous Galois representation which is pro-modular and such that $\bar{\rho}_p$ is absolutely irreducible. Then we have a $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Q}_p) \times \prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbb{Q}_l)$-equivariant isomorphism of Banach admissible E-representations.*

$$H^1(Y, E)[\mathfrak{p}] \simeq \rho \otimes_E B(\rho_p) \otimes_E \pi_{\Sigma_0}(\rho)^{K_{\Sigma_0}}$$

We recall that the cohomology group on the left is the p-adic completed cohomology of Emerton

$$H^1(Y, E) = \left(\varprojlim_{s} \varinjlim_{K_p} H^1_{et}(Y(K^p K_p), \mathcal{O}/\varpi^s \mathcal{O}) \right) \otimes_{\mathcal{O}} E$$

where K_p runs over compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let us remark that the Galois action of $G_{\mathbb{Q}_p}$ arises on Y, X, $\mathcal{M}_{LT,\infty}$ (which we treat as adic spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$) from the Galois action on the corresponding model over $\bar{\mathbb{Q}}_p$.

We also have a similar theorem for the compactification

Theorem IV.4.2. *With assumptions as in the theorem above, we have an isomorphism of Banach admissible K-representations*

$$H^1(X, E)_{\mathfrak{m}} \simeq H^1(Y, E)_{\mathfrak{m}}$$

In particular,

$$H^1(X,E)[\mathfrak{p}] \simeq \rho \otimes_E B(\rho_p) \otimes_E \pi_{\Sigma_0}(\rho)^{K_{\Sigma_0}}$$

Proof. We have assumed that $\bar{\rho}_p$ is absolutely irreducible and hence $\bar{\rho}$ is absolutely irreducible which implies that $\mathfrak{m}$ is a non-Eisenstein ideal. Now the theorem follows as in the proof of Proposition 7.7.13 of [Eme06b]. $\square$

We now come back to the exact sequence which we have obtained earlier

$$\ldots \to H^0(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1_{X_{\mathrm{ord}}}(X, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1(X, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O}) \to \ldots$$

By Theorem 2.1.5 of [Eme06c], we get that $H^1(X, \mathcal{O}/\varpi^s\mathcal{O})$ is a smooth admissible $\mathcal{O}/\varpi^s\mathcal{O}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, also $H^0(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O})$ is a smooth admissible $\mathcal{O}/\varpi^s\mathcal{O}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as $X_{\mathrm{ss}}(K_pK^p)$ has only finite number of connected components for each K_p and K^p. The category of smooth admissible $\mathcal{O}/\varpi^s\mathcal{O}$-representations is a Serre subcategory of the category of smooth (not necessarily admissible) $\mathcal{O}/\varpi^s\mathcal{O}$-representations (see Proposition 2.4.11 of [Eme10a]). Hence, as $H^1_{X_{\mathrm{ord}}}(X, \mathcal{O}/\varpi^s\mathcal{O})$ is smooth, we infer that it is also smooth admissible. By Proposition 2.4 we get that $H^1_{X_{\mathrm{ord}}}(X, \mathcal{O}/\varpi^s\mathcal{O})$ is induced from some representation $W(\mathcal{O}/\varpi^s\mathcal{O})$ of the Borel $B(\mathbb{Q}_p)$. We deduce from Lemma 3.8 (Theorem 4.4.6 in [Eme10a]) that $W(\mathcal{O}/\varpi^s\mathcal{O})$ is smooth admissible $\mathcal{O}/\varpi^s\mathcal{O}$-representation of $B(\mathbb{Q}_p)$. Thus, we can apply to it Lemma 3.7. If π is any supersingular k-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, it implies that

$$H^1_{X_{\mathrm{ord}}}(X, \mathcal{O}/\varpi^s\mathcal{O})_{(\pi)} = 0$$

because by Proposition 2.3 the representation $H^1_{X_{\mathrm{ord}}}(X, \mathcal{O}/\varpi^s\mathcal{O})$ is a smooth induction of an admissible representation, hence no supersingular representation appears as a subquotient of it.

Localising the exact sequence above at some supersingular k-representation π we get an injection

$$H^1(X, \mathcal{O}/\varpi^s\mathcal{O})_{(\pi)} \hookrightarrow H^1(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O})$$

By passing to the limit with s we get an injection

$$H^1(X,E)_{(\pi)} \hookrightarrow H^1(X_{\mathrm{ss}}, E)$$

We can now prove our main theorem

Theorem IV.4.3. *Let* $\rho : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$ *be a pro-modular representation. Assume that* $\bar{\rho}_p = \bar{\rho}_{|G_{\mathbb{Q}_p}}$ *is absolutely irreducible. Then we have a* $\mathrm{GL}_2(\mathbb{Q}_p) \times G_{\mathbb{Q}_p}$*-equivariant injection*

$$B(\rho_p) \otimes_E \rho_p \hookrightarrow H^1(\mathcal{M}_{LT,\infty}, E)$$

Proof. Let π be the mod p representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to $\bar{\rho}_p$ by the mod p local Langlands correspondence. It is a supersingular representation by our assumption that $\bar{\rho}_p$ is absolutely irreducible. Let $\mathfrak{p}$ be the prime ideal of $\mathbb{T}_\Sigma$ associated to ρ, where $\Sigma = \Sigma_0 \cup \{p\}$ is some finite set which contains p and all the primes at which ρ is ramified. As above we have

$$H^1(X,E)_{(\pi)} \hookrightarrow H^1(X_{\mathrm{ss}}, E)$$

and hence also

$$H^1(X,E)_{(\pi)}[\mathfrak{p}] \hookrightarrow H^1(X_{\mathrm{ss}}, E)[\mathfrak{p}]$$

Theorem 4.2 implies that (we keep track only of $G_{\mathbb{Q}_p}$-action instead of $G_{\mathbb{Q}}$)

$$B(\rho_p) \otimes_E \rho_p \otimes_E \pi_{\Sigma_0}(\rho)^{K_{\Sigma_0}} \hookrightarrow H^1(X_{\mathrm{ss}}, E)[\mathfrak{p}]$$

Let K'_{Σ_0} be a compact open subgroup of $\prod_{l \in \Sigma_0} \mathrm{GL}_2(\mathbb{Q}_l)$ for which we have $\dim \pi_{\Sigma_0}(\rho)^{K'_{\Sigma_0}} = 1$ where the dimension is over E. Such a subgroup always exists by classical results of Casselman (see [Cas73]). Remark that we are using here the generic local Langlands correspondence as explained in [EH11]: to generic representations we associate the same representation as in the classical correspondence up to the twist by the determinant and for non-generic ones we take Steinberg representations induced from a parabolic subgroup. This allows us to still appeal to [Cas73] for the conclusion (compare also with Section III.5). Hence we have

$$B(\rho_p) \otimes_E \rho_p \hookrightarrow H^1(X_{\mathrm{ss}}, E)[\mathfrak{p}]^{K'_{\Sigma_0}}$$

By Kunneth formula and Proposition 2.5 (the p-adic uniformisation of Rapoport-Zink) we get, in the same way as in Section III.6.1, that

$$H^1(X_{\mathrm{ss}}, E) = \left(H^1(\mathcal{M}_{LT,\infty}, E) \widehat{\otimes}_E \mathcal{S} \right)^{D^\times(\mathbb{Q}_p)}$$

where we have denoted by $\mathcal{S}$ the p-adic quaternionic forms of level K^p

$$\widehat{H}^0(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A})/K^p, E) = \left(\varprojlim_s \varinjlim_{K_p} H^0(D^\times(\mathbb{Q}) \backslash D^\times(\mathbb{A})/K_p K^p, \mathcal{O}/\varpi^s \mathcal{O}) \right) \otimes_{\mathcal{O}} E$$

where K_p runs over compact open subgroups of $D^\times(\mathbb{Q}_p)$. As $\mathrm{GL}_2(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$ act on $H^1(X_{\mathrm{ss}}, E)$ through $H^1(\mathcal{M}_{LT,\infty}, E)$ (i.e. they act trivially on $\mathcal{S}$) we conclude by the preceding discussion that

$$B(\rho_p) \otimes_E \rho_p \hookrightarrow H^1(\mathcal{M}_{LT,\infty}, E)$$

as wanted. □

IV.4.3 Cohomology with compact support

We show that the cohomology with compact support of the Lubin-Tate tower does not contain any p-adic representations which reduce to mod p supersingular representations. Recall we have morphisms

$$j : X_{\mathrm{ss}} \hookrightarrow X$$

and

$$i : X_{\mathrm{ord}} \to X$$

which give an exact sequence for any étale sheaf F on X

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

This leads to an exact sequence of the cohomology

$$... \to H^0(X_{\mathrm{ord}}, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1_c(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1(X, \mathcal{O}/\varpi^s\mathcal{O}) \to H^1(X_{\mathrm{ord}}, \mathcal{O}/\varpi^s\mathcal{O}) \to ...$$

Because $H^1(X, \mathcal{O}/\varpi^s\mathcal{O})$ is smooth admissible as a $\mathcal{O}/\varpi^s\mathcal{O}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ (by the result of Emerton) and $H^0(X_{\mathrm{ord}}, \mathcal{O}/\varpi^s\mathcal{O})$ is smooth admissible as a $\mathcal{O}/\varpi^s\mathcal{O}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ because at each finite level X_{ord} has a finite number of connected components, we infer that also $H^1_c(X_{\mathrm{ss}}, \mathcal{O}/\varpi^s\mathcal{O})$ is smooth admissible (as the category of admissible $\mathcal{O}/\varpi^s\mathcal{O}$-representations is a Serre subcategory of smooth $\mathcal{O}/\varpi^s\mathcal{O}$-representations). Passing to the limit with s, we infer that $H^1_c(X_{\mathrm{ss}}, E)$ is Banach admissible over E. This means that we can localise $H^1_c(X_{\mathrm{ss}}, E)$ at supersingular representations.

Let π be a supersingular k-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, where k is the residue field of K. Observe that if $H^1_c(X_{\mathrm{ss}}, E)_{(\pi)} \neq 0$, then also its reduction $H^1_c(X_{\mathrm{ss}}, k)_{(\pi)}$ would be non-zero. But Theorem 8.2 in Chapter III states that $H^1_c(X_{\mathrm{ss}}, k)_{(\pi)} = 0$. Hence we get

Theorem IV.4.4. *For any supersingular k-representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ we have*

$$H^1_c(X_{\mathrm{ss}}, E)_{(\pi)} = 0$$

In particular

$$H^1_c(\mathcal{M}_{LT,\infty}, E)_{(\pi)} = 0$$

Proof. The first part follows from the preceding discussion, the second part follows from the Rapoport-Zink uniformisation. □

This theorem implies that for any continuous $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ which has an absolutely irreducible reduction $\bar{\rho}_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$, the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $B(\rho_p)$ associated to ρ_p by the p-adic Local Langlands correspondence does not appear in $H^1_c(\mathcal{M}_{LT,\infty}, E)$. Nevertheless, we believe that it appears in $H^2_c(\mathcal{M}_{LT,\infty}, E)$, though we could not prove it.

IV.4.4 Analytic cohomology

Let us explain, why we do not work with the de Rham cohomology as would the folklore conjecture suggest (to be precise: we do, but we work only with the structure sheaf as all the other differentials vanish as we show below). The reason for that is that there are no good finiteness results for de Rham cohomology of adic spaces which are not of finite type (as our Lubin-Tate perfectoid $\mathcal{M}_{LT,\infty}$). Moreover, it seems that the (continuous) de Rham cohomology does not suit well perfectoid spaces. Indeed, $H^i(X, \hat{\Omega}^j_X)$ is zero for any perfectoid space X and sheaves of continuous differentials $\hat{\Omega}^j_X$, $j > 0$. We define here $\hat{\Omega}^j_X$ locally on $\mathrm{Spa}(R, R^+)$ over (K, K^+) by firstly defining

$$\hat{\Omega}^j_{R^+/K^+} = \varprojlim_n \Omega^j_{(R^+/p^n)/(K^+/p^n)}$$

and then $\hat{\Omega}^j_{\mathrm{Spa}(R,R^+)} = \hat{\Omega}^j_{R/K} = \hat{\Omega}^j_{R^+/K^+}[1/p]$. Thus, it is enough to prove the statement for affinoid perfectoids $X = \mathrm{Spa}(R, R^+)$. We can further reduce ourselves to the case $i = 0$ by using the Cech complex associated to some rational covering of X (which will be a covering by affinoid perfectoids by Corollary 6.8 of [Sch12a]). Hence, we have to show that global sections of $\hat{\Omega}^j_X$ are zero. It suffices to show that $\hat{\Omega}^j_{R^+/K^+}$ is almost zero. This follows from induction, as for $n = 1$ the sheaf $\Omega^j_{(R^+/p)/(K^+/p)}$ is identically zero, and for $n > 1$ we conclude using an exact sequence, as in the proof of Theorem 5.10 of [Sch12a]:

$$0 \to R^+/p \to R^+/p^n \to R^+/p^{n-1} \to 0$$

Let us remark that this reasoning also implies that sheaves $\hat{\Omega}^j_X$ are zero on a perfectoid space X for $j > 0$. It is enough to check it at stalks where we have $\hat{\Omega}^j_{X,x} = \varinjlim_{x \in U} \hat{\Omega}^j_X(U)$ and U runs over rational affinoid subsets of X containing x. As such subsets are perfectoid (Corollary 6.8 of [Sch12a]) we have $\hat{\Omega}^j_X(U) = 0$ and hence the result.

As $\mathcal{M}_{LT,\infty}$ is a perfectoid space by [SW13], the above reasoning applies, showing that de Rham cohomology of $\mathcal{M}_{LT,\infty}$ reduces to the study of the cohomology with values in the structure sheaf. This is exactly the analytic cohomology we consider. By using recent results of Scholze, it seems natural to work with the analytic cohomology (i.e. topology defined by open subsets). We review this below. We believe that the 'folklore conjecture' should be understood as the statement that the p-adic local Langlands correspondence appears in the analytic cohomology of the appropiate Rapoport-Zink space at infinity. We also remark that the same applies to Shimura varieties at the infinite level, which are perfectoid spaces by [Sch13].

If Z is any adic space, we denote by Z_{an} its analytic topos which arises from the topology of open subsets. For any (coherent) sheaf $\mathcal{F}$ on Z, we write $H^i_{an}(Z,\mathcal{F})$ for the i-th cohomology group of Z_{an} with values in $\mathcal{F}$.

By Theorem IV.2.1 of [Sch13] (where we pass to the limit with $\mathbb{Z}/p^n\mathbb{Z}$ and then use the reasoning from the proof of Theorem 3.20 in [Sch12b] to descent from the pro-étale site to the étale site) we have an isomorphism

$$H^1(X,E)\widehat{\otimes}_E C \simeq H^1_{an}(X,\mathcal{O}_X)$$

which is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and also equivariant with respect to the Hecke action of $\mathbb{T}_\Sigma$.

If we were to use the same reasoning as for the p-adic completed cohomology (i.e. some exact sequence of analytic cohomology and localisation at a supersingular representation) to show that the p-adic local Langlands correspondence appears in the analytic cohomology of the Lubin-Tate tower at infinity, then we would have to start by proving admissibility of the cohomology groups. Unfortunately, this is not true. By the comparison theorem of Scholze we get that $H^1_{an}(X,\mathcal{O}_X)$ is a Banach admissible E-represention, but $H^0_{an}(X_{\mathrm{ss}},\mathcal{O}_{X_{\mathrm{ss}}})$ is not admissible (and it is not even clear whether it is a Banach space). In order to prove that, it is enough to prove it for $H^0_{an}(\mathcal{M}_{LT,\infty},\mathcal{O}_{\mathcal{M}_{LT,\infty}})$ by the p-adic uniformisation of Rapoport-Zink.

Proposition IV.4.5. *The* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $H^0_{an}(\mathcal{M}_{LT,\infty},\mathcal{O}_{\mathcal{M}_{LT,\infty}})$ *is not admissible.*

Proof. In Section 2 (see especially 2.10) in [Wei13], Weinstein gives an explicit description of the geometrically connected components of $\mathcal{M}_{LT,\infty}$. Each of them is isomorphic to $\mathrm{Spa}(A\otimes_{\mathcal{O}_{K_\infty}} C, A\otimes_{\mathcal{O}_{K_\infty}}\mathcal{O}_C)$, where K_∞ is the Lubin-Tate extension of $\mathbb{Q}_p$ (see Section 2.3 of [Wei13]; we fix an embedding $K_\infty \hookrightarrow C$) and A is a perfectoid K_∞-algebra with a tilt (Corollary 2.9.11 of [Wei13])

$$A^\flat \simeq \bar{\mathbb{F}}_p[[X_1^{1/p^\infty}, X_2^{1/p^\infty}]]$$

Hence, in $H^0_{an}(\mathcal{M}_{LT,\infty},\mathcal{O}_{\mathcal{M}_{LT,\infty}})$ appears $A\otimes_{\mathcal{O}_{K_\infty}} C$ (and in fact much more as this is the set of all unbouded funtions on the Lubin-Tate perfectoid). We have an action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on A. Let K be any compact open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. If $H^0_{an}(\mathcal{M}_{LT,\infty},\mathcal{O}_{\mathcal{M}_{LT,\infty}})$ were admissible, then in particular for the lattice $A\otimes_{\mathcal{O}_{K_\infty}}\mathcal{O}_C$ in $A\otimes_{\mathcal{O}_{K_\infty}} C$, the reduction of K-invariants $(A\otimes_{\mathcal{O}_{K_\infty}}\bar{\mathbb{F}}_p)^K$ would be of finite dimension over $\bar{\mathbb{F}}_p$ (by the very definition, see Definition 2.7.1 of [Eme10a]). This is not possible. Indeed, observe that A^K contains (and probably equals to but we do not need it) the ring of integral analytic functions on the Lubin-Tate space of K-level, which is a finite ring over the ring $\mathcal{O}_C[[X_1,X_2]]$ of power-series over $\mathcal{O}_C$. □

This means that we cannot use the localisation functor and deduce our result from the global results of Emerton. Hence, for now, we can only state a conjecture, which we believe to be a correct version of the folklore conjecture.

Conjecture IV.4.6. *Let $\rho_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ be a continuous de Rham Galois representation. Then, there is a non-zero $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant injection*

$$B(\rho_p) \hookrightarrow H^1_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$$

Observe that in fact we can state a similar conjecture for $H^0_{an}(\mathcal{M}_{LT,\infty}, \mathcal{O}_{\mathcal{M}_{LT,\infty}})$ instead of H^1_{an}. A priori, it is not clear which one should be true or whether both are. The advantage of working with H^0_{an} should be the fact that it is quite explicit by the work of Weinstein.

Christophe Breuil has informed us that a similar conjecture was made by him and Matthias Strauch in 2006 (unpublished note). The difference was that on the left side they considered the locally analytic vectors of $B(\rho_p)$ while on the right side they had a cohomology of the Drinfeld tower at some finite level. Results toward this conjecture for special series appear in [Bre04].

We believe that there is also a more refined version of the folklore conjecture which truly realizes the p-adic local Langlands correspondence in the sense that in the analytic cohomology of the Lubin-Tate perfectoid should appear a tensor product of $B(\rho_p)$ with the associated (ϕ, Γ)-module of ρ_p. We do not make precise here what kind of (ϕ, Γ)-modules we consider and how the appropiate Robba ring acts on the Lubin-Tate perfectoid. We shall come back to those issues elsewhere.

IV.4.5 Final remarks

Observe that our proof of Theorem 4.3 depends on the global data as we have to start with a global pro-modular Galois representation ρ. As our result is completely local, it is natural to ask whether the same thing holds for any absolutely irreducible Galois representation ρ_p of $G_{\mathbb{Q}_p}$ which is not necessarily a restriction of some global ρ (as in Conjecture 4.6).

Another natural problem is to try to prove Theorem 4.3 without assuming that $\bar{\rho}_p$ is absolutely irreducible. This would require a more careful study of the cohomology of the ordinary locus.

The most pertaining problem is whether one can reconstruct $B(\rho_p)$ from either the p-adic completed or the analytic cohomology of the Lubin-Tate tower and hence give a different proof of the p-adic local Langlands correspondence. This might be useful in trying to prove the existence of the p-adic correspondence for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$ as well as Theorem 4.3 for Galois representations ρ_p not necessarily coming from global Galois representations.

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