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Qualitative analysis of solutions to equations of viscous incompressible fluids

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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the dissertation is ready to be reviewed

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Abstract

In this thesis we investigate a part of the theory of fluid dynamics concerning a twodimensional stationary exterior problems with slip boundary conditions. We show existence of weak solutions to linear problems and to the Navier-Stokes equations without any assumptions on smallness of data. Later we present optimal L_p–estimates for the Oseen system in an exterior domain, which allows one to prove, that for a sufficiently small data there exists a solution to the Navier-Stokes system, which tends to a prescribed vector field at infinity. As a result of a detailed analysis of the Oseen system in the halfspace we show, that on a local level there exists a substantial difference between points in front of an obstacle and behind it, in particular we show, that in front of the obstacle considered problem has a strong elliptic character, while behind the obstacle one observes a disturbance, which is characteristic for a parabolic problems. In the last Chapter we present a new approach via the Fourier transform for dealing with a Navier-Stokes system in the whole plane. This technique allows us to derive a basic asymptotics, which shows an occurance of a wake region for $t > 1$ (understood as a region behind an obstacle).

Keywords

the Navier-Stokes equations, the Oseen system, slip boundary conditions, inhomogeneous boundary data, large data, exterior domain, half plane, plane flow, maximal regularity, qualitative analysis, asymptotic behaviour, the Fourier transform.

AMS Mathematics Subject Classification

35Q30, 35Q35, 76D05, 76D07

Streszczenie

Niniejsza rozprawa poświęcona jest badaniu modeli mechaniki płynów dla nieściśliwych stacjonarnych przepływów w dwuwymiarowych obszarach zewnętrznych, rozpatrywanych z warunkami brzegowymi typu poślizgu. Pokazujemy istnienie słabych rozwiązań dla układów liniowych oraz układu równań Naviera-Stokesa bez założeń na małość danych. Następnie przedstawiamy optymalne oszacowania w przestrzeniach L_p dla układu Oseena, co pozwala na udowodnienie istnienia rozwiązań dla równań Naviera-Stokesa, które w nieskończoności dążą do ustalonego stałego pola wektorowego. Jako rezultat szczegółowej analizy układu Oseena w półprzestrzeni pokazujemy, że na poziomie lokalnym istnieje wyraźna różnica pomiędzy punktami przed i za przeszkodą, w szczególności pokazujemy, że przed przeszkodą warunki brzegowe są tej klasy regularności, jaka występuje przy silnie eliptycznych problemach, natomiast za przeszkodą obserwujemy zaburzenie, które ma charakter paraboliczny. W ostatnim rozdziale przedstawiamy nowe podejście, używające transformaty Fouriera, do zagadnienia przepływu w całej przestrzeni. Jako zastosowanie tej techniki uzyskujemy prostą asymptotykę rozwiązań, która pokazuje istnienie regionu zaburzenia w obszarze $t > 1$ (który może być interpretowany jako obszar za przeszkodą).

Słowa kluczowe

równania Naviera-Stokesa, układ Oseena, warunki poślizgu, niejednorodne warunki brzegowe, duże dane, obszar zewnętrzny, połpłaszczyzna, płaszczyzna, maksymalna regularność, analiza jakościowa, asymptotyka, transformata Fouriera.

Klasyfikacja tematyczna według AMS

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Contents

Chapter 1

Introduction

In the present thesis we are concerned with a part of the mathematical theory of viscous incompressible fluid flows. It is devoted to the stationary Navier-Stokes equations in case of an unbounded two dimensional exterior domain Ω, i.e. the domain, which is the complement of a compact set $\mathcal{B} \subset \mathbb{R}^2$, considered with slip boundary conditions. This type of constraints is complementary to the Dirichlet ones, which have been widely explored during past decades, where the velocity of the fluid on the boundary is fully prescribed. In the case of slip boundary conditions there are two equations: one is related to Newton's second law describing a friction between the fluid and the boundary, and the second one, describing a flow of the fluid across the boundary. This type of conditions can be used in approximate models of a perfect gas ([6], [25]), modeling motion of blood, polymers and liquid metals ([9], [18]).

Systems describing flows past obstacles are of main interest in the thesis. We concentrate on the two dimensional models, since from the mathematical point of view it is the most challenging case.

The thesis consists of four main parts – each of them, as a separate chapter, is a standalone result, which is a basis of an article (see [21], [20], [21]).

In Chapter 2 we are dealing with some linear problems of fluid dynamics. This is usually the first step to develop tools for nonlinear problems. We follow approach used in [22], where authors work with the Navier-Stokes equations expressed in terms of the vorticity of the fluid. This approach is justified, since slip boundary conditions result in Dirichlet constraints for the vorticity (see [33]). A standard method ([24], [17]) to show existence of solutions with finite energy, i.e. with a finite Dirichlet integral

$$\int_\Omega |\nabla v|^2 dx < \infty, \tag{1.1}$$

is to construct a vector field a, which takes into account all information about the velocity on the boundary, and present the solution v as $v = u + a$, where u is some new unknown function, which can be sought in the class $H_0^1(\Omega)$. The construction of a must be nontrivial, since it should not only be divergence free, but also satisfy an additional inequality, which is needed, if one wants to show existence without assumptions on smallness of data. Unfortunately, this standard approach cannot be applied directly for problems considered with slip boundary conditions, since in that case one is not able to get full information about the velocity on the boundary. This results in impossibility to present v in the form $v = u + a$, where $u \in H_0^1(\Omega)$. Thus, more subtle analysis is required.

As it was mentioned earlier, we repeat the approach from [22], where a more detailed construction of an auxiliary vector field has been carried out. To be more precise, the field behaves different in the direction normal to the boundary than in the tangent direction. This allows us to obtain required inequalities for a class of functions, which is different from $H^1_0(\Omega)$, but is more natural for a problem for rotation.

Besides minor difficulties, which one faces when dealing with an exterior domain, there is a more substantial one, namely the problem of a kernel of the *rot-div* operator. The question is, whether one can recover full information about the velocity from its rotation or not. In case of a simply connected domain (as in [22]) one can easily show, that the kernel of this operator is trivial. However, in case of an exterior domain we show, that under additional, but natural, assumption on the gradient of functions from the kernel, this family of functions is one dimensional (in the simplest case $\Pi_1(\Omega) = \mathbb{Z}$). Even so, taking into account slip boundary conditions, we are able to show, that this kernel part of the velocity v is trivial. This fact comes from the strong maximum principle for harmonic functions.

Chapter 3 is devoted to the existence of weak solutions to the Navier-Stokes system in exterior domains. As was mentioned before, we extensively use techniques, which were developed in Chapter 2. As a result we prove existence of solutions without assumptions on the smallness of data. This outcome is complementary to results for the Navier-Stokes system considered with Dirichlet boundary conditions and is related to a flux problem ([5]). We follow the approach from [22] via the system for the rotation of the velocity. However due to results from Chapter 2 on the kernel of the *rot-div* operator we are able to show, that the solution is in fact a solution to the original Navier-Stokes system.

Chapter 4 is the core of the thesis. We present there a thorough L_p–analysis for the Oseen system in the half plane, which is then used to show L_p–estimates for the Oseen system in an exterior domain. We follow a different approach than the standard one via the fundamental solution. For a given solution we use a localization procedure to be able to consider this problem in the whole plane and in the half plane. Results for the whole plane are well known ([11]) – one needs to use the Lizorkin multiplier theorem. In the half plane, however, one needs to use different technique. We rewrite the system using the Fourier transform in one direction and consider the other one as time. In that way we obtain a system of ordinary differential equations, which can be studied further. To obtain proper estimates in L_p–spaces we use the Marcinkiewicz theorem for multipliers and techniques, which were used in [35], also in [31], [32]. They also have a common part with the techniques from the famous papers of Agmon, Douglis and Nirenberg ([1], [2]).

Our detailed analysis of eigenvalues, which occur while solving the mentioned system of ODEs, brings substantial and interesting information about the necessity of choosing a proper regularity class for boundary data. It appears, that a considered situation, whether we are in front of the obstacle or behind the obstacle, has a strong influence on the character of the solution. We show, that in front of the obstacle one needs to have regularity of data, which is natural for strong elliptic problems, while behind the obstacle one need to consider boundary conditions, which have a disturbance typical for parabolic problems. Although our system is stationary (elliptic). One may consider this as a source of a parabolic wake region – a phenomena, which is expected from flows, which are physically reasonable. This type of results is well known ([11]), however only for exterior problems. We believe, that our novel result on a local level (the problem in the half plane) is not so widely studied. This result seems to be the most important achievement of the presented thesis.

The natural consequence of using Marcinkiewicz's theorem is the need to use not only inhomogeneous Sobolev spaces, but also the homogeneous ones. We also use results in Besov spaces, since they are a natural choice, when one deals with multipliers in Fourier space. This allows us to obtain optimal regularity results.

Finally, we apply results from the half plane to show estimates for a flow in an exterior domain. The reason why we show L_p–estimates for the Oseen system is the fact, that they are crucial to deal with another famous problem in the Navier-Stokes theory, namely the problem of prescribing the velocity at infinity. This is a very interesting and still open question, whether the velocity tends to a prescribed velocity vector field at infinity. Only some partial results, and for only Dirichlet boundary constraints were obtained ([14], [15], [4], [12]). The reason for that is that the dimension 2 coincides with the power 2 from (1.1) and one cannot assure, that $v \to v_\infty$ as $|x| \to \infty$ for some prescribed constant vector field v_∞. In fact, there are examples of solenoidal vector fields, which are unbounded as $|x| \to \infty$. Thus a different approach is needed and the Oseen system its L_p–estimates play the fundamental role. We show, that also for slip boundary conditions one is able to show existence of the solution to the Navier-Stokes equations, which approaches a prescribed velocity at infinity.

In Chapter 5 we give a different approach to study asymptotic behaviour of the solution to the nonlinear system. The starting point were the results from the paper of Wittwer [38], where the author showed existence of solutions and asymptotic behaviour for a symmetric flow in a half plane behind the obstacle and with artificial (physically unreasonable) boundary conditions. The method is to use the Fourier transform in one direction and treat the other direction as time. Hence we are able to apply theory for evolutionary systems, as techniques of the theory of semigroups, to our stationary equations. We extend these results to the case of non symmetric flows in the whole plane, but also present a significant simplification of these methods, since in [38] it is rather difficult and many technical computations are unavoidable. A motivation for a simpler technique came from results in [30]. Our approach is less complicated, since we work with the velocity itself, and not with its rotation, as it was the case in [38]. As was mentioned earlier, when one wants to show asymptotic behaviour of a flow he needs to have nonstandard tools, since the well known theory of Sobolev spaces (in particular embedding theorems) fails. That is why we introduce a different Banach space, which is more suitable for our purposes. A similar point of view has been considered in [33] and [5], which essentially simplified calculations. Although, problems considered there, are evolutionary.

We show existence of a solution by means of the contraction principle, we also give additional results on an asymptotic behaviour of the fluid. Using the formula for the solution in the Fourier space we show that in case of $t < -1$, which can be treated as a region in front of an obstacle, one may obtain a uniform estimate of the Fourier transform of the velocity with $|t(1+|\xi|)|^{-1/2}$, while behind an obstacle (i.e. for $t > 1$) we obtain, that the Fourier transform of the velocity behaves like $|(1+t|\xi|^2)|^{-1/2}$. This in particular results in the presence of the wake region behind the obstacle. Our goal here was to give basic asymptotics, however it gives us a foretaste of results, which can be obtained using these methods.

Our main contribution to the theory of the Navier-Stokes are, in our humble opinion, results from Chapters 4 and 5. In the first one, results about local occurrence of a parabolic disturbance behind the obstacle are new and interesting. Also the simplification of the whole approach, via the Marcinkiewicz theorem, might be of interest, since these techniques can be easily used to investigate different linear problems. In Chapter 5 the whole approach is quite promising and, since it is new, one may believe, that using it one may obtain in a simple way many interesting, ane maybe essentially new, results.

Chapter 2

Linear flow problems in 2D exterior domain for 2D incompressible fluid flows

2.1 Introduction

Linear problems play a crucial role in investigations of asymptotic structure of solutions to the stationary Navier-Stokes equations in exterior domains. Qualitative properties of solutions follows that the nonlinear term $v \cdot \nabla v$ does not determine the behaviour of the system in a neighbourhood of infinity. Thus analysis of a suitable linearization can give us appropriate information about the analyzed equations.

Here we investigate two linear problems. Both come from a linearization of the Navier-Stokes equations and are expressed in terms of vorticity α of the fluid. We consider these problems in two-dimensional exterior domain

$$\Omega = \mathbb{R}^2 \setminus B,$$

where $B \subset \mathbb{R}^2$ is a simply connected bounded domain with smooth boundary.

Slip boundary conditions

$$\vec{n} \cdot \mathbb{T}(v,p) \cdot \vec{\tau} + f(v \cdot \vec{\tau}) = 0 \tag{2.1}$$

(see below for notation) are assumed in both cases, which allows us to show existence of solutions without assumptions on smallness of the data (see [19]).

What is worth notice is that from (2.1) one cannot directly get full information about velocity v (only its tangential part), however in terms of vorticity α this condition rewrites as Dirichlet condition:

$$\alpha = (2\chi - f/\nu)(v \cdot \vec{\tau}),$$

where χ is the curvature of $\partial\Omega$ (see [26]).

The first considered problem, which is in fact the Stokes problem expressed in terms of vorticity of the fluid, we introduce as follows:

$$\begin{array}{rcll} -\nu\Delta\alpha & = & \mathrm{rot}\, F & \text{in } \Omega, \\ \mathrm{rot}\, v & = & \alpha & \text{in } \Omega, \\ \mathrm{div}\, v & = & 0 & \text{in } \Omega, \\ v \cdot \vec{n} & = & 0 & \text{on } \partial\Omega, \\ \alpha & = & (2\chi - f/\nu)(v \cdot \vec{\tau}) & \text{on } \partial\Omega, \\ v & \to & v_\infty & \text{as } |x| \to \infty. \end{array} \tag{2.2}$$

For this system we prove the following:

Theorem 2.1.1. *Let $\nu > 0$, $f \in L^\infty(\partial\Omega)$ and rot $F \in (H_0^2(\Omega))^*$. Then there exists a unique weak solution v to problem (2.2) in the sense of Definition 2.3.1, such that*

$$\|\nabla v\|_{L^2(\Omega)} \leq DATA. \tag{2.3}$$

One of the most important and difficult questions arising in this problem is if condition (2.2_6) is fulfilled for weak solution v, since there are examples of solenoidal vector fields, which satisfy (2.3) and are unbounded at infinity. In [11] the author assumes extra conditions on integrability of the gradient ∇v for $p \in (1, 2)$, which imply existence of solutions satisfying condition (2.2_6). In [14], [15], [3] and [4] one can find similar results for restricted version of our problem.

The second system we study is the linearization of the Navier-Stokes problem around defined vector field $\tilde{v_0}$. We introduce it as follows:

$$\begin{array}{rcll} -\nu\Delta\alpha + \tilde{v_0} \cdot \nabla\alpha &=& \text{rot } F & \text{in } \Omega \\ \text{rot } v &=& \alpha & \text{in } \Omega \\ \text{div } v &=& 0 & \text{in } \Omega \\ \alpha &=& (2\chi - f/\nu)(v \cdot \vec{\tau}) & \text{on } \partial\Omega \\ v \cdot \vec{n} &=& 0 & \text{on } \partial\Omega \\ v &\to& v_\infty & \text{as } |x| \to \infty. \end{array} \tag{2.4}$$

where $\tilde{v_0}$ is a given divergence free vector field, which we define later, satisfying $\tilde{v_0} = 0$ on $\partial\Omega$ and $\tilde{v_0} \to v_\infty$ as $|x| \to \infty$. This is a modification of the Oseen system – with $\tilde{v_0}$ in place of v_∞.

Our second theorem states the existence of solutions to (2.4):

Theorem 2.1.2. *Given $\nu > 0$, $f \in L^\infty(\partial\Omega)$ and rot $F \in (H_0^2(\Omega))^*$. There exists a unique weak solution $v \in D^1(\Omega)$ to problem (2.4) in the sense of Definition 2.4.2 such that*

$$\|\nabla u\|_{L^2(\Omega)} \leq DATA.$$

One of the problems one faces in proofs of both theorems is the question about the kernel part of the *rot-div* operator. Indeed, since we consider problems in terms of the vorticity of the fluid, then taking back information about the velocity itself is not straightforward for non-simply connected domains, since the kernel of the mentioned operator is not trivial. We deal with this problem in Section 2.2. The reasoning there shows, that in case of slip boundary conditions the kernel part of this operator may be assumed to be trivial.

Notation. In the above we use the following notation: v is a velocity vector field, p - the corresponding pressure, ν - viscous positive constant coefficient, f - nonnegative friction coefficient, $\mathbb{T}(v,p)$ is Cauchy stress tensor, i.e. $\mathbb{T}(v,p) = \nu\mathbb{D}(v) + p\mathbb{I}$, where $\mathbb{D}(v) = \{v_{i,j} + v_{j,i}\}_{i,j=1}^2$ is the symmetric part of the gradient ∇v, and $\mathbb{I}$ is the identity matrix. Moreover $\vec{n}, \vec{\tau}$ are respectively normal and tangential vector to boundary $\partial\Omega$.

Function space $\dot{H}_0^2(\Omega)$ is introduced in Section 2.2.

Our chapter is organized as follows. In Section 2.2 we deal with a *rot-div* problem, which is fundamental in our considerations, introduce basic definitions and auxiliary lemmas used in next sections. In Section 2.3 we give a weak formulation of the problem (2.2) and prove Theorem 2.1.1. Similar arrangement is for system (2.4) in Section 4. As a result of the previous considerations in Section 2.4.2 we show existence of a solution to the standard Oseen system.

2.2 The rot-div problem

In this section we focus on properties of the *rot-div* problem in two types of domain – bounded and exterior, however both non-simply connected. Namely, we consider the following system:

$$\begin{aligned} \text{rot } v &= \alpha && \text{in } \Omega \\ \text{div } v &= 0 && \text{in } \Omega \\ v \cdot \vec{n} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.5}$$

Our goal is to get precise information about the kernel of this operator. We will need to consider a proper function space for solutions.

First let us notice that from ($2.5_{2,3}$) and Poincare Lemma we may conclude existence of a scalar stream function Φ such that

$$v = \nabla^{\perp}\Phi,$$

where $\nabla^{\perp}\Phi = (-\partial_{x_2}\Phi, \partial_{x_1}\Phi)$. From ($2.5_3$) we see that

$$\nabla\Phi \cdot \vec{\tau} = \nabla^{\perp}\Phi \cdot \vec{n} = v \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega,$$

so

$$\Phi \equiv \text{const}$$

on each connected component of $\partial\Omega$.

We decompose Φ as

$$\Phi = \varphi + \psi$$

where ψ is the kernel part of the operator (2.5).

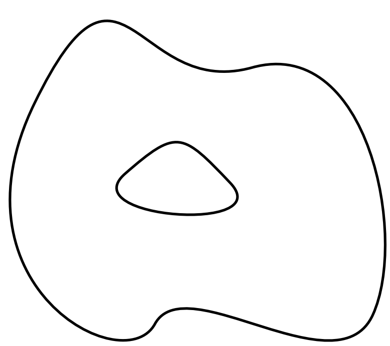
From (2.5_1) we easily see that φ and ψ fulfill the following equations in Ω:

$$\Delta\varphi = \alpha \quad \text{and} \quad \Delta\psi = 0 \qquad \text{in } \Omega.$$

Further requirements on φ and ψ, e.g. value on the boundary, will differ in case of bounded and unbounded domain, that is why we state them in separate subsections.

2.2.1 Case of a bounded non-simply connected domain

In this subsection we study system (2.5) in a bounded non-simply connected domain Ω, as in the picture:



Domain

Let us assume that boundary $\partial\Omega$ decomposes into Γ_0 and Γ_1, where Γ_0 is inner boundary and Γ_1 is outer boundary.

Since we are only interested in the gradient of φ and ψ we assume that

$$\varphi \equiv \psi \equiv 0 \qquad \text{on } \Gamma_0,$$

since $\psi \equiv \text{const}$ on Γ_1 and $\Pi^1(\Omega) = \mathbb{Z}$ we may set

$$\psi \equiv 1 \qquad \text{on } \Gamma_1$$

and then, since $\Delta\psi = 0$ in Ω, we conclude that the kernel of operator (2.5) has one dimension, i.e. every vector field $\nabla^\perp\tilde{\psi}$ from the kernel can be represented as

$$\nabla^\perp\tilde{\psi} = C_\psi \nabla^\perp \psi$$

for a proper constant C_ψ.

For a non-kernel function φ we may set

$$\varphi \equiv 0 \qquad \text{on } \Gamma_1.$$

Let us introduce the following function space:

$$\tilde{H}^2_0(\Omega) = \overline{\{f \in C^\infty(\Omega) : f_{|\Gamma_0} \equiv 0, \nabla^2 f \in L^2(\Omega)\}}^{\|\nabla^2\cdot\|_{L^2(\Omega)}}$$

which is a Banach space with respect to the norm

$$\|f\|_{\tilde{H}^2_0(\Omega)} = \|\nabla^2 f\|_{L^2(\Omega)}.$$

Since we are interested in velocity vector v we introduce the following appropriate function space:

$$\tilde{D}^1(\Omega) = \{\nabla^\perp f : f \in \tilde{H}^2_0(\Omega)\}$$

which is also a Banach space with respect to the norm $\|\nabla\cdot\|_{L^2(\Omega)}$.

It is easily seen that the class $C_0^\infty(\Omega)$ of smooth functions with compact support in Ω is not dense in this space. To be precise: no function from the kernel of the operator (2.5) can be approximated in space $\tilde{H}^2_0(\Omega)$ with functions from $C_0^\infty(\Omega)$.

2.2.2 Case of a unbounded non-simply connected domain

In this subsection we will state some results in unbounded domain, needed for further calculations.

From the case of bounded domain we see that we may assume, that φ and ψ fulfill the following system:

$$\begin{aligned} \Delta\varphi &= \alpha, & \Delta\psi &= 0 & &\text{in } \Omega,\\ \varphi &= 0, & \psi &= 0 & &\text{on } \partial\Omega, \end{aligned}$$

From the physical point of view (we recall that $v = \nabla^\perp\varphi + \nabla^\perp\psi$) we may assume that $\nabla\varphi + \nabla\psi \to 0$ as $|x| \to \infty$, however later we approximate φ with smooth functions of compact support in Ω that is why we assume that $\nabla\psi \to 0$ as $|x| \to \infty$. Finally we write:

$$\nabla\varphi \to 0 \quad \text{and} \quad \nabla\psi \to 0 \qquad \text{as } |x| \to \infty.$$

Basic function spaces. Similarly to $\tilde{H}^2_0(\Omega)$ in case of bounded domain we introduce space

$$\dot{H}^2_0(\Omega) = \overline{\{f \in C_0^\infty(\Omega) : f_{|\partial\Omega} \equiv 0, \nabla^2 f \in L^2(\Omega)\}}^{\|\nabla^2\cdot\|_{L^2(\Omega)}}$$

which is a Banach space with respect to the norm $\|\nabla^2 \cdot \|_{L^2(\Omega)}$. Similarly we introduce a proper space for velocity vector v as:

$$D^1(\Omega) = \{\nabla^{\perp} f : f \in \dot{H}^2_0(\Omega)\}$$

which is a Banach space with respect to the norm $\|\nabla \cdot \|_{L^2(\Omega)}$.

Kernel function ψ. Now we find some information about the kernel of the operator (2.5) in case of unbounded domain. Let us recall the system for ψ:

$$\begin{aligned} \Delta\psi &= 0 \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \partial\Omega, \\ \nabla\psi &\to 0 \qquad \text{as } |x| \to \infty. \end{aligned}$$

Notice that we have no explicit information about the behaviour of ψ at infinity.

We show now that in the class of tempered distributions the above system has one-parameter family of solutions.

Lemma 2.2.1. *Given system*

$$\begin{aligned} \Delta\psi &= 0 \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \partial\Omega, \\ \nabla\psi &\to 0 \qquad \text{as } |x| \to \infty. \end{aligned} \tag{2.6}$$

There exists one-dimensional space of solutions in the class of tempered distributions.

Proof . Existence of a solution is obvious. Let ψ be a solution to (2.6). Introducing a smooth cut-off function $\eta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\eta \equiv 0 \text{ in } B_{R_1}, \qquad \eta \equiv 1 \text{ in } \mathbb{R}^2 \setminus B_{R_2},$$

where $B_{R_1} \subset \mathbb{R}^2$ is a ball containing the hole Ω and $R_1 < R_2$.

Then we see that $\varphi\eta$ satisfies the following conditions:

$$\begin{aligned} \Delta(\varphi\eta) &= 2\nabla\varphi \cdot \nabla\eta + \varphi\Delta\eta \qquad \text{in } \mathbb{R}^2 \\ \nabla(\varphi\eta) &\to 0 \qquad \text{as } |x| \to \infty. \end{aligned} \tag{2.7}$$

Denoting $2\nabla\varphi \cdot \nabla\eta + \varphi\Delta\eta =: F$ we find that F has compact support in $\mathbb{R}^2$ and we may take

$$\tilde{\varphi} = E * F,$$

where E is the fundamental solution for Laplace operator, as a solution to (2.7). Properties of $\tilde{\varphi}$ are well known since support of F is compact. Thus – $\tilde{\varphi}$ has logarithmic growth and its gradient tends to 0 like $1/|x|$ as $|x| \to \infty$.

Further we get a system for $\varphi\eta - \tilde{\varphi}$:

$$\begin{aligned} \Delta(\varphi\eta - \tilde{\varphi}) &= 0 \qquad \text{in } \mathbb{R}^2, \\ \nabla(\varphi\eta - \tilde{\varphi}) &\to 0 \qquad \text{as } |x| \to \infty. \end{aligned} \tag{2.8}$$

Taking Fourier transform $(\hat{\ })$ of the first equation we get:

$$|\xi|^2 (\varphi\eta - \tilde{\varphi})\hat{\ } = 0,$$

hence

$$\operatorname{supp}\, (\varphi\eta - \tilde{\varphi})\hat{\ } \subset \{0\},$$

which implies

$$(\varphi\eta - \tilde{\varphi}) = p(x),$$

where $p(x)$ is a polynomial. Since we have assumption (2.8) we conclude that $p(x) = c$, for some constant c.

This shows that every solution in the class of tempered distributions to problem (2.6) has logarithmic growth at infinity and its gradient behaves like $1/|x|$.

To finish the proof we need to show, that two solutions ψ_1, ψ_2 to (2.6) such that

$$\lim_{|x|\to\infty} \frac{\psi_1}{\ln|x|} = \lim_{|x|\to\infty} \frac{\psi_2}{\ln|x|} = c$$

for some constant c, are equal. It is however simple, since this property (in $\mathbb{R}^2$) implies analycity of $\psi_1 - \psi_2$ at infinity and thus $\psi_1 - \psi_2 = 0$, since $\psi_1 - \psi_2 = 0$ on $\partial\Omega$. $\square$

Application to a flow problem – a trivial kernel. We would like to combine previous results with the slip boundary conditions. From the proof of Lemma 2.2.1 we know, that the kernel function has a constant sign. Indeed, since $\psi = 0$ on $\partial\Omega$ and $\lim_{|x|\to\infty} \psi/\ln|x| = c$, then ψ has the same sign in all Ω as c. This implies, that on the boundary $\partial\Omega$ function ψ has either its maximum or minimum. Using strong maximum principle for a nontrivial ψ we get, that in that points $\frac{\partial\psi}{\partial\vec{n}} \neq 0$. This however, stays in contrary with the slip boundary conditions. Indeed, since rot $v = 0$ then

$$\text{rot } v = (2\chi - f/\nu)v \cdot \vec{\tau} = 0.$$

Since it is impossible that $(2\chi - f/\nu) \equiv 0$ on all $\partial\Omega$ (because of the positivity of f and ν) it has to be

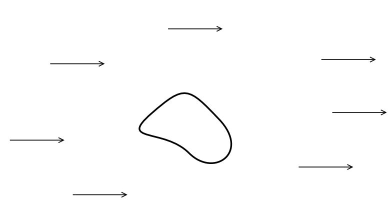
$$v \cdot \vec{\tau} = \frac{\partial\psi}{\partial\vec{n}} = 0,$$

but we have just shown, that for a nontrivial ψ this cannot be satisfied.

This conclusion might be used to extend results from [22] to the case where the domain is non simply-connected. In this paper authors are considering simply-connected domains because in this case the kernel of the rot-div operator is obviously trivial and one may recover full information about the velocity of the fluid from its rotation.

2.3 Stokes Problem

In this section we investigate the Stokes problem gathered by the system (2.2). First we give its weak formulation. The following picture shows the considered situation.



2.3.1 v_0 construction

For the sake of further considerations we need to construct a vector field $v_0 \in H^1(\Omega)$ which for given $\epsilon > 0$ fulfills the following requirements:

$$\begin{array}{ll} \text{div } v_0 = 0 & \textit{in } \ \Omega, \\ v_0 \cdot \vec{n} = -v_\infty \cdot \vec{n} & \textit{on } \partial\Omega \end{array} \tag{2.9}$$

and for every $\varphi \in \dot{H}^2_0(\Omega)$ the following inequality holds:

$$\left| \int_\Omega (v_0 \cdot \nabla\varphi)^2 \, dx \right| \leq \epsilon \|\Delta\varphi\|^2_{L^2(\Omega)}. \tag{2.10}$$

Moreover, the following estimate is valid

$$\|v_0\|_{H^1(\Omega)} \leq C\|v_\infty \cdot \vec{n}\|_{H^{1/2}(\partial\Omega)}.$$

This construction is the subject of Lemma 2.5.1 with $-v_\infty \cdot \vec{n}$ in place of d.

2.3.2 Reformulation

To show existence first we introduce a decomposition of v as:

$$v = v_0^\infty + u,$$

where $v_0^\infty = v_0 + v_\infty$. Then for u we get the following system:

$$\begin{array}{rcll} \text{rot } u &=& \alpha - \text{rot } v_0^\infty & \text{in } \Omega, \\ \text{div } u &=& 0 & \text{in } \Omega, \\ u \cdot \vec{n} &=& 0 & \text{on } \partial\Omega, \\ u &\to& 0 & \text{as } |x| \to \infty. \end{array}$$

For u satisfying this conditions we may assume existence of φ such that $\varphi \in \dot{H}^2_0(\Omega)$ and:

$$u = \nabla^\perp\varphi. \tag{2.11}$$

Here we know, that the kernel part of u is trivial.

2.3.3 Weak formulation and existence

Let us multiply (2.2_1) by a function θ, such that $\theta = 0$ on $\partial\Omega$, and integrate by parts. We get the weak formulation of our problem

Definition 2.3.1. *We say that $\varphi \in \dot{H}^2_0(\Omega)$ is a weak solution to problem (2.2) iff the following identity holds for all $\theta \in \dot{H}^2_0(\Omega)$:*

$$\begin{aligned} -\nu \int_\Omega \Delta\varphi\Delta\theta + \nu \int_{\partial\Omega} [(2\chi - f/\nu)((\nabla^\perp\varphi) \cdot \vec{\tau})] \frac{\partial\theta}{\partial\vec{n}} \\ = \nu \int_\Omega \text{rot } v_0 \Delta\theta - \int_\Omega F \cdot \nabla^\perp\theta - \nu \int_{\partial\Omega} (2\chi - f/\nu)(v_0 \cdot \vec{\tau}) \frac{\partial\theta}{\partial\vec{n}} \end{aligned} \tag{2.12}$$

Proof of Theorem 2.1.1. To show existence of weak solutions first we introduce a scalar product in $\dot{H}^2_0(\Omega)$ as follows:

$$(u,v)_{\dot{H}^2_0(\Omega)} = \int_\Omega \mathbb{D}(u) : \nabla v + \int_{\partial\Omega} f(u\cdot\vec{\tau})(v\cdot\vec{\tau}). \tag{2.13}$$

To show that this is indeed a scalar product one can use Korn inequality, which proof can be found in [26].

Next we introduce an orthonormal basis with respect to this scalar product:

$$\text{span}(\{\varphi_1, \varphi_2, \ldots\}) = \dot{H}^2_0(\Omega).$$

We look for a solution u in the form (2.11), where $\varphi \in \text{span}\{\varphi_1, \varphi_2, \ldots\}$.

Further we need the following identity:

$$\int_\Omega \alpha^2 = \int_\Omega \mathbb{D}(v) : \nabla(v) + \int_{\partial\Omega} 2\chi(v\cdot\tau)^2.$$

Using it we find that (2.12) is equivalent to the following:

$$\nu \int_\Omega \mathbb{D}(v) : \nabla\nabla^\perp\theta + \int_{\partial\Omega} f(v\cdot\vec{\tau})\left(\frac{\partial\theta}{\partial\vec{n}}\right) = -\int_\Omega F\cdot\nabla^\perp\theta.$$

Having v_0 we show existence of φ in the way of Galerkin method.

We introduce the following approximation space $V^N(\Omega) = \text{span}\{\varphi_1, \ldots, \varphi_N\}$. Then we search for approximate solution $\varphi^N \in V^N(\Omega)$ in the form:

$$\varphi^N = \sum_{i=1}^N c_i^N \varphi_i.$$

Identity (2.12) must be fulfilled for $\theta = \varphi_1, \ldots, \varphi_N$. To show existence of coefficients c_i^N such that (2.12) is valid for $\theta = \varphi_1, \ldots, \varphi_N$ we use the following Lemma:

Lemma 2.3.2. *Having continuous mapping $P : V^N(\Omega) \to V^N(\Omega)$. If the condition*

$$(P(\varphi^N), \varphi^N)_{V^N(\Omega)} > 0$$

is fulfilled for all $\varphi^N \in V^N$ such that $\|\varphi^N\|_{V^N(\Omega)} = M$, then there exists $\varphi_0^N \in V^N(\Omega)$ such that $P(\varphi_0^N) = 0$ and $\|\varphi_0^N\|_{V^N} \le M$.

To use this lemma we define mapping $P : V^N(\Omega) \to V^N(\Omega)$ as follows: first we define u^N:

$$u^N = v_0 + \sum_{i=1}^N c_i^N \varphi_i,$$

next:

$$P(\varphi^N) := \left(\sum_{i=1}^N \nu \int_\Omega \mathbb{D}(u^N) : \nabla\nabla^\perp\varphi_i + \int_{\partial\Omega} f(u^N\cdot\vec{\tau})\left(\frac{\partial\varphi_i}{\partial\vec{n}}\right) + \int_\Omega F\cdot\nabla^\perp\varphi_i\right)\cdot\varphi_i.$$

Since $\{\varphi_i\}$ is orthonormal basis we easily find that:

$$\begin{aligned}
(P(\varphi^N),\varphi^N)_{V^N(\Omega)} &= \nu\int_\Omega \mathbb{D}(u^N):\nabla\nabla^\perp\varphi^N + \\
&\int_{\partial\Omega} f(u^N\cdot\vec{\tau})\left(\frac{\partial\varphi^N}{\partial\vec{n}}\right) + \int_\Omega F\cdot\nabla^\perp\varphi^N \\
&= (u^N,\varphi^N)_{V^N(\Omega)} + \int_\Omega F\cdot\nabla^\perp\varphi^N \\
&= (\varphi^N,\varphi^N)_{V^N(\Omega)} + (v_0,\varphi^N)_{V^N(\Omega)} + \int_\Omega F\cdot\nabla^\perp\varphi^N \\
&\geq \|\varphi^N\|^2_{V^N(\Omega)} - C(v_0,F)\|\varphi^N\|_{V^N(\Omega)} > 0
\end{aligned}$$

for all φ^N such that $\|\varphi^N\|_{V^N(\Omega)} > C(v_0,F)$, where $C(v_0,F)$ does not depend on N.

Using Lemma (2.3.2) we get that there exists φ_0^N (i.e. coefficients c_i^N) such that φ_0^N is a solution to (2.12) and $\|\varphi_0^N\|_{V^N(\Omega)} \leq C(v_0,F)$. Since all these solutions are bounded we are able to choose weakly convergent in $\dot{H}_0^2$ subsequence (for simplicity we do not add another index) $\{\varphi_0^i\}_{i=1}^\infty$ for which a limit function $\varphi_0 \in \dot{H}_0^2(\Omega)$: $\varphi_0 = \text{w-}\lim_{i\to\infty}\varphi_0^i$ satisfies (2.12) in the sense of distributions.

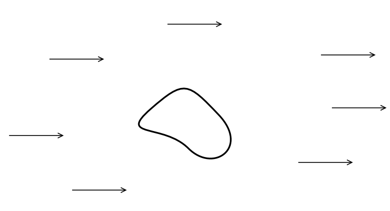
To complete the proof of Theorem 2.1.1 we need to show uniqueness of this solution. This is however straightforward, since one may rearrange previous estimates for φ, i.e. terms $\Delta\varphi$ and $\nabla^\perp\varphi$ in a way, that they will become estimates for $\Delta\varphi + \text{rot}\ v_0$ and $\nabla^\perp\varphi + v_0$. In that case one may obtain:

$$\|\nabla(\nabla^\perp\varphi + v_0)\|_{L^2(\Omega)} \leq C\|\text{rot}\ F\|_{(\dot{H}_0^2(\Omega))^*},$$

hence for $F = 0$ we get $\nabla^\perp\varphi = -v_0$, i.e. $v = 0$. This completes the proof of Theorem 2.1.1. □

2.4 Linearization of the N-S problem.

In this section we investigate problem (2.4): where $\tilde{v}_0 = v_0 + v_\infty$, v_0 has compact support and the system (2.9)-(2.16) is fulfilled. The system comes from a linearization of the Navier-Stokes system on a rotation level. The domain in this case is the same as before, i.e.



2.4.1 Weak formulation

A weak formulation of the system (2.4) can be obtained similarly as for the Stokes problem. First we recall that sought solution v can be stated in the following form:

$$v = v_0 + \nabla^\perp\varphi,$$

where v_0 is the vector field constructed in Section 2.3.1 and $\varphi \in \dot{H}_0^2(\Omega)$.

The first approach in defining weak solutions might be as follows: we say that $\varphi \in \dot{H}^2_0(\Omega)$ is a weak solution to problem (2.4) iff the following identity holds for all $\theta \in \dot{H}^2_0(\Omega)$:

$$-\nu\int_\Omega \Delta\varphi\Delta\theta + \nu\int_{\partial\Omega}(2\chi - f/\nu)(\nabla^\perp\varphi\cdot\vec{\tau})\frac{\partial\theta}{\partial\vec{n}} - \int_\Omega \tilde{v_0}\cdot\nabla\theta\alpha =$$
$$= \nu\int_\Omega \text{rot } v_0\Delta\theta - \int_\Omega F\cdot\nabla^\perp\theta - \nu\int_{\partial\Omega}(2\chi - f/\nu)(v_0\cdot\vec{\tau})\frac{\partial\theta}{\partial\vec{n}},$$

however we encounter difficulties with defining the meaning of the term $\int_\Omega \tilde{v_0}\cdot\nabla\theta\alpha$, in particular the term $v_\infty\int_\Omega \theta_{,1}\Delta\varphi$. As we shall see we can replace it by the term $-\int_\Omega \Delta\theta\varphi_{,1} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}$. In the next lemma we show this and the fact that $\varphi_{,1} \in L^2(\Omega)$.

Lemma 2.4.1. *For $\theta \in \dot{H}^2_0(\Omega)$ and $\varphi \in \dot{H}^2_0(\Omega)$ - a solution to (2.15), the following term is well defined:*

$$\int_\Omega \theta_{,1}\Delta\varphi$$

Proof . First let us assume that $\theta \in C_0^\infty(\Omega)$. Then the following calculations are valid:

$$\begin{aligned}\int_\Omega \theta_{,1}\Delta\varphi &= -\int_\Omega \nabla\theta_{,1}\nabla\varphi + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}\\ &= \int_\Omega \Delta\theta_{,1}\varphi + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n} - \int_{\partial\Omega}(\nabla\theta_{,1}\cdot\vec{n})\varphi\\ &= -\int_\Omega \Delta\theta\varphi_{,1} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n} + \int_{\partial\Omega}\Delta\theta\varphi\vec{n}^{(1)}\\ &= -\int_\Omega \Delta\theta\varphi_{,1} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}.\end{aligned}$$

We rewrite above term as follows:

$$\int_\Omega \Delta\theta\varphi_{,1} = -\int_\Omega \theta_{,1}\Delta\varphi + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}. \tag{2.14}$$

Now since φ is a solution (in the sense of distributions) we find:

$$\int_\Omega \Delta\theta\varphi_{,1} = \nu\int_\Omega(\Delta\varphi + \text{rot } v_0)\Delta\theta + \int_\Omega v_0\cdot\nabla\theta(\Delta\varphi + \text{rot } v_0) -$$
$$-\nu\int_{\partial\Omega}(2\chi - f/\nu)(v_0 + \nabla^\perp\varphi)\cdot\vec{\tau}\frac{\partial\theta}{\partial\vec{n}} + \int_{\partial\Omega}\theta_{,1}\nabla\varphi\cdot\vec{n}.$$

In this form we are able to get estimates on $L^2(\Omega)$ norm of $\varphi_{,1}$, that is using the following definition of a norm:

$$\|\varphi_{,1}\|_{L^2(\Omega)} = \sup_{f\in C_0^\infty(\Omega), \|f\|_{L^2(\Omega)}\le 1}(\varphi_{,1}, f)_{L^2(\Omega)}.$$

We use it together with (2.14). First for arbitrary $f \in C_0^\infty(\Omega)$ we solve the following system:

$$\begin{aligned}\Delta\theta &= f \qquad \text{in } \Omega,\\ \theta &= 0 \qquad \text{on } \partial\Omega.\end{aligned}$$

It is well known that there exists a solution θ to this problem in the class $\dot{H}^2_0(\Omega)$ (we do not need uniqueness) which, since $\|f\|_{L^2(\Omega)} \le 1$, satisfies the following inequality:

$$\|\theta\|_{\dot{H}^2_0(\Omega)} \le C$$

It is then easily seen that in (2.14) the right hand side is well defined and can be estimated with a constant C, independent of f. Thus

$$\varphi_{,1} \in L^2(\Omega)$$

and the term $\int_\Omega \theta_{,1}\Delta\varphi$ is well defined. □

We are now ready to give proper definition of weak solution to problem (2.4):

Definition 2.4.2. *We say that $\varphi \in \dot{H}^2_0(\Omega)$ is a weak solution to problem (2.4) iff the following identity holds for all $\theta \in \dot{H}^2_0(\Omega)$:*

$$\begin{aligned} -\nu \int_\Omega \Delta\varphi\Delta\theta + \nu \int_{\partial\Omega} (2\chi - f/\nu)(\nabla^\perp\varphi \cdot \vec{\tau})\frac{\partial\theta}{\partial\vec{n}} - \int_\Omega (\tilde{v}_0 \cdot \nabla\theta(\text{rot } v_0) + v_0 \cdot \nabla\theta\Delta\varphi) \\ + \int_\Omega \Delta\theta\varphi_{,1} - \int_{\partial\Omega} \theta_{,1}\nabla\varphi \cdot \vec{n} = \\ = \nu \int_\Omega \text{rot } v_0\Delta\theta - \int_\Omega F \cdot \nabla^\perp\theta - \nu \int_{\partial\Omega} (2\chi - f/\nu)(v_0 \cdot \vec{\tau})\frac{\partial\theta}{\partial\vec{n}}. \quad (2.15) \end{aligned}$$

Proof of Theorem 2.1.2. To show existence of this solution we proceed as earlier, i.e. we compute proper estimates for $\|\varphi\|_{\dot{H}^2_0(\Omega)}$. However in this case we have additional term $\int_\Omega \tilde{v}_0 \cdot \nabla\theta\alpha$. All steps from the proof of Theorem 2.1.1 can be repeated with simple modifications if one uses the following lemma to estimate the additional term:

Lemma 2.4.3. *For every $\epsilon > 0$ there exists compactly supported v_0, which satisfies the following conditions:*

$$\begin{aligned} \nabla \cdot v_0 &= 0 \qquad \text{in } \Omega, \\ v_0 &= -v_\infty \qquad \text{on } \partial\Omega, \end{aligned}$$

such that the following inequality holds

$$\left| \int_\Omega (v_0 + v_\infty) \cdot \nabla\varphi\Delta\varphi \; dx \right| \leq \epsilon \|\Delta\varphi\|^2_{L^2(\Omega)} \quad (2.16)$$

for every $\varphi \in \dot{H}^2_0(\Omega)$.

Proof . First we transform the term

$$\int_\Omega \widetilde{v_0} \cdot \nabla\varphi\Delta\varphi dx, \quad (2.17)$$

using integration by parts, to get a term without v_∞ in it. This is because the term

$$\int_\Omega v_\infty \cdot \nabla\varphi\Delta\varphi dx = \int_\Omega \varphi_{,1}\Delta\varphi dx$$

could cause some difficulty in estimate, since a'priori we do not know whether or not $\varphi_{,1} \in L^2(\Omega)$.

Remark: in the following calculations we boundary integrals vanish, since $\widetilde{v_0} = 0$ on $\partial\Omega$, i.e. $v_0^{(1)} = 0$ and $v_0^{(2)} + v_\infty = 0$. We split (2.17) as follows:

$$\begin{aligned} \int_\Omega \widetilde{v_0} \cdot \nabla\varphi\Delta\varphi dx &= \int_\Omega (v_0 + v_\infty) \cdot \nabla\varphi\Delta\varphi dx \\ &= \int_\Omega v_0^{(1)}\varphi_{,1}\Delta\varphi \; + \int_\Omega (v_0^{(2)} + v_\infty)\varphi_{,2}\Delta\varphi \; dx \\ &=: I_1 + I_2 \end{aligned}$$

and

$$I_2 = \int_\Omega (v_0^{(2)} + v_\infty)\varphi_{,2}\varphi_{,11}dx + \int_\Omega (v_0^{(2)} + v_\infty)\varphi_{,2}\varphi_{,22}dx =: I_{21} + I_{22}.$$

Recalling that $\widetilde{v_0} = 0$ on $\partial\Omega$ we calculate:

$$\begin{aligned} I_{21} &= -\int_\Omega (v_0^{(2)} + v_\infty)_{,1}\varphi_{,2}\varphi_{,1}dx - \int_\Omega (v_0^{(2)} + v_\infty)\varphi_{,21}\varphi_{,1}dx \\ &= -\int_\Omega v_{0,1}^{(2)}\varphi_{,2}\varphi_{,1}dx + \frac{1}{2}\int_\Omega v_{0,2}^{(2)}\varphi_{,1}^2 dx. \end{aligned}$$

Similarly I_{22}:

$$I_{22} = \frac{1}{2}\int_\Omega (v_0^{(2)} + v_\infty)(\varphi_{,2}^2)_{,2}dx = -\frac{1}{2}\int_\Omega v_{0,2}^{(2)}\varphi_{,2}^2 dx.$$

Next we split I_1:

$$I_1 = \int_\Omega v_0^{(1)}\varphi_{,1}\varphi_{,11}dx + \int_\Omega v_0^{(1)}\varphi_{,1}\varphi_{,22}dx =: I_{11} + I_{12}.$$

and use the fact that $v_0^{(1)} = 0$ on $\partial\Omega$ to obtain:

$$I_{11} = \frac{1}{2}\int_\Omega v_0^{(1)}(\varphi_{,1}^2)_{,1}dx = -\frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,1}^2 dx$$

and for I_{12}:

$$I_{12} = -\int_\Omega v_{0,2}^{(1)}\varphi_{,1}\varphi_{,2}dx - \int_\Omega v_0^{(1)}\varphi_{,12}\varphi_{,2}dx = -\int_\Omega v_{0,2}^{(1)}\varphi_{,1}\varphi_{,2}dx + \frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,2}^2 dx.$$

Finally, summing up all above calculations we get:

$$\begin{aligned} I &= -\frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,1}^2 dx - \int_\Omega v_{0,2}^{(1)}\varphi_{,1}\varphi_{,2}dx + \frac{1}{2}\int_\Omega v_{0,1}^{(1)}\varphi_{,2}^2 dx \\ &\quad - \int_\Omega v_{0,1}^{(2)}\varphi_{,1}\varphi_{,2}dx + \frac{1}{2}\int_\Omega v_{0,2}^{(2)}\varphi_{,1}^2 dx - \frac{1}{2}\int_\Omega v_{0,2}^{(2)}\varphi_{,2}^2 dx \\ &= -\frac{1}{2}\int_\Omega \varphi_{,1}^2(v_{0,1}^{(1)} - v_{0,2}^{(2)})dx + \frac{1}{2}\varphi_{,2}^2(v_{0,1}^{(1)} - v_{0,2}^{(2)})dx \\ &\quad - \int_\Omega \varphi_{,1}\varphi_{,2}(v_{0,2}^{(1)} + v_{0,1}^{(2)})dx. \end{aligned}$$

Now, since $\nabla \cdot v_0 = 0$ we may write:

$$\begin{aligned} I &= -\int_\Omega \varphi_{,1}^2 v_{0,1}^{(1)}dx - \int_\Omega \varphi_{,2}^2 v_{0,2}^{(2)}dx - \int_\Omega \varphi_{,1}\varphi_{,2}(v_{0,2}^{(1)} + v_{0,1}^{(2)})dx \\ &= -\int_\Omega \nabla\varphi \cdot \nabla v_0 \cdot \nabla\varphi dx \end{aligned}$$

In this form we see, that there is no term with v_∞, but its structure does not allow us to go into more subtle analysis of its behavior near the boundary of the domain. That is why we transform it into more appropriate term. This is not straightforward since simple calculation by parts would lead us to the point we have started with. This is because there is still information about v_∞ in this term – it occurs in v_0 on the boundary. Thus an auxiliary vector field is needed to take away v_∞ from the boundary. We proceed as follows:

$$I = -\int_\Omega \nabla\varphi \cdot \nabla v_0 \cdot \nabla\varphi = -\int_\Omega \nabla\varphi \cdot \nabla(v_0 + V_\epsilon) \cdot \nabla\varphi + \int_\Omega \nabla\varphi \cdot \nabla V_\epsilon \cdot \nabla\varphi,$$

where V_ϵ is constructed as follows: let us introduce a vector field V in (t_1, t_2) coordinates (see Appendix):

$$V(p(t_1,t_2)) := \begin{pmatrix} 0 \\ \frac{v_\infty}{2}[1+cos((\pi/\zeta)t_2)] \end{pmatrix}$$

for which the following conditions are valid:

$$V(p(t_1,0)) = \begin{pmatrix} 0 \\ v_\infty \end{pmatrix}, \qquad V(p(t_1,\zeta)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\zeta = \zeta(\Omega)$ is a constant from the construction of the mapping $p(t_1, t_2)$.

Similar conditions are fulfilled by a vector field

$$V_\epsilon(p(t_1,t_2)) := \begin{cases} V(p(t_1,t_2/\epsilon)) & \text{for } t_2 \le \zeta\epsilon \\ 0 & \text{for } \zeta\epsilon \le t_2 \le \zeta. \end{cases} \tag{2.18}$$

From the construction of V_ϵ it is easily seen that

$$\|\nabla V_\epsilon\|_{L^2(\Omega)} \le C(\Omega, V)\frac{1}{\epsilon^{1/2}}. \tag{2.19}$$

Indeed, from the definition (2.18) we calculate:

$$\begin{aligned}\int_\Omega |\nabla V_\epsilon|^2 &\le C(p)\int_0^L\int_0^{\zeta\epsilon} |\nabla_t(V(p(t_1,t_2/\epsilon)))|^2 dt_2 dt_1 \\ &\le C(p)\int_0^L\int_0^{\zeta\epsilon} |\nabla_t V(p(t_1,t_2/\epsilon))|^2 \frac{1}{\epsilon^2} dt_2 dt_1 \\ &= C(p)\int_0^L\int_0^{\zeta} |\nabla V|^2 \frac{1}{\epsilon} dt_2' dt_1 \le \frac{C(p)}{\epsilon}\|\nabla V\|^2_{L^2(\Omega)}.\end{aligned}$$

We may now estimate the integral I:

$$I = -\int_\Omega \nabla\varphi\cdot\nabla(v_0+V_\epsilon)\cdot\nabla\varphi dx + \int_\Omega \nabla\varphi\cdot\nabla V_\epsilon\cdot\nabla\varphi dx =: I_1 + I_2.$$

Since $v_0 + V_\epsilon = 0$ on $\partial\Omega$ we may integrate I_1 by parts and get:

$$I_1 = \int_\Omega \nabla\varphi(v_0+V_\epsilon)\Delta\varphi dx + \frac{1}{2}\int_\Omega ((v_0+V_\epsilon)\cdot\nabla)(|\nabla\varphi|^2)dx = I_{11} + I_{12}$$

Now, from the Stokes theorem and since $\nabla\cdot v_0 = 0$ in Ω

$$I_{12} = -\frac{1}{2}\int_\Omega (\nabla\cdot V_\epsilon)(|\nabla\varphi|^2)dx.$$

Gathering all these calculations we get:

$$\begin{aligned}I = \int_\Omega \nabla\varphi v_0\Delta\varphi dx + \int_\Omega \nabla\varphi V_\epsilon\Delta\varphi dx - \frac{1}{2}\int_\Omega (\nabla\cdot V_\epsilon)(|\nabla\varphi|^2)dx + \\ + \int_\Omega \nabla\varphi\cdot\nabla V_\epsilon\cdot\nabla\varphi dx =: J_1 + J_2 + J_3 + J_.\end{aligned}$$

Before we estimate these integrals let us introduce the following notation for a tubular neighbourhood of $\partial\Omega$:

$$\Omega_\epsilon = \{x\in\Omega : \text{dist}(x,\partial\Omega) < \epsilon\zeta\}.$$

Integrals J_3 and J_4 are similar and we estimate them first. Since supp $V_\epsilon \subset \Omega_\epsilon$:

$$J_3 + J_4 \leq C \left(\int_{\Omega_\epsilon} |\nabla V_\epsilon|^2 \right)^{1/2} \left(\int_{\Omega_\epsilon} |\nabla \varphi|^4 \right)^{1/2}. \tag{2.20}$$

We use the following interpolation inequality:

$$c\|\nabla \varphi\|_{L^4(\Omega_\epsilon)} \leq \|\varphi\|^{1/4}_{L^2(\Omega_\epsilon)} \|\nabla^2 \varphi\|^{3/4}_{L^2(\Omega_\epsilon)}. \tag{2.21}$$

Since $\varphi \equiv 0$ on $\partial\Omega$ and $\varphi \in H^2(\Omega_\epsilon)$ we use embedding theorem

$$H^2(\Omega_\epsilon) \subset C^\alpha(\Omega_\epsilon), \qquad \alpha < 1$$

to conclude that

$$\|\varphi\|_{L^2(\Omega_\epsilon)} \leq C(\Omega_\epsilon) \|\nabla^2 \varphi\|_{L^2(\Omega_\epsilon)} \cdot \epsilon^{\frac{1+2\alpha}{2}}.$$

Indeed:

$$\begin{aligned} \|\varphi\|^2_{L^2(\Omega_\epsilon)} &\leq \int_0^L \int_0^{\zeta\epsilon} |\varphi(t_1, t_2)|^2 |Jp| dt_1 dt_2 \\ &\leq C(\Omega) \int_0^L \int_0^{\zeta\epsilon} t_2^{2\alpha} \|\nabla^2 \varphi\|^2_{L^2(\Omega_\epsilon)} dt_1 dt_2 \\ &\leq C(\Omega) \|\nabla^2 \varphi\|^2_{L^2(\Omega_\epsilon)} \epsilon^{1+2\alpha}. \end{aligned}$$

Inserting this inequality to (2.21) we get:

$$\|\nabla \varphi\|^2_{L^4(\Omega_\epsilon)} \leq C(\Omega) \|\nabla^2 \varphi\|^2_{L_2(\Omega_\epsilon)} \cdot \epsilon^{\frac{1+2\alpha}{4}}. \tag{2.22}$$

Now from (2.19), (2.20) and (2.22) we conclude:

$$J_3 + J_4 \leq C(\Omega, V) \epsilon^{\frac{2\alpha - 1}{4}} \|\nabla^2 \varphi\|^2_{L_2(\Omega)}.$$

Since $0 < \alpha < 1$ (in particular α can be taken $\alpha > 1/2$) we may choose ϵ small enough to get

$$J_3 + J_4 \leq \frac{1}{8} \|\nabla^2 \varphi\|^2_{L_2(\Omega)}.$$

The estimate of the integral J_2 is similar:

$$J_2 = \int_\Omega \nabla \varphi V_\epsilon \Delta \varphi \leq \left(\int_{\Omega_\epsilon} |V_\epsilon|^4 \, dx \right)^{1/4} \left(\int_{\Omega_\epsilon} |\nabla \varphi|^4 \, dx \right)^{1/4} \left(\int_\Omega |\nabla^2 \varphi|^2 \right)^{1/2} \tag{2.23}$$

and since

$$\left(\int_{\Omega_\epsilon} |V_\epsilon|^4 \, dx \right)^{1/4} \leq \left(\int_\Omega |V|^4 \, dx \right)^{1/4} = C(V) \leq \infty \tag{2.24}$$

we may combine (2.22) with (2.24) and (2.23), to get, for ϵ small enough, desired estimate:

$$J_2 \leq \frac{1}{8} \|\nabla^2 \varphi\|^2_{L^2(\Omega)}$$

For integral J_1 we refer the Reader to Section 2.3.1, which states that for every $\epsilon > 0$ the vector field v_0 can be constructed in such a way that the following inequality holds:

$$\left| \int_\Omega (v_0 \cdot \nabla \varphi)^2 dx \right| \leq \epsilon \|\Delta \varphi\|^2_{L^2(\Omega)} \qquad \text{for all } \varphi \in \dot{H}^2_0(\Omega).$$

Thus integral J_1 can be estimated using Schwarz inequality and above lemma:

$$\left| \int_\Omega \nabla\varphi v_0 \Delta\varphi dx \right| \leq \frac{1}{4} \|\nabla^2 \varphi\|^2_{L^2(\Omega)}$$

This completes the proof. □

Note on existence: As was mentioned before, having (2.16) one can repeat all steps from the proof of Theorem 2.1.1 to obtain existence. Also the reasoning for uniqueness can be repeated.

2.4.2 The Oseen system.

Here we give a note, that above results can be used to show existence of a solution to the Oseen system. As was mentioned earlier, the kernel of the *rot-div* operator is trivial, thus we may consider the following equivalent system for the rotation of the velocity:

$$\begin{aligned} -\nu\Delta\alpha + \vec{v}_\infty \cdot \nabla\alpha &= \text{rot } F && \text{in } \Omega \\ \text{rot } v &= \alpha && \text{in } \Omega \\ \text{div } v &= 0 && \text{in } \Omega \\ \alpha &= (2\chi - f/\nu)(v \cdot \vec{\tau}) && \text{on } \partial\Omega \\ v \cdot \vec{n} &= 0 && \text{on } \partial\Omega \\ v &\to v_\infty && \text{as } |x| \to \infty. \end{aligned} \tag{2.25}$$

We add the term $v_0 \cdot \nabla\alpha$ to both sides of (2.25_1) to obtain the following equivalent identity:

$$-\nu\Delta\alpha + \tilde{v_0} \cdot \nabla\alpha = \text{rot } F + v_0 \cdot \nabla\alpha \qquad \text{in } \Omega. \tag{2.26}$$

In this form it is easy to use previous results, since on the left hand side we have the modified Oseen system, which was the subject of Theorem 2.1.2. We modify the right side of (2.26) replacing $v_0 \cdot \nabla\alpha$ with $v_0 \cdot \nabla\text{rot } w$, for some vector field w, to obtain the following system for v:

$$\begin{aligned} -\nu\Delta\alpha + \tilde{v_0} \cdot \nabla\alpha &= \text{rot } F + v_0 \cdot \nabla\text{rot } w && \text{in } \Omega \\ \text{rot } v &= \alpha && \text{in } \Omega \\ \text{div } v &= 0 && \text{in } \Omega \\ \alpha &= (2\chi - f/\nu)(v \cdot \vec{\tau}) && \text{on } \partial\Omega \\ v \cdot \vec{n} &= 0 && \text{on } \partial\Omega \\ v &\to v_\infty && \text{as } |x| \to \infty. \end{aligned} \tag{2.27}$$

This system defines a mapping $\widetilde{P}$ such that $\widetilde{P}(w) = v$. It is then straightforward, that we will show existence to the Oseen system 2.25, when we show, that the mapping $\widetilde{P}$ is a contraction and use the Banach fixed point theorem. This is however simple, since we have an estimate:

$$\|\nabla v\|_{L^2(\Omega)} \leq C \left(\|\text{rot } F + v_0 \cdot \nabla\text{rot } w\|_{(\dot{H}^2_0(\Omega))^*} \right). \tag{2.28}$$

Indeed, for given two vector fields w_1, w_2 and corresponding solutions v_1, v_2 we have:

$$\|\nabla(v_1 - v_2)\|_{L^2(\Omega)} \leq C \|v_0 \cdot \nabla\text{rot } (w_1 - w_2)\|_{(\dot{H}^2_0(\Omega))^*}. \tag{2.29}$$

The term on the right hand side estimates as follows:

$$\begin{aligned}
C\|\tilde{v_0}\cdot\nabla\text{rot }(w_1-w_2)\|_{(\dot{H}_0^2(\Omega))^*} &= C\sup_{\|\theta\|_{\dot{H}_0^2(\Omega)}\le 1}\int_\Omega v_0\cdot\nabla\text{rot }(w_1-w_2)\theta\\
&= C\sup_{\|\theta\|_{\dot{H}_0^2(\Omega)}\le 1}\int_\Omega v_0\cdot\nabla\theta(\text{rot }(w_1-w_2))\\
&\le C\sup_{\|\theta\|_{\dot{H}_0^2(\Omega)}\le 1}\left(\int_\Omega |v_0\cdot\nabla\theta|^2\right)\left(\int_\Omega \text{rot }(w_1-w_2)\right)\\
&\le C\epsilon^{1/2}\|\nabla(w_1-w_2)\|_{L^2(\Omega)},
\end{aligned}$$

where in the last step we used result from Section 2.3.1. Since the choice of v_0 did not influence constant C we may choose v_0 properly, to get that $C\epsilon<1$. This shows that $\widetilde{P}$ is a contraction mapping on a suitable space and hence has a fixed point, which is the solution to the Oseen system (2.25).

2.5 Appendix

Below we present a construction of a vector field v_0, which is used to obtain a'priori estimates for our solution. As was mentioned earlier – this construction is in a correspondence to the Hopf construction of a solenoidal vector field, which is prescribed on the boundary and has a small support. Our analysis additionally takes into account the fact, that the singularity, which occurs when one wants to have small support, should only take place in the direction tangential to the boundary.

Additionally, at the beginning of the proof of the following lemma, we introduce a (t_1,t_2)-coordinates, which were used in the previous considerations.

Lemma 2.5.1. *Let Ω be bounded domain with $\partial\Omega\in C^2$. Given $d\in H^{1/2}(\partial\Omega)$ satisfying compatibility condition*

$$\int_{\partial\Omega} d\,d\sigma=0.$$

Then for any $\epsilon>0$ there exists vector field v_0 such that $v_0\in H^1(\Omega)$ and

$$\begin{aligned}\text{div } v_0 &= 0 \ \textit{in}\ \ \Omega,\\ v_0\cdot\vec{n} &= d\ \ \textit{on}\ \partial\Omega\end{aligned}\tag{2.30}$$

and for every $\varphi\in\dot{H}_0^2(\Omega)$ the following inequality holds:

$$\left|\int_\Omega (v_0\cdot\nabla\varphi)^2\,dx\right|\le\epsilon\,\|\Delta\varphi\|^2_{L^2(\Omega)}.\tag{2.31}$$

Moreover, the following estimate is valid

$$\|v_0\|_{H^1(\Omega)}\le C\,\|d\|_{H^{1/2}(\partial\Omega)}.$$

Proof . (t_1,t_2)**-coordinates**. Let $s:[0,L]\to\mathbb{R}^2$ be a normal parameterization of boundary $\partial\Omega$, i.e.

$$s([0,L])=\partial\Omega,\quad s(0)=s(L)=x_0\in\partial\Omega,\quad\text{and}\quad s'(t)=1$$

for a fixed point x_0 and L – the length of $\partial\Omega$. Next we introduce the following map $p : [0, L] \times [0, \zeta] \to \mathbb{R}^2$ such that

$$p(t_1, t_2) = s(t_1) - t_2\vec{n}(s(t_1)),$$

where $\vec{n}$ is the outer normal vector to boundary $\partial\Omega$. If ζ is small enough (comparing to curvature χ of boundary $\partial\Omega$), then the map is one-to-one and $p \in C^1$. Moreover

$$dist(p(t_1, t_2), \partial\Omega) = t_2.$$

Using the definition we compute the gradient of map p as follows

$$p_{,t_1} = (1 - t_2\chi)\vec{\tau}(s(t_1)), \qquad p_{,t_2} = \vec{n}(s(t_1)).$$

Then we see that

$$p_{,1} \perp p_{,2} \quad \text{and} \quad (\nabla p)^{-1} = (\frac{1}{1 - t_2\chi}\vec{\tau}, \vec{n})^T. \tag{2.32}$$

By coordinates (t_1, t_2) we denote coordinates obtained using mapping p.

We proceed with the construction of the extension. Introduce a function $D : [0, L] \to \mathbb{R}$ such that $D \in H^{3/2}((0, L))$ and

$$D(t) = \frac{L}{2\pi}\int_0^t d(p(s, 0))\; ds.$$

Next, we define a function $\bar{D} : \mathcal{S}^1 \to \mathbb{R}$ (where $\mathcal{S}^1$ is the unit circle) such that

$$\bar{D}(r(x)) \equiv D(x) \quad \text{for } x \in [0, L] \quad \text{and} \quad \left\|\bar{D}\right\|_{H^{3/2}(\mathcal{S}^1)} \leq C \left\|D\right\|_{H^{3/2}(0,L)},$$

where $r : [0, L] \to [0, 2\pi]$ is a simple parameterization: $r(s) = \frac{2\pi}{L}s$. Let $E : \mathcal{S}^1 \times \mathbb{R}^+ \to \mathbb{R}$ be an extension of function $\overline{D}$ such that $E \in H^2_{(loc)}(\mathcal{S}^1 \times \mathbb{R}^+)$ (see Lemma 2.5.2 with $u_0 = \bar{D}$ below). Now we take $\xi : [0, L] \times [0, \zeta] \to \mathbb{R}$ defined as follows:

$$\xi(t_1, t_2) = E(r(t_1), t_2)\eta_\epsilon(t_2),$$

where a smooth function $\eta_\epsilon(t)$ is defined as follows:

$$\eta_\epsilon(t) = \begin{cases} 1 & \text{for } t < \gamma^2(\epsilon), \\ \epsilon \ln \frac{\gamma(\epsilon)}{t} & \text{for } \gamma^2(\epsilon) \leq t < \gamma(\epsilon), \\ 0 & \text{for } t \geq \gamma^2(\epsilon), \end{cases} \tag{2.33}$$

where $\gamma(\epsilon) = \exp\left(-\frac{1}{\epsilon}\right)$. Then $\xi \in H^2([0, L] \times \mathbb{R}^+)$.

For $\epsilon < \zeta$ we use the mapping $p : [0, L] \times [0, \zeta] \to \mathbb{R}^2$ to define ξ on Ω:

$$\xi(x) = \xi(p^{-1}(x)) \;\text{ for }\; x \in p([0, L] \times [0, \zeta]) \;\text{ and }\; \xi(x) = 0 \;\text{ otherwise.}$$

To avoid misunderstandings we will denote ∇_t as a gradient in (t_1, t_2) coordinates, and ∇_x as a gradient in (x_1, x_2) coordinates. Then we have:

$$\nabla_x\xi \cdot \vec{\tau} = d \qquad \text{on } \partial\Omega,$$

since $\nabla_x\xi \cdot \vec{\tau} = \nabla_t\xi \cdot (\nabla p)^{-1} \cdot \vec{\tau} = \nabla_t\xi \cdot [1, 0] = \xi_{,t_1} = D'(t)r'(t) = d(p(t, 0))$.

Our sought field will be given as follows

$$v_0 = \nabla_x^\perp\xi \qquad \text{in } \Omega. \tag{2.34}$$

By the construction conditions (2.30) are satisfied.

Let us show inequality (2.31). Taking $\varphi \in \bar{H}^2_0(\Omega)$ we have

$$\begin{aligned}\int_\Omega (v_0 \cdot \nabla\varphi)^2 \; dx &= \int_0^L \int_0^\zeta (v_0 \nabla\varphi)^2 |Jp| \; dt_1 dt_2 \\ &\le C \int_0^L \int_0^\zeta (v_0 \cdot p_{,1} \nabla\varphi \cdot p_{,1})^2 + (v_0 \cdot p_{,2} \nabla\varphi \cdot p_{,2})^2 dt_1 dt_2 \\ &= I_1 + I_2.\end{aligned}$$

Recalling (2.32) and (2.34) we calculate the first integral

$$I_1 = C \int_0^L \int_0^\zeta [(\nabla_x \xi \cdot (1 - t_2\chi) p_{,2})(\nabla\varphi \cdot p_{,1})]^2 dt_1 dt_2.$$

Introducing the following notation:

$$f_1(t_1, t_2) = (1 - t_2\chi)^2 \nabla_x \xi \cdot p_{,2}, \qquad g_1(t_1, t_2) = \nabla\varphi \cdot \vec{\tau}, \tag{2.35}$$

we calculate

$$\begin{aligned}f_1(t_1, t_2) &= (1 - t_2\chi)^2 \nabla_t \xi \cdot (\nabla p)^{-1} \cdot p_{,2} = (1 - t_2\chi)^2 \nabla_t \xi \cdot [0, 1] \\ &= (1 - t_2\chi)^2 \xi_{,t_2} = (1 - t_2\chi)^2 \left(\frac{\partial E}{\partial t_2} \eta_\epsilon(t_2) + E(t_1, t_2) \eta'_\epsilon(t_2) \right)\end{aligned}$$

and since $(1 - t_2\chi)^2$ is bounded we rewrite integral I_1 as follows

$$\begin{aligned}I_1 &= C \int_0^L \int_0^\zeta |f_1 g_1|^2 \; dt_1 dt_2 \\ &\le C \int_0^L \int_0^\zeta \left| \frac{\partial E}{\partial t_2} \eta_\epsilon(t_2) \right|^2 |g_1|^2 dt_1 dt_2 + C \int_0^L \int_0^\zeta |E(t_1, t_2) \eta'_\epsilon(t_2)|^2 |g_1|^2 dt_1 dt_2 \\ &= I_{11} + I_{12}.\end{aligned}$$

To estimate integral I_{11} we recall that supp $\eta_\epsilon \subset [0, L] \times [0, \epsilon]$ and the fact $H^1((0, L) \times (0, \zeta)) \subset L_\infty(0, \zeta; L_4(0, L))$, then we get

$$\begin{aligned}I_{11} &\le C \int_0^L \int_0^\epsilon \left| \frac{\partial E}{\partial t_2} \eta_\epsilon(t_2) \right|^2 |g_1|^2 dt_1 dt_2 \\ &\le \epsilon C \|E\|^2_{H^2(\Omega)} \int_\Omega |\Delta\varphi|^2 \; dx.\end{aligned}$$

To consider I_{12} we apply the Hopf estimate (see [11],[19]): for each $u \in H^1_0(\Omega)$ the following bound is valid

$$\left\| \frac{u}{\delta} \right\|_{L^2(\Omega)} \le C \|u\|_{H^1_0(\Omega)}, \tag{2.36}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and constant C does not depend on u. By (2.33) we see that

$$\eta'_\epsilon(t_2) = \frac{\epsilon}{t_2} \text{ for } t_2 \in [\gamma^2(\epsilon), \gamma(\epsilon)] \quad \text{and} \quad \eta'_\epsilon(t_2) = 0 \text{ otherwise.}$$

Moreover from the definition $g_1 \in H^1_0(\Omega)$, hence with the help of (2.36), recalling (2.35) we conclude

$$I_{12} \le \epsilon C \|E\|^2_{H^2(\Omega)} \int_\Omega |\Delta\varphi|^2 \; dx,$$

since $||g_1||_{H^1_0(\Omega)} \le C||\Delta\phi||_{L_2(\Omega)}$. The integral I_2 is estimated similarly to the integral I_{11}, choosing support of η_ϵ small enough. Lemma 2.1 is proved. □

In the above proof we had to use the following result about existence of an extension of boundary data in a proper class of functions.

Lemma 2.5.2. *Given the following problem:*

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)^2 u &= 0 \text{ in } \mathcal{S}^1\times\mathbb{R}^+, \\ \frac{\partial u}{\partial t} &= 0 \text{ on } \mathcal{S}^1\times\{0\}, \\ u(x,0) &= u_0(x) \text{ for } x\in\mathcal{S}^1, \\ u &\to 0 \text{ as } t\to\infty, \end{aligned} \tag{2.37}$$

where $u_0 \in H^{3/2}(\mathcal{S}^1)$. There exists a solution $u \in H^2_{loc}(\mathcal{S}^1\times\mathbb{R})$ to this problem, satisfying the following estimate

$$\|u\|_{H^2(\mathcal{S}^1\times(0,1))} \le C\,\|u_0\|_{H^{3/2}(\mathcal{S}^1)}\,.$$

Proof . We define $E:\mathbb{Z}\times[0,2\pi]\to\mathbb{R}$ as follows:

$$E(k,z)=\begin{cases}\sin kz & \text{for } k>0\\ \cos kz & \text{for } k\le 0.\end{cases}$$

We have $\partial_z E(k,z) = kE(-k,z)$ and $\overline{\text{span}\{\{E(k,z)\}_{k\in\mathbb{Z}}\}}^{\|\cdot\|_{L^2(\mathcal{S}^1)}} = L_2(\mathcal{S}^1)$. We require $u: \mathcal{S}^1\times\mathbb{R}^+\to\mathbb{R}$ to be in the following form:

$$u(x,t)=\sum_{k\in\mathbb{Z}} c_k(t)E(k,x), \tag{2.38}$$

where $c_k(t)$ are sought functions with initial data given by

$$u_0(x)=\sum_{k\in\mathbb{Z}} c_k(0)E(k,x). \tag{2.39}$$

Since $u_0(x)\in H^{3/2}(\mathcal{S}^1)$ and $\{E(k,x)\}_{k\in\mathbb{Z}}$ is an orthogonal basis of $L_2(\mathcal{S}^1)$ we have:

$$\sum_k c_k^2(0)(1+|k|^2)^{3/2} \le C\,\|u_0\|^2_{H^{3/2}}\,.$$

From $(2.37)_1$ and (2.38) we get ordinary differential equations for coefficients $c_k(t)$:

$$\left(\frac{\partial^2}{\partial t^2}-k^2\right)^2 c_k(t)=0 \qquad \text{for every } k\in\mathbb{Z}. \tag{2.40}$$

Taking into account the initial condition (2.39) and (2.37_4) we find a solution to (2.40)

$$c_k(t)=(1+|k|t)c_k(0)e^{-|k|t}.$$

Let us show that $u\in H^2(\mathcal{S}^1\times(0,1))$. First

$$\begin{aligned} \|u\|^2_{L^2(\mathcal{S}^1\times(0,1))} &= \int_0^1\int_0^{2\pi}\sum_{k\in\mathbb{Z}}(1+|k|t)^2c_k^2(0)E^2(k,x)e^{-2|k|t}\,dx\,dt \\ &\le C\sum_k(1+|k|^3)c_k^2(0)\le C\,\|u_0\|^2_{H^{3/2}(\mathcal{S}^1)}\,. \end{aligned}$$

Next, we prove $u_{tt} \in L^2(\mathcal{S}^1 \times (0,1))$:

$$\|u_{tt}\|_{L^2}^2 \leq C \int_0^1 \sum_k c_k^2(0)(1+|k|^6 t^2)e^{-2|k|t} dt \leq C \sum_k (1+|k|^3) c_k^2(0) = C \|u_0\|_{H^{3/2}(\mathcal{S}^1)}^2 .$$

Other derivatives can be estimated similarly. □

Chapter 3

On nonhomogeneous slip boundary conditions for 2D incompressible exterior fluid flows

3.1 Introduction

One of the most difficult problems in the theory of the Navier-Stokes equations are related to a stationary two dimensional flow in an exterior domain, namely to the problem

$$v \cdot \nabla v - \nu \Delta v + \nabla p = F \qquad \text{in } \Omega, \tag{3.1}$$

$$\nabla \cdot v = 0 \qquad \text{in } \Omega, \tag{3.2}$$

$$B(v) = 0 \qquad \text{on } \partial\Omega, \tag{3.3}$$

$$\lim_{|x|\to\infty} v(x) = v_\infty, \tag{3.4}$$

where the sought solution (v, p) is a velocity vector field and the corresponding pressure, ν is a viscous positive constant coefficient, F – an exterior force acting on the fluid, v_∞ – a prescribed constant vector field and $B(v)$ stands for a boundary conditions, e.g. Dirichlet boundary conditions, as $v = v_*$ on $\partial\Omega$ ([5]). In our case the system will be supplemented with the slip boundary conditions, namely:

$$\vec{n} \cdot \mathbb{T}(v, p) \cdot \vec{\tau} + f(v \cdot \vec{\tau}) = b \qquad \text{on } \partial\Omega, \tag{3.5}$$

$$v \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega, \tag{3.6}$$

where f is a nonnegative friction coefficient, $\mathbb{T}(v, p)$ is the Cauchy stress tensor, i.e. $\mathbb{T}(v, p) = \nu\mathbb{D}(v) + p\mathbb{I}$, where $\mathbb{D}(v) = \{v_{i,j} + v_{j,i}\}_{i,j=1}^2$ is the symmetric part of the gradient ∇v, and $\mathbb{I}$ is the identity matrix. Moreover $\vec{n}, \vec{\tau}$ are respectively the normal and tangential vector to boundary $\partial\Omega$ of an exterior domain Ω, i.e. $\Omega = \mathbb{R}^2 \setminus B$, for a bounded simply-connected domain $B \subset \mathbb{R}^2$.

The slip boundary conditions govern the motion of particles at the boundary – relation (3.5) is just Newton's second law ([7], [28]). From the physical point of view this constraint is more general than the Dirichlet boundary data, since for $f \to \infty$ and $b \equiv 0$ one can obtain relation $v_{\partial\Omega} = 0$. The case where $f = 0$ is important for applications, since then the fluid reacts with surface $\partial\Omega$ as the perfect gas ([6], [25]).

In many modern applications, like the model of motion of blood, polymers and liquid metals, this type of boundary conditions is widely used ([9], [18]). Our considerations in an exterior domain are also important for example in the field of aerodynamics, where problems with flow past an obstacle is of high interest.

There are many questions related to this problem, namely: existence of solutions, uniqueness and asymptotic behaviour. In this paper we are concerned with the first issue for arbitrary data.

We are concerned with weak solutions to the problem (3.1)-(3.4) thus it is natural to require finite Dirichlet integral

$$\int_{\Omega} |\nabla v|^2 dx < \infty. \tag{3.7}$$

Since the power 2 coincides with the dimension of the domain we are not able to use standard embedding theorems for v to get some information about the velocity at infinity and assure that (3.4) holds in any sense. Many mathematicians brought their attention to this problem. Some partial results were obtained by Gilbarg and Weinberger ([14], [15]), like the case with $B(v)$ – Dirichlet boundary condition with $v_* \equiv 0$ on $\partial\Omega$. Then the following assertions hold: a) every solution to (3.1) - (3.3) that satisfies (3.7) is necessarily bounded; b) for every solution to (3.1) - (3.3) that satisfies (3.7) there exists $\tilde{v}_\infty$ such that

$$\lim_{|x|\to\infty} \int_0^{2\pi} |v(|x|, \theta) - \tilde{v}_\infty|^2 d\theta = 0. \tag{3.8}$$

In 1988 Amick [4] published a paper, where he proved that if the body B is symmetric around the direction of v_∞, and boundary data v_* is symmetric with respect to the direction of v_∞, then there exists a symmetric solution v, p such that

$$\lim_{|x|\to\infty} v(x) = \tilde{v}_\infty, \text{uniformly.}$$

These results however give no information about the relation between v_∞ and $\tilde{v}_\infty$ (see [11] for more detailed information about this problem).

On the other hand Finn and Smith [8] and Galdi [10] showed that for small values of Reynolds number and when $v_\infty \neq 0$ there exists at least one solution to the system (3.1)-(3.4) in a proper space. This has been done by applying contraction mapping technique and proper L^p-estimates for the Oseen system in exterior domain. In this chapter we would like to point out that this technique should also work for our problem considered with slip boundary conditions. In [21] we give proper L^p-estimates for the exterior Oseen system with slip boundary conditions.

The main purpose of this paper is to show that the system (3.1)-(3.4) together with (3.5)-(3.6) admits at least one weak solution for arbitrary data. This result is gathered in the following theorem:

Theorem 3.1.1. *Let $\nu > 0$, $f \geq 0$, $F \in (\nabla \dot{H}_0^2(\Omega))^*$ and $b \in H^{-1/2}(\partial\Omega)$. Then for a properly constructed vector field v_0 there exists at least one weak solution (in the sense of Definition 3.2.1)*

$$v = (v_\infty + v_0) + \nabla^\perp \varphi$$

to the system (3.1)-(3.7) with boundary conditions $B(v)$ as in (3.5)-(3.6), for which the following inequalities holds:

$$\|\varphi\|_{\dot{H}_0^2(\Omega)} \leq C = C(\nu, f, b, F) \quad \text{and} \quad \|\nabla v\|_{L^2(\Omega)} \leq C(\nu, f, b, F).$$

Note: *We used here a simplified notation $\nabla \dot{H}_0^2(\Omega)$ which stands for:*

$$\nabla \dot{H}_0^2(\Omega) = \{\nabla\varphi : \varphi \in \dot{H}_0^2(\Omega)\}.$$

See also (3.13).

Our approach to prove this theorem is via a reformulation of the problem in terms of the vorticity α of the vector field v. Following this manner one faces a problem of a kernel of the rot-div operator. In bounded simply connected domains this kernel is trivial and full information about velocity v can be retrieved from its vorticity α.

However in case of unbounded non-simply connected domain some more precise way of finding solution should be followed. This problem has been studied earlier in Chapter 1 Section 2.2 of this thesis.

One of the classical approaches to show existence of solutions is to use the Hopf inequality (see [17], [23])

$$\left\|\frac{u}{\delta}\right\|_{L^2(\Omega)} \leq C\|u\|_{H_0^1(\Omega)} \qquad \text{for all } u \in H_0^1(\Omega),$$

where $\delta(x) = \text{dist }(x, \partial\Omega)$ is the distance from the boundary, to get a'priori estimates on a solution u. However, in case of slip boundary conditions we are not able to get full information about the velocity on the boundary $\partial\Omega$ (in particular $u \notin H_0^1(\Omega)$), as it is the case for Dirichlet constraint. Thus different methods are needed to get a'priori estimates.

We introduce a reformulation of our system in terms of the vorticity $\alpha = \text{rot } v$ of the fluid. This increase the order of equations and transforms slip boundary conditions on v into Dirichlet condition on α. This way will be cleared up in the Section 3.2.

This Chapter is organized as follows: in Section 3.2 we give a reformulation of our problem in terms of rotation of the fluid, introduce a function space $\dot{H}_0^2(\Omega)$ and give a definition of a weak solutions to our problem. The next section is the main part of our paper, where we obtain a'priori estimate for our sought vector field. This estimate allows us to show existence of solutions via the Galerkin method (see: Section 3.3).

3.2 Reformulation

In this section we give a reformulation of our problem in terms of the vorticity α of the velocity vector field. Taking rotation of (3.1) we get:

$$-\nu\Delta\alpha + v \cdot \nabla\alpha = \text{rot } F,$$

where $\alpha = \text{rot } v = v_{2,1} - v_{1,2}$. The slip boundary conditions give us information about the vorticity of the fluid on the boundary, namely from (3.7)-(3.8) we get a condition on α:

$$\begin{aligned} \alpha &= (2\chi - f/\nu)v \cdot \vec{\tau} + b \qquad \text{on } \partial\Omega, \\ v \cdot \vec{n} &= 0, \\ v &\to v_\infty \qquad \text{as } |x| \to +\infty, \end{aligned}$$

where χ is the curvature of the boundary. The exact procedure was considered in [28].

Next, we take over the information at infinity from v. Let us introduce an extension vector field $\widetilde{v_0} = v_0 + v_\infty$ satisfying the following conditions:

$$\begin{aligned} \nabla \cdot \widetilde{v_0} &= 0 \qquad \text{in } \Omega, \\ \widetilde{v_0} &\equiv 0 \qquad \text{on } \partial\Omega, \\ \widetilde{v_0} &\to v_\infty \qquad \text{as } |x| \to \infty. \end{aligned}$$

A smooth compactly supported vector field v_0 will be defined later. Its construction will fulfill requirements from Lemma 3.2.2 in order to get a'priori estimates on φ.

Having $\widetilde{v_0}$ we rewrite v as

$$v = \widetilde{v_0} + u,$$

where for u we have the following constraints:

$$\begin{aligned} \nabla \cdot u &= 0 \qquad \text{in } \Omega, && (3.9)\\ u \cdot \vec{n} &= 0 \qquad \text{on } \partial\Omega, && (3.10)\\ u &\to 0 \qquad \text{as } |x| \to \infty. && (3.11) \end{aligned}$$

Having (3.9)-(3.10) we use the Poincaré lemma to present u in the following form:

$$u = \nabla^{\perp}\varphi,$$

where $\nabla^{\perp}\varphi = (-\varphi_{,2}, \varphi_{,1})$. Since $u \cdot \vec{n} = \frac{\partial\varphi}{\partial\vec{\tau}} = 0$ on $\partial\Omega$ we may take $\varphi \equiv 0$ on $\partial\Omega$.

For further calculations let us notice that

$$\text{rot } \nabla^{\perp}\varphi = \Delta\varphi.$$

We now derive from (3.1)-(3.4) the system of equations for u. Recalling that

$$\alpha = \text{rot } v = \Delta\varphi + \text{rot } \widetilde{v_0}$$

we write:

$$\begin{aligned} -\nu\Delta\alpha + (\widetilde{v_0} + \nabla^{\perp}\varphi) \cdot \nabla\alpha &= \text{rot } F && \text{in } \Omega,\\ \alpha - \Delta\varphi &= \text{rot } \widetilde{v_0} && \text{in } \Omega,\\ \varphi &= 0 && \text{on } \partial\Omega, \qquad (3.12)\\ \alpha - (2\chi - f/\nu)\nabla^{\perp}\varphi \cdot \vec{\tau} &= (2\chi - f/\nu)\widetilde{v_0} \cdot \vec{\tau} + b && \text{on } \partial\Omega,\\ \nabla^{\perp}\varphi &\to 0 \qquad \text{as } |x| \to \infty. \end{aligned}$$

Since we want Dirichlet integral for v to be finite, i.e.

$$\int_{\Omega} |\nabla v|^2 \, dx < \infty,$$

we establish a suitable space for the solution φ. Let us introduce the following Banach space equipped with a suitable norm:

$$\dot{H}^2_0(\Omega) = \overline{C_0^{\infty}(\Omega)}^{\|\nabla^2 \cdot\|_{L^2(\Omega)}}, \qquad \|\psi\|_{\dot{H}^2_0(\Omega)} = \left(\int_{\Omega} |\nabla^2\psi|^2 \, dx\right)^{1/2}. \qquad (3.13)$$

We now define a weak solution to the problem (3.1)-(3.4).

Definition 3.2.1. *We say that v is a weak solution to problem (3.1)-(3.4) together with (3.7)-(3.8) iff there exists $\varphi \in \dot{H}^2_0(\Omega)$ such that $v = \widetilde{v_0} + \nabla^{\perp}\varphi$ and the following identity holds for every $\psi \in C_0^{\infty}(\Omega)$:*

$$\int_{\Omega} (\nabla^{\perp}\varphi + \widetilde{v_0}) \cdot \nabla\psi(\Delta\varphi + rot\ \widetilde{v_0})dx + \nu \int_{\Omega} (\Delta\varphi + rot\ \widetilde{v_0})\Delta\psi \, dx +$$
$$- \nu \int_{\partial\Omega} (2\chi - f/\nu)(\nabla^{\perp}\varphi + \widetilde{v_0}) \cdot \tau \frac{\partial\psi}{\partial\vec{n}} d\sigma = \int_{\Omega} F \cdot \nabla^{\perp}\psi \, dx. \quad (3.14)$$

To prove existence of solutions in the sense of Definition 3.2.1 we first need to show a'priori estimates on φ, what will be the case in Section 3.2.1.

3.2.1 A'priori estimate.

To show existence of a solution to our problem we use the standard Galerkin method. This is a construction of approximate solutions which converge in some sense (strong enough to pass to the limit in the equation) to the limit vector function. The construction of this sequence requires usage of a fixed point theorem. Proper converging of this sequence (or a subsequence) can be assured by showing uniform boundedness (in a proper function space) of all its elements. In both of these steps a great help is an a'priori estimate of solutions to our equation. That is why we now focus on obtaining it.

We follow the standard approach to get a'priori estimate, i.e. we multiply (3.12_1) by φ and integrate over Ω. Recall that

$$\alpha = \Delta\varphi + \mathrm{rot}\ \widetilde{v_0}. \tag{3.15}$$

The first term $-\nu \int_\Omega \Delta\alpha\varphi\ dx$ gives us:

$$-\nu \int_\Omega \Delta\alpha\varphi\ dx = -\nu \int_\Omega (\Delta\varphi + \mathrm{rot}\ \widetilde{v_0})\Delta\varphi\ dx + \nu \int_{\partial\Omega} (2\chi - f/\nu)((\widetilde{v_0} + \nabla^\perp\varphi)\cdot\vec{\tau} + b)\frac{\partial\varphi}{\partial\vec{n}}\ d\sigma.$$

For the second term from (3.12_1) it is easily seen that

$$\int_\Omega (\widetilde{v_0} + \nabla^\perp\varphi)\cdot\nabla\alpha\varphi\ dx = \int_\Omega (\widetilde{v_0} + \nabla^\perp\varphi)\cdot\nabla\varphi\alpha\ dx$$

since $\varphi \equiv 0$ on $\partial\Omega$ and $\nabla\cdot(\widetilde{v_0} + \nabla^\perp\varphi) = 0$ in Ω. Finally, since $\nabla^\perp\varphi\cdot\nabla\varphi = 0$ we may write

$$\int_\Omega (\widetilde{v_0} + \nabla^\perp\varphi)\cdot\nabla\alpha\varphi\ dx = \int_\Omega \widetilde{v_0}\cdot\nabla\varphi\alpha dx.$$

This term causes difficulties in getting a'priori estimates for the solution. The reason why is that it is of order 2 with respect to φ (see 3.15), just like $\int_\Omega |\Delta\varphi|^2 dx$, but without any information about its sign. Thus we need to prove an inequality in the form:

$$\left|\int_\Omega \widetilde{v_0}\cdot\nabla\varphi\Delta\varphi dx\right| \le \gamma\|\Delta\varphi\|^2_{L^2(\Omega)},$$

for some constant γ, which should be small enough (in our case $\nu/2$). This is done by a proper construction of the vector field $\widetilde{v_0}$.

Since the construction of the vector field $\widetilde{v_0}$ is done in a neighbourhood of $\partial\Omega$ we introduce compactly supported v_0 such that

$$\widetilde{v_0} = v_0 + v_\infty,$$

with proper constraints for v_0, which will be precised in the following lemma:

Lemma 3.2.2. *For every $\epsilon > 0$ there exists compactly supported v_0, which satisfies the following conditions:*

$$\begin{aligned} \nabla\cdot v_0 &= 0 && \text{in } \Omega,\\ v_0 &= -v_\infty && \text{on } \partial\Omega, \end{aligned}$$

and the following inequality holds

$$\left|\int_\Omega (v_0 + v_\infty)\cdot\nabla\varphi\Delta\varphi\ dx\right| \le \epsilon\|\Delta\varphi\|^2_{L^2(\Omega)}$$

for every $\varphi \in \dot{H}^2_0(\Omega)$.

Note: This Lemma has been proved in Chapter 1 (see Lemma 2.4.3).

Above inequalities allow us to get a'priori estimates. Namely φ fulfills the following identity

$$-\nu\int_\Omega |\Delta\varphi|^2\,dx+\nu\int_{\partial\Omega}((2\chi-f/\nu)((\widetilde{v_0}+\nabla^\perp\varphi)\cdot\vec{\tau})+b)\frac{\partial\varphi}{\partial\vec{n}}d\sigma+$$
$$+\int_\Omega \widetilde{v_0}\cdot\nabla\varphi(\text{rot }\widetilde{v_0}+\Delta\varphi)\,dx=\nu\int_\Omega(\text{rot }\widetilde{v_0})\Delta\varphi\,dx+\int_\Omega F\cdot\nabla^\perp\varphi\,dx. \quad (3.16)$$

We see, that there is still a term of order two with respect to φ, for which one cannot verify its sign, namely

$$\int_{\partial\Omega}2\chi(\nabla^\perp\varphi)\cdot\vec{\tau}\frac{\partial\varphi}{\partial\vec{n}}d\sigma=\int_{\partial\Omega}2\chi\left(\frac{\partial\varphi}{\partial\vec{n}}\right)^2.$$

We deal with this problem using the following identity (see [22]) for $v=\widetilde{v_0}+u$:

$$\int_\Omega\alpha^2\,dx-\int_{\partial\Omega}\alpha(v\cdot\vec{\tau})\,d\sigma=\int_\Omega\mathbb{D}^2(v)\,dx+\int_{\partial\Omega}\big((v\cdot\vec{\tau})^2f-b(v\cdot\vec{\tau})\big)\,d\sigma, \quad (3.17)$$

which comes from the well known identity for $v\in\dot{H}^1(\Omega)$ with $\nabla\cdot v=0$ in Ω (see [35]):

$$\int_\Omega\alpha^2dx=\int_\Omega|\mathbb{D}(v)|^2dx+\int_{\partial\Omega}2\chi(v\cdot\vec{\tau})^2. \quad (3.18)$$

Since

$$-\nu\int_\Omega|\Delta\varphi|^2\,dx+\nu\int_{\partial\Omega}((2\chi-f/\nu)((\widetilde{v_0}+\nabla^\perp\varphi)\cdot\vec{\tau})+b)\left(\frac{\partial\varphi}{\partial\vec{n}}\right)\,d\sigma-\int_\Omega\Delta\varphi\text{rot }\widetilde{v_0}\,dx=$$
$$-\nu\int_\Omega\alpha^2\,dx+\nu\int_{\partial\Omega}\alpha(v\cdot\vec{\tau})\,d\sigma+\int_\Omega\Delta\varphi\text{rot }\widetilde{v_0}\,dx+\int_\Omega(\text{rot }\widetilde{v_0})^2\,dx-\nu\int_{\partial\Omega}\alpha(v_0\cdot\vec{\tau})\,d\sigma,$$

we may derive from (3.16) using (3.17) the following identity:

$$\begin{aligned}-\nu\int_\Omega\mathbb{D}^2(\nabla^\perp\varphi+\widetilde{v_0})\,dx&+\\+\int_\Omega\Delta\varphi\text{rot }\widetilde{v_0}\,dx+\int_\Omega(\text{rot }\widetilde{v_0})^2\,dx&\\-\nu\int_{\partial\Omega}((\nabla^\perp\varphi+\widetilde{v_0})\cdot\vec{\tau})^2f-b((\nabla^\perp\varphi+\widetilde{v_0})\cdot\vec{\tau})\,d\sigma&-\\-\nu\int_{\partial\Omega}\alpha(v_0\cdot\vec{\tau})\,d\sigma+\int_\Omega\widetilde{v_0}\cdot\nabla\varphi(\text{rot }\widetilde{v_0}+\Delta\varphi)\,dx&=\int_\Omega F\cdot\nabla^\perp\varphi\,dx.\end{aligned} \quad (3.19)$$

To get a'priori estimate from (3.19) we need to use the Korn's inequality (see Lemma 3.4.1 in Appendix):

$$\int_\Omega\mathbb{D}^2(\nabla^\perp\varphi)dx\geq K\int_\Omega|\nabla^2\varphi|^2dx, \quad (3.20)$$

where K is a constant dependent only on Ω, which allows us to get $\int_\Omega|\nabla^2\varphi|^2dx$ in the estimate, namely:

$$-\nu\int_\Omega\mathbb{D}^2(\nabla^\perp\varphi)dx-\nu\int_{\partial\Omega}((\nabla^\perp\varphi+\widetilde{v_0})\cdot\vec{\tau})^2f\leq-K\nu\int_\Omega|\nabla^2\varphi|^2dx. \quad (3.21)$$

Here we also used the fact that $f\geq 0$. Combining (3.19) with (3.21) we are able to get an estimate of the form:

$$\int_\Omega|\nabla^2\varphi|^2dx\leq C(\text{DATA})\left(\int_\Omega|\nabla^2\varphi|^2\right)^{1/2}+\frac{1}{K\nu}\left|\int_\Omega\widetilde{v_0}\cdot\nabla\varphi\Delta\varphi dx\right|.$$

This is because $\left|\int_\Omega \widetilde{v_0}\cdot\nabla\varphi\Delta\varphi dx\right|$ is the only term of order 2 with respect to φ, and all other terms are of order 1 can be estimated by $C(\text{DATA})\left(\int_\Omega|\nabla^2\varphi|^2\right)^{1/2}$ (see the Remark below).

We now use Lemma 3.2.2 with $\epsilon = K\nu/2$ to estimate remaining term of order 2 and get the following inequality:

$$\|\nabla^2\varphi\|^2_{L^2(\Omega)} \le C(\text{DATA})\|\nabla^2\varphi\|_{L^2(\Omega)},$$

where in $C(\text{DATA})$ one includes all constants dependent on Ω, F, ν, etc. This inequality gives us of course a'priori estimate on $\|\nabla^2\varphi\|_{L^2(\Omega)}$:

$$\|\nabla^2\varphi\|_{L^2(\Omega)} \le C(\text{DATA}).$$

Remark: It is not hard to estimate terms in (3.20), where u (i.e. $\nabla^\perp\varphi$) appears in a form different than $\nabla^2\varphi$ – one must recall that $\varphi\equiv 0$ on $\partial\Omega$, which gives us, together with $\varphi\in H^2_{\text{loc}}(\Omega)$, the fact, that all local estimates (the only needed) can be obtained using $\|\nabla^2\varphi\|_{L^2(\Omega)}$.

3.3 Existence.

In this section we use the standard Galerkin method to prove the existence of a solution to

$$\begin{aligned}
-\nu\Delta\alpha + v\cdot\nabla\alpha &= \text{rot } F,\\
v\cdot\vec{n} &= 0,\\
\text{rot } v &= \alpha, \qquad \text{in } \Omega\\
\nabla\cdot v &= 0, \qquad \text{in } \Omega\\
\alpha &= (2\chi - f/\nu)v\cdot\vec{\tau} + b \qquad \text{on } \partial\Omega
\end{aligned}$$

in the sense of distributions, i.e. in the sense of Definition 3.2.1. This means that we prove the existence of a function $\varphi\in\dot{H}^2_0(\Omega)$, which satisfies (3.14) for all $\psi\in C^\infty(\Omega)$ with compact support in Ω. As was mentioned before, we seek for a solution in the form

$$v = \widetilde{v_0} + \nabla^\perp\varphi.$$

Since $\dot{H}^2_0(\Omega)$ is Hilbertian and separable we take a base of compactly supported functions $\{w_i\}_{i=1}^\infty$:

$$\dot{H}^2_0(\Omega) = \overline{\{w_1, w_2, \ldots\}}^{\|\cdot\|_{\dot{H}^2_0(\Omega)}}.$$

Next we introduce a finite dimensional subspace $V^N(\Omega)\subset\dot{H}^2_0(\Omega)$:

$$V^N(\Omega) = span\{w_1,\ldots,w_N\}.$$

We additionally assume that $(w_i,w_j)_{V^N} = \delta_{ij}$, where $(\cdot,\cdot)_{V^N}$ is the inner product in V^N, which comes from the inner product in $\dot{H}^2_0(\Omega)$. We search for an approximation φ^N of the function φ in the form:

$$\varphi^N(x) = \sum_{j=0}^N c_j^N w_j \in V^N.$$

To find coefficients c_j^N we solve the following system:

$$\int_\Omega(\nabla^\perp\varphi^N + \widetilde{v_0})\cdot\nabla w_i(\Delta\varphi^N + \text{rot }\widetilde{v_0})dx + \nu\int_\Omega(\Delta\varphi^N + \text{rot }\widetilde{v_0})\Delta w_i\, dx +$$
$$-\nu\int_{\partial\Omega}\Big((2\chi - f/\nu)(\nabla^\perp\varphi^N + \widetilde{v_0})\cdot\vec{\tau} + b\Big)\frac{\partial w_i}{\partial\vec{n}}d\sigma = \int_\Omega F\cdot\nabla^\perp w_i\, dx$$

for $i = 1, \ldots, N$.

Let us introduce a mapping $P : V^N \to V^N$ as follows:

$$\begin{aligned} P(\varphi^N) = \sum_{i=1}^{N} \Bigg(& \int_\Omega (\nabla^\perp \varphi^N + \widetilde{v_0}) \cdot \nabla w_i (\Delta \varphi^N + \text{rot } \widetilde{v_0}) dx \\ & + \nu \int_\Omega (\Delta \varphi^N + \text{rot } \widetilde{v_0}) \Delta w_i \; dx \\ & - \nu \int_{\partial\Omega} \Big((2\chi - f/\nu)(\nabla^\perp \varphi^N + \widetilde{v_0}) \cdot \vec{\tau} + b \Big) \frac{\partial w_i}{\partial \vec{n}} d\sigma \\ & - \int_\Omega F \cdot \nabla^\perp w_i \; dx \Bigg) \cdot w_i. \end{aligned}$$

From the definition of the mapping P and for $(\cdot, \cdot)_{V^N}$ – the inner product in V^N, we easily calculate that:

$$\begin{aligned} \big(P(\varphi^N), \varphi^N\big)_{V^N} = & \int_\Omega (\nabla^\perp \varphi^N + \widetilde{v_0}) \cdot \nabla \varphi^N (\Delta \varphi^N + \text{rot } \widetilde{v_0}) dx \\ & + \nu \int_\Omega (\Delta \varphi^N + \text{rot } \widetilde{v_0}) \Delta \varphi^N \; dx \\ & - \nu \int_{\partial\Omega} \Big((2\chi - f/\nu)(\nabla^\perp \varphi^N + \widetilde{v_0}) \cdot \vec{\tau} + b \Big) \frac{\partial \varphi^N}{\partial \vec{n}} d\sigma \\ & - \int_\Omega F \cdot \nabla^\perp \varphi^N \; dx \end{aligned}$$

To get proper estimate for the term $(P(\varphi^N), \varphi^N)_{V^N}$ first we must use the following identity (see (3.17) or (3.18)):

$$\int_\Omega \alpha^2 \quad dx \; - \; \int_{\partial\Omega} \alpha(v \;\cdot\; \tau) d\sigma \quad = \quad \int_\Omega \mathbb{D}^2(v) \quad dx \; + \; \int_{\partial\Omega} ((v \;\cdot\; \tau)^2 f \; - \; b(v \;\cdot\; \vec{\tau})) d\sigma.$$

Hence, since $\nabla^\perp \varphi^N \cdot \nabla \varphi^N = 0$:

$$\begin{aligned} \big(P(\varphi^N), \varphi^N\big)_{V^N} = & \int_\Omega \widetilde{v_0} \cdot \nabla \varphi^N (\Delta \varphi^N + \text{rot } \widetilde{v_0}) dx - \nu \int_\Omega \alpha \text{rot } \widetilde{v_0} \\ & \nu \left(\int_\Omega \mathbf{D}^2(v) dx + \int_{\partial\Omega} ((v \cdot \vec{\tau})^2 f - b(v \cdot \vec{\tau})) \right) d\sigma - \int_\Omega F \cdot \nabla^\perp \varphi^N \; dx \end{aligned}$$

In this form it is easily seen, that one can obtain the following estimate

$$\big(P(\varphi^N), \varphi^N\big)_{V^N} > 0 \qquad \text{for } \|\varphi^N\|_{V^N} > k$$

for some constant $k > 0$. One must just repeat the reasoning from a'priori estimates.

Such condition gives us (see [19]) existence of φ_N such that $\|\varphi^N\|_{V^N} \le k$ and moreover $P(\varphi^N) = 0$, which solves our problem for coefficients c_j^N. Thus we get sequence of approximating solutions $\varphi^N \in V^N$ such that $\|\varphi^N\|_{V^N} \le k$ for some constant $k > 0$ independent of N.

Passing to the limit. Since we have uniform bound on $\|\varphi^N\|_{\dot{H}_0^2(\Omega)}$, i.e. $\|\varphi^N\|_{\dot{H}_0^2(\Omega)} < k$, we can take a subsequence which is weakly convergent to some limit. However, for the sake of passing to the limit in nonlinear terms of (3.14) we must use diagonal technique.

Let us denote $\Omega_R = \Omega \cap B_R$. In bounded domain Ω_R we have $\varphi^N \in H^2(\Omega_R)$, since $\varphi^N, \nabla\varphi^N \in L^2_{\text{loc}}(\Omega)$. Moreover, since $\varphi^N \equiv 0$ on $\partial\Omega$ we have

$$\|\varphi^N\|_{H^2(\Omega_R)} \le C(R)\|\varphi^N\|_{\dot{H}^2(\Omega)},$$

hence we may choose a subsequence $\varphi^{N'}$, which we further denote for simplicity as φ^N, which is convergent on Ω_R to φ in the following sense:

$$\begin{array}{llll} \nabla^2\varphi^N \to \nabla^2\varphi & \text{weakly } L^2 \text{ on } \Omega_R, & \varphi^N \to \varphi & \text{strongly } L^2 \text{ on } \Omega_R, \\ \nabla\varphi^N \to \nabla\varphi & \text{strongly } L^2 \text{ on } \Omega_R, & \varphi^N \to \varphi & \text{strongly } L^2 \text{ on } \partial\Omega. \end{array}$$

We repeat this treatment for $R \to \infty$ to get subsequence φ^N, which is convergent in above sense on all bounded domains. Since a test function in 3.14 has compact support we may pass to the limit. Thus φ is a solution in the sense of Definition 3.2.1.

3.4 Appendix

Korn's inequality. In what follows we give a proof of Korn's inequality – a result, which is widely used in fluid dynamics to show equivalence of the L_2-norm of the gradient of the velocity and its symmetric part:

Lemma 3.4.1. *For an exterior domain $\Omega \subset \mathbb{R}^2$, which is not spherically symmetric, there exists a constant $K > 0$, dependent on the domain Ω such that the following inequality*

$$\int_\Omega \mathbb{D}^2(u)\ dx \ge K \int_\Omega |\nabla u|^2\ dx$$

holds for every $u \in \overline{C_0^\infty}^{\|\nabla\cdot\|_{L^2(\Omega)}}$ satisfying

$$\nabla\cdot u = 0 \quad \text{in } \Omega, \qquad u\cdot\vec{n} = 0 \quad \text{on } \partial\Omega. \tag{3.22}$$

Proof . Let us state that $u \in C_0^\infty(\Omega)$ need not be zero on the $\partial\Omega$ - it is only required to have bounded support in Ω.

Let us take $u \in C_0^\infty(\Omega)$. It is easily seen that

$$\begin{aligned} \int_\Omega \mathbb{D}^2(u)\ dx &= \int_\Omega \sum_{i,j}(u_{i,j}+u_{j,i})^2\ dx \\ &= 2\int_\Omega |\nabla u|^2\ dx + \int_\Omega \sum_{i,j} u_{i,j}u_{j,i}\ dx. \end{aligned}$$

Integration by parts of the term $I = \int_\Omega \sum_{i,j} u_{i,j}u_{j,i}\ dx$ gives us:

$$\begin{aligned} I &= -\int_\Omega \sum_{i,j} u_{i,ji}u_j\ dx + \int_{\partial\Omega} u_{i,j}u_jn_i\ d\sigma \\ &= \int_\Omega \sum_{i,j} u_{i,i}u_{j,j}\ dx + \int_{\partial\Omega} u_{i,i}u_jn_j\ d\sigma + \int_{\partial\Omega} u_{i,j}u_jn_i\ d\sigma. \end{aligned}$$

Now since (3.22) we have $I = -\int_{\partial\Omega} u_{i,j}u_jn_i\ d\sigma$. Moreover, the condition $u\cdot\vec{n}$ gives us:

$$0 = (u\cdot\nabla)(u\cdot\vec{n}) = \sum_{i,j} u_ju_{i,j}n_i + \sum_{i,j} u_ju_in_{i,j}.$$

Gathering all these calculations together we get the following inequality:

$$\int_{\Omega} |\nabla u|^2 \; dx \leq K_1 \left(\int_{\Omega} \mathbb{D}^2(u) \; dx + \int_{\partial\Omega} u^2 \; d\sigma \right). \tag{3.23}$$

To finish our proof we need to show, that there exists constant K_2 such that the following inequality holds:

$$\int_{\partial\Omega} u^2 \; d\sigma \leq \frac{1}{2K_1} \|\nabla u\|^2_{L^2(\Omega)} + K_2 \|\mathbb{D}(u)\|^2_{L^2(\Omega)}.$$

Let us assume by contrary, that such constant does not exist. Then we are able to find a bounded sequence $\{u_n\} \subset C_0^\infty(\Omega)$ satisfying

$$\int_{\partial\Omega} u_n^2 \; d\sigma \geq \frac{1}{2K_1} \|\nabla u\|_{L^2(\Omega)} + n \|\mathbb{D}(u)\|^2_{L^2(\Omega)}$$

together with $\|\nabla u_n\|_{L^2(\Omega)} = 1$. Now taking $w_n = u_n / \|u_n\|_{L^2(\partial\Omega)}$ we get

$$1 \geq \frac{1}{2K_1} \|\nabla w_n\|^2_{L^2(\Omega)} + n \|\mathbb{D}(w_n)\|^2_{L^2(\Omega)}.$$

Thus $\|\mathbb{D}(w_n)\|^2_{L^2(\Omega)} \leq 1/n$, but also $\|\nabla w_n\|_{L^2(\Omega)} \leq M$, so w_m has weakly convergent subsequence in $H^1_{\text{loc}}(\Omega)$ since

$$\|w_m\|_{L^2(\Omega_\epsilon)} \leq C(\Omega_\epsilon)(\|w_m\|_{L^2(\partial\Omega)} + \|\nabla w_m\|_{L^2(\Omega)}).$$

This weak convergence gives us strong convergence of w_n in $L^2(\partial\Omega)$. Let us notice that inequality (3.23) and above convergence and estimate on $\|\mathbb{D}(w_n)\|_{L^2(\Omega)}$ allow us to conclude that ∇w_n is a Cauchy sequence in $L^2(\Omega_\epsilon)$. This suffices to pass to the limit in the term $\int_\Omega \mathbb{D}^2(w_n) \; dx$ locally. Thus the limit vector field v_*, for which $\nabla v_* \in L^2_{\text{loc}}$, satisfies

$$\int_{\Omega_\epsilon} \mathbb{D}^2(v_*) \; dx = 0$$

which implies that $v_* = (ax_2, -ax_1)$ for some constant a. But our domain Ω is not spherically symmetric, thus the condition $v_* \cdot \vec{n} = 0$ on $\partial\Omega$ cannot be fulfilled. This completes the proof of Lemma 3.4.1.

□

Chapter 4

L_p-estimates and thorough analysis of the Oseen system in 2D exterior domains.

4.1 Introduction

One of the main problems in the theory of the Navier-Stokes equations studied in exterior domains is the question about the behaviour of the velocity vector field of the fluid at infinity. The typical system in a bounded domain:

$$v \cdot \nabla v - \Delta v + \nabla p = F \qquad \text{in } \Omega, \tag{4.1}$$

$$\operatorname{div} v = 0 \qquad \text{in } \Omega, \tag{4.2}$$

$$B(v,p) = b \qquad \text{on } \partial\Omega, \tag{4.3}$$

where $B(v,p)$ stands for the boundary constraints (e.g. Dirichlet boundary condition), is complemented with the condition on the velocity vector field at infinity, namely

$$v \to \vec{v}_\infty \qquad \text{as } |x| \to \infty \tag{4.4}$$

for some prescribed constant vector field v_∞. There are classical results of Leray about existence of solutions with the finite Dirichlet integral to the system (4.1)-(4.3). However one cannot predict that these solutions satisfy (4.4). Indeed, in two dimensions we cannot use standard embedding theorems, since the dimension of the domain coincides with the power 2 in the integral, that is why the condition

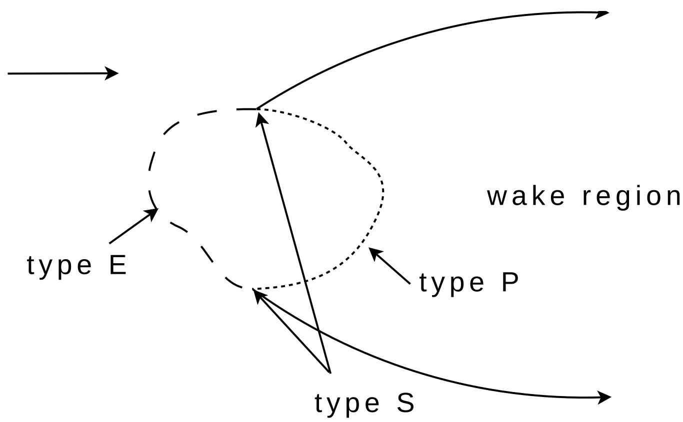
$$\int_\Omega |\nabla v|^2 dx < \infty \tag{4.5}$$

itself is insufficient even to assure that $v \in L^\infty(\Omega)$. The condition (4.5) implies that $v \in BMO(\Omega)$ only, hence we are not able to deduce information about the behaviour of v at infinity.

The Oseen system has this advantage over the Stokes system that one can obtain better information of the solution v at infinity, because of the presence of the additional term $v_{,1}$ (see [8], [10]).

While existence of solutions to this problem itself is very interesting also investigating their behaviour, both close to the obstacle and at infinity, brings up many substantial questions. For example what is the decay rate for the velocity at infinity and if there exists a wake region behind the obstacle. Both of these questions have answers, depending on a proper information of the solution (see [13]). One can expect that the decay rate of the solution to the Navier-Stokes system will be similar to the decay rate of the Oseen fundamental solution, however we do not want to address this question in our paper.

Our analysis of the Oseen system shows that the behaviour of the solution depends strongly on the angle between the surface and the vector $\vec{v}_\infty$. In a simplified case of a convex obstacle it can be shown, that the character of the system is elliptic in front of the obstacle, while its character changes into a parabolic degeneration behind the obstacle. This is presented on the following figure:



The core of the paper is the thorough analysis of the Oseen system in the half plane. We show that proper L_p–estimates are valid for the second derivatives of the velocity and for the gradient of the pressure, under assumption, that boundary constraints are in a suitable class of regularity. What is substantial is the class of regularity required by the boundary problem. It turns out that the choice of boundary data should depend on the sign of $\vec{v}_\infty \cdot \vec{n}$ ($\vec{n}$ is the normal vector to the boundary), which corresponds to the position of the obstacle. In the case $\vec{v}_\infty \cdot \vec{n} < 0$ (points E), which analyzes the system in front of the obstacle, the class of regularity of boundary data is of the elliptic type. For $\vec{v}_\infty \cdot \vec{n} > 0$ (points P — behind the obstacle) it appears that the system loses its purely elliptic character in favour of a parabolic degeneration. This feature corresponds to the appearance of the wake region behind the obstacle. As $\vec{v}_\infty \cdot \vec{n} = 0$ (points S) we obtain a transition area.

As an application to this analysis we show L_p–estimates for the Oseen system in exterior domain, which allows one to obtain also existence results for the Navier-Stokes system, which by results of Galdi and Sohr [13] describes the structure of solutions at infinity.

We would like to emphasize that our approach does not require explicit form of the fundamental solution. A similar approach has been examined by Solonnikov ([34]) and later by Zajączkowski and Mucha ([31], [32]).

Let us precise our problem. We consider the system:

$$v_\infty v_{,1} - \Delta v + \nabla p = F \qquad \text{in } \Omega, \tag{4.6}$$

$$\operatorname{div} v = G \qquad \text{in } \Omega, \tag{4.7}$$

$$\vec{n} \cdot \mathbb{T}(v,p) \cdot \vec{\tau} + f(v \cdot \vec{\tau}) = b \qquad \text{on } \partial\Omega, \tag{4.8}$$

$$v \cdot \vec{n} = d \qquad \text{on } \partial\Omega, \tag{4.9}$$

together with the condition (4.4) at infinity. The pair (v,p) is the sought solution – respectively the velocity vector field and the corresponding pressure, F is an external force acting on the fluid, G is the function describing compressibility of the fluid, v_∞ is a constant describing the velocity of the fluid at infinity, f is a nonnegative friction coefficient, $\mathbb{T}(v,p)$ is the Cauchy stress tensor, i.e. $\mathbb{T}(v,p) = \nu\mathbb{D}(v) + p\mathbb{I}$, where $\mathbb{D}(v) = \{v_{i,j} + v_{j,i}\}_{i,j=1}^2$ is the symmetric part of the gradient ∇v, and $\mathbb{I}$ is the identity matrix. Moreover $\vec{n}, \vec{\tau}$ are, respectively, the normal and tangential vector to boundary $\partial\Omega$ of an exterior domain Ω, where $\Omega = \mathbb{R}^2 \setminus B$, for a bounded simply-connected domain $B \subset \mathbb{R}^2$.

As a direct result of our analysis of the system in the half plane we prove the following theorem:

Theorem 4.1.1. *Let $1 < p < \infty$, $F \in L^p(\Omega) \cap H^{-1}(\Omega)$, $G \in W_p^1(\Omega)$ with F and G of compact support in Ω, $b \in W_p^{1-1/p}(\partial\Omega)$ and $d \in W_p^{2-1/p}(\partial\Omega)$, for which the following compatibility condition is fulfilled:*

$$\int_{\partial\Omega} d(x) d\sigma = \int_\Omega G dx.$$

Moreover let $f > 0$ be a positive constant and $v_\infty \neq 0$. Then there exists a solution (v,p) to the system (4.6)-(4.9), for which the following estimate holds:

$$\begin{aligned} v_\infty \|v_{,1}\|_{L^p(\Omega)} &+ \|\nabla^2 v\|_{L^p(\Omega)} + \|\nabla p\|_{L^p(\Omega)} \leq \\ &\leq C(\Omega, f, v_\infty) \Big(\|F\|_{L^p(\Omega)\cap H^{-1}(\Omega)} + \|G\|_{W_p^1(\Omega)} + \\ &+ \|b\|_{W_p^{1-1/p}(\Omega)} + \|d\|_{W_p^{2-1/p}(\Omega)} \Big). \end{aligned} \tag{4.10}$$

Denoting the term on the right hand side of (4.10) as $C(DATA)$ we also have:

- *for $1 < p < 3$: $v_\infty^{1/3} \|\nabla v\|_{L^{3p/(3-p)}} \leq C(DATA)$*
- *for $1 < p < 3/2$: $v_\infty^{2/3} \|v\|_{L^{3p/(3-2p)}} \leq C(DATA)$.*

As was mentioned before, we may use this result together with the techniques from the work of Galdi ([10]) to obtain the following result:

Theorem 4.1.2. *Considering the system (4.1)-(4.2) of Navier-Stokes equations in an exterior domain Ω, together with slip boundary conditions (4.8)-(4.9). where $b \in W_p^{1-1/p}(\partial\Omega)$ and $d \in W_p^{2-1/p}(\partial\Omega)$ and $v_\infty \neq 0$. If $d(x)$ satisfies the compatibility condition $\int_{\partial\Omega} d(x) d\sigma = 0$, then for $1 < p < 6/5$ and sufficiently small data (i.e. $\|b\|_{W_p^{1-1/p}(\partial\Omega)}$, $\|d\|_{W_p^{2-1/p}(\partial\Omega)}$, v_∞) there exists a unique solution (v,p) such that:*

$$v - v_\infty \in L^{3p/(3-2p)}(\Omega), \quad \nabla v \in L^{3p/(3-p)}(\Omega), \quad \nabla^2 v \in L^p(\Omega),$$

and suitable estimates hold.

We refer the Reader to [11] for a detailed discussion of this and similar results (for example uniqueness of solution (v,p) under suitable condition).

Notation. Throughout the paper we use standard notation ([37]): $W_p^k(\Omega)$ for Sobolev spaces and the following definition of the norm in Slobodeckii spaces $W_p^s(\mathbb{R}^n)$:

$$\|f\|^p_{W_p^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^p dx + \sum_{0 \leq |s'| \leq [|s|]} \int_{\mathbb{R}^n} |D^{s'} f(x)|^p dx + \|f\|^p_{\dot{W}_p^s(\mathbb{R}^n)},$$

where $[\alpha]$ stands for the integral part of α and

$$\|f\|^p_{\dot{W}_p^s(\mathbb{R}^n)} = \sum_{|s'| = [|s|]} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^{s'} f(x) - D^{s'} f(x')|^p}{|x - x'|^{n + p(|s| - [|s|])}}. \tag{4.11}$$

We also introduce the following notation for intersected spaces. Let $X_r(U)$ be a Banach space, dependent of a constant r, equipped with the norm $\|\cdot\|_{X_r(U)}$. Let $A \subset \mathbb{R}$ be a nonempty set. Then we introduce the following function space:

$$X_{r,A}(U) = \bigcap_{r\in A} X_r(U),$$

equipped with the norm

$$\|f\|_{X_{r,A}(U)} = \sup_{r\in A} \|f\|_{X_r(U)}.$$

In our case the set A will be always of finite elements.

The structure of this Chapter is as follows: the core part, Section 4.2, is devoted to the case of a flow in the half plane. At the beginning we give some preliminary considerations, we state main results in Theorems 4.2.2 and 4.2.3 and then we give some results about consistency of the boundary conditions. Later we derive a solution to our problem and give estimates for the pressure, i.e. we prove Theorem 4.2.2. Section 4.2.3 consists of introducing auxiliary problem for the velocity of the fluid. This result is used to prove estimates for the second derivatives of u in Section 4.2.4 and Section 4.2.5, i.e. $u_{,11}$ and $u_{,22}$ respectively, which proves Theorem 4.2.3. At the end of Section 4.2 we give a brief summary about the choice of boundary conditions. Section 4.3 is devoted to results in the whole space $\mathbb{R}^2$, which were being used in the previous section. As a consequence of results from Section 4.2 in part 4 we give a proof of Theorem 4.1.1. In Appendix we present two multiplier theorems of Marcinkiewicz type. We also give some additional results, which are needful for our considerations, but are connected with a general theory of function spaces rather than with a theory of fluid dynamics.

4.2 The Oseen system in the half space $\mathbb{R}^2_+$

The localization procedure obviously changes not only the domain our problem is considered in, but also affects its structure. The substantial difference is that the term $v_\infty v_{,1}$ from (4.6) transforms into $a_1 v_{,1} + a_2 v_{,2} = (a_1, a_2)\cdot\nabla v$, where $a_1^2 + a_2^2 = v_\infty^2 > 0$. We emphasize this because the sign of a_2 will be crucial in our considerations, since it is the same as a sign of $\vec{v}_\infty \cdot \vec{n}$, which, for a convex obstacle, reflects the region of the considered situation, namely the case $a_2 < 0$ corresponds to a region of the boundary in front of the obstacle, while $a_2 > 0$ stands for the situation behind the obstacle.

In this section we consider the following system:

$$a_1 v_{,1} + a_2 v_{,2} - \Delta v + \nabla q = F \qquad \text{in } \mathbb{R}^2_+, \tag{4.12}$$

$$\nabla \cdot v = G \qquad \text{in } \mathbb{R}^2_+, \tag{4.13}$$

$$\vec{n}\cdot \mathbb{T}(v,p)\cdot\vec{\tau} + f(v\cdot\tau) = \underline{b} \qquad \text{on } \partial\mathbb{R}^2_+, \tag{4.14}$$

$$\vec{n}\cdot v = \underline{d} \qquad \text{on } \partial\mathbb{R}^2_+, \tag{4.15}$$

$$v \to 0 \qquad \text{as } |x|\to\infty. \tag{4.16}$$

We assume that F and G have compact support in $\mathbb{R}^2_+$, since this system comes from the localization procedure.

To simplify the problem we remove the inhomogeneity from (4.12) and (4.13) using results in the whole space $\mathbb{R}^2$, i.e. Theorem 4.3.4. Then we need to use Lemma 4.3.5 to see, in which class of regularity on the boundary of $\mathbb{R}^2_+$ the obtained solution is. We gather this in the following Lemma:

Lemma 4.2.1. *Let $q > 3$. Given $F \in L^q(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ and $G \in W^1_q(\mathbb{R}^2) \cap W^1_1(\mathbb{R}^2)$. Considering the following Oseen system in the whole space:*

$$\begin{aligned} a_1\widetilde{v}_{,1} + a_2\widetilde{v}_{,2} - \Delta\widetilde{v} + \nabla\widetilde{q} &= \widetilde{F} \qquad \text{in } \mathbb{R}^2, \\ \operatorname{div} \widetilde{v} &= \widetilde{G} \qquad \text{in } \mathbb{R}^2, \end{aligned}$$

there exists a solution $(\widetilde{v}, \widetilde{q})$ for this system, for which the following conditions are satisfied:

$$\widetilde{v} \in W^2_r(\mathbb{R}^2) \qquad \text{for all } r \in (3, q], \tag{4.17}$$

$$\nabla\widetilde{v} \in W^1_r(\mathbb{R}^2) \qquad \text{for all } r \in (3/2, q], \tag{4.18}$$

and for all $r \in (3, q]$:

$$\vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q}) \cdot \vec{\tau}_{|x_2=0} + f(\widetilde{v} \cdot \vec{\tau})_{|x_2=0} \in W^{1-1/r}_r(\mathbb{R}), \tag{4.19}$$

$$\vec{n} \cdot \widetilde{v}_{|x_2=0} \in W^{2-1/r}_r(\mathbb{R}), \tag{4.20}$$

like also for all $r \in (3/2, q]$:

$$\vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q}) \cdot \vec{\tau}_{|x_2=0} + f(\widetilde{v} \cdot \vec{\tau})_{|x_2=0} \in \dot{W}^{1-1/r}_r(\mathbb{R}), \tag{4.21}$$

$$\vec{n} \cdot \widetilde{v}_{|x_2=0} \in \dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-1/r}_r(\mathbb{R}). \tag{4.22}$$

Proof . Existence of a solution $(\widetilde{v}, \widetilde{q})$ is straightforward from Theorem 4.2.2. Conditions (4.17)-(4.18) come from (4.55) and (4.56). (4.17) immediately imply (4.19)-(4.20). Condition (4.18) implies that $\vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q}) \cdot \vec{\tau} \in \dot{W}^{1-1/r}_r(\mathbb{R})$. However to show $f(\widetilde{v} \cdot \vec{\tau}) \in \dot{W}^{1-1/r}_r(\mathbb{R})$ one has to use (4.57). Condition (4.22) also comes from (4.57). □

Using the above Lemma we are able to simplify the system (4.23)-(4.27). Denoting $v = u + \widetilde{v}$ and $q = p + \widetilde{q}$ we get a system for (u, p)

$$a_1u_{,1} + a_2u_{,2} - \Delta u + \nabla p = 0 \qquad \text{in } \mathbb{R}^2_+, \tag{4.23}$$

$$\nabla \cdot u = 0 \qquad \text{in } \mathbb{R}^2_+, \tag{4.24}$$

$$\vec{n} \cdot \mathbb{T}(u, p) \cdot \vec{\tau} + f(v \cdot \tau) = b \qquad \text{on } \partial\mathbb{R}^2_+, \tag{4.25}$$

$$\vec{n} \cdot u = d \qquad \text{on } \partial\mathbb{R}^2_+, \tag{4.26}$$

$$u \to 0 \qquad \text{as } |x| \to \infty. \tag{4.27}$$

where for the readability we denoted the term $(\underline{b} - \vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q}) \cdot \vec{\tau} - f(\widetilde{v} \cdot \vec{\tau}))$ as b and $\underline{d} - \vec{n} \cdot \widetilde{v}$ as d.

The main result concerns estimates for the pressure and for the velocity. For the readability of the paper we split it into two theorems within each we consider some cases. We emphasize that we use homogeneous spaces $\dot{W}^s_p$ and inhomogeneous spaces W^s_p.

Theorem 4.2.2. Estimates for the pressure. *Let $f > 0$ be a constant friction coefficient and $p > 3/2$. Given the solution (u, p) to the system (4.23)-(4.27). Considering the following cases:*

- *for $a_2 \leq 0$: let $b \in \dot{W}^{1-1/r}_r(\mathbb{R})$ and $d \in \dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-1/r}_r(\mathbb{R})$ for all $3/2 < r \leq p$. Then $\nabla p \in L^r(\mathbb{R}^2_+)$ for all $3/2 < r \leq p$ and the following inequality holds:*

$$\|\nabla p\|_{L^r(\mathbb{R}^2_+)} \leq C(f, a_1, a_2) \left(\|b\|_{\dot{W}^{1-1/r}_r(\mathbb{R})} + \|d\|_{\dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-1/r}_r(\mathbb{R})} \right).$$

- *for* $a_2 > 0$*: let* $b \in \dot W_r^{1-1/r}(\mathbb{R})$ *and* $d \in \dot W_r^{2-1/r}(\mathbb{R})$ *for all* $3/2 < r \le p$*. Then* $\nabla p \in L^r(\mathbb{R}^2_+)$ *for all* $3/2 < r \le p$ *and the following inequality holds:*

$$\|\nabla p\|_{L^r(\mathbb{R}^2_+)} \le C(f, a_1, a_2)\left(\|b\|_{\dot W_r^{1-1/r}(\mathbb{R})} + \|d\|_{\dot W_r^{2-1/r}(\mathbb{R})}\right).$$

Theorem 4.2.3. Estimates for the velocity. *Let* $f > 0$ *be a constant friction coefficient and* $p > 3$*. Given the solution* (u, p) *to the system (4.23)-(4.27). Considering the following cases:*

- *for* $a_2 < 0$*: let* $b \in W_r^{1-1/r}(\mathbb{R})$ *and* $d \in W_r^{2-1/r}(\mathbb{R})$ *for all* $3/2 < r \le p$*. Then* $\nabla^2 u \in L^s(\mathbb{R}^2_+)$ *for all* $s \in (3, p]$ *and the following inequality holds:*

$$\|\nabla^2 u\|_{L^s(\mathbb{R}^2_+)} \le C(f, a_1, a_2)\left(\|b\|_{W^{1-1/s}_{s,A_s}(\mathbb{R})} + \|d\|_{W^{2-1/s}_{s,A_s}(\mathbb{R})}\right),$$

  where $A_s = \{3s/(3+s), s\}$

- *for* $a_2 = 0$*: let* $b \in \dot W_r^{1-1/r}(\mathbb{R})$ *and* $d \in \dot W_r^{2-1/r}(\mathbb{R}) \cap \dot W_r^{1-1/r}(\mathbb{R})$ *for all* $3/2 < r \le p$*. Then* $\nabla^2 u \in L^s(\mathbb{R}^2_+)$ *for all* $s \in (3, q]$ *and the following inequality holds:*

$$\|\nabla^2 u\|_{L^s(\mathbb{R}^2_+)} \le C(f, a_1, a_2)\left(\|b\|_{\dot W^{1-1/s}_{s,A_s}(\mathbb{R})} + \|d\|_{\dot W^{2-1/s}_{s,A_s}(\mathbb{R}) \cap \dot W^{1-1/s}_{s,A_s}(\mathbb{R})}\right),$$

  where $A_s = \{3s/(3+s), s\}$.

- *for* $a_2 > 0$*: let* $b \in \dot W_r^{1-1/r}(\mathbb{R}) \cap \dot W_r^{1-2/r}(\mathbb{R})$ *and* $d \in \dot W_r^{2-1/r}(\mathbb{R}) \cap \dot W_r^{2-2/r}(\mathbb{R})$ *for all* $2 < r \le p$*. Then* $\nabla^2 u \in L^s(\mathbb{R}^2_+)$ *for all* $s \in (3, q]$ *and the following inequality holds:*

$$\|\nabla^2 u\|_{L^s(\mathbb{R}^2_+)} \le C(f, a_1, a_2)\left(\|b\|_{\dot W^{1-1/s}_{s,A_s}(\mathbb{R}) \cap \dot W^{1-2/s}_{s,A_s}(\mathbb{R})} + \|d\|_{\dot W^{2-1/s}_{s,A_s}(\mathbb{R}) \cap \dot W^{2-2/s}_{s,A_s}(\mathbb{R})}\right),$$

  where $A_s = \{3s/(3+s), s\}$.

Remark: We would like to emphasize, that above assumptions on b and d are consistent with Lemma 4.2.1 in the sense, that the procedure of subtracting the inhomogeneity from the right hand side of (4.12)-(4.13) does not determine the regularity of boundary conditions b and d. We discuss this in details: let us recall that b and d from the right hand side of (4.25) and (4.26) come from the subtraction of terms $\vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q}) \cdot \vec{\tau} + f(\widetilde{v} \cdot \vec{\tau})$ and $\vec{n} \cdot \widetilde{v}$ from $\underline{b}$ and $\underline{d}$ from the original boundary constraints (4.14) and (4.15). Let us thus denote $\vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q}) \cdot \vec{\tau} + f(\widetilde{v} \cdot \vec{\tau})$ as $\widetilde{b}$ and $\vec{n} \cdot \widetilde{v}$ as $\widetilde{d}$ and check if assumptions on boundary conditions in Theorem 4.2.2 and Theorem 4.2.3 are consistent with regularity of $\widetilde{b}$ and $\widetilde{d}$.

In Theorem 4.2.2 we assume $b \in \dot W_r^{1-1/r}(\mathbb{R})$ and $d \in \dot W_r^{2-1/r}(\mathbb{R})$ for all $3/2 < r \le p$ in case $a_2 > 0$, and $b \in \dot W_r^{1-1/r}(\mathbb{R})$ and $d \in \dot W_r^{2-1/r}(\mathbb{R}) \cap \dot W_r^{1-1/r}(\mathbb{R})$ in case $a_2 \le 0$, but due to (4.21)-(4.22) we have $\widetilde{b} \in \dot W_r^{1-1/r}(\mathbb{R})$ and $\widetilde{d} \in \dot W_r^{2-1/r}(\mathbb{R}) \cap \dot W_r^{1-1/r}(\mathbb{R})$ for all $r \in (3/2, p]$, and hence subtraction of $\widetilde{b}$ and $\widetilde{d}$ does not influence regularity of b and d and one has the following inequalities:

- for $a_2 > 0$ and all $r \in (3/2, p]$:

$$\|b\|_{\dot W_r^{1-1/r}(\mathbb{R})} \le C(\|\underline{b}\|_{\dot W_r^{1-1/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$
$$\|d\|_{\dot W_r^{2-1/r}(\mathbb{R})} \le C(\|\underline{d}\|_{\dot W_r^{1-1/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

  where $A_r = \{3r/(3+r), r\}$,

- for $a_2 \le 0$ and all $r \in (3/2, p]$:

$$\|b\|_{\dot W_r^{1-1/r}(\mathbb{R})} \le C(\|\underline{b}\|_{\dot W_r^{1-1/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

$$\|d\|_{\dot W_r^{2-1/r}(\mathbb{R})\cap \dot W_r^{1-1/r}(\mathbb{R})} \le C(\|\underline{d}\|_{\dot W_r^{2-1/r}(\mathbb{R})\cap \dot W_r^{1-1/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

where $A_r = \{3r/(3+r), r\}$.

Similarly, using properties (4.19) and (4.20), we are able to show the following inequalities:

- for $a_2 > 0$ and all $r \in (3, p]$:

$$\|b\|_{\dot W_r^{1-1/r}(\mathbb{R})\cap \dot W_r^{1-2/r}(\mathbb{R})} \le C(\|\underline{b}\|_{\dot W_r^{1-1/r}(\mathbb{R})\cap \dot W_r^{1-2/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

$$\|d\|_{\dot W_r^{2-1/r}(\mathbb{R})\cap \dot W_r^{2-2/r}(\mathbb{R})} \le C(\|\underline{d}\|_{\dot W_r^{2-1/r}(\mathbb{R})\cap \dot W_r^{2-2/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

where $A_r = \{3r/(3+2r), 3r/(3+r), r\}$,

- for $a_2 = 0$ and all $r \in (3, p]$:

$$\|b\|_{\dot W_r^{1-1/r}(\mathbb{R})} \le C(\|\underline{b}\|_{\dot W_r^{1-1/r}(\mathbb{R})\cap \dot W_r^{1-2/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$
$$\|d\|_{\dot W_r^{2-1/r}(\mathbb{R})\cap \dot W_r^{1-1/r}(\mathbb{R})} \le C(\|\underline{d}\|_{\dot W_r^{2-1/r}(\mathbb{R})\cap \dot W_r^{2-2/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

where $A_r = \{3r/(3+r), r\}$,

- for $a_2 < 0$ and all $r \in (3, p]$:

$$\|b\|_{W_r^{1-1/r}(\mathbb{R})} \le C(\|\underline{b}\|_{W_r^{1-1/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

$$\|d\|_{W_r^{2-1/r}(\mathbb{R})} \le C(\|\underline{d}\|_{W_r^{2-1/r}(\mathbb{R})} + \|F\|_{L^r_{A_r}(\mathbb{R}^2_+)} + \|G\|_{W^1_{r,A_r}(\mathbb{R}^2_+)}),$$

where $A_r = \{3r/(3+2r), 3r/(3+r), r\}$.

4.2.1 Derivation of the solution.

Regularity results from Theorem 4.2.2 come from the formula for the solution, while to show estimates from Theorem 4.2.3 we will consider auxiliary system.

In this section we derive the solution using the Fourier transform and solving algebraically the obtained system of ODEs.
Let

$$v(\xi_1, x_2) = \mathcal{F}_{x_1}(u) \quad \text{and} \quad \pi(\xi_1, x_2) = \mathcal{F}_{x_1}(p),$$

where $\mathcal{F}_{x_1}$ is the Fourier transform with respect to x_1, i.e.:

$$v(\xi_1, x_2) = \int_{\mathbb{R}} e^{-i\xi_1 x_1} u(x_1, x_2) dx_1.$$

First two equations of (4.23) give us the following system:

$$\begin{aligned}
a_1 i\xi_1 v_1 + a_2 v_{1,2} + \xi_1^2 v_1 - v_{1,22} + i\xi_1 \pi &= 0 && \text{in } \mathbb{R}^2_+,\\
a_1 i\xi_1 v_2 + a_2 v_{2,2} + \xi_1^2 v_2 - v_{2,22} + \pi_{,2} &= 0 && \text{in } \mathbb{R}^2_+,\\
i\xi_1 v_1 + v_{2,2} &= 0 && \text{in } \mathbb{R}^2_+.
\end{aligned}$$

Denoting ∂_{x_2} as $\dot{}$ $(x_2 \to t)$ and ξ_1 as k we get:

$$\begin{aligned} a_1ikv_1 + a_2\dot{v}_1 + k^2v_1 - \ddot{v}_1 + ik\pi &= 0 && \text{in } \mathbb{R}^2_+, \\ a_1ikv_2 + a_2\dot{v}_2 + k^2v_2 - \ddot{v}_2 + \dot{\pi} &= 0 && \text{in } \mathbb{R}^2_+, \\ ikv_1 + \dot{v}_2 &= 0 && \text{in } \mathbb{R}^2_+. \end{aligned}$$

Differentiating the last equation with respect to t and adding the result to the second one we get:

$$a_1ikv_2 + a_2\dot{v}_2 + k^2v_2 + ik\dot{v}_1 + \dot{\pi} = 0 \qquad \text{in } \mathbb{R}^2_+.$$

We now introduce a new function $w = \dot{v}_1$ to get the following system of ordinary differential equation of order one:

$$\begin{aligned} \dot{v}_1 &= w, \\ \dot{w} &= (k^2 + a_1ik)v_1 + a_2w + ik\pi, \\ \dot{\pi} &= ika_2v_1 - ikw - (k^2 + a_1ik)v_2, \\ \dot{v}_2 &= -ikv_1, \end{aligned}$$

which can be written as:

$$\dot{\begin{bmatrix} v_1 \\ w \\ \pi \\ v_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (k^2 + a_1ik) & a_2 & ik & 0 \\ ika_2 & -ik & 0 & -(k^2 + a_1ik) \\ -ik & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ w \\ \pi \\ v_2 \end{bmatrix} \tag{4.28}$$

The matrix above we denote as A.

Calculating the characteristic polynomial of matrix A we end up with:

$$w_A(\lambda) = \lambda^4 - a_2\lambda^3 - 2k^2\lambda^2 + a_2k^2\lambda + k^4 + ia_1k(k^2 - \lambda^2),$$

for which polynomial we have the following roots:

$$\begin{aligned} \lambda_1 &= -|k|, \\ \lambda_2 &= |k|, \\ \lambda_3 &= \frac{1}{2}(a_2 - \sqrt{a_2^2 + 4(k^2 + ia_1k)}), \\ \lambda_4 &= \frac{1}{2}(a_2 + \sqrt{a_2^2 + 4(k^2 + ia_1k)}). \end{aligned}$$

Since we are interested in a solution which tends to zero as $|t| \to \infty$ we consider only λ_1 and λ_3, because their real parts $\Re\lambda_1$ and $\Re\lambda_3$ are negative. Since $k = \frac{k}{|k|}|k|$ we introduce $\sigma(k) = \frac{k}{|k|}$ as an abbreviation of $sign(k)$, and write eigenvectors for these eigenvalues as follows:

$$\varphi_1 = \left[1;\ \lambda_1;\ -(a_1 + \sigma(k)ia_2);\ -\tfrac{ik}{\lambda_1}\right] \quad \text{and} \quad \varphi_3 = \left[1;\ \lambda_3;\ 0;\ -\tfrac{ik}{\lambda_3}\right].$$

Introducing matrix $S = [\varphi_1, \varphi_2, \varphi_3, \varphi_4]$ of eigenvectors we have the following identity:

$$S^{-1}AS = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

where $\mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the diagonal matrix with the proper diagonal values.

Introducing notation $V := [v_1, w, v_2, \pi]$ and a new vector U such that $V = SU$, we may rewrite equation (4.28) in the form $S\dot{U} = ASU$. Multiplying it by S^{-1} from the left side we get: $\dot{U} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and then

$$U_i(t,k) = e^{t\lambda_i} U_{0i}(k)$$

for the proper vector function $U_0(k)$. As mentioned before we are interested only in stable modes, i.e. those λ_i for which $\Re(\lambda_i) < 0$. These are λ_1 and λ_3.

Going back to the original $V = [v_1, w, \pi, v_2] = SU$ we have:

$$v_1(t,k) = e^{t\lambda_1} U_{01}(k) + e^{t\lambda_3} U_{03}(k), \tag{4.29}$$

$$v_2(t,k) = -\frac{ik}{\lambda_1(k)} e^{t\lambda_1} U_{01}(k) - \frac{ik}{\lambda_3(k)} e^{t\lambda_3} U_{03}, \tag{4.30}$$

$$\pi(t,k) = -(a_1 + \sigma(k) i a_2) e^{t\lambda_1} U_{01}(k). \tag{4.31}$$

We now calculate $U_{0i}(k)$ from the boundary conditions $(4.23)_{3,4}$ after applying the Fourier transform. Since we are in $\mathbb{R}^2_+$ we find that $(4.23)_4$ reads

$$v \cdot \vec{n} = -v_2 = \hat{d} \qquad \text{on } \partial\mathbb{R}^2_+.$$

and slip boundary condition transforms into:

$$-\dot{v}_1 - ikv_2 + fv_1 = \hat{b}(k) \qquad \text{on } \partial\mathbb{R}^2_+.$$

This system and the previous one give us the following system for (U_{01}, U_{03}):

$$\begin{aligned} \frac{ik}{\lambda_1(k)} U_{01}(k) + \frac{ik}{\lambda_3(k)} U_{03}(k) &= \hat{d}(k), \\ (f - \lambda_1) U_{01}(k) + (f - \lambda_3) U_{03}(k) &= \hat{b}(k) - ik\hat{d}(k). \end{aligned}$$

This system has the following solution (U_{01}, U_{03}):

$$\begin{aligned} U_{01}(k) &= \frac{\lambda_1(\hat{d}(k)(\lambda_3(-f + \lambda_3) + k^2) + ik\hat{b}(k))}{ik(\lambda_3 - \lambda_1)(-f + \lambda_3 + \lambda_1)}, \\ U_{03}(k) &= \frac{\hat{b}(k) - ik\hat{d}(k) - (f - \lambda_1) U_{01}(k)}{(f - \lambda_3)}. \end{aligned}$$

Immediately, having this solution, we formulate the following result, which gives us the reason to consider only Dirichlet boundary conditions:

Lemma 4.2.4. *Given a vector field u such that $v = \mathcal{F}_{x_1}(u)$ satisfies (4.29)-(4.30) for some functions $U_{01}(\xi)$ and $U_{03}(\xi)$. If u satisfies the following slip boundary conditions*

$$\begin{aligned} \vec{n} \cdot \mathbb{D}(v) \cdot \vec{\tau} + f(v \cdot \vec{\tau}) &= b, \\ \vec{n} \cdot v &= d, \end{aligned} \tag{4.32}$$

where $b \in \dot{W}_p^{1-1/p}(\mathbb{R})$, $d \in \dot{W}_p^{2-1/p}(\mathbb{R})$, then this vector field satisfies also Dirichlet boundary constraints on $\partial\mathbb{R}^2_+$:

$$u(x_1, 0) = D(x_1),$$

where $D \in \dot{W}_p^{2-1/p}(\mathbb{R})$ *and is given by:*

$$\begin{aligned} D_1(x_1) &= \mathcal{F}_\xi^{-1}\left(\tfrac{\hat{b}(\xi)-\hat{d}(\xi)(i\sigma(\xi)\lambda_3(\xi)-i\xi)}{f-\lambda_1(\xi)-\lambda_3(\xi)}\right)(x_1) \\ D_2(x_1) &= -d(x_1). \end{aligned}$$

and satisfies the following inequality:

$$\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})} \leq C(\|b\|_{\dot{W}_p^{1-1/p}(\mathbb{R})} + \|d\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}).$$

Moreover, for $b \in \dot{W}_p^{1-1/p}(\mathbb{R})$ *and* $d \in \dot{W}_p^{2-1/p}(\mathbb{R}) \cap \dot{W}_p^{1-1/p}(\mathbb{R})$ *one has:*

$$D \in \dot{W}_p^{2-1/p}(\mathbb{R}) \cap \dot{W}_p^{1-1/p}(\mathbb{R}), \tag{4.33}$$

and the following inequality holds:

$$\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{1-1/p}(\mathbb{R})} \leq C(\|b\|_{\dot{W}_p^{1-1/p}(\mathbb{R})} + \|d\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{1-1/p}(\mathbb{R})}).$$

Remark: *Above homogeneous spaces can be replaced with inhomogeneous ones without any additional assumptions.*

Proof of Lemma 4.2.4. Taking the Fourier transform in x_1 direction in boundary conditions (4.32) one gets:

$$\begin{aligned} -\dot{v}_1(\xi,0) - ikv_2(\xi,0) + fv_1(\xi,0) &= \hat{b}(\xi), \\ -v_2(\xi,0) &= \hat{d}(\xi). \end{aligned}$$

After inserting the form (4.29)-(4.30) of v into the above equations it is then an algebraic calculation to obtain that

$$\begin{aligned} v_1(\xi,0) &= \frac{\hat{b}(\xi)-\hat{d}(\xi)(i\sigma(\xi)\lambda_3(\xi)-i\xi)}{f-\lambda_1(\xi)-\lambda_3(\xi)}, \\ v_2(\xi,0) &= -\hat{d}(\xi). \end{aligned}$$

Obviously $u_2(x_1,0) = -d(x_1)$ and hence $u_2 \in \dot{W}_p^{2-1/p}(\mathbb{R})$. To show proper regularity of $u_1(x_1,0)$ we split v_1 into two terms:

$$\frac{\hat{b}(\xi)}{f-\lambda_1(\xi)-\lambda_3(\xi)} \quad \text{and} \quad \frac{-\hat{d}(\xi)(i\sigma(\xi)\lambda_3(\xi)-i\xi)}{f-\lambda_1(\xi)-\lambda_3(\xi)}.$$

The inverse Fourier transform of the second term is in the same class of regularity as $d(x_1)$, since $\frac{-i\sigma(\xi)\lambda_3(\xi)-i\xi}{f-\lambda_1(\xi)-\lambda_3(\xi)}$ is a proper bounded multiplier to use with multiplier theorems in Besov spaces. Similarly the first term — since $\frac{ik}{f-\lambda_1-\lambda_3}$ is a smooth bounded function in $\mathbb{R}\setminus\{0\}$ then $\frac{\hat{b}(k)}{f-\lambda_1-\lambda_3}$ corresponds to a term in $\dot{W}_p^{2-1/p}(\mathbb{R})$. This proves that $\mathcal{F}_{x_1}^{-1}(v_1(0,\cdot)) \in \dot{W}_p^{2-1/p}(\mathbb{R})$.

To be more precise, the term connected with b is in fact of higher regularity, namely if $b \in \dot{W}_p^{1-1/p}(\mathbb{R})$, then $\mathcal{F}_\xi^{-1}(\frac{\hat{b}(\xi)}{f-\lambda_1(\xi)-\lambda_3(\xi)}) \in \dot{W}_p^{1-1/p}(\mathbb{R}) \cap \dot{W}_p^{2-1/p}(\mathbb{R})$. This proves (4.33). □

We have a proper regularity of v on $\partial\mathbb{R}^2_+$ and its exact correspondence to slip boundary constraints, thus we can treat our system as the one with Dirichlet boundary conditions, namely:

$$\begin{aligned} v_1(0,k) &= \frac{\hat{b}(k)-ik\hat{d}(k)}{f-\lambda_1-\lambda_3} - \frac{\hat{d}(k)i\lambda_3\sigma(k)}{f-\lambda_1-\lambda_3} \\ v_2(0,k) &= -\hat{d}(k), \end{aligned}$$

which we will denote as $v(0,k) = \hat{D}(k)$ on $\partial\mathbb{R}^2_+$. We want to emphasize that this fact will be used later on during the estimate of $\nabla^2 v$.

4.2.2 Estimate of the pressure.

In this part of the paper we give a proof of Theorem 4.2.2. Let us start now with estimates for the pressure
$$p(x_1,x_2) = \mathcal{F}_k^{-1}\left(\pi(k,x_2)\right).$$
Recall (4.31):
$$\pi(t,k) = -(a_1+\sigma(k)ia_2)e^{t\lambda_1}U_{01}(k).$$
We want to estimate the gradient ∇p, i.e. $\partial_{x_1}p$ and $\partial_{x_2}p$. These two terms correspond, after the Fourier transform, to terms $ik\pi(t,k)$ and $-|k|\pi(t,k)$, which, from the point of view of our approach, are equivalent, since they differ only by a function $\sigma(k)$, which, as we shall see, makes no difference in our estimates.

Let us thus focus on the term
$$ik\pi(t,k) = -e^{t\lambda_1}\frac{ik(a_1+\sigma(k)ia_2)\lambda_1(\hat{d}(k)(\lambda_3(-f+\lambda_3)+k^2)+ik\hat{b}(k))}{ik(\lambda_3-\lambda_1)(-f+\lambda_3+\lambda_1)}.$$
We remove $(a_1+\sigma(k)ia_2)$ using the following calculation:
$$\begin{aligned}\frac{ik}{\lambda_3-\lambda_1} &= \frac{2ik}{(a_2-\sqrt{a_2^2+4(k^2+a_1ik)})+2|k|}\\ &= \frac{2ik(a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)})}{(a_2^2+4a_2|k|+4|k|^2-(a_2^2+4(k^2+a_1ik))}\\ &= \frac{2ik(a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)})}{-4ik(\sigma(k)a_2i+a_1)}\\ &= -\frac{a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)}}{a_1+\sigma(k)a_2i}.\end{aligned}$$
Thus we present $ik\pi(t,k)$ as follows:
$$ik\pi(t,k) == e^{t\lambda_1}\frac{(a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)})\lambda_1(\hat{d}(k)(\lambda_3(-f+\lambda_3)+k^2)+ik\hat{b}(k))}{ik(-f+\lambda_3+\lambda_1)}.$$
First we focus on the term involving $\hat{b}(k)$. Let
$$I_1(t,k) = e^{t\lambda_1}ik\frac{(a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)})\lambda_1}{ik(-f+\lambda_3+\lambda_1)}\hat{b}(k).$$
and we denote as I_2 the remaining part of $ik\pi(t,k)$:
$$I_2(t,k) = e^{t\lambda_1}ik\frac{(a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)})\sigma(k)(\lambda_3(-f+\lambda_3)+k^2)}{ik(-f+\lambda_3+\lambda_1)}\hat{d}(k). \tag{4.34}$$
We present $I_1(t,k)$ as
$$I_1(t,k) = e^{t\lambda_1}ik\varphi_1(k)\hat{b}(k),$$
where
$$\varphi_1(k) = \frac{(a_2+2|k|+\sqrt{a_2^2+4(k^2+a_1ik)})\lambda_1}{ik(-f+\lambda_3+\lambda_1)}.$$
Since $\Re(\lambda_3+\lambda_1)\leq 0$ and $f>0$ we have $\Re(-f+\lambda_3+\lambda_1)\leq -f<0$ and a function $\varphi_1(k)$ is a proper multiplier in the sense of Theorem 4.5.1 – indeed, since $\lambda_1 = -|k|$ our multiplier

is bounded and smooth for $k \in \mathbb{R} \setminus \{0\}$. Moreover its derivative has a good decay rate, which guarantees that $|k| \cdot \varphi_1'(k)$ is bounded for all $k \in \mathbb{R}$.

The above considerations justify the following inequality:

$$\|\mathcal{F}_k^{-1}\left(I_1(t,\cdot)\right)\|_{L^p(\mathbb{R})} \leq C\|\mathcal{F}_k^{-1}\left(e^{t\lambda_1} ik\hat{b}(k)\right)\|_{L^p(\mathbb{R})}.$$

We now estimate the term:

$$\mathcal{F}_k^{-1}\left(-e^{t\lambda_1} ik\hat{b}(k)\right) = \mathcal{F}_k^{-1}\left(-e^{t\lambda_1} ik\right) * \mathcal{F}_k^{-1}\left(\hat{b}(k)\right),$$

where $*$- is a convolution with respect to x_1. Now since

$$\mathcal{F}_k^{-1}\left(-ike^{-|k|t}\right) = \left(\sqrt{\frac{2}{\pi}}\frac{t}{t^2+x_1^2}\right)_{,x_1} = -\sqrt{\frac{2}{\pi}}\frac{2x_1 t}{(t^2+x_1^2)^2},$$

we rewrite our term as follows:

$$\mathcal{F}_k^{-1}\left(-e^{t\lambda_1} ik\right) * \mathcal{F}_k^{-1}\left(\hat{b}(k)\right) = \sqrt{\frac{2}{\pi}}\frac{x_1 t}{(t^2+x_1^2)^2} * b(x_1) =: \sqrt{\frac{2}{\pi}} J_1(t,x_1).$$

Now since

$$\int_{\mathbb{R}} b(x)\frac{-2yt}{(t^2+y^2)^2}dy = 0,$$

we write:

$$J_1(t,x) = \int_{\mathbb{R}} \frac{2yt}{(t^2+y^2)^2}\left[b(x-y)-b(x)\right]dy.$$

First we focus on:

$$\|J_1(t,\cdot)\|^p_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} \frac{2yt}{(t^2+y^2)^2}\left[b(x-y)-b(x)\right]dy\right|^p dx\right).$$

After an application of the Hölder's inequality to the internal integral we get:

$$\begin{aligned}\int_{\mathbb{R}} \frac{t^{1/q}}{(t^2+y^2)^{1/q}}\frac{2yt^{1/p}}{(y^2+t^2)^{2-1/q}}\left[b(x-y)-b(x)\right]dy \leq \\ \leq \left(\int_{\mathbb{R}}\frac{t}{(t^2+y^2)}dy\right)^{1/q}\left(\int_{\mathbb{R}}\frac{2y^p t}{(y^2+t^2)^{1+p}}\left|b(x-y)-b(x)\right|^p dy\right)^{1/p} \\ \leq \pi^{1/q}\left(\int_{\mathbb{R}}\frac{2y^p t}{(y^2+t^2)^{1+p}}\left|b(x-y)-b(x)\right|^p dy\right)^{1/p}\end{aligned}$$

and thus

$$\|J_1\|^p_{L^p(\mathbb{R}\times\mathbb{R}_+)} \leq \pi^{p/q}\int_{\mathbb{R}_+} dt \int\limits_{\mathbb{R}\times\mathbb{R}} \frac{2y^p t}{(y^2+t^2)^{1+p}}\left|b(x-y)-b(x)\right|^p dydx.$$

Now since

$$\int_{\mathbb{R}_+}\frac{t}{(y^2+t^2)^{1+p}}dt = \frac{y^{-2p}}{2p},$$

we get, using (4.11):

$$\|J_1\|^p_{L^p(\mathbb{R}\times\mathbb{R}_+)} \leq \frac{\pi^{p/q}}{p}\int\limits_{\mathbb{R}\times\mathbb{R}}\frac{|b(x-y)-b(x)|^p}{|y|^p}dydx \leq \frac{\pi^{p/q}}{p}\|b\|^p_{\dot{W}_p^{1-1/p}(\mathbb{R})}.$$

Of course, since $b \in \dot{W}_r^{1-1/r}(\mathbb{R})$ for any $r \in (3/2, p]$ we actually have

$$\|J_1\|_{L^r(\mathbb{R})} \leq \frac{\pi^{(r-1)/r}}{r^{1/r}} \|b\|_{\dot{W}_r^{1-1/r}(\mathbb{R})},$$

which implies:

$$\|\mathcal{F}_k^{-1}(I_1)\|_{L^r(\mathbb{R})} \leq 2^{1/2r} \frac{\pi^{(r-3/2)/r}}{r^{1/r}} \|b\|_{\dot{W}_r^{1-1/r}(\mathbb{R})},$$

This type of calculations is known since the famous papers by Agmon, Douglis and Nirenberg (see [1], [2]), where authors showed optimal estimates for solution to elliptic problems.

To finish our estimates for the gradient of the pressure we must deal with the term I_2 from (4.34), namely:

$$I_2(t,k) = e^{t\lambda_1} ik \frac{(a_2 + 2|k| + \sqrt{a_2^2 + 4(k^2 + a_1 ik)})\sigma(k)(\lambda_3(-f + \lambda_3) + k^2)}{ik(-f + \lambda_3 + \lambda_1)} \hat{d}(k).$$

Estimates differ depending on the sign of a_2. We will be interested in a behaviour of particular terms for $k \to 0$ and for $k \to \infty$, and we emphasize this by introducing a smooth cut-off function $\zeta(k)$ such that $\zeta(k) = 1$ for $|k| \leq 1$ and $\zeta(k) = 0$ for $|k| > 1$. Then we split integral I_2 as follows:

$$I_2(t,k) = \zeta(k) I_2 + (1 - \zeta(k)) I_2 = I_{21}(t,k) + I_{22}(t,k), \tag{4.35}$$

and estimate it separately.

First, let $a_2 > 0$. In this case we have

$$\Re(a_2 + 2|k| + \sqrt{a_2^2 + 4(k^2 + a_1 ik)}) > a_2 > 0$$

and $\Re(\lambda_3(-f + \lambda_3) + k^2) \sim k^2$ for small $|k|$, thus we present I_{21} as:

$$I_{21}(t,k) = e^{t\lambda_1} ik \varphi_{21}(k)(ik\hat{d}(k)), \tag{4.36}$$

where

$$\varphi_{21}(k) = \zeta(k) \frac{(a_2 + 2|k| + \sqrt{a_2^2 + 4(k^2 + a_1 ik)})\sigma(k)(\lambda_3(-f + \lambda_3) + k^2)}{-k^2(-f + \lambda_3 + \lambda_1)}.$$

It is straightforward that a function $\varphi_{21}(k)$ is a proper multiplier in the sense of the Marcinkiewicz theorem. Moreover, since $d \in \dot{W}_p^{2-1/p}(\mathbb{R})$ then $\mathcal{F}_k^{-1}(ik\hat{d}(k)) \in \dot{W}_p^{1-1/p}(\mathbb{R})$, and we reuse techniques exploited earlier to estimate terms connected with $b(k)$, to obtain:

$$\|\mathcal{F}_k^{-1}(I_{21}(t,k))\|_{L^p(\mathbb{R}^2_+)} \leq C \|\mathcal{F}_k^{-1}\left(ik\hat{d}(k)\right)\|_{\dot{W}_p^{1-1/p}(\mathbb{R})} = \|d\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}.$$

To estimate $I_{22}(t,k)$ we present it in the same form as in (4.36), i.e.:

$$I_{22}(t,k) = e^{t\lambda_1} ik \varphi_{22}(k)(ik\hat{d}(k)),$$

where

$$\varphi_{22}(k) = (1 - \zeta(k)) \frac{(a_2 + 2|k| + \sqrt{a_2^2 + 4(k^2 + a_1 ik)})\sigma(k)(\lambda_3(-f + \lambda_3) + k^2)}{-k^2(-f + \lambda_3 + \lambda_1)}.$$

Again, it is easily seen that $\varphi_{22}(k)$ is a proper multiplier in the sense of the Marcinkiewicz theorem, since in denominator and numerator of this fraction appear terms of order $|k|^3$ (for large $|k|$), and the neighbourhood of 0 is cutted of by $(1-\zeta(k))$. Once this has been noticed we can estimate I_{22} as follows:

$$\|\mathcal{F}_k^{-1}\left(I_{22}(t,k)\right)\|_{L^p(\mathbb{R}^2_+)} \leq C\|\mathcal{F}_k^{-1}\left(ik\hat{d}(k)\right)\|_{\dot{W}_p^{1-1/p}(\mathbb{R})} = C\|d\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}.$$

We would like to emphasize, that the estimate of I_{22} does not depend on the sign of a_2, and hence we can use it again in cases $a_2 = 0$ and $a_2 > 0$, provided $d \in \dot{W}_p^{2-1/p}(\mathbb{R})$.

Let us now consider the case $a_2 = 0$. We have

$$2|k| + \sqrt{4(k^2 + a_1 ik)} \sim \sqrt{k},$$

but also

$$(\lambda_3(-f+\lambda_3)+k^2) \sim \sqrt{k} \text{ for small } |k|.$$

This does not allow us to present I_{21} in the form (4.36), but the following one:

$$I_{21}(t,k) = e^{t\lambda_1} ik\varphi_{21}(k)\hat{d}(k),$$

where $\varphi_{21}(k) = \zeta(k)\frac{(2|k|+\sqrt{4(k^2+a_1ik)})\sigma(k)(\lambda_3(-f+\lambda_3)+k^2)}{-ik(-f+\lambda_3+\lambda_1)}$ is a valid multiplier in the sense of the Marcinkiewicz theorem. Hence, a proper estimate is the following:

$$\|\mathcal{F}_k^{-1}\left(I_{21}(t,k)\right)\|_{L^p(\mathbb{R}^2_+)} \leq C\|d\|_{\dot{W}_p^{1-1/p}(\mathbb{R})}.$$

Thus, in the case $a_2 = 0$ integral I_2 can be estimated as follows:

$$\|\mathcal{F}_k^{-1}\left(I_2\right)\|_{L^p(\mathbb{R}^2_+)} \leq C\|d\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{1-1/p}(\mathbb{R})}.$$

Let us now assume that $a_2 < 0$. In this case the term $(\lambda_3(-f+\lambda_3)+k^2) < a_2 < 0$, however $a_2 + 2|k| + \sqrt{a_2^2 + 4(k^2 + a_1 ik)} \sim |k|$, and thus we can present term I_{21} in the form

$$I_{21}(t,k) = e^{t\lambda_1} ik\varphi_{21}(k)\hat{d}(k),$$

where $\varphi_{21}(k) = \zeta(k)\frac{(2|k|+\sqrt{4(k^2+a_1ik)})\sigma(k)(\lambda_3(-f+\lambda_3)+k^2)}{-ik(-f+\lambda_3+\lambda_1)}$ is a proper multiplier in the sense of the Marcinkiewicz theorem, and we obtain:

$$\|\mathcal{F}_k^{-1}\left(I_{21}(t,k)\right)\|_{L^p(\mathbb{R}^2_+)} \leq C\|d\|_{\dot{W}_p^{1-1/p}(\mathbb{R})},$$

which, together with the standard estimate of I_{22} gives us:

$$\|\mathcal{F}_k^{-1}\left(I_2\right)\|_{L^p(\mathbb{R}^2_+)} \leq \|d\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{1-1/p}(\mathbb{R})}.$$

Gathering all above estimates we have proved the following inequalities:

- for $a_2 \leq 0$ and all $r \in (3/2, p]$:

$$\|\nabla p\|_{L^r(\mathbb{R}^2_+)} \leq C(f, a_1, a_2)\left(\|b\|_{\dot{W}_r^{1-1/r}(\mathbb{R})} + \|d\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap\dot{W}_r^{1-1/r}(\mathbb{R})}\right).$$

- for $a_2 > 0$ and all $r \in (3/2, p]$:

$$\|\nabla p\|_{L^r(\mathbb{R}^2_+)} \leq C(f, a_1, a_2)\left(\|b\|_{\dot{W}_r^{1-1/r}(\mathbb{R})} + \|d\|_{\dot{W}_r^{2-1/r}(\mathbb{R})}\right).$$

This completes the proof of Theorem 4.2.2.

4.2.3 Second derivatives of the velocity – reduction of the system.

In this section we introduce a homogeneous system for the velocity, from which it will be easier to obtain proper regularity of $\nabla^2 v$. Once we have a simplified system we derive the solution and show estimates for it.

First, let us recall that our solution to the system (4.23)-(4.27) satisfies (independently of the sign of a_2) the following system:

$$\begin{aligned} a_1 u_{,1} + a_2 u_{,2} - \Delta u &= -\nabla p \quad \text{in } \mathbb{R}^2_+, \\ u &= D \quad \text{on } \partial\mathbb{R}^2_+, \end{aligned} \tag{4.37}$$

where D is in a proper class, which depends on the sign of a_2, namely:

- for $a_2 < 0$: $D \in W_r^{2-1/r}(\mathbb{R})$,
- for $a_2 = 0$: $D \in \dot{W}_r^{2-1/r}(\mathbb{R}) \cap \dot{W}_r^{1-1/r}(\mathbb{R})$,
- for $a_2 > 0$: $D \in \dot{W}_r^{2-1/r}(\mathbb{R}) \cap \dot{W}_r^{2-2/r}(\mathbb{R})$,

for all $r \in (3, p]$, where we emphasize, that in case $a_2 < 0$ we need the full norm, not only an homogeneous one.

We subtract inhomogeneity from the right hand side of (4.37) without changing the regularity of boundary condition D in each of the cases of signum of a_2. To obtain this we use Theorem 4.3.1 and Theorem 4.3.3.

We present the solution u as

$$u = v + w,$$

where w is a truncation to the half space $\mathbb{R}^2_+$ of the solution to the system in the whole $\mathbb{R}^2$ space:

$$a_1 w_{,1} + a_2 w_{,2} - \Delta w = -\widetilde{\nabla p},$$

where $\widetilde{\nabla p}$ on the right hand side stands for its standard extension on the whole $\mathbb{R}^2$ with a preservation of its norm. Theorem 4.3.1 guarantees that the solution exists, thus v:

$$\begin{aligned} a_1 v_{,1} + a_2 v_{,2} - \Delta v &= 0 \quad \text{in } \mathbb{R}^2_+, \\ v &= D - w = \widetilde{D} \quad \text{on } \partial\mathbb{R}^2_+. \end{aligned} \tag{4.38}$$

The question is, does $\widetilde{D}$ have the same regularity as D. Since for $r > 3$ we have $3r/(3+r) > 3/2$ and we have $\nabla p \in L^s(\mathbb{R}^2_+)$ for all $s \in (3/2, p]$ we are in position to use Theorem 4.3.1 and Lemma 4.3.2 to get:

$$\|\nabla w\|_{W_r^1(\mathbb{R}^2)} \leq \|\nabla p\|_{L^r(\mathbb{R}^2) \cap L^{3r/(3+r)}(\mathbb{R}^2)},$$

for all $r \in (3, p]$ and

$$\|w_{|x_2=0}\|_{\dot{W}_r^{2-1/r}(\mathbb{R}) \cap \dot{W}_r^{2-2/r}(\mathbb{R})} \leq \|\nabla p\|_{L^r(\mathbb{R}^2) \cap L^{3r/(3+r)}(\mathbb{R}^2)}.$$

This implies in particular, that for $a_2 \geq 0$ subtraction of w does not change the regularity of boundary conditions, hence $\widetilde{D}$ has the same regularity as D.

For $a_2 < 0$ we have different behaviour of eigenvalues and we may use Theorem 4.3.3. Since $\widetilde{\nabla p} \in L^r(\mathbb{R}^2)$ for all $r \in (3/2, p]$ we get

$$\|w_{|x_2=0}\|_{W_r^{2-1/r}(\mathbb{R})} \leq \|\nabla p\|_{L^r(\mathbb{R}^2)},$$

which again implies that $\widetilde{D}$ has the same regularity as D, namely: $\widetilde{D} \in W_r^{2-1/r}(\mathbb{R})$.

Above considerations justify the following set of inequalities:

- for $a_2 < 0$:
$$\|\widetilde{D}\|_{W_r^{2-1/r}(\mathbb{R})} \leq C\left(\|D\|_{W_r^{2-1/r}(\mathbb{R})} + \|w_{|x_2=0}\|_{W_r^{2-1/r}(\mathbb{R})}\right), \tag{4.39}$$

- for $a_2 = 0$:
$$\|\widetilde{D}\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap \dot{W}_r^{1-1/r}(\mathbb{R})} \leq C\left(\|D\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap \dot{W}_r^{1-1/r}(\mathbb{R})} + \|w_{|x_2=0}\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap \dot{W}_r^{1-1/r}(\mathbb{R})}\right),$$

- for $a_2 > 0$:
$$\|\widetilde{D}\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap \dot{W}_r^{2-2/r}(\mathbb{R})} \leq C\left(\|D\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap \dot{W}_r^{2-2/r}(\mathbb{R})} + \|w_{|x_2=0}\|_{\dot{W}_r^{2-1/r}(\mathbb{R})\cap \dot{W}_r^{2-2/r}(\mathbb{R})}\right). \tag{4.40}$$

Derivation of solution. For the readability we again denote $\widetilde{D}$ as D, since in a view of (4.39)-(4.40) this does not affect any estimates. We solve the system (4.38) in the same manner as the one considered earlier – first we apply Fourier Transform and then solve ordinary differential equations with initial data coming from the boundary constraints.

Taking the Fourier transform of (4.38_1) we get:

$$a_1 ikv_l + a_2\dot{v}_l + k^2 v_l - \ddot{v}_l = 0, \qquad \text{where } l = 1, 2,$$

where again we denoted ∂_{x_2} as $\dot{}$. Introducing $w_l = \dot{v}_l$ we get a system of ordinary differential equations

$$\dot{\begin{bmatrix} v_l \\ w_l \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ (k^2 + a_1 ik) & a_2 \end{bmatrix} \cdot \begin{bmatrix} v_l \\ w_l \end{bmatrix},$$

from which we easily compute a solution:

$$v_l(t,k) = e^{t\lambda_-(k)} V_{0l}(k), \qquad \text{where} \qquad \lambda_-(k) = \frac{1}{2}(a_2 - \sqrt{a_2^2 + 4(k^2 + a_1 ik)}),$$

and V_{0l} is an initial function – we get it from the boundary data. From (4.38_2) we have:

$$v_i(0,k) = \hat{D}_i(k) \qquad \text{on } \partial\mathbb{R}^2_+ \qquad \text{and thus} \qquad V_{0i}(k) = \hat{D}_i(k).$$

Remark: The behaviour of $\lambda_-(k)$ at $|k| \to 0$ and $|k| \to \infty$ will be crucial for our considerations. It is straightforward, that $\lambda_-(k) \sim |k|$ for large $|k|$ independently of a_1 and a_2, however its behaviour at 0 changes depending on a_1 and a_2, namely, for small k:

- for $a_2 < 0$: $\Re\lambda_-(k) < a_2 < 0$,
- for $a_2 = 0$: $\Re\lambda_-(k) \sim -\sqrt{|k|}$,
- for $a_2 > 0$: $\Re\lambda_-(k) \sim -|k|^2$.

We emphasize that if $a_2 = 0$, then $a_1^2 = a_1^2 + a_2^2 = v_\infty^2 \neq 0$.

4.2.4 Estimate of $u_{,11}$

We start with the estimate of $u_{i,11}$, which brings down to an estimate of

$$|k|^2 e^{t\lambda_-(k)} \hat{D}_i(k).$$

To use previous estimate techniques for ∇p we notice, that above term equals

$$-e^{t\lambda_-(k)}(ik)^2 \hat{D}_i(k). \tag{4.41}$$

Since we consider multiple cases it is thus reasonable to present them in separate lemmas.

Lemma 4.2.5. *Given $u_{i,11}$ in the form (4.41). Assuming $a_2 \le 0$ and $\dot{D} \in W_p^{2-1/p}(\mathbb{R})$ one has:*

$$\|u_{i,11}\|_{L^p(\mathbb{R}^2_+)} \le C\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}.$$

Proof . We see that $ik\hat{D}_i(k)$ has a good regularity (namely $\mathcal{F}_k^{-1}(ik\hat{D}_i(k)) \in \dot{W}_p^{1-1/p}$) to get L_p-estimates for this term (repeating the procedure for the gradient ∇p), however we need to show that one can change $e^{-t\lambda_-(k)}$ into $e^{-t|k|}$. Since $a_2 \le 0$ there exists a constant $c(a_2)$ such that $\lambda_-(k) + c(a_2)|k| \le 0$ for all $k \in \mathbb{R}$ and thus the following multiplier is valid in the sense of the Marcinkiewicz theorem:

$$\varphi(k) = e^{t(\lambda_-(k)+c(a_2)|k|)}, \tag{4.42}$$

and can be estimated independently of t. Using it we are able to bring down estimate of $\|u_{i,11}\|_{L^p(\mathbb{R}^2_+)}$ to estimate of a term $\mathcal{F}_k^{-1}\left(-e^{-tc(a_2)|k|}ik(ik\hat{D}_i(k))\right)$, since

$$\mathcal{F}_x(u_{i,11}) = e^{t(\lambda_-(k)+c(a_2)|k|)}\left(-e^{-tc(a_2)|k|}ik(ik\hat{D}_i(k))\right).$$

Thus, in case $a_2 \le 0$,

$$\|u_{i,11}\|_{L^p(\mathbb{R}^2_+)} \le C\|\mathcal{F}_k^{-1}\left(e^{-tc(a_2)|k|}ik(ik\hat{D}_i(k))\right)\|_{L^p(\mathbb{R}^2_+)} \le C(a_2)\|D_i(k)\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}.$$

□

We estimate term $u_{i,11}$ for the case $a_2 > 0$ using the following Lemma:

Lemma 4.2.6. *Given $u_{i,11}$ in the form (4.41). Assuming $a_2 > 0$ and $D \in \dot{W}_p^{2-1/p}(\mathbb{R}) \cap \dot{W}_p^{2-2/p}(\mathbb{R})$ one has:*

$$\|u_{i,11}\|_{L^p(\mathbb{R}^2_+)} \le C\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{2-2/p}(\mathbb{R})}$$

Proof . Since the behaviour of $\lambda_-(k)$ is different in a neighbourhood of 0 and in a neighbourhood of ∞ we cannot use the technique use in the previous proof, because there exists no constant $c(a_2)$ such that $\varphi(k)$ from (4.42) is a valid multiplier. We thus introduce a cut-off function $\pi(k)$ as follows:

$$\pi(k) \equiv 1 \quad \text{for } |k| \le L, \qquad \pi(k) \equiv 0 \quad \text{for } |k| \ge L+1, \tag{4.43}$$

for some positive constant L, which we describe later.
We split our term $e^{t\lambda_-(k)}ik(ik\hat{D}_i(k))$ as

$$e^{t\lambda_-(k)}ik\pi(k)(ik\hat{D}_i(k)) + e^{t\lambda_-(k)}ik(1-\pi(k))(ik\hat{D}_i(k)) =: I_1 + I_2. \tag{4.44}$$

Let us first estimate I_1. We consider here the worst case, when $a_1 = 0$. All further estimates can be repeated for $a_1 \ne 0$. From the basic properties of $\lambda_-(k)$ we see that for a proper constant $\tilde{L}$ a multiplier φ_1:

$$\varphi_1(k) = e^{t(\lambda_-(k)+\tilde{L}k^2)}\pi(k)$$

is a good multiplier for all $t \ge 0$ in the sense of the Marcinkiewicz Theorem, with a proper estimate not dependent of t (the case when $\lambda_-(k) + \tilde{L}k^2 \le 0$ for all k). Thus we may write

$$\|\mathcal{F}_k^{-1}\left(I_1(t,\cdot)\right)\|_{L^p(\mathbb{R})} \le \left\|\mathcal{F}_k^{-1}\left(e^{-\tilde{L}k^2t}ik(ik\pi(k)\hat{D}_i(k))\right)\right\|_{L^p(\mathbb{R})}.$$

The constant $\tilde{L}$ does not affect any estimates, so for the readability of the paper we assume that $\tilde{L} = 1$. We also denote $ik\pi(k)\hat{D}_i(k)$ as $\hat{B}_i(k)$ and $J_1(t,x)$ as

$$J_1(t,x) = \mathcal{F}_k^{-1}\left(e^{-tk^2} ik\hat{B}_i(k)\right)$$

Since

$$\mathcal{F}_k^{-1}\left(e^{-tk^2}\right) = \frac{e^{-x^2/4t}}{\sqrt{2}\sqrt{t}} \qquad \text{we have:} \qquad \mathcal{F}_k^{-1}\left(e^{-tk^2} ik\right) = \frac{e^{-x^2/4t}x}{2\sqrt{2}t^{3/2}}. \tag{4.45}$$

Above term is integrable and odd with respect to x so we may write:

$$J_1(t,x) = \int_{\mathbb{R}} e^{-y^2/4t}\frac{y}{2\sqrt{2}t^{3/2}}[B_i(x-y) - B_i(x)]dy.$$

Using Hölder inequality we get:

$$\begin{aligned}\|J_1(t,\cdot)\|^p_{L^p(\mathbb{R})} &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-y^2/4t}\frac{y}{t^{3/2}}[B_i(x-y) - B_i(x)]dy\right|^p dx \\ &\leq \frac{1}{2\sqrt{2}}\left(\int_{\mathbb{R}} e^{-qy^2/8t}\frac{1}{t^{1/2}}\right)^{1/q}\left(\int_{\mathbb{R}} e^{-py^2/8t}\frac{y^p}{t^{p+1/2}}|B_i(x-y) - B_i(x)|^p dy\right).\end{aligned}$$

Now since $\int_{\mathbb{R}} e^{-qy^2/8t}\frac{1}{t^{1/2}}dy = \frac{2\sqrt{2\pi}}{\sqrt{q}}$ and

$$\int_0^\infty e^{-py^2/8t}\frac{y^p}{t^{p+1/2}}dt = 2^{3(p-1/2)}p^{p-1/2}\Gamma(p-1/2)|y|^{1-p} = c_1(p)|y|^{1-p}$$

we can write:

$$\|J_1(\cdot,\cdot)\|^p_{L^p(\mathbb{R}^2_+)} \leq 2^{3(p-1/2)}p^{p-1/2}\Gamma(p-1/2)\frac{\sqrt{\pi}}{\sqrt{q}}\int_{\mathbb{R}\times\mathbb{R}}\frac{|B_i(x-y) - B_i(x)|^p}{|y|^{p-1}},$$

where the right hand side can be estimated (right from the definition (4.11)) by

$$2^{3(p-1/2)}p^{p-1/2}\Gamma(p-1/2)\frac{\sqrt{\pi}}{\sqrt{q}}\|B_i\|^p_{\dot{W}_p^{1-2/p}(\mathbb{R})}.$$

This term, however, can be estimated by $C(q)\|D_i\|^p_{\dot{W}_p^{2-2/p}(\mathbb{R})}$, since multiplication by a smooth function $\pi(k)$ does not change the class of the function $ik\hat{D}_i$ and $\|\mathcal{F}_k^{-1}(ik\hat{D}_i)\|_{\dot{W}_p^{1-2/p}(\mathbb{R})} = \|\mathcal{F}_k^{-1}(\hat{D})_i\|_{\dot{W}_p^{2-2/p}(\mathbb{R})}$.

Before we make further estimates we would like to emphasize, that this type of estimates and appearance of terms like (4.45) are characteristic to a parabolic problem. We see, that a change of a sign of the coefficient a_2 results in the different behaviour of the eigenvalue, which brings in this parabolic disturbance to our estimates and might be the cause of the presence of the wake region behind the obstacle.

Let us now return to the second term from (4.44), i.e. I_2:

$$I_2 = e^{t\lambda_-(k)}ik(1-\pi(k))(ik\hat{D}_i(k)).$$

In this case we introduce a multiplier $\varphi(k) = e^{t(\lambda_-(k)+|k|/2)}$. Since $\lambda_-(k) \sim -|k|$ for large $|k|$ we see, that

$$\lambda_-(k) + |k|/2 \sim -|k|/2 < 0 \qquad \text{for } |k| \text{ large enough.} \tag{4.46}$$

Now we may go back to the definition of function π, i.e. (4.43), and set L large enough (and in fact also $\widetilde{L}$ small enough) to ensure, that for $|k| > L+1$ inequality (4.46) holds. Then our multiplier can be estimated independently of t. Summing up:

$$\|\mathcal{F}_k^{-1}\left(I_2(t,\cdot)\right)\|_{L^p(\mathbb{R})} \leq \left\|\mathcal{F}_k^{-1}\left(e^{-t|k|/2} ik(1-\pi(k))(ik\hat{D}_i(k))\right)\right\|_{L^p(\mathbb{R})}$$

and all estimates for the gradient of p can be applied directly for this term, since

$$\mathcal{F}_k^{-1}\left(ik(1-\pi(k))\hat{D}_i(k)\right) \in \dot{W}_p^{1-1/p}(\mathbb{R}).$$

This estimate completes the case of $u_{,11}$.

□

Remark: In above lemmas we used an assumption that $D_i \in \dot{W}_p^{2-1/p}(\mathbb{R}) \cap \dot{W}_p^{2-2/p}(\mathbb{R})$, but in fact $D_i \in \dot{W}_r^{2-1/r}(\mathbb{R}) \cap \dot{W}_r^{2-2/r}(\mathbb{R})$ for all $r \in (3,p]$ and thus our estimate also holds for $\|\nabla^2 u\|_{L^r(\mathbb{R}^2_+)}$.

4.2.5 Estimate of $u_{,22}$.

To complete the proof of Theorem 4.2.3 we now estimate $u_{,22}$, which corresponds to estimate of

$$u_{,22} = \mathcal{F}_k^{-1}(v_{,22}(k,t)) = \mathcal{F}_k^{-1}\left(\lambda_-^2(k)e^{t\lambda_-(k)}\hat{D}_i(k)\right). \tag{4.47}$$

Again, we treat all cases of signum of a_2 in a separate Lemma. The first one will be for the case $a_2 < 0$:

Lemma 4.2.7. *Given $u_{i,22}$ in the form (4.47). Assuming $a_2 < 0$ and $D \in W_p^{2-1/p}(\mathbb{R})$ one has the following inequality:*

$$\|u_{i,22}\|_{L^p(\mathbb{R}^2_+)} \leq C\|D\|_{W_p^{2-1/p}(\mathbb{R})}.$$

Proof . The problem one encounters is that for $a_2 < 0$ we have $\lambda_-^2(k) \geq a_2^2$, which obviously does not behave like $|k|^2$ for small k and hence we cannot write this term in a form like in (4.41), that is why a different approach is needed and we will investigate the case $a_2 < 0$ more thoroughly.

As usual we introduce a smooth cut off function $\pi(k)$ such that $\pi(k) \equiv 1$ for $|k| \leq L$, for some constant $L > 0$, which will described later, and $\pi(k) \equiv 0$ for $|k| \geq L+1$. As we have seen many times, multiplication by a smooth bounded function of compact support does not influence essential estimates. Keeping this in mind we may write:

$$v_{i,11}(k,t) = c(a_2)e^{-tc(a_2)}\pi(k)\hat{D}_i(k) + |k|^2 e^{-t|k|c(a_2)}(1-\pi(k))\hat{D}_i(k) =: I_1(t,k) + I_2(t,k) \tag{4.48}$$

where a constant $c(a_2)$ may differ from one occurrence to another.

Integral I_1 is easy to estimate, since $D \in W_p^{2-1/p}(\mathbb{R})$ for $a_2 < 0$ (and of course $\mathcal{F}_k^{-1}(\pi(k)\hat{D}) \in W_p^{2-1/p}(\mathbb{R})$), and in particular $D \in L^p(\mathbb{R})$, which gives us:

$$\|\mathcal{F}_k^{-1}(I_1(k,t))\|^p_{L^p(\mathbb{R}^2_+)} \leq \int_0^\infty c(a_2)e^{-tc(a_2)}\|D\|^p_{L^p(\mathbb{R})}dt \leq c(a_2)\|D\|^p_{L^p(\mathbb{R})} \leq c(a_2)\|D\|^p_{W_p^{2-1/p}(\mathbb{R})}.$$

Integral $I_2(t,k)$ can be estimated in the same way as it was made in case of $u_{,11}$, i.e. one presents $I_2(t,k)$ as

$$I_2(t,k) = -e^{-t|k|c(a_2)}ik\left(ik(1-\pi(k))\hat{D}_i(k)\right),$$

and estimates as follows:

$$\|\mathcal{F}_k^{-1}(I_2(t,k))\|_{L^p(\mathbb{R}^2_+)} \leq C\|D_i\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}, \qquad \text{thus} \qquad \|v_{i,11}\|_{L^p(\mathbb{R}^2_+)} \leq C\|D_i\|_{W_p^{2-1/p}(\mathbb{R})},$$

and the proof of Lemma 4.2.7 is complete. □

For the case of $a_2 = 0$ we have the following lemma:

Lemma 4.2.8. *Given $u_{i,22}$ in the form (4.47). Assuming $a_2 = 0$ and $D \in \dot{W}_p^{2-1/p}(\mathbb{R}) \cap \dot{W}_p^{1-1/p}(\mathbb{R})$ one has the following inequality:*

$$\|u_{i,22}\|_{L^p(\mathbb{R}^2_+)} \leq C\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{1-1/p}(\mathbb{R})}.$$

Proof . In case $a_2 = 0$ one has $\lambda_-^2 \sim k^2 + aik = ik(a - ik)$ for small k (we will treat this term as a part of derivative, i.e. ik, and part of a multiplicator, i.e. $a - ik$) and thus, proceeding as earlier (introducing a cut-off function $\pi(k)$):

$$v_{i,11}(k,t) = e^{-t\sqrt{|k|}}ik(-ik+a)\pi(k)\hat{D}_i(k) - e^{-t|k|c(a_2)}ik(1-\pi(k))ik\hat{D}_i(k) = \\ =: I_1(t,k) + I_2(t,k).$$

Integrals like $I_2(t,k)$ we have already seen how to estimate – since $\mathcal{F}_k^{-1}(ik(1-\pi(k))\hat{D}_i(k)) \in \dot{W}_p^{1-1/p}(\mathbb{R})$ we get:

$$\|\mathcal{F}_k^{-1}(I_2(t,k))\|_{L^p(\mathbb{R}^2_+)} \leq C\|D_i\|_{\dot{W}_p^{2-1/p}(\mathbb{R})}.$$

To estimate $I_1(t,k)$ we notice, that since there exists a constant c_{a_1} such that $-\sqrt{|k|} + c_{a_1}|k| \leq 0$ for small k, we may use Marcinkiewicz theorem for a multiplier $\varphi(k) = \pi(k)e^{t(|k|-\sqrt{|k|})}$ to get that

$$\|\mathcal{F}_k^{-1}(I_1(t,k))\|_{L^p(\mathbb{R}^2_+)} \leq \|\mathcal{F}_k^{-1}(e^{-t|k|c(a_2)})ik(a-ik)\pi(k)\hat{D}_i(k)\|_{L^p(\mathbb{R}^2_+)} \leq C\|D\|_{\dot{W}_p^{1-1/p}(\mathbb{R})}.$$

□

For the case of $a_2 > 0$ we have the following lemma:

Lemma 4.2.9. *Given $u_{i,22}$ in the form (4.47). Assuming $a_2 > 0$ and $D \in \dot{W}_p^{2-1/p}(\mathbb{R}) \cap \dot{W}_p^{2-2/p}(\mathbb{R})$ one has the following inequality:*

$$\|u_{i,22}\|_{L^p(\mathbb{R}^2_+)} \leq C\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{2-2/p}(\mathbb{R})}$$

Proof . To estimate $u_{,22}$ for $a_2 > 0$ we proceed as earlier (introducing a cut-off function $\pi(k)$): since $\Re\lambda_-(k) \sim -|k|^2$ for small $|k|$ we may write $v_{,22}$ as follows:

$$v_{i,22}(t,k) = -ike^{-t|k|^2}\pi(k)|k|^2ik\hat{\hat{D}}_i(k) - ike^{-t|k|c(a_2)}(1-\pi(k))ik\hat{\hat{D}}_i(k) =: I_1(t,k) + I_2(t,k).$$

Integral $I_2(t,k)$ can be estimated as follows:

$$\|\mathcal{F}_k^{-1}(I_2(t,k))\|_{L^p(\mathbb{R}^2_+)} \leq C\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})},$$

while for $I_1(t,k)$ one has:

$$\|\mathcal{F}_k^{-1}(I_1(t,k))\|_{L^p(\mathbb{R}^2_+)} \leq C\|D\|_{\dot{W}_p^{2-2/p}(\mathbb{R})},$$

repeating estimates for $u_{,11}$ and keeping in mind, that $\pi(k)|k|^2$ is a proper multiplier in the sense of the Marcinkiewicz theorem, since $\pi(k)$ has bounded support.

These estimates prove the following inequality:

$$\|v_{i,22}\|_{L^p(\mathbb{R}^2_+)} \le C\|D\|_{\dot{W}_p^{2-1/p}(\mathbb{R})\cap\dot{W}_p^{2-2/p}(\mathbb{R})},$$

which completes the proof of this lemma. □

Remark: As was the case for $u_{i,11}$ – since $D(x)$ is in a family of spaces, i.e. not only for p but also for all $r \in (3,p]$, all above estimates are valid also for $\|u_{i,22}\|_{L^r(\mathbb{R}^2_+)}$. This completes the proof of Theorem 4.2.3.

4.2.6 Summary.

As we have seen in the proof of Theorem 4.2.3 different regularity of boundary condition is needed in case of $u_{,11}$ and $u_{,22}$, however for the readability of the paper we did not differentiated it in the statement of the theorem, however now we can set together all these requirements. The following array shows, what regularity on D is required in particular cases:

	$a_2 < 0$	$a_2 = 0$	$a_2 > 0$
$v_{,11}$	$D \in \dot{W}_p^{2-1/p}$	$D \in \dot{W}_p^{2-1/p}$	$D \in \dot{W}_p^{2-1/p} \cap \dot{W}_p^{2-2/p}$
$v_{,22}$	$D \in W_p^{2-1/p}$	$D \in \dot{W}_p^{2-1/p} \cap \dot{W}_p^{1-1/p}$	$D \in \dot{W}_p^{2-1/p} \cap \dot{W}_p^{2-2/p}$

In this table the Reader can see, what is the connection between the class of regularity for the boundary conditions and the sign of a_2, which corresponds to the type of points on the boundary (i.e. type E (elliptic) for $a_2 < 0$, type P (parabolic) for $a_2 > 0$ and type S for $a_2 = 0$). In front of the obstacle it is required that the boundary conditions are in the inhomogeneous class $W_p^{2-1/p}(\mathbb{R})$, a typical for a strongly elliptic problems. The situation behind the obstacle $a_2 > 0$ appears to have also a parabolic disturbance, which can be seen by the need of the space $\dot{W}_p^{2-2/p}(\mathbb{R})$. Such class of regularity corresponds to the trace space for the standard heat equation.

4.2.7 Some basic properties for $\lambda_-(k)$

We present properties of $\lambda_-(k)$. Let us denote by A the following term

$$A = a_2^2 + 4(k^2 + a_1 ik).$$

Then

$$\begin{aligned}
|A|^2 &= (a_2^2+4k^2)^2+(4a_1k)^2\\
\cos\theta_A &= \frac{\Re A}{|A|}=\frac{a_2^2+4k^2}{|A|}\\
\sqrt{A} &= \sqrt{|A|}e^{\frac{-i\theta_A}{2}}\\
\cos x &= \frac{1}{\sqrt{2}}\sqrt{1+\cos 2x}\\
\cos\frac{\theta_A}{2} &= \frac{1}{\sqrt{2}}\left(\frac{a_2^2+4k^2+|A|}{|A|}\right)^{1/2}\\
\Re\sqrt{A} &= |A|^{1/2}\frac{1}{\sqrt{2}}\left(\frac{a_2^2+4k^2+|A|}{|A|}\right)^{1/2}\\
&= \frac{1}{\sqrt{2}}\left(a_2^2+4k^2+|A|\right)^{1/2}\\
\Re\lambda_-(k) &= \frac{1}{2}\left(a_2-\frac{1}{\sqrt{2}}\left(a_2^2+4k^2+[(a_2^2+4k^2)^2+(4a_1k)^2]^{1/2}\right)^{1/2}\right)\\
&= \frac{1}{2}\frac{a_2^2-\frac{1}{2}\left(a_2^2+4k^2+[(a_2^2+4k^2)^2+(4a_1k)^2]^{1/2}\right)}{a_2+\frac{1}{\sqrt{2}}\left(a_2^2+4k^2+[(a_2^2+4k^2)^2+(4a_1k)^2]^{1/2}\right)^{1/2}}.
\end{aligned}$$

If $a_2>0$ we also have:

$$\Re\lambda_-(k)\le\frac{-2k^2}{a_2+\frac{1}{\sqrt{2}}\left(a_2^2+4k^2+[(a_2^2+4k^2)^2+(4a_1k)^2]^{1/2}\right)^{1/2}}$$

Finally, if $|k|\le L$ we have

$$\Re\lambda_-(k)\le\frac{-2k^2}{C(a_1,a_2,L)},$$

and we may denote $1/C(a_1,a_2,L)$ as $\tilde{L}$:

$$\tilde{L}=1/C(a_1,a_2,L)$$

to get:

$$\Re\lambda_-(k)\le-2\tilde{L}k^2$$

4.3 The system in the whole space $\mathbb{R}^2$

In this part we would like to present results, which were used in the previous section.

The standard approach to whole space linear problems is the technique of the Fourier transform together with a multiplier theorem, for example Lizorkin Theorem (see Theorem 4.5.2). Using it are able to show the following theorems. We would like to mention that for our purposes not all estimates in this theorem are needed. Some of them are however necessary to show existence of solutions to the Navier-Stokes system (4.1)-(4.4) in an exterior domain that is why we state them and give a proof of some of them. If the Reader is interested in this problem we refer him to [11] and [33].

Theorem 4.3.1. *Let $F \in L^q(\mathbb{R}^2)$ and $1 < q < \infty$. Then there exists a solution $u = (u_1, u_2)$ to the system:*

$$a_1 u_{,1} + a_2 u_{,2} - \Delta u = F \qquad \text{in } \mathbb{R}^2,$$

for which the following inequality holds:

$$\|\nabla^2 u\|_{L^q(\mathbb{R}^2)} \le C\|F\|_{L^q(\mathbb{R}^2)}. \tag{4.49}$$

If $q < 3$ then also the following inequality holds:

$$\|\nabla u\|_{L^{3q/(3-q)}(\mathbb{R}^2)} \le C\|F\|_{L^q(\mathbb{R}^2)}. \tag{4.50}$$

Moreover, as a direct result of previous statements, if $q > 3$ and $F \in L^s(\mathbb{R}^2)$ for all $s \in (3/2, q]$, then for all $r \in (3, q]$

$$\|\nabla u\|_{W^1_r(\mathbb{R}^2)} \le C(r)\|F\|_{L^r(\mathbb{R}^2)\cap L^{3r/(3+r)}(\mathbb{R}^2)}.$$

Proof . After rotating the coordinate system this problem corresponds to the problem

$$\lambda u_1 - \Delta u = F \qquad \text{in } \mathbb{R}^2.$$

After applying the Fourier transform to the above equation and gets:

$$\hat{u}(\xi) = \frac{\hat{F}(\xi)}{i\xi_1 + |\xi|^2}.$$

Using Theorem 4.5.2 one immediately gets (4.49), since a multiplier $\frac{-\xi_i\xi_j}{i\xi_1+|\xi|^2}$, which stands for a derivative $u_{,ij}$, is a proper bounded multiplier.

To show (4.50) we again use Theorem 4.5.2 with $\beta = 1/3$. We must show that the multiplier

$$\frac{|\xi_1|^{4/3}|\xi_2|^{1/3} + |\xi_2|^{4/3}|\xi_1|^{1/3}}{|\xi|^2 + \lambda|\xi_1|}$$

is bounded for all $\xi \in \mathbb{R}^2$. Since

$$|\xi_1|^{4/3}|\xi_2|^{1/3} + |\xi_2|^{4/3}|\xi_1|^{1/3} \le |\xi_1|^{1/3}|\xi|^{4/3}$$

and

$$(|\xi|^{2/3} + \lambda^{1/3}|\xi_1|^{1/3})^3 \le C(|\xi|^2 + \lambda|\xi_1|)$$

we get

$$\lambda^{1/3}|\xi_1|^{1/3}|\xi|^{4/3} \le |\xi|^2 + \lambda|\xi_1|$$

thus

$$\lambda^{1/3}\|\nabla u\|_{L^{3p/(3-p)}(\mathbb{R}^2)} \le C\left(\|F\|_{L^p(\mathbb{R}^2)} + \|G\|_{W^{1,p}(\mathbb{R}^2)}\right).$$

□

As a direct result of this theorem we have the following:

Lemma 4.3.2. *Given $3 < q < \infty$. If $F \in L^q(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ then the solution $u = (u_1, u_2)$ to the system from Theorem 4.3.1 satisfies the following estimates:*

$$\|u\|_{W^2_r(\mathbb{R}^2)} \leq C(\lambda, r)\|F\|_{L^r_{A_r}} \quad \text{for all } r \in (3, q], \tag{4.51}$$

where $A_r = \{3r/(3+2r), 3r/(3+r), r\}$,

$$\|\nabla u\|_{W^1_r(\mathbb{R}^2)} \leq C(\lambda, r)\|F\|_{L^q_{B_r}} \quad \text{for all } r \in (3/2, q], \tag{4.52}$$

where $B_r = \{3r/(3+r), r\}$.
Moreover for all $r \in (3/2, q]$ one has:

$$u_{|x_2=0} \in \dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-1/r}_r(\mathbb{R}). \tag{4.53}$$

Proof . Let $3/2 < r \leq q$. We take $r_1 = 3r/(3+r)$ and $r_2 = 3r/(3+2r)$. Using previous theorem we immediately get $\|\nabla^2 u\|_{L^r} \leq \|F\|_{L^r}$, since $\|F\|_{L^r} \leq C(r)\|F\|_{L^q \cap L^1}$. Since $F \in L^q \cap L^1$ and we also have $F \in L^{r_1}$ and $F \in L^{r_2}$. Since $r_1 < 3$ this implies that $\|\nabla u\|_{L^r} = \|\nabla u\|_{L^{3r_1/(3-r_1)}} \leq \|F\|_{L^{r_1}}$, which is the desired estimate (4.52). The same thing we can make with r_2 and u, since $r_2 < 3/2$ and thus $\|u\|_{L^r} = \|u\|_{L^{3r_2/(3-2r_2)}} \leq \|F\|_{L^{r_2}}$ and hence the proof of (4.51) and (4.52) is complete.

To show (4.53) one must notice, that since $\nabla u_{|x_2=0} \in W^{1-1/r}_r(\mathbb{R})$ we get $u_{|x_2=0} \in \dot{W}^{2-1/r}_r(\mathbb{R})$ (straightforward from definition (4.11)). The fact that $u_{|x_2=0} \in \dot{W}^{1-1/r}_r(\mathbb{R})$ can be shown using Lemma 4.5.3 for $s = 3 + \epsilon$ and $m = r$.

Remark: bounds from (4.51) and (4.52) come from the inequalities $3s/(3-s) > 3/2$ and $3s/(3-2s) > 3$ for all $s > 1$. □

Using different techniques than those presented in the proof of Theorem 4.3.1 we are able to show the following result:

Theorem 4.3.3. *Let $f \in L^p(\mathbb{R}^2)$ such that supp $f \subset \mathbb{R}^2_+$. Given a solution to the following system:*

$$a_1 u_{,1} + a_2 u_{,2} - \Delta u = f \qquad \text{in } \mathbb{R}^2, \tag{4.54}$$

with a condition at infinity $|u| \to 0 \qquad$ as $|x| \to \infty$. Provided $a_2 < 0$, the following estimate is valid:

$$\|u_{|x_2=0}\|_{W^{2-1/p}_p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

Proof . From Lemma 4.3.2 we have immediately

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)},$$

so to prove Theorem 4.3.3 we need to show only L_p- estimate for the function u, namely we prove the following inequality:

$$\|u_{|x_2=0}\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

We apply the Fourier transform in x_1 variable to (4.54) to obtain the following differential equation:

$$a_1 i\xi v + a_2 \dot{v} + \xi^2 v - \ddot{v} = \hat{f}(t, \xi),$$

where $v(\xi, t) = \mathcal{F}_{x_1}(u)(\xi, t)$, and we denoted x_2 coordinate as t.

With this system two eigenvalues are connected: stable $\lambda_- = (a_2 - \Delta)/2$ and unstable $\lambda_+ = (a_2 + \Delta)/2$, where $\Delta = \sqrt{a_2^2 + 4(\xi^2 + a_1 i\xi)}$. Let us observe, that $\Re\lambda_- < a_2$, which will be crucial for our considerations. The solution satisfies the following equation:

$$v(\xi, t) = \int_{-\infty}^{t} \frac{1}{\Delta} e^{\lambda_+(s-t)} \hat{f}(s, \xi) ds + \int_{t}^{\infty} \frac{1}{\Delta} e^{\lambda_-(s-t)} \hat{f}(s, \xi) ds.$$

Since the support of $\hat{f}$ is a subset of $\mathbb{R}_2^+$ we have

$$v(\xi, 0) = \int_{0}^{\infty} \frac{1}{\Delta} e^{\lambda_- s} \hat{f}(s, \xi) ds.$$

To estimate $\|u_{|x_2=0}\|_{L^p(\mathbb{R})}$ we use Marcinkiewicz theorem, i.e.

$$\begin{aligned} \|u_{|x_2=0}\|_{L^p(\mathbb{R})} &= \|\mathcal{F}_\xi^{-1}(v(\xi, 0))\|_{L^p(\mathbb{R})} = \left\| \mathcal{F}_\xi^{-1} \left(\int_0^\infty \frac{1}{\Delta} e^{\lambda_- s} \hat{f}(\xi, s) ds \right) \right\|_{L^p(\mathbb{R})} \\ &\leq \int_0^\infty \|\mathcal{F}_\xi^{-1} \left(\frac{1}{\Delta} e^{\lambda_- s} \hat{f}(\xi, s) \right) \|_{L^p(\mathbb{R})} \\ &\leq \int_0^\infty C_M(s) \|f(\cdot, s)\|_{L^p(\mathbb{R})} ds, \end{aligned}$$

where the term $C_M(s)$ comes from the term $\frac{1}{\Delta} e^{\lambda_- s}$, which, for convenience, we denote as $\Psi(\xi, s)$. An estimate of the constant $C_M(s)$, which comes from from the Marcinkiewicz theorem, is crucial for our estimate. Since we are in one dimension the constant $C(s)$ is estimated by:

$$C_M(s) \leq \sup_{\xi \in \mathbb{R} \setminus \{0\}} (|\Psi(\xi, s)| + |\xi \partial_\xi \Psi(\xi, s)|).$$

We now use the assumption that $a_2 < 0$ to get that:

$$\begin{aligned} |\Psi(\xi, s)| &= \left| \frac{1}{\sqrt{a_2^2 + 4(\xi^2 + a_1 i\xi)}} e^{s(a_2 - \Delta)/2} \right| \\ &\leq e^{a_2 s/2} \left| \frac{1}{\sqrt{a_2^2 + 4(\xi^2 + a_1 i\xi)}} e^{-s\Delta/2} \right| \\ &\leq \frac{1}{|a_2|} e^{a_2 s/2}, \end{aligned}$$

since $\Re\Delta > 0$.

To estimate $|\xi \partial_\xi \Psi(\xi, s)|$ first we write write it in a full form:

$$|\xi \partial_\xi \Psi(\xi, s)| = \left| -\frac{2\xi(a_1 i + 2\xi)}{\Delta^3} e^{s(a_2 - \Delta)/2} - \frac{2\xi(a_1 i + 2\xi)s}{\Delta^2} e^{s(a_2 - \Delta)/2} \right|.$$

The first term on the right hand side can be estimated similarly to $|\Psi(\xi, s)|$, i.e. since $2\xi(a_1 i + 2\xi)/\Delta^3 \leq C(a_1, a_2)$ and an estimate for the term $e^{s(a_2 - \Delta/2)}$ is known we get:

$$\left| -\frac{2\xi(a_1 i + 2\xi)}{\Delta^3} e^{s(a_2 - \Delta)/2} \right| \leq C(a_1, a_2) e^{a_2 s/2}.$$

The second term can be estimated similarly, however we need to estimate also s using $e^{sa_2/4}$, i.e. $se^{sa_2/4} \leq C(a_2)$, to get:

$$\left| \frac{2\xi(a_1 i + 2\xi)s}{\Delta^2} e^{s(a_2 - \Delta)/2} \right| \leq C(a_2) e^{sa_2/4}.$$

All these estimates assure that:

$$C_M(s) \leq C(a_1,a_2)e^{sa_2/4}.$$

We may now go back to estimate of $\|u\|_{L^p(\mathbb{R})}$, keeping in mind that $a_2<0$:

$$\begin{aligned}
\|u_{|x_2=0}\|_{L^p(\mathbb{R})} &\leq \int_0^\infty C_M(s)\|f(\cdot,s)\|_{L^p(\mathbb{R})}ds = \\
&= \lim_{M\to\infty}\int_0^M e^{a_2 s/4}\|f(\cdot,s)\|_{L^p(\mathbb{R})}ds \leq \\
&\leq \lim_{M\to\infty}\left(\int_0^M \|f(\cdot,s)\|^p_{L^p(\mathbb{R})}\right)^{1/p}(1-e^{a_2Mq/4})^{1/q} \\
&= \lim_{M\to\infty}\|f\|_{L^p(\mathbb{R}\times[0,M])}(1-e^{a_2Mq/4})^{1/q}) = \|f\|_{L^p(\mathbb{R}^2_+)},
\end{aligned}$$

which is the desired estimate. □

The following Theorem is well known (see [11]):

Theorem 4.3.4. ***Oseen system in the full space*** $\mathbb{R}^2$**.** *Let $F\in L^q(\mathbb{R}^2)$, $G\in W^{1,q}(\mathbb{R}^2)$ and $1<q<\infty$. Then there exists a solution $u=(u_1,u_2)$ and p to the following inhomogeneous Oseen system:*

$$\begin{aligned}
\lambda u_{,1} - \Delta u + \nabla p &= F,\\
\nabla\cdot u &= G,
\end{aligned}$$

which satisfies the following estimates:

- *for all $1<q<\infty$: $\lambda\|\nabla u_2\|_{L^q} + \lambda\left\|\frac{\partial u}{\partial x_1}\right\|_{L^q} + \|\nabla^2 u\|_{L^q} + \|\nabla p\|_{L^q} \leq c(\|F\|_{L^q}+\|G\|_{W^{1,q}})$,*
- *for all $1<q<3$: $\lambda^{1/3}\|\nabla u\|_{L^{3q/(3-q)}} \leq c(\|F\|_{L^q}+\|G\|_{W^{1,q}})$,*
- *for all $1<q<3/2$: $\lambda^{2/3}\|u\|_{L^{3q/(3-2q)}} \leq c(\|F\|_{L^q}+\|G\|_{W^{1,q}})$.*

Proof . Below we give a sketch of a proof, i.e. not all estimates from the theorem are shown. If the Reader is interested in this gaps we kindly refer him to [11].

Taking the Fourier transform of these equations we get:

$$\begin{aligned}
i\lambda\xi_1\hat{u}_1 + |\xi|^2\hat{u}_1 + i\xi_1\hat{p} &= \hat{F}_1,\\
i\lambda\xi_1\hat{u}_2 + |\xi|^2\hat{u}_2 + i\xi_2\hat{p} &= \hat{F}_2,\\
i\xi_1\hat{u}_1 + i\xi_2\hat{u}_2 &= \hat{G}.
\end{aligned}$$

For readability we omit the hat ˆ.

The above algebraic system has a unique solution (u,p) in the form:

$$\begin{aligned}
u_1 &= \frac{\xi_2(-F_2\xi_1+F_1\xi_2)}{|\xi|^2\left(|\xi|^2+i\xi_1\lambda\right)} - G\frac{i\xi_1}{|\xi|^2},\\
u_2 &= \frac{\xi_1(F_2\xi_1-\xi_2F_1)}{|\xi|^2\left(|\xi|^2+i\xi_1\lambda\right)} - G\frac{i\xi_2}{|\xi|^2}\\
p &= \frac{-iF_1\xi_1 - iF_2\xi_2 + G\left(|\xi|^2+i\xi_1\lambda\right)}{|\xi|^2}.
\end{aligned}$$

First we show that

$$\nabla^2 u \in L^q(\mathbb{R}^2).$$

In order to do this we need to check proper Fourier multipliers, whether or not they fulfill requirements from the Theorem 4.5.2, namely multipliers from the following term (we focus on u_1):

$$\xi_i\xi_j\left(\frac{\xi_2(-F_2\xi_1+F_1\xi_2)}{|\xi|^2\,(|\xi|^2+i\xi_1\lambda)}-G\frac{i\xi_1}{|\xi|^2}\right)$$

We see right away that $\frac{\xi_i\xi_j}{|\xi|^2}$ fulfills these requirements and thus the part $i\xi_1 G\frac{\xi_i\xi_j}{|\xi|^2}$ gives proper estimate for $u_{1,ij}$, since $G \in W^{1,q}(\mathbb{R}^2)$. Similarly the part:

$$\xi_i\xi_j\left(\frac{\xi_2(-F_2\xi_1+F_1\xi_2)}{|\xi|^2\,(|\xi|^2+i\xi_1\lambda)}\right)=\frac{\xi_i\xi_j}{|\xi|^2}\frac{\xi_2(-F_2\xi_1+F_1\xi_2)}{(|\xi|^2+i\xi_1\lambda)}$$

is easily seen to fulfill these requirements.

In a similar way one can show desired estimates for u_2. Summing this up we get:

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^2)} \le C(\|F\|_{L^q(\mathbb{R}^2)}+\|G\|_{W^{1,q}(\mathbb{R}^2)}).$$

Now we would like to estimate $\|u_{,1}\|_{L^q(\mathbb{R}^2)}$. It is not hard to see that the part with G, namely $\frac{\xi_i\xi_1}{|\xi|^2}G$, does not cause any difficulties. Let us thus focus on:

$$\xi_1\frac{\xi_2(-F_2\xi_1+F_1\xi_2)}{|\xi|^2\,(|\xi|^2+i\xi_1\lambda)}=F_1\frac{\xi_1\xi_2^2}{|\xi|^2\,(|\xi|^2+i\xi_1\lambda)}-F_2\frac{\xi_1^2\xi_2}{|\xi|^2\,(|\xi|^2+i\xi_1\lambda)}.$$

The denominator in both terms is of order $\lambda|\xi|^2\xi_1$ near 0, that is why one can immediately get proper estimates of both these terms – similarly for u_2. Summing up:

$$\|u_{,1}\|_{L^p(\mathbb{R}^2)} \le C\left(\frac{1}{\lambda}\|F\|_{L^p(\mathbb{R}^2)}+\|G\|_{W^{1,p}(\mathbb{R}^2)}\right)$$

Remark: the reasoning for $u_{,1}$ cannot be repeated for the case with $u_{1,2}$, however may be repeated for the case with $u_{2,2}$. In this manner we get another estimate:

$$\|u_{2,2}\|_{L^p(\mathbb{R}^2)} \le C\left(\frac{1}{\lambda}\|F\|_{L^p(\mathbb{R}^2)}+\|G\|_{W^{1,p}(\mathbb{R}^2)}\right)$$

Above considerations can be easily applied to ∇p – multipliers in this case are also easy to estimate. Thus one gets:

$$\|\nabla p\|_{L^p(\mathbb{R}^2)} \le C\left(\|F\|_{L^p(\mathbb{R}^2)}+\|G\|_{W^{1,p}(\mathbb{R}^2)}\right)$$

Estimate

$$\lambda^{1/3}\|\nabla u\|_{L^{3p/(3-p)}} \le C\left(\|F\|_{L^p(\mathbb{R}^2)}+\|G\|_{W^{1,p}(\mathbb{R}^2)}\right).$$

is exactly the same, as it was in the proof of Theorem 4.3.1 – one faces the same multiplier. As was mentioned earlier – other terms can be estimated similarly – for the details we refer the Reader to [11]. □

As a direct application of the above Theorem we have the following Lemma:

Lemma 4.3.5. *Given* $3 < q < \infty$. *If* $F \in L^q(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *and* $G \in W^1_q(\mathbb{R}^2) \cap W^1_1(\mathbb{R}^2)$ *then the solution* $u = (u_1, u_2)$ *and* p *to the system from Theorem 4.3.4 satisfies the following estimates:*

$$\|u\|_{W^2_r(\mathbb{R}^2)} \leq C(\lambda, r) \left(\|F\|_{L^q \cap L^1} + \|G\|_{W^1_q \cap W^1_1} \right) \quad \text{for all } r \in (3, q], \tag{4.55}$$

$$\|\nabla u\|_{W^1_r(\mathbb{R}^2)} \leq C(\lambda, r) \left(\|F\|_{L^q \cap L^1} + \|G\|_{W^1_q \cap W^1_1} \right) \quad \text{for all } r \in (3/2, q]. \tag{4.56}$$

Moreover for all $r \in (3/2, q]$ *one has:*

$$u_{|x_2=0} \in \dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-1/r}_r(\mathbb{R}), \tag{4.57}$$

and for all $r \in (3, q]$ *one has:*

$$u_{|x_2=0} \in \dot{W}^{1-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-2/r}_r(\mathbb{R}). \tag{4.58}$$

Proof . The proof of this lemma is analogous to the proof of Lemma 4.3.2. Property (4.58) is a direct consequence of (4.55). □

4.4 Proof of Theorem 4.1.1

In this section we give a proof of Theorem 4.1.1. We extensively use results for the whole space $\mathbb{R}^2$ and for the half space $\mathbb{R}^2_+$.

To prove Theorem 4.1.1 we use a standard approach. We consider two auxiliary problems: one in the whole space and the second one in a bounded domain (some neighbourhood of the boundary of the original domain). With the former we deal with in Section 4.3. To solve the latter one may use the standard technique of partition of unity, namely, splitting a neighbourhood of the boundary into parts U_i small enough to introduce a proper curvilinear system in each of them. In this curvilinear coordinates the original problem transforms into a similar problem in a half space. Moreover – the support of a corresponding solution is contained in U_i.

Existence of solutions is assured thanks to our assumptions ($F \in H^{-1}(\Omega)$, etc.), since then one may use standard techniques for Hilbert spaces. We refer the Reader to Section 2.4.2, where existence for the Oseen system in shown, which satisfies $\nabla u \in L^2(\Omega)$ and $u, p \in L^2_{\text{loc}}(\Omega)$.

Once we have a solution we may use mentioned technique of partition of unity and show additional regularity.

Results in the full space $\mathbb{R}^2$ apply directly, however in the case of the half space $\mathbb{R}^2_+$ it cannot be made without an effort, since assumptions in the half space require that $p > 3/2$ in case of the pressure p and $p > 3$ in case of the velocity v, however for Theorem 4.1.1 to be applicable as a tool to prove Theorem 4.1.2 one has to have this type of results for $p < 6/5$.

We assume only, that $p > 1$. Since in applications we are interested in $p < 6/5$, we will focus on the case $p \in (1, 3/2)$. Before we continue we would like to mention two simple but important properties: if $p \in (1, 3/2)$ then $3p/(3-p) \in (3/2, 3)$ and $3p/(3-2p) \in (3, \infty)$.

We start with estimates on ∇p. Recalling the Remark to Theorem 4.2.2 we know, that estimates on ∇p are valid not only for $p > 3/2$, but for all $p > 1$ – the constraint $p > 3/2$ came from the fact, that we wanted to remove inhomogeneity from the right hand side while keeping proper estimates on boundary conditions. A similar condition $p > 3$ was necessary in case of the velocity v.

In this section we will not only use stated theorems and lemmas but we will go into the details of their proofs.

As was mentioned before, after a localization procedure we end up with system (4.12)-(4.16), where $F \in L^r(\mathbb{R}^2_+)$, $G \in W^1_r(\mathbb{R}^2_+)$, $\underline{b} \in W^{1-1/r}_r(\mathbb{R})$ and $\underline{d} \in W^{2-1/r}_r(\mathbb{R})$ for all $r \in (1,p]$. The next step is to solve, in a similar way to Lemma 4.2.1, an auxiliary system in the full space $\mathbb{R}^2$ obtaining the solution $(\widetilde{v}, \widetilde{q})$. Of course, since $p < 3/2$ we are not able to obtain the same conditions on traces of v and ∇v. Using Theorem 4.3.4 we get:

$$\vec{n} \cdot \mathbb{T}(\widetilde{v}, \widetilde{q})_{|x_2=0} \cdot \vec{\tau} \in \dot{W}^{1-1/r}_r(\mathbb{R}), \tag{4.59}$$

$$f(v \cdot \vec{\tau})_{|x_2=0} \in \dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-r_1}_{r_1}(\mathbb{R}), \tag{4.60}$$

$$\vec{n} \cdot v_{|x_2=0} \in \dot{W}^{2-1/r}_r(\mathbb{R}) \cap \dot{W}^{1-r_1}_{r_1}(\mathbb{R}), \tag{4.61}$$

where $r_1 = 3r/(3-r)$ where two last properties come from the fact, that $\nabla^2 v \in L^r(\mathbb{R}^2_+)$ and $\nabla v \in L^{r_1}(\mathbb{R}^2_+)$.

In such a case, a subtraction $u = v - \widetilde{v}$ and $p = q - \widetilde{q}$ implies that we obtain the system (4.23)-(4.27) for u, but b and d are of different regularity, namely:

$$b = \underline{b} - \widetilde{b} \in W^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R}) \tag{4.62}$$

$$d = \underline{d} - \widetilde{d} \in W^{2-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R}), \tag{4.63}$$

where $r_1 = 3r/(3-r)$ and $r \in (1,p]$. In the proof of Theorem 4.2.2 we used an assumption $b \in \dot{W}^{1-1/r}_r(\mathbb{R})$ and we see, that (4.62) is strong enough to obtain the following inequality:

$$\|b\|_{\dot{W}^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R})} \leq \|\underline{b} - \widetilde{b}\|_{W^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R})}.$$

In the case of d we are able to derive from (4.63) the following inequality:

$$\|d\|_{\dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r}_r(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R})} \leq \|\underline{d} - \widetilde{d}\|_{W^{2-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R})}.$$

These two inequalities imply that $\nabla p \in L^r(\mathbb{R}^2_+) + L^{r_1}(\mathbb{R}^2_+)$ and the following inequality is valid:

$$\|\nabla p\|_{L^r(\mathbb{R}^2_+) + L^{r_1}(\mathbb{R}^2_+)} \leq C(\|b\|_{\dot{W}^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R})} + \|d\|_{\dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r}_r(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R})}). \tag{4.64}$$

Indeed, to see this result we need to go into the details of the proof of Theorem 4.2.2. Since our problem is linear we may treat influence of b and d separately, say $p = p_b + p_d$. As we have mentioned before, during an estimate of p we used a seminorm $\|b\|_{\dot{W}^{1-1/r}_r(\mathbb{R})}$, that is why we present b as $b = b_1 + b_2$, where $b_1 \in \dot{W}^{1-1/r}_r(\mathbb{R})$ and $b_2 \in \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R})$ to get:

$$\|\nabla p_b\|_{L^r(\mathbb{R}^2_+) + L^{r_1}(\mathbb{R}^2_+)} \leq \|\nabla p_{b_1}\|_{L^r(\mathbb{R}^2_+)} + \|\nabla p_{b_2}\|_{L^{r_1}(\mathbb{R}^2_+)} \leq \|b\|_{\dot{W}^{1-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R})}.$$

The case with d is a little bit different. In the proof of Theorem 4.2.2, during the estimate of ∇p connected with a term d we splitted its Fourier transform into $I_{21}(t,k) + I_{22}(t,k)$ (see 4.35). The part $\mathcal{F}^{-1}_k(I_{21})$ was estimated by $\|d\|_{\dot{W}^{2-1/p}_p(\mathbb{R})}$, and the part $\mathcal{F}^{-1}_k(I_{22})$ was estimated by $\|d\|_{\dot{W}^{1-1/p}_p(\mathbb{R})}$. Now since in our case we have

$$d \in \dot{W}^{1-1/r_1}_{r_1}(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R}) + \dot{W}^{1-1/r}_r(\mathbb{R}) \cap \dot{W}^{2-1/r}_r(\mathbb{R})$$

this implies the following inequality:

$$\|\nabla p\|_{L^r(\mathbb{R}^2_+)+L^{r_1}(\mathbb{R}^2_+)} \leq \|\mathcal{F}_k^{-1}(I_{21}))\|_{L^r(\mathbb{R}^2_+)} + \|\mathcal{F}_k^{-1}(I_{22})\|_{L^{r_1}(\mathbb{R}^2_+)} \\ \leq \|d\|_{\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap \dot{W}_r^{2-1/r}+\dot{W}_r^{1-1/r}(\mathbb{R})\cap\dot{W}_r^{2-1/r}(\mathbb{R})}.$$

These considerations justify (4.64).

In an estimate of the velocity v we need not only homogeneous norm, but also L^q-norms on the boundary. That is why we must check, to which spaces our boundary conditions $\widetilde{b}$, $\widetilde{d}$ belong to.

Lemma 4.5.4 together with Theorem 4.3.4 (see also previous estimates (4.59)-(4.61)) give us the following properties:

$$\vec{n}\cdot\mathbb{T}(\widetilde{v},\widetilde{q})_{|x_2=0}\cdot\vec{\tau} \in \dot{W}_r^{1-1/r}(\mathbb{R})\cap(L^r(\mathbb{R})+L^{r_1}(\mathbb{R})),$$
$$f(v\cdot\vec{\tau})_{|x_2=0} \in \dot{W}_r^{2-1/r}(\mathbb{R})\cap\dot{W}_{r_1}^{1-r_1}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R})),$$
$$\vec{n}\cdot v_{|x_2=0} \in \dot{W}_r^{2-1/r}(\mathbb{R})\cap\dot{W}_{r_1}^{1-r_1}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R})),$$

where $r_1 = 3r/(3-r)$ and $r_2 = 3r/(3-2r)$. Using this properties we have:

$$b = \underline{b}-\widetilde{b} \in W_r^{1-1/r}(\mathbb{R})+\dot{W}_r^{1-1/r}(\mathbb{R})\cap(L^r(\mathbb{R})+L^{r_1}(\mathbb{R})) \\ +\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap\dot{W}_r^{2-1/r}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))$$

and

$$d = \underline{d}-\widetilde{d} \in W_r^{2-1/r}(\mathbb{R}) + \dot{W}_{r_1}^{1-1/r_1}(\mathbb{R}) \cap \dot{W}_r^{2-1/r}(\mathbb{R}) \cap (L^{r_1}(\mathbb{R}) + L^{r_2}(\mathbb{R})).$$

For our purposes we will need the following inequalities, which are a consequence of the above properties:

$$\|b\|_{\dot{W}_r^{1-1/r}(\mathbb{R})\cap(L^r(\mathbb{R})+L^{r_1}(\mathbb{R}))+\dot{W}_{r_1}^{1-r_1}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))} \leq \\ \|\underline{b}-\widetilde{b}\|_{W_r^{1-1/r}(\mathbb{R})+\dot{W}_r^{1-1/r}(\mathbb{R})\cap(L^r(\mathbb{R})+L^{r_1}(\mathbb{R}))+\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap\dot{W}_r^{2-1/r}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))} \tag{4.65}$$

and

$$\|d\|_{W_r^{2-1/r}(\mathbb{R})+\dot{W}_r^{2-1/r}(\mathbb{R})\cap\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))} \leq \\ \|\underline{d}-\widetilde{d}\|_{W_r^{2-1/r}(\mathbb{R})+\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap\dot{W}_r^{2-1/r}(\mathbb{R})\cap(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))}. \tag{4.66}$$

We are now in position to use Lemma 4.2.4 to derive proper class for Dirichlet boundary conditions. Using (4.65) and (4.66) it is not hard to see, that:

$$D(x_1) \in (L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))\cap\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap\dot{W}_r^{2-1/r}(\mathbb{R})+W_r^{2-1/r}(\mathbb{R}) \\ +(L^{r_1}(\mathbb{R})+L^{r_2}(\mathbb{R}))\cap\dot{W}_{r_1}^{1-1/r_1}(\mathbb{R})\cap\dot{W}_{r_1}^{2-1/r_1}(\mathbb{R})+ \\ (L^r(\mathbb{R})+L^{r_1}(\mathbb{R}))\cap\dot{W}_r^{1-1/r}(\mathbb{R})\cap\dot{W}_r^{2-1/r}(\mathbb{R}). \tag{4.67}$$

The important thing in the space from (4.67) is that it is a sum of spaces of a particular form:

$$(L^{s_1}(\mathbb{R})+L^{s_2}(\mathbb{R}))\cap\dot{W}_{s_3}^{1-1/s_3}(\mathbb{R})\cap\dot{W}_{s_4}^{2-1/s_4}(\mathbb{R}), \tag{4.68}$$

where $s_i \in \{r, r_1, r_2\}$. This form will be used during the estimate of $\nabla^2 v$.

The previous procedure of estimate the second derivatives of the velocity required introducing simplified problem and subtracting inhomogeneity, which was connected to ∇p. In our case $\nabla p \in L^r(\mathbb{R}) + L^{r_1}(\mathbb{R})$. Let us denote as w the solution to this simplified system, with the right hand side equal ∇p.

The first space $L^r(\mathbb{R})$ is more convenient for us in a sense, that since $r < 3/2$ then Theorem 4.3.1 assures that $w \in L^{r_2}(\mathbb{R}^2)$, $\nabla w \in L^{r_1}(\mathbb{R}^2)$ and $\nabla^2 w \in L^r(\mathbb{R}^2)$, which gives us that $w_{|x_2=0}$ is in a sum of space of the form (4.68), as we have seen during previous considerations connected with $\widetilde{d}$. This implies, that subtraction of w essentially will not change the class, where D belongs to.

To deal with the part of the gradient of the pressure ∇p, which belongs to L^{r_1} we will have to distinguish the case $a_2 < 0$ and $a_2 \geq 0$. Before we do this we want to notice, that since $r_1 \in (3/2, 3)$ we have:

$$\nabla^2 w \in L^{r_1}(\mathbb{R}^2) \quad \text{and} \quad \nabla^2 w \in L^{r_2}(\mathbb{R}^2),$$

since $3r_1/(3 - r_1) = 3r/(3 - 2r) = r_2$. This assures that, independently of the signum of a_2, we have:

$$w_{|x_2=0} \in \dot{W}^{1-1/r_2}_{r_2}(\mathbb{R}) \cap \dot{W}^{2-1/r_1}_{r_1}(\mathbb{R}).$$

In case of $a_2 < 0$ we may additionally use Theorem 4.3.3 to obtain, that $w \in L^{r_1}(\mathbb{R})$.

Summarizing – the subtraction of inhomogeneity using vector field w sets $\widetilde{D} = D - w$ in the following function spaces:

- for $a_2 < 0$:

$$\widetilde{D} \in \sum_{s_1,s_2,s_3,s_4 \in \{r,r_1,r_2\}} (L^{s_1}(\mathbb{R}) + L^{s_2}(\mathbb{R})) \cap \dot{W}^{1-1/s_3}_{s_3}(\mathbb{R}) \cap \dot{W}^{2-1/s_4}_{s_4}(\mathbb{R}),$$

- for $a_2 \geq 0$:

$$\widetilde{D} \in \sum_{s_3,s_4 \in \{r,r_1,r_2\}} \dot{W}^{1-1/s_3}_{s_3}(\mathbb{R}) \cap \dot{W}^{2-1/s_4}_{s_4}(\mathbb{R}),$$

with appropriate estimates.

We are now in position to obtain estimates on $\nabla^2 u$. We proceed as in the case of ∇p, i.e. we estimate particular parts of $\nabla^2 v$ by a proper part of the norm of $\widetilde{D}$. For example, in case $a_2 < 0$ estimate of $u_{,22}$ would look like follows: we recall I_1 and I_2 from (4.48). Since I_1 can be estimated by the L_p-norm of $\widetilde{D}$ and I_2 can be estimated by the $\dot{W}^{2-1/p}_p$-norm of $\widetilde{D}$, then for the part of $\widetilde{D}$, which belongs to, say, $(L^{r_1} + L^{r_2}) \cap \dot{W}^{1-1/r_1}_{r_1} \cap \dot{W}^{2-1/r}_r$ we get an estimate for $u_{,22}$ in the space $L^r + (L^{r_1} + L^{r_2})$. Similarly we may estimate other terms. The Reader immediately notice, that in the case $a_2 = 0$ exactly the same procedure works, since all necessary requirements on $\widetilde{D}$ are satisfied. We may thus summarize this with the following inequality:

$$\|\nabla^2 u\|_{L^r(\mathbb{R}^2_+)+L^{r_1}(\mathbb{R}^2_+)+L^{r_2}(\mathbb{R}^2_+)} \leq \sum_{s_1,s_2,s_3,s_4 \in \{r,r_1,r_2\}} \|\widetilde{D}\|_{(L^{s_1}(\mathbb{R})+L^{s_2}(\mathbb{R})) \cap \dot{W}^{1-1/s_3}_{s_3}(\mathbb{R}) \cap \dot{W}^{1-1/s_4}_{s_4}(\mathbb{R})}, \tag{4.69}$$

which we shown to be valid for $a_2 \leq 0$.

For $a_2 > 0$ we encounter a small obstacle, namely during estimates we need the $\dot{W}^{2-2/p}_p$-norm, which does not explicitly appear in the norm of $\widetilde{D}$. To deal with this we notice, that the $\dot{W}^{2-2/p}$-norm is required in terms, which come from the multiplication in a Fourier space by a smooth function with bounded support (see for example I_1 from (4.44)). Once this is known we can use Lemma 4.5.5 to estimate the $\dot{W}_s^{2-2/s}$-norm with the $\dot{W}_s^{1-1/s}$-norm, which in our case might be written as:

$$\|\widetilde{D}\|_{\dot{W}_{s_1}^{2-2/s_1}(\mathbb{R})\cap\dot{W}_{s_2}^{2-1/s_2}(\mathbb{R})} \leq \|\widetilde{D}\|_{\dot{W}_{s_1}^{1-1/s_1}(\mathbb{R})\cap\dot{W}_{s_2}^{2-1/s_2}(\mathbb{R})},$$

where $s_1, s_2 \in \{r, r_1, r_2\}$. Once we have estimate of this norm we may estimate terms in case $a_2 > 0$ in an exactly the same way it was made earlier to obtain, that (4.69) is valid also for $a_2 > 0$.

Summarizing, we have proved the following inequality:

$$\|\nabla^2 u\|_{L^r(\mathbb{R}^2_+)+L^{r_1}(\mathbb{R}^2_+)+L^{r_2}(\mathbb{R}^2_+)} \leq C \sum_{s_1,s_2,s_3,s_4\in\{r,r_1,r_2\}} \|\widetilde{D}\|_{(L^{s_1}(\mathbb{R})+L^{s_2}(\mathbb{R}))\cap\dot{W}_{s_3}^{1-1/s_3}(\mathbb{R})\cap\dot{W}_{s_4}^{1-1/s_4}(\mathbb{R})},$$

which, together with previous estimates, gives us the following inequality for the solution (v, q) to the system (4.12)-(4.16):

$$\|\nabla q\|_{L^r(\mathbb{R}^2_+)+L^{r_1}(\mathbb{R}^2_+)} + \|\nabla^2 v\|_{L^r(\mathbb{R}^2_+)+L^{r_1}(\mathbb{R}^2_+)+L^{r_2}(\mathbb{R}^2_+)} \leq C\left(\|F\|_{L^r(\mathbb{R}^2_+)} + \|G\|_{W^1_r(\mathbb{R}^2_+)} + \|\underline{b}\|_{W_r^{1-1/r}(\mathbb{R})} + \|\underline{d}\|_{W_r^{2-1/r}(\mathbb{R})}\right).$$

We now recall the fact, that $r \in (1, 3/2)$, which implies that $r_1 = 3r/(3-r) > 3/2$ and $r_2 = 3r/(3-2r) > 3$. We also know, that the support of q and v is compact, since this came from the localization procedure, hence L^{r_1} and L^{r_2} norm majorize L^r norm, with a coefficient dependent only on the size of the support of q and v, thus the following inequality holds:

$$\|\nabla q\|_{L^r(\mathbb{R}^2_+)} + \|\nabla^2 v\|_{L^r(\mathbb{R}^2_+)} \leq C\left(\|F\|_{L^r(\mathbb{R}^2_+)} + \|G\|_{W^1_r(\mathbb{R}^2_+)} + \|\underline{b}\|_{W_r^{1-1/r}(\mathbb{R})} + \|\underline{d}\|_{W_r^{2-1/r}(\mathbb{R})}\right).$$

This estimate allows us to complete the proof of Theorem 4.1.1, since, as we have shown earlier, this proof requires estimates in the whole space, which is guaranteed due to Theorem 4.3.4, and local estimates near the boundary, which we have just proved. Thus, the proof of Theorem 4.1.1 is completed.

4.5 Appendix

In this section we give statements of lemmas and theorems, which were used in proofs of the previous results. The following two Theorems are extensively used in our paper. The first one is due to Marcinkiewicz:

Theorem 4.5.1. *Suppose that the function $\Phi : \mathbb{R}^m \to \mathbb{C}$ is smooth enough and there exists $M > 0$ such that for every point $x \in \mathbb{R}^m$ we have*

$$|x_{j_1}\ldots x_{j_k}| \left|\frac{\partial^k \Phi}{\partial x_{j_1}\ldots\partial x_{j_k}}\right| \leq M, \quad 0 \leq k \leq m, 1 \leq j_1 < \ldots < j_k \leq m.$$

Then the operator

$$Pg(x) = (2\pi)^{-m}\int_{\mathbb{R}^m} dy e^{ix\cdot y}\Phi(y)\int_{\mathbb{R}^m} e^{-iy\cdot z}g(z)dz$$

is bounded in $L_p(\mathbb{R}^m)$ and

$$\|Pg\|_{L_p(\mathbb{R}^m)} \leq A_{p,m}M\|g\|_{L_p(\mathbb{R}^m)}$$

The next theorem is due to Lizorkin:

Theorem 4.5.2. *Let*

$$Tu \equiv h(x) = \frac{1}{2\pi} \int_{R^2} e^{ix\cdot\xi} \Phi(\xi) \hat{u}(\xi) d\xi, \tag{4.70}$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ *is continuous together with the derivatives*

$$\frac{\partial \Phi}{\partial \xi_1}, \frac{\partial \Phi}{\partial \xi_2}, \frac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2},$$

for $|\xi_i| > 0$, $i = 1, 2$. *Then, if for some* $\beta \in [0, 1)$ *and* $M > 0$

$$|\xi_1|^{\kappa_1+\beta} |\xi_2|^{\kappa_2+\beta} \left| \frac{\partial^\kappa}{\partial \xi_1^{\kappa_1} \partial \xi_2^{\kappa_2}} \right| \le M,$$

where κ_i *is zero or one and* $\kappa = \kappa_1 + \kappa_2$, *the integral transform (4.70) defines a bounded linear operator from* $L^q(\mathbb{R}^2)$ *into* $L^r(\mathbb{R}^2)$, $1 < q < \infty$, $1/r = 1/q - \beta$, *and we have:*

$$\|Tu\|_{L^r} \le C\|u\|_{L^q},$$

with a constant $C = c(q, r)M$.

The following Lemma allows us to estimate a homogeneous norm of a function on a boundary:

Lemma 4.5.3. *Let* $f \in L^s(\mathbb{R}^2)$ *and* $\nabla f \in L^m(\mathbb{R}^2)$. *For* $s \in (1, 2]$ *we assume* $m \in (1, s)$, *and for* $s > 2$ *we assume* $m \in (\frac{2s}{2+s}, s)$. *Then* $f_{|x_2=0} \in \dot{W}_m^{1-1/m}(\mathbb{R})$ *and the following inequality holds:*

$$\|f\|_{\dot{W}_m^{1-1/m}(\mathbb{R})} \le C\|\nabla f\|_{L^m(\mathbb{R}^2)}.$$

Proof . We construct a sequence of functions, which converge to f appropriately and their trace is in a proper function space. Let us introduce a smooth cut-off function $\eta(x)$ such that: $\eta(x) = 1$ for all $x \in B(0, 1)$ and $\eta(x) = 0$ for all $x \in \mathbb{R}^2 \setminus B(0, 2)$, together with sequence of cut-off functions $\eta_k(x)$, defined as $\eta_k(x) = \eta(x/k)$.

Let $f_k(x) = f(x)\eta_k(x)$. Since $\eta_k(x)$ has a bounded support we have $f_k(x) \in W_m^1(\mathbb{R}^2)$ and hence $f_k(x)_{|x_2=0} \in W_m^{1-1/m}(\mathbb{R})$. Moreover, from the standard scaling argument it is easy to see, that $\|f_k(x)_{|x_2=0}\|_{\dot{W}_m^{1-1/m}(\mathbb{R})} \le \|\nabla f_k(x)\|_{L^m(\mathbb{R}^2)}$.

Of course $f_k \to f$ in $L^s(\mathbb{R}^2)$ as $k \to \infty$. To prove our lemma we need to show that $f_{k|x_2=0}$ is a Cauchy sequence in $\dot{W}_m^{1-1/m}(\mathbb{R})$. From the definition of f_k we get:

$$\begin{aligned} \|f_k - f_l\|_{\dot{W}_m^{1-1/m}(\mathbb{R})} &\le \|\nabla f_k - \nabla f_l\|_{L^m(\mathbb{R}^2)} \\ &\le \|\nabla f(\eta_k - \eta_l)\|_{L^m(\mathbb{R}^2)} + \|f\nabla(\eta_k - \eta_l)\|_{L^m(\mathbb{R}^2)}. \end{aligned}$$

The first term on the right hand side is obviously small for large k and l. The second is also small for k and l large enough, since

$$\|f\nabla(\eta_k - \eta_l)\|_{L^m(\mathbb{R}^2)} \le \|f\|_{L^s(\mathbb{R}^2)} \|\nabla(\eta_k - \eta_l)\|_{L^{ms/(s-m)}(\mathbb{R}^2)},$$

and $\|\nabla \eta_k\|_{L^{ms/(s-m)}(\mathbb{R}^2)} \to 0$ as $k \to \infty$. Indeed, $|\mathrm{supp} \nabla \eta_k| \sim k^2$ and $|\nabla \eta_k| \sim 1/k$, hence $\|\nabla \eta_k\|_{L^{ms/(s-m)}(\mathbb{R}^2)} \le C(\eta) k^{(2-ms/(s-m))/r} \to 0$ as $k \to \infty$, since under our assumptions $2 - ms/(s-m) < 0$. □

We use the following lemma to set a function space, where the trace of a function belongs to:

Lemma 4.5.4. *Let $f \in L^{p_1}(\mathbb{R}^2_+)$ and $\nabla f \in L^{p_2}(\mathbb{R}^2_+)$, then $f_{|x_2=0} \in L^{p_1}(\mathbb{R}) + L^{p_2}(\mathbb{R})$ and the following estimate is valid:*

$$\|f_{|x_2=0}\|_{L^{p_1}(\mathbb{R})+L^{p_2}(\mathbb{R})} \le C(\|f\|_{L^{p_1}(\mathbb{R}^2_+)} + \|\nabla f\|_{L^{p_1}(\mathbb{R}^2_+)}).$$

Proof . Introducing a smooth cut-off function $\eta(x_2)$ such that $\eta(x_2) = 1$ for $x_2 < 1$ and $\eta(x_2) = 0$ for $x_2 > 2$ we can write:

$$f(0,x') = \eta(0)f(0,x') = \int_0^2 \partial_{x_2}(\eta(s)f(s,x'))ds = \int_0^2 \eta'(s)f(s,x')ds + \int_0^2 \eta\partial_{x_2}f(s,x')ds.$$

This proves, that $f(0,x')$ is a sum of two functions from $L^{p_1}(\mathbb{R}^2_+)$ and $L^{p_2}(\mathbb{R}^2_+)$, which completes the proof of the Lemma. □

The following Lemma is substantial to estimate higher homogeneous norms of a function with a bounded support in Fourier space:

Lemma 4.5.5. *Let $f \in \dot{W}^s_r(\mathbb{R})$, $s \notin \mathbb{Z}$. Given a smooth function $\pi(k)$ such that $\pi(k) = 1$ for $|k| \le L$ and $\pi(k) = 0$ for $|k| \ge L+1$. Then $\mathcal{F}_k^{-1}\left(\pi(k)\hat{f}\right) \in \dot{W}^{s+\epsilon}_r(\mathbb{R})$ and the following inequality holds:*

$$\|\mathcal{F}_k^{-1}\left(\pi(k)\hat{f}\right)\|_{\dot{W}^{s+\epsilon}_r(\mathbb{R})} \le C(\epsilon)\|f\|_{\dot{W}^s_r(\mathbb{R})},$$

where $\epsilon > 0$ is an arbitrary positive constant.

Proof . In case of $s \notin \mathbb{Z}$ we have $\dot{W}^s_r(\mathbb{R}) = \dot{B}^s_{r,r}(\mathbb{R})$ ([37]), where $\dot{B}^s_{r,r}(\mathbb{R})$ stands for the homogeneous Besov space equipped with a norm:

$$\|f\|_{\dot{B}^s_{r,r}(\mathbb{R})} = \left(\sum_{j=-\infty}^{\infty} 2^{jsr}\left\|\mathcal{F}_k^{-1}\left(\varphi_j\hat{f}\right)\right\|^r_{L^r(\mathbb{R})}\right)^{1/r}, \tag{4.71}$$

where $\{\varphi_j\}_{j=-\infty}^{\infty}$ is a set of smooth functions, each of them of bounded support supp $\varphi_j \subset \{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}$ and such that $\sum_{j=-\infty}^{\infty}\varphi_j(\xi) = 1$ for every $\xi \in \mathbb{R}\setminus\{0\}$ (see [37]).

Multiplication by the function π implies that the sum in (4.71), corresponding to the function $\mathcal{F}_k^{-1}\left(\pi\hat{f}\right)$, has infinite number of elements with nonpositive j, and finite number of elements of elements with positive j. Without loss in generality we can assume, that $L > 1$. Then, in the case of negative j we have:

$$\sum_{j=-\infty}^{0} 2^{j(s+\epsilon)r}\|\mathcal{F}_k^{-1}\left(\varphi_j\pi\hat{f}\right)\|^r_{L^r(\mathbb{R})} = \sum_{j=-\infty}^{0} 2^{j\epsilon r}2^{jsr}\|\mathcal{F}_k^{-1}\left(\varphi_j\pi\hat{f}\right)\|^r_{L^r(\mathbb{R})} \le$$

$$\le \sum_{j=-\infty}^{0} 2^{jsr}\|\mathcal{F}_k^{-1}\left(\varphi_j\pi\hat{f}\right)\|^r_{L^r(\mathbb{R})} = \sum_{j=-\infty}^{0} 2^{jsr}\|\mathcal{F}_k^{-1}\left(\varphi_j\hat{f}\right)\|^r_{L^r(\mathbb{R})} \le \|f\|^r_{\dot{B}^s_{r,r}},$$

since $\pi(\xi) = 1$ for $\xi \in \cup_{j=-\infty}^{0}$supp φ_j. Remaining terms (finite number) can be estimated using Marcinkiewicz theorem:

$$\sum_{j=1}^{\lceil 1+\log_2(L+1)\rceil} 2^{j(s+\epsilon)r}\|\mathcal{F}_k^{-1}\left(\varphi_j\pi\hat{f}\right)\|^r_{L^r(\mathbb{R})} \le$$

$$\sum_{j=1}^{\lceil 1+\log_2(L+1)\rceil} 2^{j\epsilon r}C(\pi)2^{jsr}\|\mathcal{F}_k^{-1}\left(\varphi_j\hat{f}\right)\|^r_{L^r(\mathbb{R})} \le (2L+2)^{\epsilon r}C(\pi)\|f\|^r_{\dot{B}^s_{r,r}(\mathbb{R})}.$$

This completes the proof of the following inequality:

$$\left\| \mathcal{F}_k^{-1} \left(\pi \hat{f} \right) \right\|_{\dot{W}_r^{s+\epsilon}(\mathbb{R})} \leq C(\pi)(2L+3)^{\epsilon} \|f\|_{\dot{W}_r^{s}(\mathbb{R})},$$

and the proof of Lemma 4.5.5. □

Chapter 5

A new approach to study asymptotic behaviour of a fluid in $\mathbb{R}^2$

5.1 Introduction

We investigate the following problem in $\mathbb{R}^2$:

$$u \cdot \nabla u - \Delta u - \nabla p = F \qquad \text{in } \mathbb{R}^2, \tag{5.1}$$

$$\text{div } u = 0 \qquad \text{in } \mathbb{R}^2, \tag{5.2}$$

$$u \to u_\infty \qquad \text{as } |x| \to \infty, \tag{5.3}$$

where a velocity vector field u and a pressure p are unknown functions, F is a given exterior force and u_∞ is a prescribed vector field.

This problem is strictly connected with a problem of a flow around an obstacle, i.e. considered in a domain Ω which is the exterior of a compact set $\mathcal{B}$. In such case one need to add boundary constraints on the boundary of the domain, let say $B(u) = 0$. In a case of Dirichlet boundary conditions, i.e. $u = u_*$ on $\partial\Omega$, provided u_* satisfies zero-outflow condition $\int_{\partial\Omega} u_* \cdot \vec{n} = 0$ and $\Omega \neq \mathbb{R}^2$ there is a classical work of Leray [24], where he shows, that there exists a smooth solution (u, p) with finite Dirichlet integral, i.e.

$$\int_\Omega |\nabla v|^2 dx \leq M, \tag{5.4}$$

for some constant M, but it is unknown, whether (5.3) holds or not. This is connected with the fact, that the power 2 coincides with the dimension 2, which does not allow to use embedding theorems in Sobolev spaces. In fact, condition (5.4) does not even assure that $v \in L^\infty(\Omega)$.

There are some partial, but very interesting results to this problem, but we do not address them here – for more information we refer the Reader to [12].

In this chapter we consider the case of a full plane flow with a goal to develop new methods to deal with a problem in exterior domain. Similar techniques were used by Wittwer in [38], however for a symmetric flow. In [16] authors omitted this condition, however these techniques are still very technical. Our approach are much simpler – we do not consider system for the rotation of the fluid, but we operate on the velocity vector field itself. Moreover, we give analysis of the flow not only in the half plane, but in the full plane. In such case one can obtain not only asymptotic behaviour behind the obstacle, but also in front of it. The class of functions, where we look for a solution, is different from standard Sobolev spaces. This is due to the fact, that our analysis is carried through in a Fourier space only in one direction. Of course, one may interpret these results in Sobolev spaces, but we do not address this question here.

Throughout the chapter we use the following Banach spaces:

Definition 5.1.1. *Let $\beta \in \mathbb{R}$. Space $\mathcal{X}_\beta$ consists of these functions, for which the following norm is finite:*

$$\|a\|_{\mathcal{X}_\beta} = \sup_{(t,\xi)\in\mathbb{R}^2} (1+|t\xi^2|)^\beta |a(t,\xi)|.$$

Function space $\mathcal{Y}$ consists of these functions, for which the following norm is finite:

$$\|b\|_{\mathcal{Y}} = \sup_{(t,\xi)\in\mathbb{R}^2} |t|^{1/2}|b(t,\xi)|.$$

The technique used in this work is to consider parallel to u_∞ coordinate as time and then use the Fourier transform to obtain system of ordinary differential equations, which can be solved and analyzed to obtain information about its asymptotic behaviour.

Taking (5.3) into account we may introduce a new vector field $v = u_\infty + u$, for which we have the following system:

$$u_\infty v + v\cdot\nabla v - \Delta v - \nabla p = F \qquad \text{in } \mathbb{R}^2, \tag{5.5}$$

$$\text{div } v = 0 \qquad \text{in } \mathbb{R}^2, \tag{5.6}$$

$$v \to 0 \qquad \text{as } |x| \to \infty. \tag{5.7}$$

Our main result states the following:

Theorem 5.1.2. *Considering system (5.1)-(5.3). Given F and u_∞ such that $\|\hat{F}/|\xi|\|_{\mathcal{Y}}$ and $|u_\infty|$ are small enough. Then there exists a solution (u,p) such that $\hat{u} \in \mathcal{X}_\beta$, $\hat{p} \in \mathcal{Y}$ and the following estimate is valid:*

$$\|\hat{u}\|_{\mathcal{X}_\beta} + \|p\|_{\mathcal{Y}} \le C(u_\infty^{-2\beta+1/2} + \|\hat{F}/|\xi|\|_{\mathcal{Y}}).$$

The proof of this theorem is given in Section 5.3.1. Our approach to show existence of such solution is the following: first we split this problem into two auxiliary ones, define a proper mapping, for which the solution to our problem is a fixed point. Then we give suitable estimates to show, that the mapping is a contraction. The last step, together with the Banach fixed point theorem, gives us existence of the solution.

The chapter is organized as follows: in the next section we investigate auxiliary problems, which have been introduce earlier. First, we derive a solution by means of the Fourier transform. Then we provide suitable estimates, which play fundamental role in the proof of existence of a solution to the main problem (5.5)-(5.7). This proof is a subject of the proceeding section, which is followed by a part, where we show some basic asymptotic properties, together with a discussion about the presence of a wake region behind an obstacle. Section 5.4 is devoted to technical lemmas, which play fundamental role in mentioned estimates.

5.2 Auxiliary systems.

In this section we introduce two mentioned auxiliary systems. The first one is for the pressure. Taking div from (5.5) we have:

$$\Delta p = \text{div } F - \text{div div } (v \otimes v), \tag{5.8}$$

since div $v = 0$ and $v \cdot \nabla v = \text{div } (v \otimes v)$. We introduce function G as the right hand side of (5.8), i.e.

$$G = \text{div } F - \text{div div } (v \otimes v).$$

The second one is for the velocity. We transform (5.5) into the following system:

$$u_\infty v_{,1} - \Delta v = \nabla p - \operatorname{div}(v \otimes v) \qquad \text{in } \mathbb{R}^2, \tag{5.9}$$

$$v \to 0 \qquad \text{as } |x| \to \infty. \tag{5.10}$$

As earlier – we introduce function H as the right hand side of (5.9), i.e.

$$H = F + \nabla p - \operatorname{div}(v \otimes v).$$

Our mapping used in fixed point theorem will be considered as follows: we start with v in a proper Banach space, then we calculate pressure p, from (5.8). Having v and p we may use (5.9) to calculate new $\tilde{v}$. We show that this mapping maps a ball small enough into itself assuring, that there exists v for which $\tilde{v} = v$.

In this following sections we deal with our two problems (5.8) and (5.9) using the Fourier Transform in x_2 space variable and transforming them into ordinary differential equations. Similar procedure was used in [38], however not for the velocity directly, but for the rotation of the fluid.

5.2.1 Derivation of the solution.

Let us focus on (5.8) first. Taking Fourier transform in x_2 variable and denoting the new variable as ξ and x_1 as t we get:

$$\ddot{\widehat{p}} - \xi^2 \widehat{p} = \widehat{G}. \tag{5.11}$$

For the simplicity we omit the hat$\widehat{\ }$.

Introducing $w = \dot{p}$ we can rewrite (5.11) as:

$$\dot{\begin{bmatrix} p \\ w \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ \xi^2 & 0 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix}. \tag{5.12}$$

Eigenvalues and eigenvectors can be easily computed: $\lambda_1 = -|\xi|$, $\lambda_2 = |\xi|$ and $\varphi_1 = [-1/|\xi|, 1]$, $\varphi_2 = [1/|\xi|, 1]$. Introducing matrix $P = [\varphi_1, \varphi_2]$, i.e.

$$P = \begin{bmatrix} -\frac{1}{|\xi|} & \frac{1}{|\xi|} \\ 1 & 1 \end{bmatrix},$$

and new variables $[U_1, U_2] = P^{-1}[p, w]$ we rewrite (5.12) as:

$$\dot{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}} = \begin{bmatrix} -|\xi| & 0 \\ 0 & |\xi| \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + P^{-1} \begin{bmatrix} 0 \\ G \end{bmatrix}$$

A solution to this system is:

$$U_1(t,\xi) = \frac{1}{2} \int_{-\infty}^{t} e^{|\xi|(s-t)} G(s) ds$$

$$U_2(t,\xi) = -\frac{1}{2} \int_{t}^{\infty} e^{-|\xi|(s-t)} G(s) ds.$$

which gives us:

$$p(t,\xi) = -\frac{1}{2}\frac{1}{|\xi|} \left(\int_{-\infty}^{t} e^{-|\xi||t-s|} G(s,\xi) ds + \frac{1}{2} \int_{t}^{\infty} e^{-|\xi||t-s|} G(s,\xi) ds, \right) \tag{5.13}$$

$$= -\frac{1}{2|\xi|} \int_{-\infty}^{+\infty} e^{-|\xi||t-s|} G(s,\xi) ds. \tag{5.14}$$

For the equation 5.9 we proceed similarly, i.e. we apply the Fourier Transform in x_2 direction. The corresponding system of ordinary differential equations is

$$\begin{aligned}\dot{v}(t,\xi) &= w(t,\xi)\\ \dot{w}(t,\xi) &= u_\infty w(t,\xi) + \xi^2 v(t,\xi) - H(t,\xi).\end{aligned}$$

Proceeding as earlier we can solve this system using diagonalization. Introducing

$$\Delta = \sqrt{u_\infty^2 + 4\xi^2}, \quad \lambda_1 = \frac{1}{2}(u_\infty - \sqrt{u_\infty^2 + 4\xi^2}), \quad \lambda_2 = \frac{1}{2}(u_\infty + \sqrt{u_\infty^2 + 4\xi^2})$$

we may write a solution v as follows:

$$v(t,\xi) = -\frac{1}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}H(s,\xi)ds + \frac{1}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}H(s,\xi)ds \tag{5.15}$$

Let us now focus on detailed information about $G(s)$ and $H(s)$. We start with the former. Since

$$G(s) = \mathcal{F}_{x_2}(\text{div } F - \text{div div } (v\otimes v))$$

we see that

$$G(s,\xi) = \partial_s\widehat{F_1}(s,\xi) + i\xi\widehat{F_2}(s,\xi) - \partial_s^2\widehat{(v_1^2)} + 2\partial_s i\xi\widehat{(v_1 v_2)} - \xi^2\widehat{(v_2^2)}.$$

First we integrate by parts the term from (5.13) corresponding to $\partial_s^2\widehat{(v_1^2)}$, namely:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|\xi||t-s|}\partial_s^2\widehat{(v_1^2)}ds &= -\int_{-\infty}^{t}|\xi|e^{-|\xi|(t-s)}\partial_s\widehat{(v_1^2)}ds + \partial_s\widehat{(v_1^2)}(t)\\ &\quad -\int_{t}^{\infty}(-|\xi|)e^{-|\xi|(s-t)}\partial_s\widehat{(v_1^2)}ds - \partial_s\widehat{(v_1^2)}(t)\\ &= \int_{-\infty}^{t}(|\xi|^2)e^{-|\xi|(t-s)}\widehat{(v_1^2)}ds - |\xi|\widehat{(v_1^2)}(t)\\ &\quad -\int_{t}^{\infty}(-|\xi|^2)e^{-|\xi|(s-t)}\widehat{(v_1^2)}ds - |\xi|\widehat{(v_1^2)}(t)\\ &= \int_{-\infty}^{\infty}|\xi|^2 e^{-|\xi||t-s|}\widehat{(v_1^2)}ds - 2|\xi|\widehat{(v_1^2)}(t),\end{aligned}$$

hence

$$\frac{1}{2|\xi|}\int_{-\infty}^{\infty} e^{-|\xi||t-s|}\partial_s^2\widehat{(v_1^2)}ds = \frac{1}{2}\int_{-\infty}^{\infty}|\xi|e^{-|\xi||t-s|}\widehat{(v_1^2)}ds - \widehat{(v_1^2)}(t). \tag{5.16}$$

In the same manner, terms from (5.13) corresponding to $2i\xi\partial_s\widehat{(v_1v_2)}$, $\xi^2\widehat{(v_2^2)}$ are of the same structure as the first term on the right hand side of (5.16). Similarly $\partial_s\widehat{F_1}$ and $i\xi\widehat{F_2}$ can be considered as one term. Summarizing, p can be presented as

$$\begin{aligned}p(t,\xi) = -\frac{1}{2}\int_{-\infty}^{\infty}|\xi|e^{-|\xi||t-s|}\left(\sum_{ij} c_{ij}\widehat{(v_iv_j)}\right)ds\\ -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|\xi||t-s|}\left(\sum_i b_i\widehat{F_i}\right)ds + \widehat{(v_1)^2}(t),\end{aligned} \tag{5.17}$$

for some constants c_{ij} and b_i such that $|c_{ij}| = |b_i| = 1$ (which is irrelevant for our purposes).

The same calculations can be repeated for v and H, i.e. for (5.15) to get that

$$v_1(t,\xi) = \frac{\lambda_1}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}(\widehat{p(\xi,s) - v_1^2})ds - \frac{\lambda_2}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}(\widehat{p(\xi,s) + v_1^2})ds$$
$$+ \frac{i\xi}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}(\widehat{(v_1v_2)} - \frac{\hat{F}_1}{i\xi})ds + \frac{i\xi}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}(\widehat{(v_1v_2)} + \frac{\hat{F}_1}{i\xi})ds, \quad (5.18)$$

and

$$v_2(t,\xi) = \frac{\lambda_1}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}\widehat{(v_1v_2)}ds - \frac{\lambda_2}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}\widehat{(v_1v_2)}ds$$
$$+ \frac{i\xi}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}(\widehat{(p - v_2^2)} - \frac{\hat{F}_2}{i\xi})ds + \frac{i\xi}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}(\widehat{(p - v_2^2)} + \frac{\hat{F}_2}{i\xi})ds.$$

5.2.2 Main estimates.

We are now in position to formulate Lemmas which play fundamental role in showing, that our mapping is a contraction.

Lemma 5.2.1. *Let $\widehat{v} \in \mathcal{X}_\beta$, such that $\|\widehat{v}\|_{\mathcal{X}_\beta} \leq M$, and $\widehat{F}/\xi \in \mathcal{Y}$, such that $\|\widehat{F}/\xi\|_{\mathcal{Y}} \leq N_F$. Given p in the form:*

$$\widehat{p}(t,\xi) = -\frac{1}{2}\int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|}\left(\sum_{ij} c_{ij}\widehat{(v_iv_j)}\right) ds$$
$$- \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\xi||t-s|}\left(\sum_{i} b_i\widehat{F}_i\right) ds + \widehat{(v_1)^2}(t). \quad (5.19)$$

Then $\widehat{p} \in \mathcal{Y}$ and

$$\|\widehat{p}\|_{\mathcal{Y}} \leq Cu_\infty^{-2\beta+1/2}M^2 + N_F,$$

for some constant C independent of u_∞ and F.

Proof . To prove this Lemma we extensively use results from Section 5.4, that is: recalling that $|c_{ij}| = |b_i| = 1$ we estimate $\widehat{p}$ as follows:

$$|\widehat{p}(t,\xi)| \leq \frac{1}{2}\int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|}\left(\sum_{ij} |\widehat{(v_iv_j)}|\right) ds$$
$$+ \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\xi||t-s|}\left(\sum_{i} |\widehat{F}_i|\right) ds + |\widehat{(v_1)^2}|(t). \quad (5.20)$$

Since $\widehat{v} \in \mathcal{X}_\beta$ we use Lemma 5.4.2 to get that:

$$\|\widehat{(v_iv_j)}\|_{\mathcal{Y}} = \|\widehat{v_i} * \widehat{v_j}\|_{\mathcal{Y}} \leq Cu_\infty^{-2\beta+1/2}M^2.$$

It estimates the last term on the right hand side of (5.20), but also, with a help of Lemma 5.4.3, gives us the following estimate:

$$\left\|\frac{1}{2}\int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|}\left(\sum_{ij} |\widehat{(v_iv_j)}|\right) ds\right\|_{\mathcal{Y}} \leq 4Cu_\infty^{-2\beta+1/2}M^2.$$

To finish estimate of $\widehat{p}$ we present the remaining term and estimate it as follows

$$\left\| \frac{1}{2}\int_{-\infty}^{\infty} e^{-|\xi||t-s|}\left(\sum_i |\widehat{F}_i|\right) ds \right\|_{\mathcal{Y}} \leq \left\| \frac{1}{2}\int_{-\infty}^{\infty} |\xi| e^{-|\xi||t-s|}\left(\sum_i |\widehat{F}_i|/|\xi|\right) ds \right\|_{\mathcal{Y}} \leq \|\widehat{F}_i/|\xi|\|_{\mathcal{Y}} = N,$$

where again we used Lemma 5.4.3. □

The second Lemma we need to proof is the following:

Lemma 5.2.2. *Let $\widehat{w} \in \mathcal{X}_\beta$, such that $\|\widehat{w}\|_{\mathcal{X}_\beta} \leq M$, and $\widehat{p} \in \mathcal{Y}$, such that $\|\widehat{p}\|_{\mathcal{Y}} \leq N$. Then, for the following terms v_1 and v_2:*

$$v_1(t,\xi) = \frac{\lambda_1}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}(\widehat{p(\xi,s) - w_1^2})ds - \frac{\lambda_2}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}(\widehat{p(\xi,s) + w_1^2})ds$$
$$+ \frac{i\xi}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}(\widehat{(w_1w_2)} - \frac{\hat{F}_1}{i\xi})ds + \frac{i\xi}{\Delta}\int_{-\infty}^{t} e^{-\lambda_2(s-t)}(\widehat{(w_1w_2)} + \frac{\hat{F}_1}{i\xi})ds, \quad (5.21)$$

and

$$v_2(t,\xi) = \frac{\lambda_1}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}\widehat{(w_1w_2)}ds - \frac{\lambda_2}{\Delta}\int_{t}^{\infty} e^{-\lambda_2(s-t)}\widehat{(w_1w_2)}ds$$
$$+ \frac{i\xi}{\Delta}\int_{-\infty}^{t} e^{-\lambda_1(s-t)}(\widehat{(p - w_2^2)} - \frac{\hat{F}_2}{i\xi})ds + \frac{i\xi}{\Delta}\int_{-\infty}^{t} e^{-\lambda_2(s-t)}(\widehat{(p - w_2^2)} + \frac{\hat{F}_2}{i\xi})ds. \quad (5.22)$$

the following estimate is valid:

$$\|v_i\|_{\mathcal{X}_\beta} \leq C(N + u_\infty^{-2\beta+1/2}M^2 + \|\hat{F}/\xi\|_{\mathcal{Y}}).$$

Proof . The proof is analogous to the proof of the previous lemma. First we notice, that

$$\|\widehat{(w_iw_j)}\|_{\mathcal{Y}} = \|\widehat{w_i} * \widehat{w_j}\|_{\mathcal{Y}} \leq Cu_\infty^{-2\beta+1/2}M^2,$$

like also $\|p\|_{\mathcal{Y}} \leq N$.

Then, we find, that in the form of v_i, i.e. in (5.21) and (5.22), these integrals in a sequence are respectively in the form of integrals $\tilde{B}$, $\tilde{D}$, $\tilde{A}$, $\tilde{C}$ from Lemma 5.4.4. Thus, applying this Lemma we obtain:

$$\|v_i\|_{\mathcal{X}_\beta} \leq C(\|p\|_{\mathcal{Y}} + \|\widehat{(w_iw_j)}\|_{\mathcal{Y}} + \|\hat{F}/\xi\|_{\mathcal{Y}}) \leq C(N + \|\hat{F}/\xi\|_{\mathcal{Y}}) + u_\infty^{-2\beta+1/2}M^2).$$

□

5.3 Main results.

In this section we gather our main results – existence and basic asymptotic behaviour of the fluid. They can be a subject to a more detailed analysis, for example one can consider an exterior domain, use a cut-off function to extend the system into the full plane and treat obtained additional terms as a force F. We do not address this problem here to maintain the simplicity of this part of the thesis and to emphasize the technique of the Fourier transform itself.

5.3.1 The proof of Theorem 5.1.2.

In this section we would like to prove existence of a solution to our problem. We use standard approach, namely Banach's fixed point theorem for a contraction mapping.

We recall, that our mapping is defined as follows: having a vector field w and force F we calculate the pressure p using formula (5.19), then we calculate a vector field v using p, w and formula (5.21)-(5.22). Let us denote this mapping as $G : \mathcal{X}_\beta \to \mathcal{X}_\beta$. First we would like to show, that there exists a constant ϵ such that for sufficiently small $\|F/|\xi|\|_{\mathcal{Y}}$ the mapping G maps a ball of radius ϵ in the space $\mathcal{X}_\beta$ into itself, namely:

$$G(\mathcal{B}_\alpha(\epsilon)) \subset \mathcal{B}_\alpha(\epsilon). \tag{5.23}$$

We take $w \in \mathcal{X}_\beta$ such that $\|w\|_{\mathcal{X}_\beta} \leq M$ and F such that $\|\hat{F}/|\xi|\|_{\mathcal{Y}} \leq N_F$. From Lemma 5.2.1 we have:

$$\|p\|_{\mathcal{Y}} \leq C u_\infty^{-2\beta+1/2} M^2 + N_F.$$

We may now use Lemma 5.2.2 to obtain:

$$\|v\|_{\mathcal{X}_\beta} \leq C(\|p\|_{\mathcal{Y}} + \|\hat{F}/\xi\|_{\mathcal{Y}} + u_\infty^{-2\beta+1/2} M^2) \leq C(u_\infty^{-2\beta+1/2} M^2 + N_F).$$

To find ϵ in (5.23) we have to solve an inequality:

$$C(u_\infty^{-2\beta+1/2} M^2 + N_F) \leq M.$$

We get that for $\epsilon = C^{-1} u_\infty^{2\beta-1/2}$ and $N_F \leq \frac{\epsilon - C u_\infty^2}{C}$ the mapping G maps a ball $\mathcal{B}_\alpha(\epsilon)$ into itself.

In a similar way we show, that on a smaller ball the mapping G is a contraction. For $w_1, w_2 \in \mathcal{B}_\beta(\epsilon/2)$ and corresponding v_1, v_2 we have:

$$\|v_1 - v_2\|_{\mathcal{X}_\beta} \leq C u_\infty^{-2\beta+1/2} \|w_1 - w_2\|^2_{\mathcal{X}_\beta} \leq \gamma \|w_1 - w_2\|_{\mathcal{X}_\beta},$$

where $\gamma < 1$. Using the Banach fixed point theorem we get, that there exists a vector field $v \in \mathcal{B}_\beta(\epsilon/2)$, such that $v = G(v)$.

5.3.2 Asymptotic behaviour.

In this section we would like to show different behaviour of the solution for $t < -1$ and for $t > 1$. These cases reflects situation behind the obstacle and in front of the obstacle, which is present within the region $t \in (-1, 1)$. We would like to show, that behind the obstacle one can observe a parabolic wake region.

We focus here on estimate of asymptotic behaviour of $\hat{v}_1(t,\xi)$. Our solution $v \in \mathcal{B}_\beta(\epsilon/2) \subset \mathcal{X}_\beta$ satisfies $\|v\|_{\mathcal{X}_\beta} \leq \epsilon/2$. The corresponding pressure is in $\mathcal{Y}$ satisfying a similar estimate. Thus, using (5.17) and (5.18) we may repeat reasoning from the existence section to estimate v_1 as follows:

$$\begin{aligned}|v_1(t,\xi)| \leq \frac{|\lambda_1|}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} |s|^{-1/2} ds + \frac{\lambda_2}{\Delta} \int_{t}^{\infty} e^{-\lambda_2(s-t)} |s|^{-1/2} ds \\ + \frac{|\xi|}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} |s|^{-1/2} ds + \frac{|\xi|}{\Delta} \int_{t}^{\infty} e^{-\lambda_2(s-t)} |s|^{-1/2} ds.\end{aligned} \tag{5.24}$$

We denote integrals on the right hand side of (5.24) as I_1, I_2, I_3, I_4 respectively.

The behaviour of integrals I_1 and I_3 is similar, thus we give estimates only for I_1. Similarly, I_2 and I_4 are in a strong correspondence, thus we consider here only I_2.

Let us assume now that $t > 1$. First we focus on integral I_2. Using a substitution $\lambda_2 s = u + t\lambda_2$ we may present I_2 as:

$$I_2 = \frac{1}{\Delta}\int_0^\infty e^{-u}|u + t\lambda_2|^{-1/2}\lambda_2^{1/2}du. \tag{5.25}$$

Now since $t > 1$ and $\lambda_2 \geq u_\infty$ we may estimate $|u + t\lambda_2|^{-1/2}$ by $(t\lambda_2)^{-1/2}$ and hence:

$$I_2 \leq \frac{1}{t^{1/2}\Delta} = \frac{1}{t^{1/2}\sqrt{u_\infty^2 + 4\xi^2}}.$$

The substantial thing is that for small $|\xi|$ one has $I_2(t,\xi) \sim Ct^{-1/2}$, while for large $|\xi|$ one has $I_2(t,\xi) \sim C(t|\xi|^2)^{-1/2}$.

Different behaviour is observed for I_1, namely:

$$I_1(t,\xi) = \frac{|\lambda_1|^{1/2}}{\Delta}\int_0^\infty e^{-u}|u + t\lambda_1|^{-1/2}du.$$

Since $t\lambda_1 \leq 0$ we split this integral into three parts:

$$I_1(t,\xi) = \int_0^{|t\lambda_1|/2} + \int_{|t\lambda_1|/2}^{2|t\lambda_1|} + \int_{2|t\lambda_1|}^\infty =: I_{11} + I_{12} + I_{13}.$$

We use similar estimates to those from Section 5.4. We write $a \sim b$ if there $a \leq c_1 b$ and $b \leq c_2 a$ for some positive constants c_1 and c_2. We have:

$$I_{11}(t,\xi) \sim \frac{|\lambda_1|^{1/2}}{\Delta}|t\lambda_1|^{-1/2}\int_0^{|t\lambda_1|/2} e^{-u}du = \frac{t^{-1/2}}{\Delta}(1 - e^{-|t\lambda_1|/2}),$$

hence for small $|t\lambda_1|$ we get:

$$I_{11}(t,\xi) \sim |t\xi^2|^{1/2},$$

while for large $|t\lambda_1|$ one has $I_{11}(t,\xi) \sim |t\xi^2|^{-1/2}$.
For $I_{12}(t,\xi)$ we proceed similarly to obtain:

$$I_{12}(t,\xi) \sim e^{-|t\lambda_1|}\sqrt{t\lambda_1},$$

thus $I_{12} \sim I_{11}$. For I_{13} one has:

$$I_{13}(t,\xi) = \frac{\lambda_1}{\Delta}\int_{2|t\lambda_1|}^\infty e^{-u}|u + t\lambda_1|^{-1/2} \sim (1 + |t\xi^2|)^{-1/2}, \tag{5.26}$$

since for small $|\xi|$ one has $I_{13}(t,\xi) \sim (1 + |t\lambda_1|)^{-1/2}$ and for large $|\xi|$ one has $I_{13}(t,\xi) \sim \frac{\lambda_1}{\Delta}(1 + |t\lambda_1|)^{-1/2} \sim (1 + |t\xi^2|)^{-1/2}$. Combining all these estimates for I_{11}, I_{12} and I_{13} we get:

$$v_1(t,\xi) \sim (1 + |t\xi^2|)^{-1/2}. \tag{5.27}$$

Note: We would like to emphasize, that since $t \geq 1$ and $\lambda_1 = (u_\infty - \sqrt{u_\infty^2 + 4\xi^2})$, then smallness of $|t\lambda_1|$ is strictly connected with the smallness of $|\xi|$.

For comparison we must now derive estimates for $|v_1(t,\xi)|$ for $t \leq -1$. Here we use the same notation as earlier. In integral I_2 from (5.25) we now have different behaviour, since $t\lambda_2 < 0$. We need to split it into three parts:

$$I_2(t,\xi) = \int_0^{|t\lambda_2|/2} + \int_{|t\lambda_2|/2}^{2|t\lambda_2|} + \int_{2|t\lambda_2|}^\infty =: I_{21} + I_{22} + I_{23}.$$

For I_{21} we have:

$$I_{21}(t,\xi) \sim |t\lambda_2|^{-1/2} \int_0^{|t\lambda_2|/2} e^{-u} du = |t\lambda_2|^{-1/2}(1 - e^{-|t\lambda_2|/2}).$$

Now since $\lambda_2 > u_\infty$ one has $I_{21} \sim |t\lambda_2|^{-1/2}$. For I_{22} we get:

$$I_{22} \sim e^{-|t\lambda_2|}\sqrt{|t\lambda_2|} \sim |t\lambda_2|^{-1/2}.$$

Integral I_{23} estimates easily:

$$I_{23} \leq |t\lambda_2|^{-1/2},$$

thus

$$I_2 \sim |t\lambda_2|^{-1/2}. \tag{5.28}$$

We finish our estimates considering I_1. Since $t < -1$ and $\lambda_1 < 0$ we have:

$$I_1 \sim \frac{\lambda_1^{1/2}}{\Delta}|t\lambda_1|^{-1/2} = |t\Delta|^{-1/2},$$

which, together with (5.28) and the fact, that $\lambda_2 \sim \Delta$, implies, that:

$$v_1(t,\xi) \sim |t\Delta|^{-1/2} \sim |t|^{-1/2}(1 + |\xi|)^{-1/2}. \tag{5.29}$$

Summary: The main difference in behaviour of v_1 in front of and behind the obstacle is the following: in front of the obstacle one may estimate v_1 by $|t|^{-1/2}$ for small $|\xi|$ and by $|t\xi|^{-1/2}$ for large $|\xi|$. Behind the obstacle one cannot obtain similar estimate, because integral I_1 cannot be estimated uniformly by $|t|^{-1/2}$, since $|\xi|^2$ can be chosen small enough to level the influence of $|t|^{-1/2}$ (see 5.26). This is strictly connected with the first eigenvalue λ_1, for which one has $\lambda_1 \sim |\xi|^2$ for small $|\xi|$, resulting in estimates, which strongly depend on the term $t|\xi|^2$ – this is the occurrence of the mentioned parabolic wake region behind the obstacle. Indeed, one may use the inverse Fourier transform to obtain, that in front of the obstacle:

$$\mathcal{F}_\xi^{-1}(v_1(t,\xi)) \sim |t|^{-1/2}\Phi_-(x), \tag{5.30}$$

for some function $\Phi_-(x)$, while behind the obstacle one has:

$$\mathcal{F}_\xi^{-1}(v_1(t,\xi)) \sim |t|^{-1/2}\Phi_+\left(\frac{x^2}{t}\right), \tag{5.31}$$

for some function $\Phi_+(x)$.

5.4 Main Lemmas

The main auxiliary lemma, which will be used many times is the following:

Lemma 5.4.1. *Given $\theta > 0$. Then the following estimate is valid:*

$$\int_0^\infty e^{-u}|u - \theta|^{-1/2} du \leq (1 + \theta)^{-1/2}. \tag{5.32}$$

Proof . Let $I = \int_0^\infty e^{-u}|u-\theta|^{-1/2}du$. We split this integral into three parts:

$$I = I_1 + I_2 + I_3 = \int_0^{\theta/2} + \int_{\theta/2}^{2\theta} + \int_{2\theta}^{\infty},$$

and estimate them separately. Let us focus on I_1:

$$\begin{aligned} I_1 &= \int_0^{\theta/2} e^{-u}|u-\theta|^{-1/2}du \\ &\leq 2\int_0^{\theta/2} e^{-u}|\theta|^{-1/2}du = 2|\theta|^{-1/2}(1-e^{-\theta}) \\ &\leq (1+\theta)^{-1/2}, \end{aligned}$$

since $(1-e^{-\theta}) \sim \theta/(1+\theta)$.

For I_2 we proceed as follows:

$$I_2 = \int_{\theta/2}^{2\theta} e^{-u}|u-\theta|^{-1/2}du \leq C\int_0^{2\theta} e^{-\theta}|u|^{-1/2}du \leq Ce^{-\theta}|\theta|^{1/2} \leq \frac{C}{(1+\theta)^{1/2}},$$

which is a desired estimate.

Integral I_3 we estimate for large and small θ ($\theta > 1$ and $\theta < 1$ respectively). In the first case

$$I_3 = \int_\theta^1 + \int_1^\infty = (1-\theta^{1/2}) + C \leq C,$$

while in the second one we have:

$$I_3 \leq \int_\theta^\infty e^{-u}u^{-1/2} \leq \int_\theta^\infty e^{-u}|\theta|^{-1/2} \leq (1-\theta)^{-1/2}$$

□

Lemma 5.4.2. *Let $a, b \in \mathcal{X}_\beta$ and $4\beta > 1$. Then $a * b \in \mathcal{Y}$ and the following estimate holds:*

$$\|a * b\|_{\mathcal{Y}} \leq u_\infty^{-2\beta+1/2}\|a\|_{\mathcal{X}_\beta}\|b\|_{\mathcal{X}_\beta}.$$

Proof . Without any loss we may assume that $\|a\|_{\mathcal{X}_\beta} = \|b\|_{\mathcal{X}_\beta} = 1$. Therefore

$$|a(t,\xi)| \leq (u_\infty + |t\xi^2|)^{-\beta} \qquad \text{for all } (t,\xi) \in \mathbb{R}^2,$$

and we may write:

$$|(a*b)(t,\xi)| \leq \int_{\mathbb{R}} \frac{1}{(u_\infty + |t||y-\xi|^2)^\beta}\frac{1}{(u_\infty + |ty^2|)^\beta}dy. \tag{5.33}$$

By I we denote the right hand side of (5.33). Using the substitution $u = |t|^{1/2}y$ we have:

$$I = |t|^{-1/2}\int_{\mathbb{R}} \frac{1}{(u_\infty + \left|u - |\xi||t|^{1/2}\right|^2)^\beta(u_\infty + |u|^2)^\beta}du. \tag{5.34}$$

Because of the presence of the term $|t|^{-1/2}$ it is sufficient to show, that the integral on the right hand side of 5.34 is bounded by some constant M independent of ξ.

We split domain $\mathbb{R}$ in the integral into three parts:

- $A_1 = \{u : |u| \leq \frac{1}{2}|t|^{1/2}|\xi|\}$,
- $A_2 = \{u : \frac{1}{2}|t|^{1/2}|\xi| < |u| \leq 2|t|^{1/2}|\xi|\}$,
- $A_3 = \{u : 2|t|^{1/2}|\xi| < |u|\}$,

and we denote by J_1, J_2, J_3 corresponding integrals. First we estimate J_1:

$$\begin{aligned} J_1 &= \int_{A_1} \frac{1}{(u_\infty + \left|u - |\xi||t|^{1/2}\right|^2)^\beta (u_\infty + |u|^2)^\beta} du \\ &\leq \int_{A_1} \frac{1}{(u_\infty + |\xi|^2|t|/4)^\beta (u_\infty + |u|^2)^\beta} du \\ &\leq (u_\infty + |t||\xi|^2)^{-\beta} \int_{A_1} \frac{1}{(\sqrt{u_\infty} + |u|)^{2\beta}} du \\ &= (u_\infty + |t||\xi|^2)^{-\beta} (\sqrt{u_\infty} + |u|)^{-2\beta+1} \Big|_0^{|t|^{1/2}|\xi|/2} \end{aligned}$$

We thus have a condition:

$$-\beta + \frac{-2\beta + 1}{2} \leq 0,$$

which is fulfilled for $4\beta \geq 1$, providing that J_1 is bounded independently of $|\xi|$ and t. Moreover, since $\beta > 0$ we may estimate J_1 as follows:

$$J_1 \leq C u_\infty^{-2\beta+1/2}$$

For J_2 we proceed in a similar way:

$$J_2 = \int_{A_2} \frac{1}{(u_\infty + \left|u - |\xi||t|^{1/2}\right|^2)^\beta (u_\infty + |u|^2)^\beta} du \leq \\ C \int_{A_2} \frac{1}{(u_\infty + \left|u - |\xi||t|^{1/2}\right|^2)^\beta (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{2\beta}} du,$$

using a substitution $t^{1/2}\xi - u = y$ we can write:

$$J_2 \leq 2 \int_{-t^{1/2}|\xi|}^{\frac{1}{2}t^{1/2}|\xi|} \frac{1}{(\sqrt{u_\infty} + |y|)^{2\beta} (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{2\beta}} dy \leq C(\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{-4\beta+1},$$

which is also finite and can be estimated independently of t and $|\xi|$ for $4\beta \geq 1$, and the same estimate is valid, namely:

$$J_2 \leq C u_\infty^{-2\beta+1/2}.$$

Finally we estimate J_3:

$$J_3 = \int_{A_3} \frac{1}{(u_\infty + \left|u - |\xi||t|^{1/2}\right|^2)^\beta (u_\infty + |u|^2)^\beta} du \\ \leq \int_{|t|^{1/2}|\xi|}^{\infty} \frac{1}{\sqrt{u_\infty} + |u|)^{4\beta}} du = (\sqrt{u_\infty} + |t|^{1/2}|\xi|)^{-4\beta+1},$$

for which we get the same condition $4\beta \geq 1$ and the same estimate:

$$J_3 \leq u_\infty^{-2\beta+1/2}.$$

Since $|t|^{1/2}I(t,\xi) = (J_1+J_2+J_3)(t,\xi)$ and each of components J_i is uniformly estimated provided $4\beta \geq 1$ the proof of Lemma 5.4.2 is finished. $\square$

Next lemma, which we need to estimate the pressure p, is the following:

Lemma 5.4.3. *Let $f \in \mathcal{Y}$. Then we have*

$$\tilde{I} := \xi \int_{\mathbb{R}} e^{-|\xi||t-y|} f(y,\xi)dy \in \mathcal{Y}$$

and the following estimate holds:

$$\|\tilde{I}\|_{\mathcal{Y}} \leq \|f\|_{\mathcal{Y}}$$

Proof . Since $f \in \mathcal{Y}$ we have

$$|f(t,\xi)| \leq |t|^{-1/2}\|f\|_{\mathcal{Y}}$$

for all t and ξ. Hence we can assume without any loss that $\|f\|_{\mathcal{Y}} = 1$ and we can focus on estimate of the following integral:

$$I = |\xi| \int_{\mathbb{R}} e^{-|\xi||t-y|}|y|^{-1/2}dy.$$

We use a substitution $u = \xi y$ to get:

$$I = |\xi|^{1/2} \int_{\mathbb{R}} e^{-||\xi t|-|u||}u^{-1/2}du. \tag{5.35}$$

We split this integral into three parts: $A_1 = \{u : |u| < |\xi||t|/2\}$, $A_2 = \{u : |\xi||t|/2 \leq u \leq 2|\xi||t|\}$, $A_3 = \{u : 2|\xi||t| \leq u\}$ and we introduce I_1, I_2 and I_3 as the corresponding parts of integral I from (5.35).
For I_1 we have:

$$I_1 \leq \frac{1}{2}|\xi|^{1/2} \int_{A_1} e^{-|\xi||t|/2}u^{-1/2} = |\xi|^{1/2}e^{-|\xi||t|/2}|\xi t|^{1/2}.$$

Now since

$$|\xi t|e^{-|\xi t|} \leq C, \tag{5.36}$$

where C is independent of ξt, we have

$$I_1 \leq |\xi|^{1/2}\frac{|\xi t|^{1/2}}{(1+|\xi t|)} = |t|^{-1/2}\frac{|\xi t|}{(1+|\xi t|)} \leq |t|^{-1/2},$$

which is the desired estimate.

For I_2 we proceed similarly:

$$I_2 \leq |\xi|^{1/2} \int_{|\xi t|/2}^{2|\xi t|} e^{-||\xi t|-u|}u^{-1/2}du \leq |\xi|^{1/2}|\xi t|e^{-|\xi t|}|\xi t|^{-1/2} \leq |t|^{-1/2}\frac{|\xi t|}{(1+|\xi t|)}.$$

For I_3 we must distinguish two cases: one if $|\xi t| < 1$ and the other one for $|\xi t| \geq 1$. For the first case we have:

$$I_3 \leq |\xi|^{1/2} \int_{|\xi t|}^{\infty} e^{-u}u^{-1/2}du = \int_{|\xi t|}^{1} + \int_{1}^{\infty} \leq |\xi|^{1/2} \int_{|\xi t|}^{1} u^{-1/2}du + |\xi|^{1/2} \int_{1}^{\infty} e^{-u}u^{-1/2}du$$
$$\leq |\xi|^{1/2}(1-|\xi t|) + |\xi|^{1/2} \leq |\xi|^{1/2} \leq |t|^{-1/2},$$

since $|\xi t| < 1$. For $|\xi t| \geq 1$ we have:

$$I_3 \leq |\xi|^{1/2} \int_{|\xi t|}^{\infty} e^{-u} du = |\xi|^{1/2} e^{-|\xi t|},$$

and again using (5.36) we get

$$I_3 \leq C|t|^{-1/2}.$$

Since for all I_1, I_2 and I_3 the proper estimate holds the proof of our Lemma is finished. □

Lemma 5.4.4. *Let $f \in \mathcal{Y}$. Given the following terms:*

$$\tilde{A} := \tfrac{|\xi|}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} f(s,\xi) ds, \qquad \tilde{B} := \tfrac{\lambda_1}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} f(s,\xi) ds,$$
$$\tilde{C} := \tfrac{|\xi|}{\Delta} \int_{t}^{\infty} e^{-\lambda_2(s-t)} f(s,\xi) ds, \qquad \tilde{D} := \tfrac{\lambda_2}{\Delta} \int_{t}^{\infty} e^{-\lambda_2(s-t)} f(s,\xi) ds.$$

Then $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathcal{X}_\beta$ provided $2\beta \leq 1$ and the following estimate is valid:

$$\|\tilde{A}\|_{\mathcal{X}_\beta} + \|\tilde{B}\|_{\mathcal{X}_\beta} + \|\tilde{C}\|_{\mathcal{X}_\beta} + \|\tilde{D}\|_{\mathcal{X}_\beta} \leq c\|f\|_{\mathcal{Y}}.$$

$$\tilde{I} := \frac{|\xi|}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} f(s,\xi) ds$$

belongs to the function space $\mathcal{X}_\beta$ provided $2\beta \leq 1$ and the following estimate is valid:

$$\|\tilde{I}\|_{\mathcal{X}_\beta} \leq \|f\|_{\mathcal{Y}}.$$

***Remark**: The same estimate is valid for*

$$\tilde{I} := \frac{\lambda_1}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} f(s,\xi) ds$$

Proof . Let us start with integral $\tilde{A}$. Without loss of generality we may assume that $t > 0$. As in previous lemmas we may also assume that $\|f\|_{\mathcal{Y}} = 1$ and consider integral

$$A := \frac{|\xi|}{\Delta} \int_{-\infty}^{t} e^{-\lambda_1(s-t)} |s|^{-1/2} ds.$$

Since $\lambda_1 = (u_\infty - \Delta)/2$ and $\Delta = \sqrt{u_\infty^2 + 4\xi^2}$ we see, that the behaviour of λ_1 is different for small ξ and large ξ.

Let us assume, that $|\xi| < u_\infty$. In this case we have $\Delta \sim u_\infty$ and $\lambda_1 \sim -|\xi|^2/u_\infty$, thus:

$$A \leq C \frac{|\xi|}{u_\infty} \int_{-\infty}^{t} e^{|\xi|^2/u_\infty (s-t)} |s|^{-1/2} ds.$$

Using a substitution $-u = |\xi|^2(s-t)$ we get:

$$A \leq C \frac{1}{u_\infty} \int_0^{\infty} e^{-u/u_\infty} |u - t\xi^2|^{-1/2} du = C u_\infty^{-1/2} \int_0^{\infty} e^{-y} |y - t\xi^2|^{-1/2} dy.$$

We use inequality (5.32) to obtain:

$$A \leq C u_\infty^{-1/2} (1 + t\xi^2/u_\infty)^{-1/2} = C(u_\infty + t\xi^2)^{-1/2}, \tag{5.37}$$

which is the desired estimate, since for $2\beta \le 1$ we have:

$$(u_\infty + t\xi^2)^\beta A \le (u_\infty + t\xi^2)^{\beta-1/2} \le c,$$

where constant c does not depend on t, ξ and u_∞, thus $A \in \mathcal{X}_\beta$.

Let us now assume that $|\xi| \ge u_\infty$. In this case we have $\lambda_1 \sim -|\xi|$ and $\Delta \sim |\xi|$, thus:

$$A \le C \int_{-\infty}^{t} e^{|\xi|(s-t)} |s|^{-1/2} ds. \tag{5.38}$$

Using a substitution $|\xi|s = -u + t|\xi|$ we end up with:

$$\begin{aligned} A &\le |\xi|^{-1/2} | \int_0^\infty e^{-u} |t|\xi| - u|^{-1/2} du \\ &\le |\xi|^{-1/2}(1 + t|\xi|)^{-1/2} = (|\xi| + t|\xi|^2)^{-1/2} \\ &\le (u_\infty + t|\xi|^2)^{-1/2}, \end{aligned}$$

where we used again inequality (5.32) with $\theta = t|\xi|$ and the fact, that $|\xi| > u_\infty$. This is the same inequality as (5.37), thus we have proved results of Lemma 5.4.4 for $\tilde{A}$.

To prove estimate for $\tilde{B}$ we notice that λ_1 can be estimated by $|\xi|$, since for small $|\xi|$, i.e. $|\xi| \le u_\infty$ one has $\lambda_1 \sim |\xi|^2/u_\infty \le |\xi|$, and for large $|\xi|$ one has $\lambda_1 \sim |\xi|$.

To estimate $\tilde{C}$ we first notice, that without loss of generality we may assume $t < 0$, i.e. $t = -|t|$. It is easy to see that we may show this inequality for $t < 0$, i.e. As earlier – the behaviour of $\lambda_2 = (u_\infty + \Delta)/2$ is different for small $|\xi|$ and large $|\xi|$, in particular $\lambda_2 \sim u_\infty$ for small $|\xi|$ and $\lambda_2 \sim |\xi|$ for large $|\xi|$. Term Δ behave exactly the same.

Let us first consider the case $|\xi| \le u_\infty$. We have

$$\begin{aligned} C &= \frac{|\xi|}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} |s|^{-1/2} ds \le \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} ds \\ &= u_\infty^{1/2} \int_0^\infty e^{-u_\infty u} |u - |t||^{-1/2} du = u_\infty^{-1/2} \int_0^\infty e^{-y} |y - u_\infty |t||^{-1/2} dy. \end{aligned}$$

We may now use (5.32) to obtain:

$$C \le u_\infty^{-1/2}(1 + u_\infty|t|)^{-1/2} = (u_\infty + |t|u_\infty^2)^{-1/2},$$

but since $|\xi| < u_\infty$ the last term can be estimated by $(u_\infty + |t||\xi|^2)^{-1/2}$, which, as we have seen earlier, is the desired estimate.

Let us now assume that $|\xi| > u_\infty$. In this case we have:

$$C = \frac{|\xi|}{\Delta} \int_t^\infty e^{-\lambda_2(s-t)} |s|^{-1/2} ds \le \int_t^\infty e^{-|\xi|(s-t)} |s|^{-1/2} ds,$$

and the last term estimates exactly in the same was as in (5.38), thus:

$$C \le C(u_\infty + t|\xi|^2)^{-1/2}.$$

Estimate of $\tilde{D}$ is analogous to the previous one, since for large $|\xi|$ both integrals behave exactly the same, and for small $|\xi|$ one has $\lambda_2 \sim u_\infty$, hence we must estimate integral:

$$D = \frac{\lambda_2}{\Delta} \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} ds \le \int_t^\infty e^{-u_\infty(s-t)} |s|^{-1/2} ds,$$

but exactly the same integral has been estimated during the proof of estimate for C. $\square$

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