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# Algebraic and homological properties of strict polynomial functors

*PhD dissertation*

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*Author's declaration*
I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

15 September 2025, ________________________

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*Supervisor's declaration*
The dissertation is ready to be reviewed.

15 September 2025, ________________________

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# Abstract

This doctoral dissertation is devoted to the homological algebra of the category $\mathcal{P}_d$ of strict polynomial functors of degree $d \leq 2p$, where $p$ is a characteristic of a ground field. This category, introduced by Friedlander and Suslin, is equivalent to the category of finite-dimensional left modules over the Schur algebra $S(n,d)$ with $n \geq d$. Hence, it is deeply related to the category of polynomial representations of the algebraic group $\mathrm{GL}_n$. It turns out that homological computations are much easier in the category under consideration due to phenomena, which are non-visible at the level of $\mathrm{GL}_n$-modules.

The dissertation contains the results from the following articles

- M. Chałupnik, P. Jaśniewski, *On strict polynomial functors with bounded domain*, Homol. Homotopy Appl. (1) **26** (2024), 87–104,

- P. Jaśniewski, *On homological properties of strict polynomial functors of degree p*, J. Algebra **618** (2023), 141–164

as well as the results not yet published. In the case of the results of the first article, only those obtained by the author of the dissertation are presented.

The first nontrivial case from the homological point of view is the case $d = p$. Indeed, for $d < p$ the category $\mathcal{P}_d$ is equivalent to the category of left modules over the group algebra of the symmetric group $\Sigma_d$, which is semisimple by Maschke's theorem. The first part of the dissertation is devoted to the case $d = p$. We describe the injective envelopes and projective covers of simple functors, which allows us to determine the decomposition matrix. We also introduce the minimal resolutions of Schur functors, which are used to determine the Ext-groups between functors important from the representation-theoretic point of view. An important corollary of these computations is the existence of a Kazhdan-Lusztig theory for $\mathcal{P}_p$. We then study the Ext-algebras of simple and Schur functors and prove the formality of these algebras. The obtained results are generalized to the case of degrees $d < 2p$ by proving the equivalence between the principal block of $\mathcal{P}_p$ and a block of $\mathcal{P}_d$ corresponding to the set of Young diagrams with the same $p$-core obtained from them in a single step.

We then turn to the computation of the Ext-groups between Schur functors in $\mathcal{P}_{2p}$. We apply the previously obtained homological results and the decomposition formula arising from the sum-diagonal adjunction, which is specific to functor categories. The approach used to computing of the Ext-groups substantially differs from that in case $d = p$. The results indicate that the homological algebra of $\mathcal{P}_{2p}$ is significantly more complex than that of $\mathcal{P}_d$ for $d < 2p$; some of them seem to be quite unexpected.

In the last part of the dissertation we define the category of strict polynomial functors of degree $d$ with bounded domain. We discuss the structural properties of that category and we prove the existence of a recollement setup connecting the introduced category with $\mathcal{P}_d$. Using this recollement diagram and the results on the category $\mathcal{P}_p$, we study the homological category of the new category in case $d = p$.

**Keywords:** block, Ext-group, formality, polynomial representation, Schur algebra, strict polynomial functor

**AMS MSC 2020 classification:** 16E30, 16E35, 18A25, 20G15

# Streszczenie

Niniejsza rozprawa doktorska dotyczy algebry homologicznej kategorii $\mathcal{P}_d$ funktorów ściśle wielomianowych stopnia $d \leqslant 2p$, gdzie $p$ jest charakterystyką ciała bazowego. Kategoria ta, wprowadzona przez Friedlandera i Suslina, jest równoważna kategorii skończenie wymiarowych lewostronnych modułów nad algebrą Schura $S(n,d)$ dla $n \geqslant d$, a więc jest głęboko związana z reprezentacjami wielomianowymi grupy algebraicznej $\mathrm{GL}_n$. Okazuje się, że obliczenia homologiczne są istotnie prostsze w rozważanej kategorii ze względu na zjawiska niewidoczne z poziomu $\mathrm{GL}_n$-modułów.

Rozprawa zawiera wyniki pochodzące z następujących artykułów:

- M. Chałupnik, P. Jaśniewski, *On strict polynomial functors with bounded domain*, Homol. Homotopy Appl. (1) **26** (2024), 87–104,
- P. Jaśniewski, *On homological properties of strict polynomial functors of degree p*, J. Algebra **618** (2023), 141–164,

a także te jeszcze nieopublikowane. W przypadku wyników z pierwszego artykułu zaprezentowano jedynie te rezultaty, które zostały otrzymane przez autora rozprawy.

Pierwszym nietrywialnym przypadkiem z punktu widzenia algebry homologicznej jest przypadek $d = p$. W istocie, dla $d < p$ kategoria $\mathcal{P}_d$ jest równoważna kategorii lewostronnych modułów nad algebrą grupową grupy symetrycznej $\Sigma_d$, która jest półprosta na mocy twierdzenia Maschke. Pierwsza część rozprawy jest poświęcona przypadkowi $d = p$. Opisujemy powłoki injektywne i nakrycia projektywne funktorów prostych, co pozwala opisać macierz rozkładu. Wprowadzamy także rezolwenty minimalne funktorów Schura, za pomocą których wyznaczamy grupy Ext pomiędzy funktorami istotnymi w teorii reprezentacji. Istotnym wnioskiem z tych obliczeń jest istnienie teorii Kazhdana-Lusztiga w $\mathcal{P}_p$. W dalszej kolejności badamy Ext-algebry funktorów prostych i Schura, a także dowodzimy formalności tych algebr. Otrzymane wyniki uogólniamy na przypadek stopni $d < 2p$, dowodząc równoważności bloku głównego kategorii $\mathcal{P}_p$ z blokiem kategorii $\mathcal{P}_d$ odpowiadającemu zbiorowi diagramów Younga o wspólnym $p$-rdzeniu, który można uzyskać z tych diagramów w jednym kroku.

Następnie zajmujemy się wyznaczaniem grup Ext pomiędzy funktorami Schura w kategorii $\mathcal{P}_{2p}$. W poświęconym temu rozdziale wykorzystujemy wcześniej otrzymane wyniki homologiczne oraz formułę rozkładu pochodzącą z dołączenia funktora diagonalnego do funktora sumy prostej, które jest specyficznym narzędziem dla kategorii funktorów. Metoda wyznaczania grup Ext znacząco różni się od tej stosowanej w przypadku $d = p$. Otrzymane rezultaty wskazują, że algebra homologiczna kategorii $\mathcal{P}_{2p}$ jest istotnie bardziej złożona niż w sytuacji $d < 2p$; niektóre z nich okazują się być dość zaskakujące.

W ostatniej części rozprawy definiujemy kategorię funktorów ściśle wielomianowych stopnia $d$ z ograniczoną dziedziną. Omawiamy strukturalne własności tej kategorii oraz uzasadniamy istnienie „recollement” łączącego wprowadzoną kategorię z kategorią $\mathcal{P}_d$. W oparciu o to „recollement” i o wyniki otrzymane dla $\mathcal{P}_p$, badamy algebrę homologiczną wprowadzonej kategorii w przypadku $d = p$.

**Słowa kluczowe:** algebra Schura, blok, formalność, funktor ściśle wielomianowy, grupa Ext, reprezentacje wielomianowe

**Klasyfikacja AMS MSC 2020:** 16E30, 16E35, 18A25, 20G15

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# Contents

# Introduction

In this dissertation we undertake a systematic study of the homological algebra of the category $\mathcal{P}_d$ of homogeneous strict polynomial functors of degree $d$ over a field $\mathbf{k}$ of characteristic $p > 0$ by considering degrees $d \leq 2p$. This category was introduced in [FS97], where the finite generation of the rational cohomology ring of a finite group scheme was established. Using this category, the authors obtained impressive homological results demonstrating that $\mathcal{P}_d$ is a valuable tool in the study of representations of the general linear group $\mathrm{GL}_n$ regarded as an algebraic group. Intuitively, a strict polynomial functor of degree $d$ is an endofunctor of the category of finite-dimensional vector spaces over $\mathbf{k}$, which acts on Hom-sets via polynomial homogeneous maps of degree $d$. The category of ordinary endofunctors $\mathcal{F}$ already turned out to be effective in the study of representations of the finite group $\mathrm{GL}_n(\mathbf{k})$ for a finite field $\mathbf{k}$ (see [Bet99], [FFSS99, Appendix], [FLS94]). However, $\mathcal{P}_d$ carries a richer structure than $\mathcal{F}$. Let us recall that the Schur algebra $S(n,d)$ is the linear dual of the subcoalgebra of the coordinate bialgebra $\mathbf{k}[M_n(\mathbf{k})]$ consisting of homogeneous polynomials of degree $d$, where $M_n(\mathbf{k})$ is the vector space of $n \times n$ matrices over $\mathbf{k}$. It is known that $\mathcal{P}_d$ is equivalent to the category $S(n,d)$-mod of finite-dimensional left modules over the Schur algebra $S(n,d)$ for $n \geq d$ (see [FS97, Theorem 3.2]). Therefore, it has nice algebraic features like the highest weight structure and the block decomposition, which is not the case for $\mathcal{F}$. The functorial approach relies on the observation that for $F \in \mathcal{P}_d$, $F(\mathbf{k}^n)$ is a polynomial representation of $\mathrm{GL}_n$ of degree $d$. It is known that the category $S(n,d)$-mod is equivalent to the category $\mathrm{GL}_n^{\mathrm{Pol},d}$ $-$mod of finite-dimensional polynomial $\mathrm{GL}_n$-modules of degree $d$ (cf. [Mar93, Theorem 2.2.7]), hence $\mathcal{P}_d \simeq \mathrm{GL}_n^{\mathrm{Pol},d}$ $-$mod for $n \geq d$. The equivalence is given by the evaluation functor $F \mapsto F(\mathbf{k}^n)$. Moreover, it induces isomorphisms on the Ext-groups:

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F,G) \simeq \mathrm{Ext}^*_{\mathrm{GL}_n-\mathrm{mod}}(F(\mathbf{k}^n), G(\mathbf{k}^n)) \text{ for } n \geq d,$$

where $\mathrm{GL}_n$ $-$mod denotes the category of rational $\mathrm{GL}_n$-modules (cf. [FS97, Corollary 3.13]). Investigating the homological algebra of $\mathcal{P}_d$ is more beneficial than a direct study of $\mathrm{GL}_n$-modules due to some crucial tools specific to functor categories such as the sum-diagonal adjunction [FFSS99].

In many papers there were computed the Ext-groups between particular functors for a general degree $d$. Before we give examples of such functors, let us recall that the Frobenius twist functor $I^{(1)}$ is a strict polynomial functor of degree $p$, which assigns to a space $V$ the space with the same underlying set as $V$ and the multiplication by scalars affected by the Frobenius morphism. The Frobenius twist of $F \in \mathcal{P}_d$ is the functor $F \circ I^{(r)} \in \mathcal{P}_{p^r d}$ for some $r \geq 1$, where $I^{(r)}$ is the $r$-fold composition of $I^{(1)}$. The importance of this class of functors comes from the role they play in the theorem comparing Ext-groups in $\mathcal{P}_d$ and $\mathcal{F}$ (cf. [FFSS99, Theorem 3.10]). Now we turn to the examples promised above. The classical exponential functors (symmetric, divided and exterior) and their Frobenius twists were considered in, e.g., [Aki89, Cha08, FFSS99, FS97]. The computations of the Ext-groups between Frobenius twists of Weyl and Schur functors were performed in, e.g., [Cha05, Cha09, Cha15, Tou12]. Some Ext-groups in the category of strict polynomial functors compute rational cohomology groups of the general linear, orthogonal and symplectic groups (see, e.g., [FF08, Tou10a, Tou10b, Pha22]). There were also established deep "partial formality" phenomena known as the Collapsing Conjecture (see [Cha15, Cha17, Tou12]). Certain homological problems concerning $S(n,d)$-module, e.g., the computation of $\mathrm{Ext}^1$ and $\mathrm{Ext}^2$-groups between Weyl modules corresponding to hook Young diagrams, were studied by using direct representation-theoretic methods (see, e.g., [Kul05, Kul99, MS22, MS24, Ste21]). In the dissertation we perform homological computations regardless of the degree of Ext-groups and involving various functors such as simple, Schur and Weyl functors (in the sense of [ABW82]), which are important from the representation-theoretic point of view. By using these results we get some insight into the structure of $\mathcal{P}_d$ for degrees $d < 2p$. For instance, we establish the existence of a Kazhdan-Lusztig theory and formality phenomena. General tools of homological algebra and the functorial approach seem to be promising for the study of the homological algebra of $\mathcal{P}_d$ for a general degree $d$. Under the aforementioned equivalences, the results obtained in the dissertation translate into theorems on modules over Schur algebras, and therefore on polynomial representations of $\mathrm{GL}_n$.

We now turn to a brief discussion of the main results and the organization of the dissertation. After introducing in Chapter 1.1 the formal definition of $\mathcal{P}_d$, its basic properties and different tools useful in the later considerations, we begin our study with the case $d = p$. The Schur algebra $S(n,d)$, and hence $\mathcal{P}_d$, is semisimple for $d < p$ (cf. [Mar93, Theorem 2.2.8]). In consequence, the first nontrivial case from a homological point of view is $d = p$, which we investigate in Chapter 2. We begin by describing the injective envelopes and projective covers of simple functors (see Proposition 2.1.1), which turn out to be significant in the construction of resolutions crucial for later computations. As a corollary of Proposition 2.1.1, we get the decomposition matrix of the only non-simple block $\mathcal{P}_p^{\varnothing}$ corresponding to the set of hooks (see Corollary 2.1.2). It is is already known, but obtained by different methods (see [DEN04, DM09]). We also give an alternative and more direct proof of the main result of [Xi92], which establishes the Morita equivalence between the Schur algebra $S(n,p)$ for $n \geq p$ and a quotient of a path algebra. In Section 2.2 we compute the groups $\mathrm{Ext}^*(G_1, G_2)$, where $G_i$ for $i \in \{1,2\}$ is a simple, a Schur or a Weyl functor (both functors are not necessarily of the same type). The most significant results on Ext-groups are Proposition 2.2.1, Corollary 2.2.4 and Proposition 2.2.6. Let us note that the Ext-groups between Schur functors are known in the literature as "the intertwining numbers", and a longstanding question is to determine them (see, e.g., [AB88, CL74]). An immediate and important consequence of Proposition 2.2.1 is Corollary 2.2.3, which implies that $\mathcal{P}_p$ has a Kazhdan–Lusztig theory. A Kazhdan–Lusztig theory in a highest weight category can be understood as a generalization of the celebrated Lusztig conjecture [Lus80]. Although it was shown in [WKM17] that this conjecture does not hold in general, it is interesting and conceptually challenging to understand the role of the "even-odd vanishing" often occurring in homological computations in modular representation theory. Under the equivalence $\mathcal{P}_d \simeq \mathrm{GL}_n^{\mathrm{Pol},d}-\mathrm{mod}$ for $n \geq d$, Corollary 2.2.3 shows that a certain variant of the Lusztig conjecture holds for polynomial representations of $\mathrm{GL}_n$ of degree $p$ for $n \geq p$. In Chapter 4 we extend this result to the case $n < p$. We then study the Yoneda algebras $\mathrm{Ext}^*(S,S)$ and $\mathrm{Ext}^*(F,F)$, where $S$ (resp. $F$) is the direct sum of Schur (resp. simple) functors belonging to the block $\mathcal{P}_p^{\varnothing}$ (see Theorems 2.3.2 and 2.3.5). In the case of the Yoneda algebra of Schur functors, we prove that it is isomorphic to the square-zero extension of the algebra of $p \times p$ upper triangular matrices over $\mathbf{k}$ by its bimodule of $p \times p$ strictly upper triangular matrices over $\mathbf{k}$. We also describe the Yoneda algebra of simple functors explicitly as a graded algebra. Let us recall that a DG algebra $A$ is formal if $A$ is quasi-isomorphic to its cohomology algebra $H^*(A)$ treated as a DG algebra with zero differential. We show that the endomorphism algebras computing the Yoneda algebras under consideration are formal. These results imply equivalences of triangulated categories $\mathcal{D}^b\mathcal{P}_p^{\varnothing} \simeq \mathcal{D}^b(\mathrm{Ext}^*(S,S)) \simeq \mathcal{D}^b(\mathrm{Ext}^*(F,F))$, where the second equivalence is by no means obvious from the explicit forms of the involved graded algebras (see Corollaries 2.3.3, 2.3.6 and 2.3.7). We conclude Chapter 2 with Theorem 2.4.1, which establishes an equivalence of abelian categories (even of highest weight categories) between $\mathcal{P}_p^{\varnothing}$ and a block of $\mathcal{P}_d$ of $p$-weight

1, i.e., corresponding to a Young diagram whose $p$-core is obtained in a single step. It was proved in [HTY14] that two blocks of $\mathcal{P} = \bigoplus_{d\geq 0} \mathcal{P}_d$ of the same $p$-weight are derived equivalent. Theorem 2.4.1 shows that in the case of $p$-weight 1 we have an equivalence of abelian categories. However, our approach does not work for larger $p$-weights. This theorem generalizes the results of Chapter 2 to the case $p < d < 2p$, since in this situation all blocks of $\mathcal{P}_d$ are of $p$-weight at most 1. It plays an important role in the computations performed in the next part of the dissertation.

Chapter 3 is devoted to the complete computation of the Ext-groups between Schur functors in $\mathcal{P}_{2p}$. Theorems 3.2.4, 3.3.4 and 3.4.2 indicate that the homological algebra of $\mathcal{P}_{2p}$ is more complicated than that of $\mathcal{P}_d$ for $d < 2p$. For instance, we encounter a completely new and surprising phenomenon: the vanishing of Ext-groups between Schur functors corresponding to some hook Young diagrams, though there is no reason coming from the partial order (see Theorem 3.2.4). There are also cases in which the Ext-groups between Schur functors are four-dimensional, in contrast to $\mathcal{P}_p$, where the overall picture is significantly simpler. We obtain these Ext-groups by quite different methods than those in case $d = p$. We make an extensive use of results of the first part of the dissertation and the decomposition formula arising from the sum-diagonal adjunction and the decomposition of the bifunctor $S_\lambda(- \oplus -)$ (cf. [ABW82, Theorem II.4.11]). After the submission of the doctoral thesis, I will continue the study of the homological algebra of $\mathcal{P}_{2p}$. I plan to compute the Ext-groups between simple functors and Schur functors, which will be used to answer the question on the existence of a Kazhdan-Lusztig theory in $\mathcal{P}_{2p}$. The next natural direction is to investigate the similar objects and phenomena as in the case of $\mathcal{P}_p$, which could be then used to study the homological algebra of $\mathcal{P}_d$ for $d > 2p$ in a structured manner.

In the final chapter we introduce the category $\mathcal{P}_{d,n}$ of strict polynomial functors of degree $d$ with bounded domain: we restrict the domain of a strict polynomial functor of degree $d$ to vector spaces of dimension at most $n$. We then discuss basic properties of the introduced category and relate it to the Schur algebra $S(n, d)$ (see Proposition 4.1.1 and Theorem 4.1.2). In particular, one can use the new category to study polynomial representations of $\mathrm{GL}_n$ of degree $d$ for small $n$, i.e., for $n < d$. We also establish the recollement diagram linking $\mathcal{P}_{d,n}$ and $\mathcal{P}_d$, which allows one to study the category $\mathcal{P}_{d,n}$ in terms of $\mathcal{P}_d$ (see Theorem 4.1.4). In Section 4.2 we study the homological algebra of $\mathcal{P}_{p,n}$ for $n < p$ using the recollement setup and the results of the first part of the dissertation, adapting constructions in $\mathcal{P}_p$ to the new situation. We observe the similarities to the case of $\mathcal{P}_p$, e.g., the existence of a Kazhdan-Lusztig theory (see Theorems 4.2.1 and 4.2.2). However, we also notice notable differences. For instance, there are examples of computations, which are not just truncations of those in $\mathcal{P}_p$ (see Theorem 4.2.3). We conclude the chapter by investigating the derived functor of the classical Schur functor $\mathrm{Hom}(I^{\otimes p}, -) : \mathcal{P}_p \to \mathbf{k}[\Sigma_p]$, where $\mathbf{k}[\Sigma_p]$ denotes the group algebra of the symmetric group $\Sigma_p$ (see Theorem 4.2.5 and Corollary 4.2.6). In Corollary 4.2.7 we obtain a surprisingly close connection between $\mathcal{P}_{p,p-1}$ and the category of $\mathbf{k}[\Sigma_p]$-modules at the level of K-theory.

The results of Chapters 2 and 4 were published in [Jas23] and a joint work [CJ24], respectively. I declare that the results of Chapter 4 of the dissertation, which correspond to Sections 1, 2 in [CJ24], are solely my contribution. The results of Chapter 3 have not yet been published; the manuscript is currently under preparation.

# Chapter 1

# The category of strict polynomial functors

We assume throughout that $\mathbf{k}$ is a field of characteristic $p > 0$. Whenever we refer to the general linear group $\mathrm{GL}_n$, we regard it as an algebraic group. The category of left (resp. right) modules over an algebra $A$ is denoted by $A - \mathrm{Mod}$ (resp. $\mathrm{Mod} - A$). We denote the category of finite-dimensional left (resp. right) modules over an algebra $A$ by $A - \mathrm{mod}$ (resp. $\mathrm{mod} - A$). $A - \mathrm{mod}^{\mathrm{gr}}$ denotes the category of finite-dimensional left graded modules over a graded algebra $A$.

In this chapter we introduce the category $\mathcal{P}_d$. Then we recall various properties of $\mathcal{P}_d$ and tools useful in the thesis such as the highest weight structure of our category, the Koszul and de Rham complexes, the decomposition formula and the block structure of $\mathcal{P}_d$.

## 1.1 The category $\mathcal{P}_d$. Definition, tools and important facts

We begin by recalling some standard constructions of linear algebra ( [Lan02, Rob63]). Let $V$ be a vector space over a field $\mathbf{k}$ of positive characteristic $p$. Denote the $d$-th tensor power functor by $I^{\otimes d}$ (for $d = 1$ we omit the superscript). There is a natural left action of the symmetric group $\Sigma_d$ on $I^{\otimes d}(V) = V^{\otimes d}$ by permuting tensor factors: $\sigma(v_1 \otimes \ldots \otimes v_d) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(d)}$. Then the $d$-th divided power $\Gamma^d(V)$ and the $d$-th symmetric power $S^d(V)$ are defined by

$$\Gamma^d(V) = (V^{\otimes d})^{\Sigma_d}, \quad S^d(V) = (V^{\otimes d})_{\Sigma_d},$$

i.e., the invariants and coinvariants spaces of the $\Sigma_d$–action, respectively. The image of $v_1 \otimes \cdots \otimes v_d$ under the canonical projection $V^{\otimes d} \twoheadrightarrow S^d(V)$ is denoted by $v_1 \ldots v_d$. It follows from the definition that $\Gamma^d(V^*) \simeq (S^d(V))^*$; hence $\Gamma^d(V) \simeq (S^d(V^*))^*$. The assignments $V \mapsto \Gamma^d(V)$ and $V \mapsto S^d(V)$ give the functors $\Gamma^d$ and $S^d$. Let us note that $\Gamma^d$ and $S^d$ are not isomorphic for $d \geq p$. The multiplication map $m_V : S^i(V) \otimes S^j(V) \to S^{i+j}(V)$ and the comultiplication map $c_V : S^{i+j}(V) \to S^i(V) \otimes S^j(V)$ are given on pure tensors by

$$\begin{aligned} m(v_1 \ldots v_i \otimes w_1 \ldots w_j) &= v_1 \ldots v_i w_1 \ldots w_j, \\ c(u_1 \ldots u_{i+j}) &= \sum_{\sigma \in \Sigma^i_{i+j}} u_{\sigma(1)} \ldots u_{\sigma(i)} \otimes u_{\sigma(i+1)} \ldots u_{\sigma(i+j)}, \end{aligned}$$

where $\Sigma^i_{i+j} = \{\sigma \in \Sigma_{i+j} : \sigma(1) < \ldots < \sigma(i) \text{ and } \sigma(i+1) < \ldots < \sigma(i+j)\}$. Under the identification $\Gamma^d(V) \simeq S^d(V^*)^*$, the dual of $m_{V^*}$ (resp. $c_{V^*}$) is identified with the comultiplication map $\Gamma^{i+j}(V) \to \Gamma^i(V) \otimes \Gamma^j(V)$ (resp. the multiplication map $\Gamma^i(V) \otimes \Gamma^j(V) \to \Gamma^{i+j}(V)$). We now introduce the $d$-th exterior power:

$$\Lambda^d(V) = V^{\otimes d} / \operatorname{span}\{v_1 \otimes \ldots \otimes v_d : v_i = v_j \text{ for some } i \neq j\}.$$

The image of $v_1 \otimes \ldots \otimes v_d$ under the canonical projection $V^{\otimes d} \twoheadrightarrow \Lambda^d(V)$ is denoted by $v_1 \wedge \ldots \wedge v_d$. There is a natural isomorphism $\Lambda^d(V^*) \simeq \Lambda^d(V)^*$, i.e., $\Lambda^d(V) \simeq (\Lambda^d(V^*))^*$. The assignment $V \mapsto \Lambda^d(V)$ gives the functor $\Lambda^d$. The multiplication map $m : \Lambda^i(V) \otimes \Lambda^j(V) \to \Lambda^{i+j}(V)$ and the comultiplication map $c : \Lambda^{i+j}(V) \to \Lambda^i(V) \otimes \Lambda^j(V)$ are given on basis elements by

$$m(v_1 \wedge \ldots \wedge v_i \otimes w_1 \wedge \ldots w_j) = v_1 \wedge \ldots v_i \wedge w_1 \wedge \ldots \wedge w_j,$$
$$c(u_1 \ldots u_{i+j}) = \sum_{\sigma \in \Sigma^i_{i+j}} \operatorname{sgn}(\sigma) u_{\sigma(1)} \wedge \ldots \wedge u_{\sigma(i)} \otimes u_{\sigma(i+1)} \wedge \ldots \wedge u_{\sigma(i+j)}.$$

One has an embedding $\Lambda^d(V) \hookrightarrow V^{\otimes d}$ given by the antisymmetrization map

$$v_1 \wedge \ldots \wedge v_d \mapsto \sum_{\sigma \in \Sigma_d} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}.$$

Let $\mathcal{V}ect_{\mathbf{k}}$ be the category of finite-dimensional vectors spaces over $\mathbf{k}$. We define the $\mathbf{k}$-linear category $\Gamma^d\mathcal{V}ect_{\mathbf{k}}$ as follows. The objects of $\Gamma^d\mathcal{V}ect_{\mathbf{k}}$ are finite-dimensional vector spaces over $\mathbf{k}$. We set $\operatorname{Hom}_{\Gamma^d\mathcal{V}ect_{\mathbf{k}}}(V, W) = \Gamma^d \operatorname{Hom}_{\mathbf{k}}(V, W)$, where $\operatorname{Hom}_{\mathbf{k}}(V, W)$ denotes the vector space of $\mathbf{k}$-linear maps $V \to W$. The composition of morphisms is determined by the natural identification $\Gamma^d \operatorname{Hom}_{\mathbf{k}}(V, W) \simeq \operatorname{Hom}_{\mathbf{k}}(V^{\otimes d}, W^{\otimes d})^{\Sigma_d}$ with $\Sigma_d$ acting on $\operatorname{Hom}_{\mathbf{k}}(V^{\otimes d}, W^{\otimes d})$ via $(\sigma f)(v) = \sigma f(\sigma^{-1} v)$ for $f \in \operatorname{Hom}_{\mathbf{k}}(V^{\otimes d}, W^{\otimes d})$ and $\sigma \in \Sigma_d$.

We now introduce the category of strict polynomial functors, which is the principal object of study in this dissertation. Instead of the original description given in [FS97], we follow here the approach due to Pirashvili [Pir02].

**Definition.** The category $\mathcal{P}_d$ of homogeneous strict polynomial functors over $\mathbf{k}$ of degree $d$ is the category of $\mathbf{k}$-linear functors $\Gamma^d\mathcal{V}ect_{\mathbf{k}} \to \mathcal{V}ect_{\mathbf{k}}$. The category $\mathcal{P}$ of strict polynomial functors (over $\mathbf{k}$) is given by $\mathcal{P} = \bigoplus_{d \geq 0} \mathcal{P}_d$.

The standard examples of strict polynomial functors of degree $d$ are $I^{\otimes d}, \Gamma^d, S^d$ and $\Lambda^d$. Another important example is the Frobenius twist functor $I^{(1)}$, which is of degree $p$ (cf. [FS97, pp. 3-4, 14]). It assigns to a space $(V, +, \cdot)$ the space $(V, +, \odot)$ with the scalar multiplication $\odot$ induced by the Frobenius morphism, i.e., $\lambda \odot v = \lambda^p \cdot v$.

Let us now recollect the well-known facts on strict polynomial functors (cf. [FS97]). $\mathcal{P}_d$ is an abelian category with enough injectives and projectives. The Kuhn duality is a contravariant exact equivalence of categories $(-)^{\#} : (\mathcal{P}_d)^{\mathrm{op}} \to \mathcal{P}_d$ defined as follows: $G^{\#}(V) = G(V^*)^*$. The tensor product of strict polynomial functors is a biexact functor $\otimes : \mathcal{P}_d \times \mathcal{P}_e \to \mathcal{P}_{d+e}$ given on objects by $(F \otimes G)(V) = F(V) \otimes G(V)$. Let us recall that the Schur algebra $S(n, d)$ is the linear dual of the subcoalgebra of the coordinate bialgebra $\mathbf{k}[M_n(\mathbf{k})]$ consisting of homogeneous polynomials of degree $d$. There is an isomorphism of $\mathbf{k}$-algebras: $S(n, d) \simeq \Gamma^d(\operatorname{End}_{\mathbf{k}}(\mathbf{k}^n))$ (cf. [Mar93, Theorem 2.1.3]). If $n \geq d$ then $\mathcal{P}_d \simeq S(n, d) - \mathrm{mod} \simeq \mathrm{GL}_n^{\mathrm{Pol},d}$ $-$mod, where $\mathrm{GL}_n^{\mathrm{Pol},d}$ $-$mod denotes the category of finite-dimensional homogeneous polynomial $\mathrm{GL}_n$-modules of degree $d$ (cf. [FS97, Theorem 3.2 and Lemma 3.4]).

We now recall the definition of a highest weight category in the sense of [CPS88] with a minor change. The authors assume that a poset is interval-finite, but we do not need such generality and will assume that it is finite.

**Definition.** Let $\mathcal{C}$ be an artinian abelian $\mathbf{k}$-linear category and let $(P, <)$ be a finite poset whose elements will be called weights. The category $\mathcal{C}$ is a highest weight category if

- there is a collection of non-isomorphic simple objects $\{F_\lambda : \lambda \in P\}$ indexed by elements of the poset $P$;

- for each $\lambda \in P$ there exists an object $\nabla_\lambda$ called the costandard object corresponding to $\lambda$ and an embedding $F_\lambda \hookrightarrow \nabla_\lambda$ such that all the composition factors $F_\mu$ of $\nabla_\lambda / F_\lambda$ satisfy $\mu < \lambda$;
- each simple object $F_\lambda$ has the injective envelope $I_\lambda$, which admits a filtration
$$0 \subset F_1 \subset \ldots \subset F_k = I_\lambda$$
such that $F_1 = \nabla_\lambda$ and $F_i/F_{i-1} \simeq \nabla_{\mu_i}$ for $2 \leq i \leq k$ and $\lambda < \mu_i$.

It was shown in [CPS88] that if a poset is finite, then each simple object in $\mathcal{C}$ has the projective cover. Let $\Delta_\lambda$ be the largest quotient of the projective cover $P_\lambda$ of $F_\lambda$ whose composition factors $F_\mu$ satisfy $\mu \leq \lambda$. We call $\Delta_\lambda$ the standard object corresponding to $\lambda$. It turns out that the collections $\{F_\lambda\}_{\lambda \in P}, \{\Delta_\lambda\}_{\lambda \in P}$ and $\{P_\lambda\}_{\lambda \in P}$ satisfy the dual conditions to those in the above definition. We say that $M \in \mathcal{C}$ admits a good filtration if $M$ has a filtration with the associated graded object consisting of costandard objects. We denote the multiplicity of $\nabla_\lambda$ in a good filtration of $M$ by $(M : \nabla_\lambda)$ and the multiplicity of $F_\lambda$ in a composition series of $M$ by $[M : F_\lambda]$. We now record several facts on highest weight categories.

**Proposition 1.1.1** ([CPS88])**.** *Let $\mathcal{C}$ be a highest weight category and let $P$ be its defining poset. Let $\lambda, \mu \in P$. Then:*

*(a) $F_\lambda$ is the socle of $\nabla_\lambda$ and the top of $\Delta_\lambda$.*

*(b) If $\lambda$ is minimal, then $\nabla_\lambda \simeq \Delta_\lambda \simeq F_\lambda$.*

*(c) If $\lambda$ is maximal, then $\nabla_\lambda$ is injective and $\Delta_\lambda$ is projective.*

*(d) If $\mathrm{Ext}^n_{\mathcal{C}}(F_\lambda, \nabla_\mu) \neq 0$ or $\mathrm{Ext}^n_{\mathcal{C}}(\nabla_\lambda, \nabla_\mu) \neq 0$ for some $n \geq 0$, then $\mu \leq \lambda$. Moreover, if $n > 0$ then $\mu < \lambda$.*

*(e) If $\mathrm{Ext}^n_{\mathcal{C}}(F_\lambda, F_\mu) \neq 0$ for some $n > 0$, then $\lambda < \mu$ or $\mu < \lambda$.*

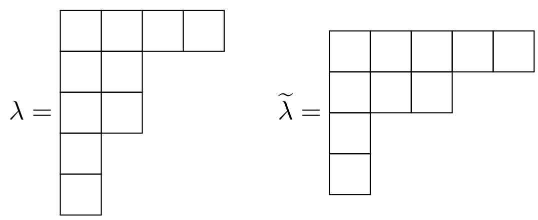
*(f) One has* $\mathrm{Ext}^q_{\mathcal{C}}(\Delta_\lambda, \nabla_\mu) = \begin{cases} \mathrm{End}_{\mathcal{C}}(F_\lambda) & \textit{if } q = 0 \textit{ and } \lambda = \mu, \\ 0 & \textit{otherwise.} \end{cases}$

*(g) (Brauer-Humphreys reciprocity) One has $(I_\mu : \nabla_\lambda) = [\Delta_\lambda : F_\mu]$.*

*(h) If $\mathrm{Ext}^1(W_\lambda, M) = 0$ for all weights $\lambda$, then $M$ admits a good filtration. In this situation $(M : S_\mu) \cdot \dim \mathrm{End}_{\mathcal{C}}(F_\mu) = \dim \mathrm{Hom}_{\mathcal{C}}(W_\mu, M)$.*

It is known that $\mathcal{P}_d$ is a highest weight category with poset of Young diagrams of weight $d$ with the reversed dominance order (cf. [Kra22, Theorem 8.6.6]). We now recall the definition of this poset and the construction of costandard and standard objects of this category called, respectively, Schur and Weyl functors. Let $\Lambda(d)$ be the set of partitions of $d$, i.e., non-increasing sequences of positive integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying the condition $\sum_{i=1}^n \lambda_i = d$. If the number $r$ appears consecutively $i$ times in a partition, then we write it as $r^i$ in the partition. We identify a partition of $d$ with a Young diagrams of weight $d$: for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ the $i$-th row has $\lambda_i$ boxes. The weight of a diagram $\lambda$ will be denoted by $|\lambda|$. The conjugate $\widetilde{\lambda}$ of $\lambda$ is the Young diagram whose the $i$-th row has the same length as the $i$-th column of $\lambda$.

**Example.** This example shows the Young diagrams $\lambda = (4, 2^2, 1^2)$ and $\widetilde{\lambda} = (5, 3, 1^2)$:



.

A partial order on the set $\Lambda(d)$ is the reversed dominance order: $\lambda = (\lambda_1, \ldots, \lambda_n)$ dominates $\mu = (\mu_1, \ldots, \mu_m)$ if and only if $\sum_{j\leq i} \lambda_j \leq \sum_{j\leq i} \mu_j$ for all $1 \leq i \leq \max\{m, n\}$. One can describe this order combinatorially: $\lambda$ dominates $\mu$ if and only if $\mu$ is obtained from $\lambda$ by a sequence of moves, each moving a single box from the end of a row to the end of a higher row (cf. [JK81, Lemma 1.4.10]). This fact can be used to show that if $\lambda$ dominates $\mu$, then $\widetilde{\mu}$ dominates $\widetilde{\lambda}$ (cf. [JK81, Lemma 1.4.11]). We are now ready to describe the costandard and standard objects in $\mathcal{P}_d$ with poset $\Lambda(d)$ defined above. We begin by recalling the definitions of Schur and Weyl functors in the sense of [ABW82]. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\widetilde{\lambda} = (\lambda_1', \ldots, \lambda_m')$. Each number $1 \leq r \leq d$ can be expressed in the unique way as $r = \lambda_1 + \lambda_{i_r-1} + j_r$ with $1 \leq i_r \leq n$ and $1 \leq j_r \leq \lambda_{i_r}$. Let $\tau_\lambda \in \Sigma_d$ be such that $\tau(r) = \lambda_1' + \ldots + \lambda_{j_r-1}' + i_r$ for $1 \leq r \leq d$. We define the linear map $t_\lambda : V^{\otimes d} \to V^{\otimes d}$ on pure tensors as follows: $t_\lambda(v_1 \otimes \ldots \otimes v_d) = v_{\tau(1)} \otimes \ldots \otimes v_{\tau(d)}$. Finally, let

$$\begin{aligned}
&\Lambda^\lambda = \Lambda^{\lambda_1} \otimes \ldots \otimes \Lambda^{\lambda_n},\\
&S^\lambda = S^{\lambda_1} \otimes \ldots \otimes S^{\lambda_n},\\
&c_\lambda = c_{\lambda_1} \otimes \ldots \otimes c_{\lambda_n} : \Lambda^\lambda \to I^{\otimes d},\\
&m_\lambda = m_{\lambda_1} \otimes \ldots \otimes m_{\lambda_n} : I^{\otimes d} \to S^\lambda,
\end{aligned}$$

where $c_r : \Lambda^r \hookrightarrow I^{\otimes r}$ is the antisymmetrization map and $m_r : I^{\otimes r} \twoheadrightarrow S^r$ is the canonical projection. Then the Schur functor $S_\lambda$ and the Weyl functor $W_\lambda$ are defined by

$$S_\lambda = \mathrm{im}(m_{\widetilde{\lambda}} \circ t_{\widetilde{\lambda}} \circ c_\lambda), \quad W_\lambda = S_\lambda^{\#}.$$

By the definition of $S_\lambda$, the functor $W_\lambda$ can be directly expressed as the composition of the Kuhn duals of the maps given above: $W_\lambda = \mathrm{im}(\Gamma^{\widetilde{\lambda}} \to I^{\otimes d} \to \Lambda^\lambda)$. The category $\mathcal{P}_d$ is a highest weight category with poset $\Lambda(d)$ in the following way: the simple functor $F_\lambda$ is the image of the composition of the natural maps $W_\lambda \hookrightarrow \Lambda^\lambda \twoheadrightarrow S_\lambda$, $\nabla_\lambda = S_\lambda$ and $\Delta_\lambda = W_\lambda$. As examples, one derives from the above definitions that $S_{(1^d)} = S^d$, $W_{(1^d)} = \Gamma^d$ and $S_{(d)} = W_{(d)} = F_{(d)} = \Lambda^d$. We should explain why we use the *reversed* dominance order: it is caused by the fact that we label the family of Schur functors as in [ABW82, Wey03] and simultaneously follow the general conventions concerning highest weight categories.

If a field $\mathbf{k}$ is algebraically closed then it is well-known that $\mathrm{End}_{\mathcal{P}_d}(F_\lambda) = \mathbf{k}$. Assume now that $\mathbf{k}$ is not algebraically closed. Let $\overline{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$ and consider the extension of scalars functor $(-)_{\overline{\mathbf{k}}}$. It sends a strict polynomial functor $F$ over $\mathbf{k}$ to a strict polynomial functor over $\overline{\mathbf{k}}$ satisfying $F_{\overline{\mathbf{k}}}(V \otimes_{\mathbf{k}} \overline{\mathbf{k}}) = F(V) \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ (cf. [BFS97, §2]). By using the extension of scalars functor and [BFS97, Proposition 2.6] we conclude that $\mathrm{End}_{\mathcal{P}_d}(F_\lambda) = \mathbf{k}$ for any ground field $\mathbf{k}$. Then, by Proposition 1.1.1(f), we have

$$\mathrm{Ext}^q_{\mathcal{P}_d}(W_\lambda, S_\mu) = \begin{cases} \mathbf{k} & \text{if } q = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

By applying the axioms of a highest weight category and Proposition 1.1.1(d) it can be deduced that $\mathrm{End}_{\mathcal{C}}(S_\lambda) \simeq \mathrm{End}_{\mathcal{C}}(F_\lambda)$ (see the proof of [CPS88, Lemma 3.2(c)]). Then, by that and Proposition 1.1.1(d), we have

$$\mathrm{Ext}^q_{\mathcal{P}_d}(S_\lambda, S_\lambda) = \begin{cases} \mathbf{k} & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

We now present the definition of the Koszul duality and its properties studied in [Cha08]. The Koszul duality is an exact functor $\Theta : \mathcal{D}^b\mathcal{P}_d \to \mathcal{D}^b\mathcal{P}_d$ given by

$$\Theta(F)(V) = \mathrm{RHom}_{\mathcal{D}^b\mathcal{P}_d}(\Lambda^d \circ \mathrm{Hom}_{\mathbf{k}}(V, -), F).$$

We have $\Theta(S_\lambda) = W_{\widetilde{\lambda}}$ and this is also a self-equivalence of $\mathcal{D}^b\mathcal{P}_d$ (cf. [Cha08, Proposition 2.2 and Corollary 2.4]). It preserves tensor products, i.e., $\Theta(F \otimes G) = \Theta(F) \otimes \Theta(G)$ (cf. [Cha08, Proposition 2.6]). Using these properties and the Kuhn duality, we get isomorphisms:

$$\mathrm{Ext}^*_{\mathcal{P}_d}(S_\lambda, S_\mu) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(W_{\widetilde{\lambda}}, W_{\widetilde{\mu}}) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(S_{\widetilde{\mu}}, S_{\widetilde{\lambda}}), \tag{1.3}$$

$$\mathrm{Ext}^*_{\mathcal{P}_d}(S_\lambda, S_{\eta_1} \otimes S_{\eta_2}) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(W_{\widetilde{\lambda}}, W_{\widetilde{\eta_1}} \otimes W_{\widetilde{\eta_2}}) \simeq \mathrm{Ext}^*_{\mathcal{P}_d}(S_{\widetilde{\eta_1}} \otimes S_{\widetilde{\eta_2}}, S_{\widetilde{\lambda}}). \tag{1.4}$$

It is worth noting that the isomorphism (1.3) was already established in [AB88, Theorem 7.7].

We now recall the definitions of the Koszul and de Rham complexes introduced and studied in [FLS94] and [FS97]. For $e \geq 1$ and $0 \leq i \leq e$ set $\Omega^i_e = S^{e-i} \otimes \Lambda^i$. We refer to $i$ as the cohomological degree. The Koszul differential $\kappa_i : \Omega^i_e \to \Omega^{i-1}_e$ and the de Rham differential $d_i : \Omega^i_e \to \Omega^{i+1}_e$ are defined by using the coproduct and product operations in the symmetric and exterior algebras:

$$\kappa_i : S^{e-i}(V) \otimes \Lambda^i(V) \to S^{e-i}(V) \otimes V \otimes \Lambda^{i-1}(V) \to S^{e-i+1}(V) \otimes \Lambda^{i-1}(V),$$
$$d_i : S^{e-i}(V) \otimes \Lambda^i(V) \to S^{e-i-1}(V) \otimes V \otimes \Lambda^i(V) \to S^{e-i-1}(V) \otimes \Lambda^{i+1}(V).$$

The complex $(\Omega^\bullet_e, \kappa)$ is called the Koszul complex and it is well-known that this complex is acyclic. The complex $(\Omega^\bullet_e, d)$ is called the de Rham complex. The homology of the de Rham complex is described by the Cartier isomorphism $H^*((\Omega^*_*(V), d)) \simeq \Omega^*_*(V^{(1)})$, where $V^{(1)}$ is the image of a vector space $V$ under the Frobenius twist functor (cf. [FS97, Theorem 4.1]). The de Rham and Koszul differentials are related by the formula $d\kappa + \kappa d = e \cdot \mathrm{id}$. In particular, if $e$ is divisible by $p$, then

$$d\kappa + \kappa d = 0. \tag{1.5}$$

Let $K^i_e = \ker \kappa_i$. The equality (1.5) shows that $(K^\bullet_e, d)$ is a complex if and only if $e$ is divisible by $p$. In this case it is a subcomplex of $(\Omega^\bullet_e, d)$. It will be called the Koszul kernel complex. More important for us than the Cartier isomorphism recalled above will be the restriction of this isomorphism to the Koszul kernel complex:

$$H^*(K^\bullet_{pe}(V)) \simeq K^*_e(V^{(1)}). \tag{1.6}$$

Consider now $d_{j-1}\kappa_j \in \mathrm{End}_{\mathcal{P}_e}(\Omega^j_e)$ for $1 \leq j \leq d-1$. One has

$$d_{j-1}\kappa_j \neq 0 \tag{1.7}$$

Indeed, let $V$ be a vector space with a basis $\{v_1, \ldots, v_e\}$. Then ($\widehat{x}$ means here that $x$ is omitted)

$$\begin{aligned}
&v_1 \ldots v_{e-j} \otimes v_{e-j+1} \wedge \ldots \wedge v_e \overset{\kappa_j}{\longmapsto} \\
&\sum_{m=1}^{j} (-1)^{m-1} v_1 \ldots v_{e-j} v_{e-j+m} \otimes v_{e-j+1} \wedge \ldots \wedge \widehat{v_{e-j+m}} \wedge \ldots \wedge v_e \overset{d_{j-1}}{\longmapsto} \\
&j \cdot v_1 \ldots v_{e-j} \otimes v_{e-j+1} \wedge \ldots \wedge v_e + \\
&\sum_{m=1}^{j} \sum_{n=1}^{e-j} v_1 \ldots \widehat{v_n} \ldots v_{e-j} v_{e-j+m} \otimes v_{e-j+1} \wedge \ldots v_{e-j+m-1} \wedge v_n \wedge \ldots \wedge v_e \neq 0.
\end{aligned}$$

The important class of Young diagrams, which is considered in the dissertation, is the set of hooks, i.e., Young diagrams $(m, 1^{d-m})$. Let $S_i = S_{(i+1,1^{d-i-1})}$, $W_i = W_{(i+1,1^{d-i-1})}$ and $F_i = F_{(i+1,1^{d-i-1})}$ for $0 \leq i \leq d-1$. It is known that $S_i$ is the cokernel of the Koszul differential $\kappa_{i+2} : \Omega^{i+2}_d \to \Omega^{i+1}_d$ for all $0 \leq i \leq d-2$ (cf. [Wey03, (2.1.3) Example (h)]). For $i = d-1$ we have $S_{d-1} = \Lambda^d = \Omega^d_d$. By the acyclicity of the Koszul complex $(\Omega^\bullet_d, \kappa)$ we see that $S_i$ is indeed the kernel of the Koszul differential $\kappa_i : \Omega^i \to \Omega^{i-1}$ and, by the acyclicity of the Koszul complex, we obtain the short exact sequence

$$0 \to S_m \to \Omega^m \to S_{m-1} \to 0 \tag{1.8}$$

for $0 \leq m \leq d$, assuming $S_{-1} = S_d = 0$.

## 1.2 The Littlewood-Richardson rule

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\eta = (\eta_1, \dots, \eta_m)$ be Young diagrams of weights $d_1$ and $d_2$, respectively. By defintion, the diagram $\eta$ is contained in $\lambda$ if and only if $m \leq n$ and $\eta_i \leq \lambda_i$ for $1 \leq i \leq m$. In this case one has $d_1 \geq d_2$. We define the skew Young diagram $\lambda/\eta$ to be the figure obtained from $\lambda$ by removing the first $\eta_i$ boxes in the $i$-th row for $1 \leq i \leq m$. The skew Young diagram $\lambda/\eta$ is disconnected if $\eta_i \geq \lambda_{i+1}$ for some $1 \leq i \leq \min\{m, n-1\}$; otherwise, it is connected.

One can also consider Schur functors corresponding to skew Young diagrams. These functors will not be the main object of interests in this dissertation and will play only the technical role. They appear mainly in the decomposition formula, which will be introduced in the next section. The Schur functor corresponding to $\lambda/\eta$ is defined in the similar way as the Schur functor corresponding to $\lambda$: one replaces the length of rows of $\lambda$ and $\widetilde{\lambda}$ by the length of rows of $\lambda/\eta$ and $\widetilde{\lambda}/\widetilde{\eta}$, respectively (cf. [ABW82, Definition II.1.4]). It can be deduced from the definition that if $\lambda/\eta$ is disconnected and decomposes as the disjoint union of two (ordinary or skew) Young diagrams $\lambda_1$ and $\lambda_2$, then $S_{\lambda/\eta} \simeq S_{\lambda_1} \otimes S_{\lambda_2}$.

Before formulating the Littlewood–Richardson rule, we recall the necessary combinatorial definitions. A tableau of shape $\lambda$ (resp. $\lambda/\eta$) is a filling of $\lambda$ (resp. $\lambda/\eta$), i.e., a function, which assigns to a box of $\lambda$ (resp. $\lambda/\eta$) a positive integer. A tableau is semistandard if and only if each row is a strictly increasing sequence and each column is a weakly increasing sequence. The content of a tableau $T$ is the finite sequence $(a_1, \dots, a_n)$ such that $a_n$ is the number of the occurrences of the number $1 \leq i \leq k$ in $T$. A finite sequence $(b_1, \dots, b_n)$ of positive integers is a Yamanouchi word if for each $1 \leq m \leq n$ and each positive integer $i$ the number of occurrences of $i$ in the subsequence $(b_1, \dots, b_m)$ is not smaller than the number of occurrences of $i+1$ there. A tableau satisfies the Yamanouchi condition if the sequence obtained by reading each column from the bottom up, starting from the leftmost column and moving column by column to the right, is a Yamanouchi word. For given Young diagrams $\alpha, \eta, \lambda$ with $\eta \subset \lambda$ let $c(\alpha, \eta; \lambda)$ denote the number of tableaux of shape $\lambda/\eta$ and content $\widetilde{\alpha}$ that satisfy the Yamanouchi condition.

**Example.** Let $\lambda = (4, 3, 2, 1), \eta = (2, 1), \alpha = (3^2, 1)$. Then there are exactly two semistandard tableaux of shape $\lambda/\eta$ with content $\widetilde{\alpha} = (3, 2^2)$, which satisfy the Yamanouchi condition, i.e., $c(\alpha, \eta; \lambda) = 2$:

$$\begin{array}{cccc} & & 2 & 3 \\ & 1 & 3 & \\ 1 & 2 & & \\ 1 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 3 \\ & 2 & 3 & \\ 1 & 2 & & \\ 1 & & & \end{array} .$$

Now we turn to the Littlewood–Richardson rule.

**Theorem 1.2.1** (Littlewood–Richardson rule [Bof88])**.**

*(a) Let $\lambda$ and $\eta$ be Young diagrams such that $\eta \subset \lambda$. Then $S_{\lambda/\eta}$ has a good filtration such that*

$$S_{\lambda/\eta} = \sum_{\substack{\alpha \\ |\alpha| = |\lambda| - |\eta|}} c(\alpha, \eta; \lambda) \cdot S_\alpha.$$

*(b) Let $\lambda$ and $\eta$ be Young diagrams. Then $S_\lambda \otimes S_\eta$ has a good filtration such that*

$$S_\lambda \otimes S_\eta = \sum_{\substack{\lambda \subset \alpha \\ |\alpha| = |\lambda| + |\eta|}} c(\eta, \lambda; \alpha) \cdot S_\alpha.$$

Let us note that the part (b) follows from (a). Indeed, for $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\eta = (\eta_1, \dots, \eta_n)$ let $\alpha = (\eta_1 + \lambda_1, \dots, \eta_1 + \lambda_m, \eta_1, \dots, \eta_n)$ and $\beta = (\eta_1^m)$. Then the part (b) follows from (a) and the fact that $c(\gamma, \beta; \alpha) = c(\eta, \lambda; \gamma)$ (see, e.g., [Ful97, Lemma 1 and Corollary 2 in Chapter 5]).

## 1.3 The decomposition and exponential formulas

In this section we present the sum-diagonal adjunction discussed in [FFSS99, Proof of Theorem 1.7] and its consequences like the decomposition and exponential formulas.

The category $\mathcal{P}(2)_d$ of strict polynomial bifunctors of total degree $d$ is the category of $\mathbf{k}$-linear functors $\Gamma^d(\mathcal{V}ect_{\mathbf{k}} \times \mathcal{V}ect_{\mathbf{k}}) \to \mathcal{V}ect_{\mathbf{k}}$ (cf. [Tou10a, pp. 37–39]). Loosely speaking, a strict polynomial bifunctor $F$ of degree $d$ is a bifunctor such that for finite-dimensional vector spaces $V$ and $W$, $F(-, W)$ and $F(V, -)$ are (covariant) strict polynomial functors. Moreover, $F$ decomposes as the direct sum of homogeneous strict polynomial bifunctors, where each summand is of degree $d_1$ in the first variable, degree $d_2$ in the second variable and $d_1 + d_2 = d$.

Let $D : \mathcal{V}ect_{\mathbf{k}} \to \mathcal{V}ect_{\mathbf{k}} \times \mathcal{V}ect_{\mathbf{k}}$ be the diagonal functor given by $D(V) = (V, V)$. Let $\Pi : \mathcal{V}ect_{\mathbf{k}} \otimes \mathcal{V}ect_{\mathbf{k}} \to \mathcal{V}ect_{\mathbf{k}}$ be the direct sum functor given by $\Pi(V, W) = V \oplus W$. By precomposing with these functors we get a pair of adjoints on both sides between $\mathcal{P}_d$ and $\mathcal{P}(2)_d$. Since $D$ and $\Pi$ are exact, we have for $F \in \mathcal{P}_d$ and $G \in \mathcal{P}(2)_d$ the equalities

$$\mathrm{Ext}^*_{\mathcal{P}_d}(F, G \circ D) = \mathrm{Ext}^*_{\mathcal{P}(2)_d}(F \circ \Pi, G),$$
$$\mathrm{Ext}^*_{\mathcal{P}_d}(G \circ D, F) = \mathrm{Ext}^*_{\mathcal{P}(2)_d}(G, F \circ \Pi).$$

For $F \in \mathcal{P}_d, G \in \mathcal{P}_e$ we define the (external) tensor product bifunctor $- \boxtimes - \in \mathcal{P}(2)_{d+e}$ via $(F \boxtimes G)(V, W) = F(V) \otimes G(W)$. We now introduce the decomposition formula. It is known that there is a filtration of $S_\lambda(V \oplus W)$ with the associated graded object $\bigoplus_{\alpha \subset \lambda} S_\alpha(V) \otimes S_{\lambda/\alpha}(W)$ (cf. [ABW82, Theorem II.4.11]). Then, for $F \in \mathcal{P}_d$ and $G \in \mathcal{P}_e$, by using the above adjunction isomorphisms and the Künneth formula (cf. [BFS97, Proposition 3.6]) we obtain the spectral sequences, both called the decomposition formula, whose first pages are:

$$E_1^{**} = \sum_{\substack{\alpha \subset \lambda \\ |\alpha| = d}} \mathrm{Ext}^*_{\mathcal{P}_d}(F, S_\alpha) \otimes \mathrm{Ext}^*_{\mathcal{P}_e}(G, S_{\lambda/\alpha}) \Rightarrow \mathrm{Ext}^*_{\mathcal{P}(2)_{d+e}}(F \boxtimes G, \sum_{\substack{\alpha \subset \lambda \\ |\alpha| = d}} S_\alpha \boxtimes S_{\lambda/\alpha}) =$$
$$\mathrm{Ext}^*_{\mathcal{P}(2)_{d+e}}(F \boxtimes G, S_\lambda(- \oplus -)) = \mathrm{Ext}^*_{\mathcal{P}_{d+e}}(F \otimes G, S_\lambda),$$
$$E_1^{**} = \sum_{\substack{\alpha \subset \lambda \\ |\alpha| = d}} \mathrm{Ext}^*_{\mathcal{P}_d}(S_\alpha, F) \otimes \mathrm{Ext}^*_{\mathcal{P}_e}(S_{\lambda/\alpha}, G) \Rightarrow \mathrm{Ext}^*_{\mathcal{P}_{d+e}}(S_\lambda, F \otimes G) \tag{1.9}$$

(the variant with $S_\lambda$ in the first variable is obtained exactly as the one with $S_\lambda$ in the second variable). More precisely, the term $E_1^{st}$ is $E_1^{st} = \bigoplus_{s_1+s_2=s+t} \mathrm{Ext}^{s_1}_{\mathcal{P}_d}(F, S_\alpha) \otimes \mathrm{Ext}^{s_2}_{\mathcal{P}_e}(G, S_{\lambda/\alpha})$ in the first spectral sequence and $E_1^{st} = \bigoplus_{s_1+s_2=s+t} \mathrm{Ext}^{s_1}_{\mathcal{P}_d}(S_\alpha, F) \otimes \mathrm{Ext}^{s_2}_{\mathcal{P}_e}(S_{\lambda/\alpha}, G)$ in the second one, where $t$ is the place of $\alpha$ in the reversed lexicographic order defined as follows: $\lambda$ is smaller than $\eta$ in the reversed lexicographic order if $\lambda_i > \eta_i$ for the smallest $i$ such that $\lambda_i \neq \eta_i$.

Fix $H_* = S^*$, $H_* = \Lambda^*$ or $H_* = \Gamma^*$. If $\lambda \in \{(i), (1^i)\}$ then the aforementioned filtration for $S_\lambda(- \oplus -)$ gives the well-known formula (in case of $\Gamma^i$ we also use the Kuhn duality): $H_i(V \oplus W) \simeq \bigoplus_{m=0}^i H_m(V) \otimes H_{i-m}(W)$. In this case the decomposition formula (1.9) reduces to the following, which we call the exponential formula (see also [FFSS99, Theorem 1.7]):

$$\mathrm{Ext}^q_{\mathcal{P}_{d+e}}(F \otimes G, H_{d+e}) = \bigoplus_{m=0}^{q} \mathrm{Ext}^m_{\mathcal{P}_d}(F, H_d) \otimes \mathrm{Ext}^{q-m}_{\mathcal{P}_e}(G, H_e),$$
$$\mathrm{Ext}^q_{\mathcal{P}_{d+e}}(H_{d+e}, F \otimes G) = \bigoplus_{m=0}^{q} \mathrm{Ext}^m_{\mathcal{P}_d}(H_d, F) \otimes \mathrm{Ext}^{q-m}_{\mathcal{P}_e}(H_e, G). \tag{1.10}$$

## 1.4 The block structure of $\mathcal{P}_d$ and the related combinatorics

In this section we recall the block structure of $\mathcal{P}_d$, which is inherited from that of the Schur algebra, since $\mathcal{P}_d \simeq S(n,d) - \text{mod}$ for $n \geq d$. We begin with the necessary combinatorial terminology and results, whose exposition can be found in, e.g., [JK81, Section 2.7].

A skew $p$-hook is a connected skew Young diagram with $p$ boxes that does not contain the Young diagram $(2,2)$. By definition, the foot of a skew $p$-hook is the last box in its first column and the hand is the last box in its first row. If the first row of a skew $p$-hook has length $m$, then the skew $p$-hook corresponds to the hook $(m, 1^{p-m})$.

We now describe the procedure of the removal of skew $p$-hooks from a given Young diagram $\lambda$. Whenever possible, we remove a skew $p$-hook from the rim of $\lambda$ so that its foot and hand are, respectively, the last boxes of the column and row of $\lambda$ that contain them. The shape obtained by this removal is a (non-skew) Young diagram. A $p$-core is a Young diagram from which no skew $p$-hook can be removed. The $p$-core of $\lambda$ is a $p$-core obtained from $\lambda$ by successively removing skew $p$-hooks until none remain. It is well-known that the $p$-core of a Young diagram does not depend on the order of removals (cf. [JK81, Theorem 2.7.16]). The $p$-weight of $\lambda$ is the total number of skew $p$-hooks removed in this process.

We are now ready to describe the block structure of $\mathcal{P}_d$. It turns out to be similar to that of the group algebra $\mathbf{k}[\Sigma_d]$, as described by the Nakayama conjecture.

**Theorem 1.4.1** ( [Don87, Theorem (2.12)])**.** *The simple functors $F_\lambda, F_\eta \in \mathcal{P}_d$, and therefore the Schur functors $S_\lambda, S_\mu$, lie in the same block of $\mathcal{P}_d$ if and only if $\lambda$ and $\eta$ have the same $p$-core.*

We note that the part of the theorem for Schur functors follows from that for simple functors and the fact that the socle of $S_\alpha$ is $F_\alpha$ for a Young diagram $\alpha$ (see Proposition 1.1.1(a)). The block of $\mathcal{P}_d$ corresponding to a $p$-core $\lambda$ is denoted by $\mathcal{P}_d^\lambda$.

Let us now introduce a convenient tool for recording the process of the removal of skew $p$-hooks from a Young diagram. Let $\lambda = (\lambda_1, \dots, \lambda_j, 0, \dots)$ be a Young diagram (extended by zeros beyond its last nonzero row). The $\beta$-sequence of $\lambda$ is defined by $\beta_l = \lambda_l - l$ for $l \geq 1$. Now consider an infinite abacus with the $p$ vertical runners, numbered $0, \dots, p-1$ from left to right, and infinitely many horizontal rows indexed by all integers from top to bottom. We label the position in the $m$-th runner and $n$-th row by the integer $np + m$. To represent $\lambda$, we put a bead at every position whose label is an element of the $\beta$-sequence of $\lambda$. Sliding a bead one space up is equivalent to removing a skew $p$-hook. Indeed, if it is possible to remove a skew $p$-hook in $\lambda$ and $(\beta_l)$ is its $\beta$-sequence, then one checks that $(\beta_1, \dots, \beta_{l-1}, \beta_{l+1}, \dots, \beta_{l+p-i-1}, \beta_l - p, \beta_{l+p-i}, \dots)$ is the $\beta$-sequence of the diagram arising from $\lambda$ by removing a skew $p$-hook corresponding to the hook $(i+1, 1^{p-i-1})$ whose hand lies in the $l$-th row of $\lambda$. Consequently, the resulting diagram is represented by the abacus configuration obtained from that of $\lambda$ by sliding the bead labeled $\beta_l$ one space upward. The following observation, which we record as a lemma for later convenience, follows from comparing the abacus configurations of $\lambda$ and the diagram obtained by removing the skew $p$-hook.

**Lemma 1.4.2.** *Let $\lambda$ be a Young diagram. If it is possible to remove a skew $p$-hook corresponding to the hook $(i+1, 1^{p-i-1})$ from $\lambda$, then there are exactly $p-i-1$ beads in the abacus configuration of $\lambda$ between the bead that is slid one space up to remove the given skew $p$-hook and its new position.*

The number of Young diagrams of $p$-weight $k$ is equal to

$$\sum_{(\alpha_0, \dots, \alpha_{p-1})} p(\alpha_0) \cdot \ldots \cdot p(\alpha_{p-1}),$$

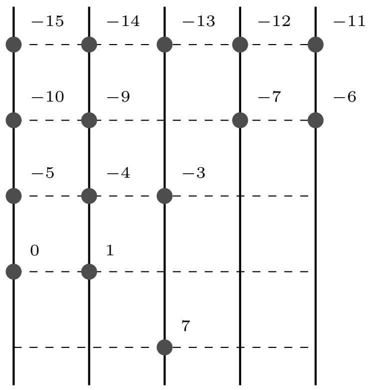
where the sum is taken over the set of sequences of $p$ nonnegative integers whose sum is $k$ and $p(\alpha_i)$ denotes the number of Young diagrams of weight $\alpha_i$.

We end this section by explaining the way how to encode the abacus configuration of $\lambda$ into a sequence of Young diagrams. Let $0 \leq m \leq p-1$. For the $m$-th runner we start with the bead having the lowest label and the empty space immediately above it. We slide this bead upward as long as possible, recording the number of empty spaces between its initial position and the next bead above. Then we repeat this process with the updated runner configuration, continuing until there is no bead that can be slid upward. The sequence of the recorded numbers, taken in the reverse order, is a weakly decreasing sequence $\lambda^{(m)}$, i.e., a Young diagram. If no bead has an empty space above it, then we set $\lambda^{(m)}$ to be the empty diagram. The $p$-quotient of $\lambda$ is defined as the sequence $(\lambda^{(0)}, \ldots, \lambda^{(p-1)})$. It follows from the discussion in this and the previous paragraphs that each Young diagram is uniquely determined by its $p$-core and $p$-quotient.

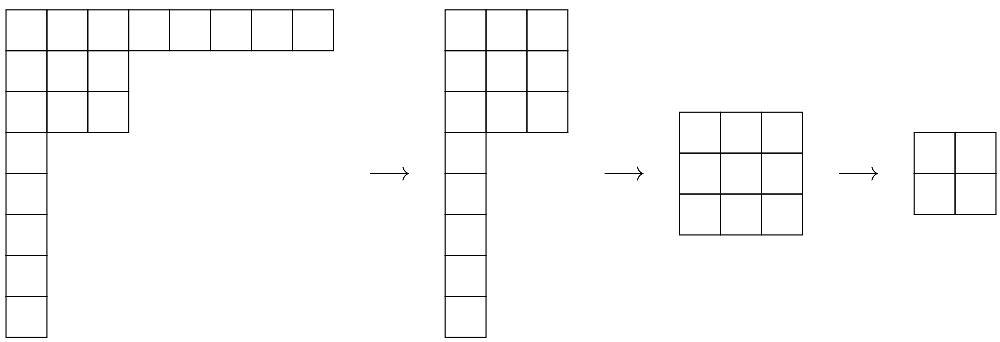
**Example.** Let $\lambda = (8, 3^2, 1^5)$ and $p = 5$. The $\beta$-sequence of $\lambda$ is

$$(7, 1, 0, -1, -2, -3, -4, -5, -6, -7, -9, -10, \ldots).$$

The following illustration shows the abacus configuration of $\lambda$:




The 5-core of $\lambda$ is $(2, 2)$. Indeed, sliding all beads upward to eliminate gaps in each runner, we obtain the abacus configuration corresponding to the sequence $(1, 0, -3, -4, \ldots)$, which is the $\beta$-sequence of the 5-core $\lambda = (2, 2)$. It is clear from the abacus configuration of $\lambda$ that it is of 5-weight 3 and its 5-quotient is $(\varnothing, \varnothing, (2, 1), \varnothing, \varnothing)$. Note that there are two ways to remove skew 5-hooks. Here, we give an example of the removal corresponding to the bead slides starting from the bead with label 7.



# Chapter 2

# Homological considerations in $\mathcal{P}_p$

## 2.1 Injective envelopes of simples. Resolutions of Schur functors

By Theorem 1.4.1 the simple functors $F_\lambda$ and $F_\eta$ belong to the same block if and only if $\lambda$ and $\eta$ have the same $p$-core. This theorem also holds for Schur functors and Weyl functors. It is straightforward that a non-hook is a $p$-core and all hooks have the same $p$-core, namely the empty diagram. Since the Ext-groups between functors belonging to different blocks vanish, we restrict our attention to the homological algebra of the block $\mathcal{P}_p^{\varnothing}$. Since all computations of Ext-groups are performed in this block, we ease our notation by omitting subscripts in our Ext-groups. Now we turn to the injective envelopes and projective covers of simples in $\mathcal{P}_p^{\varnothing}$.

**Proposition 2.1.1.** *Let $0 \leq i \leq p-1$ and let $I_i$ and $P_i$ be, respectively, the injective envelope and the projective cover of the simple functor $F_i$. Then $I_i = \Omega^i$ and $P_i = (\Omega^i)^\#$. In particular, $I_i \simeq P_i$ unless $i = 0$.*

*Proof.* Fix $0 \leq i \leq p-1$. Since $F_i$ is a subfunctor of $S_i$ and $S_i = \ker(\kappa_i : \Omega^i \to \Omega^{i-1})$, $F_i$ is a subfunctor of $\Omega^i$. We observe that $\Lambda^i$ is a direct summand of $I^{\otimes i}$ and $S^{(p-i,1^i)} \simeq S^{p-i} \otimes I^{\otimes i}$ is an injective functor as a direct summand of the injective functor $S^p(\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}^p, -))$ (cf. [FS97, Theorem 2.10]). Hence $\Omega^i = S^{p-i} \otimes \Lambda^i$ is also injective. To end the proof, it is sufficient to show that $\Omega^i$ is an indecomposable functor. Then $I_i = \Omega^i$ and, by the Kuhn duality, $P_i = (\Omega^i)^\#$. The last statement follows from the first part and the isomorphism $S^i \simeq \Gamma^i$ for $i < p$.

The functor $\Omega^0 = S^p = S_0$ is indecomposable by Proposition 1.1.1(a). We now assume that $1 \leq i \leq p-1$ and we suppose that $\Omega^i$ is a decomposable functor, i.e., there exist nonzero subfunctors $G_1, G_2$ of $\Omega^i$ such that $\Omega^i = G_1 \oplus G_2$. Since a direct summand of an injective object is injective, the functors $G_1$ and $G_2$ are injective. Then it follows that $\mathrm{Ext}^1(W_\mu, G_k) = 0$ for $k = 1, 2$ and a Young diagram $\mu$ of weight $p$. Therefore, by Proposition 1.1.1(h), $G_1$ and $G_2$ are filtered by Schur functors. By the long exact sequence for $\mathrm{Ext}^*(W_\mu, -)$ applied to the short exact sequence $0 \to G_1 \to \Omega^i \to G_2 \to 0$ and the injectivity of $G_1, G_2$ and $\Omega^i$ we obtain the short exact sequence $0 \to \mathrm{Hom}(W_\mu, G_1) \to \mathrm{Hom}(W_\mu, \Omega^i) \to \mathrm{Hom}(W_\mu, G_2) \to 0$. It follows from the short exact sequence above, the fact that $\Omega^i$ has the filtration by Schur functors given by (1.8) and Proposition 1.1.1(h) (together with the equality $\mathrm{End}(F_\lambda) = \mathbf{k}$) that

$$(G_1 : S_\mu) + (G_2 : S_\mu) = (\Omega^i : S_\mu) = \begin{cases} 1 & \text{if } \mu = (i+1, 1^{p-i-1}) \text{ or } \mu = (i, 1^{p-i}), \\ 0 & \text{otherwise.} \end{cases}$$

Then we conclude that

$$G_1 = S_i, G_2 = S_{i-1} \quad \text{or} \quad G_1 = S_{i-1}, G_2 = S_i. \tag{2.1}$$

We claim that the only injective Schur functor in $\mathcal{P}_p^{\varnothing}$ is $S^p$. Indeed, we have seen above that $S^p = \Omega^0$ is injective. By the long exact sequence for $\mathrm{Ext}^*(I^{(1)}, -)$ applied to the short exact sequence (1.8) and the Vanishing Theorem (cf. [FS97, Theorem 2.13], this is also a special case of the sum-diagonal adjunction introduced in Section 1.3) we see that $\mathrm{Ext}^i(I^{(1)}, S_i) \simeq \mathrm{Ext}^{i-1}(I^{(1)}, S_{i-1})$. Since $\mathrm{Hom}(I^{(1)}, S^p) = k$ (cf. [FS97, Theorem 2.10]), we conclude, by induction on $i$, that $S_i$ is not injective for $i > 0$. Thus, $S^p$ is the only injective functor in $\mathcal{P}_p^{\varnothing}$. By (2.1) it leads to the contradiction with the injectivity of both the functors $G_1$ and $G_2$. In particular, $\Omega^i$ is indecomposable. □

**Corollary 2.1.2** (The decomposition matrix).
*The decomposition matrix $D = (d_{lm}) \in M_{p\times p}(\mathbb{Z})$ of the category $\mathcal{P}_p^{\varnothing}$ is given by*

$$d_{lm} := [W_{l-1} : F_{m-1}] = \begin{cases} 1 & \text{if } m = l \text{ or } m = l+1, \\ 0 & \text{otherwise.} \end{cases}$$

*Moreover, for all $0 \le j \le p-1$ the following short exact sequences hold, assuming $F_p = 0$:*

$$0 \to F_{j+1} \to W_j \to F_j \to 0 \tag{2.2}$$

$$0 \to F_j \to S_j \to F_{j+1} \to 0. \tag{2.3}$$

*Proof.* It follows from Proposition 2.1.1 and the short exact sequence (1.8) that $I_0 \simeq S_0$ and $I_j$ is filtered by the Schur functors $S_j$ and $S_{j-1}$ for $1 \le j \le p-1$. By the Brauer-Humphreys reciprocity (see Theorem 1.1.1(g)) $W_j$ has the composition series containing only two quotients, which are isomorphic to $F_j$ and $F_{j+1}$, i.e.,

$$[W_j : F_i] = \begin{cases} 1 & \text{if } i = j \text{ or } i = j+1, \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le i, j \le p-1$. It gives us immediately the decomposition matrix of $\mathcal{P}_p^{\varnothing}$ and implies the following short exact sequence

$$0 \to F_{j+1} \to W_j \to F_j \to 0$$

for $0 \le j \le p-1$. By the Kuhn duality we obtain the second short exact sequence given in the assertion of the corollary. □

**Remark.** The decomposition matrix of $S(n, p)$ for $n \ge p$, and therefore of $\mathcal{P}_p$, is already known (cf. [DEN04, DM09]). We obtain this result by the different method and it is a straightforward corollary of Proposition 2.1.1.

We now give the alternative proof of the main result of [Xi92], which we obtain as another corollary of Proposition 2.1.1.

**Corollary 2.1.3.** *Let $Q$ be the quiver*

$$1 \underset{w_0}{\overset{v_0}{\rightleftarrows}} 2 \underset{w_1}{\overset{v_1}{\rightleftarrows}} 3 \rightleftarrows \cdots \rightleftarrows p-1 \underset{w_{p-2}}{\overset{v_{p-2}}{\rightleftarrows}} p$$

*and let $I \subset \mathbf{k}Q$ be the ideal of the path algebra of $Q$ generated by $v_i v_{i+1}$, $w_{i+1} w_i$, $v_{i+1} w_{i+1} - w_i v_i$, $v_0 w_0$ for $0 \le i \le p-3$. Then there is an equivalence of categories*

$$\mathcal{P}_p^{\varnothing} \simeq (\mathbf{k}Q/I) - \textit{mod}.$$

*Proof.* Let $\Omega = \bigoplus_{0\le i\le p-1}\Omega^i$. By Proposition 2.1.1 the functor $\Omega^\#$ is a projective generator of $\mathcal{P}_p^\varnothing$. Let $\mathsf{P_d}$ be the category of $\mathbf{k}$-linear functors from $\Gamma^d\mathcal{V}ect_{\mathbf{k}}$ to the category of all vector spaces over $\mathbf{k}$. It is known that $\mathsf{P_d} \simeq S(n,d) - \mathrm{Mod}$ for $n \ge d$ (cf. [Kra22, Proposition 8.3.11]). Under this equivalence, let $\mathsf{P}_\mathsf{p}^\varnothing$ be the block of $\mathsf{P_p}$ corresponding to the empty diagram. Then by the Gabriel theorem (cf. [Mit65, Theorem 4.1]) we have $\mathsf{P}_\mathsf{p}^\varnothing \simeq \mathrm{Mod} - \mathrm{End}_{\mathsf{P}_\mathsf{p}^\varnothing}(\Omega^\#)$. Since this equivalence is given by $\mathrm{Hom}_{\mathsf{P}_\mathsf{p}^\varnothing}(\Omega^\#,-)$, it preserves finite-dimensionality. Therefore, we obtain

$$\mathcal{P}_p^\varnothing \simeq \mathrm{mod} - \mathrm{End}_{\mathcal{P}_p^\varnothing}(\Omega^\#) \simeq \mathrm{End}_{\mathcal{P}_p^\varnothing}(\Omega^\#)^{\mathrm{op}} - \mathrm{mod} \simeq \mathrm{End}_{\mathcal{P}_p^\varnothing}(\Omega) - \mathrm{mod}.$$

From now on we omit the subscript in the End-space, since all computations are performed in $\mathcal{P}_p^\varnothing$.

Now our goal is to describe the structure of the algebra $\mathrm{End}(\Omega)$. Let us recall that $S^d = S_0$ and $\Lambda^d = S_{d-1}$. For $d = 1$ we have $S^1 = \Lambda^1$, hence $\mathrm{Hom}(S^1,\Lambda^1) = \mathrm{Hom}(\Lambda^1,S^1) = \mathbf{k}$ by (1.2). For $1 < d < p$ the Young diagrams $(d)$ and $(1^d)$ are different $p$-cores. In consequence, $\mathrm{Hom}(S^d,\Lambda^d) = 0$ and $\mathrm{Hom}(\Lambda^d,S^d) = 0$. Let $0 \le i,j \le p-1$. By the computation similar to that in (1.9), the exponential formula (1.10), the above equalities and (1.2) we obtain

$$\begin{aligned}
&\mathrm{Hom}(\Omega^i,\Omega^j) = \mathrm{Hom}_{\mathcal{P}(2)_p}(\Omega^i(-\oplus-), S^{p-j}\boxtimes\Lambda^j) = \\
&\mathrm{Hom}_{\mathcal{P}(2)_p}\left(\bigoplus_{\substack{0\le r\le p-i\\ 0\le s\le i}} (S^r\otimes\Lambda^s)\boxtimes(S^{p-i-r}\otimes\Lambda^{i-s}), S^{p-j}\boxtimes\Lambda^j\right) = \\
&\bigoplus_{\substack{0\le r\le p-i\\ 0\le s\le i}} \mathrm{Hom}_{\mathcal{P}}(S^r\otimes\Lambda^s, S^{p-j})\otimes\mathrm{Hom}_{\mathcal{P}}(S^{p-i-r}\otimes\Lambda^{i-s},\Lambda^j) = \\
&\bigoplus_{\substack{0\le r\le p-i\\ 0\le s\le i}} \mathrm{Hom}_{\mathcal{P}}(S^r,S^r)\otimes\mathrm{Hom}_{\mathcal{P}}(\Lambda^s,S^{p-j-r})\otimes\mathrm{Hom}_{\mathcal{P}}(S^{p-i-r},\Lambda^{j-i+s})\otimes\mathrm{Hom}_{\mathcal{P}}(\Lambda^{i-s},\Lambda^{i-s}) = \\
&\begin{cases}\mathbf{k} & \text{if } j-i\in\{-1,1\} \text{ or } i=j=0,\\ \mathbf{k}^2 & \text{if } i>0 \text{ and } i=j,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{2.4}$$

Denote $\theta_i = d_i$, $\overline{\theta_i} = (-1)^i\kappa_{i+1}$ and $\widetilde{\theta_i} = (-1)^i d_i\kappa_{i+1}$ for $0 \le i \le p-2$. It follows from (1.7) that $\overline{\theta_i}$ and $\widetilde{\theta_i}$ are linearly independent. Using this observation and the formula (2.4), we deduce that

$$\mathcal{A} = \{\theta_i : 0\le i\le p-2\}\cup\{\overline{\theta_i} : 0\le i\le p-2\}\cup\{\widetilde{\theta_i} : 0\le i\le p-2\}\cup\{\mathrm{id}_{\Omega^i} : 0\le i\le p-1\}$$

is a basis of $\mathrm{End}(\Omega) = \bigoplus_{0\le i,j\le p-1}\mathrm{Hom}(\Omega^i,\Omega^j)$. By (1.5) we observe that for $0 \le i \le p-2$ we have

$$\kappa_1 d_0 = 0, \quad \kappa_{i+1}d_i = -d_{i-1}\kappa_i \text{ for } i\ge 1,$$
$$\kappa_{i+1}d_i\kappa_{i+1} = \kappa_{i+1}(-\kappa_{i+2}d_{i+1}) = 0, \quad d_i\kappa_{i+1}d_i = d_i(-d_{i-1}\kappa_i) = 0.$$

Therefore, we obtain the following equalities, which describe the multiplication on the algebra $\mathrm{End}(\Omega)$:

$$\alpha\cdot\mathrm{id}_{\Omega^k} = \mathrm{id}_{\Omega^k}\cdot\alpha = \alpha \text{ for } \alpha\in\mathcal{A},$$
$$\theta_i\cdot\theta_j = \overline{\theta_i}\cdot\overline{\theta_j} = \theta_i\cdot\widetilde{\theta_i} = \widetilde{\theta_i}\cdot\theta_j = \widetilde{\theta_i}\cdot\overline{\theta_i} = \overline{\theta_i}\cdot\widetilde{\theta_i} = \widetilde{\theta_i}\cdot\widetilde{\theta_j} = 0,$$
$$\theta_i\cdot\overline{\theta_j} = \begin{cases}\widetilde{\theta_i} & \text{if } i=j,\\ 0 & \text{otherwise,}\end{cases} \qquad \overline{\theta_i}\cdot\theta_j = \begin{cases}\widetilde{\theta_{i-1}} & \text{if } i\ge 1 \text{ and } i=j,\\ 0 & \text{otherwise}\end{cases}$$

for $0 \le i,j,k \le p-2$. Under the identification of $\theta_i$ with the path $w_i$ and of $\overline{\theta_i}$ with the path $v_i$ in the quiver $Q$ for $0 \le i \le p-2$, the algebra $\mathrm{End}(\Omega)$ is the algebra described in the corollary. □

We now establish several resolutions crucial in the later computations. By the short exact sequences (2.2) and (2.3) we obtain, respectively, the long exact sequence

$$0 \leftarrow F_j \leftarrow W_j \leftarrow W_{j+1} \leftarrow \ldots \leftarrow W_{p-1} \leftarrow 0, \tag{2.5}$$

$$0 \to F_j \to S_j \to S_{j+1} \to \ldots \to S_{p-1} \to 0 \tag{2.6}$$

for all $0 \leq j \leq p-1$.

Fix $0 \leq i \leq p-1$. The long exact sequence

$$0 \to S_i \to \Omega^i \xrightarrow{\kappa_i} \Omega^{i-1} \xrightarrow{\kappa_{i-1}} \ldots \xrightarrow{\kappa_2} \Omega^1 \xrightarrow{\kappa_1} \Omega^0 \to 0 \tag{2.7}$$

is a minimal injective resolution of $S_i$. Indeed, we have $\operatorname{coker}\kappa_j = S_{j-2}$ for $1 \leq j \leq p-1$ (assuming $S_{-1} = 0$) by the acyclicity of the Koszul complex. The injective envelope of $S_j$ is $\Omega^j$ by the fact that $F_j$ is the socle of $S_j$ and Proposition 2.1.1. Thus, (2.7) is indeed the minimal injective resolution of $S_i$.

Now let us consider the following long exact sequence, which is constructed by gluing the Kuhn dual complex of the Koszul complex and the truncated Koszul complex:

$$0 \to (\Omega^0)^\# \xrightarrow{\kappa_1^\#} (\Omega^1)^\# \to \ldots \to (\Omega^{p-2})^\# \xrightarrow{\kappa_{p-1}^\#} (\Omega^{p-1})^\# \xrightarrow{\kappa_p\kappa_p^\#} \Omega^{p-1} \xrightarrow{\kappa_{p-1}} \Omega^{p-2} \to \ldots \to$$
$$\Omega^{i+2} \xrightarrow{\kappa_{i+2}} \Omega^{i+1} \to S_i \to 0. \tag{2.8}$$

We claim that (2.8) is the minimal projective resolution of $S_i$. Indeed, the projective cover of $S_{p-1} = F_{p-1}$ is $(\Omega^{p-1})^\# \simeq \Omega^{p-1}$ by Proposition 2.1.1. For $0 \leq j < p-1$ we have an epimorphism $\Omega^{j+1} \to S_j$ by (1.8) and we see from the proof of Proposition 2.1.1 that $\Omega^{j+1} \simeq (\Omega^{j+1})^\#$ is projective and indecomposable, hence $\Omega^{j+1}$ is the projective cover of $S_j$. We observe that $\ker\kappa_j^\# = (\operatorname{coker}\kappa_j)^\# = (S^{j-2})^\# = W_{j-2}$ for $1 \leq j \leq p$ (assuming $W_{-1} = 0$), by the Kuhn duality and the acyclicity of the Koszul complex, and $\ker\kappa_p\kappa_p^\# = \ker\kappa_p^\#$, since $\kappa_p$ is a monomorphism. The projective cover of $W_j$ is $(\Omega^j)^\#$ by the fact that $F_j$ is the top of $W_j$ and Proposition 2.1.1. Thus, (2.8) is indeed the minimal projective resolution of $S_i$.

## 2.2 Additive Ext-computations

In this section we compute the groups $\operatorname{Ext}^*(G_1, G_2)$, where $G_i$ for $i \in \{1, 2\}$ is a simple, a Schur or a Weyl functor (both functors are not necessarily of the same type).

**Proposition 2.2.1.** *If* $0 \leq m, n \leq p-1$ *then*

$$\operatorname{Ext}^q(F_m, S_n) = \begin{cases} \mathbf{k} & \text{if } n \geq m \text{ and } q = n-m, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We have by (1.1) that $R^qG(W_j) = 0$ for $q > 0$, $0 \leq j \leq p-1$ and $G = \operatorname{Hom}(-, S_n)$. In other words, the functors $W_j$ are $G$-acyclic. By that and the long exact sequence (2.5) we deduce that $\operatorname{Ext}^q(F_m, S_n) = R^q\operatorname{Hom}(W_\bullet, S_n)$, where $\operatorname{Hom}(W_\bullet, S_n)$ is the complex obtained by applying the functor $\operatorname{Hom}(-, S_n)$ to the resolution of $F_m$ (2.5) and erasing the first term. Then the proposition follows from (1.1). □

**Remark.** One can also use in the proof of Proposition 2.2.1 the injective resolution of $S_n$ (2.7) instead of the resolution of $F_m$ by Weyl functors (2.5).

**Corollary 2.2.2.** *If* $0 \leq m, n \leq p-1$ *then*

$$\operatorname{Ext}^q(W_m, F_n) = \begin{cases} \mathbf{k} & \text{if } m \geq n \text{ and } q = m-n, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The corollary follows from Proposition 2.2.1 by the Kuhn duality. □

We recall that a highest weight category $\mathcal{C}$ with poset $P$ has a Kazhdan-Lusztig theory relative to a length function $l : P \to \mathbb{Z}$ if and only if the following conditions hold:

$$\operatorname{Ext}^i(F_\lambda, S_\mu) \neq 0 \implies i \equiv l(\lambda) - l(\mu) \bmod 2,$$
$$\operatorname{Ext}^i(W_\lambda, F_\mu) \neq 0 \implies i \equiv l(\lambda) - l(\mu) \bmod 2$$

for any $\lambda, \mu \in P$ (cf. [CPS93, Theorem 2.4]).

**Corollary 2.2.3.** *The category $\mathcal{P}_p$ has a Kazhdan-Lusztig theory relative to the function $l$ given by $l((i+1, 1^{p-i-1})) = i$ and $l(\lambda) = 0$ for a Young diagram $\lambda$ not being a hook.*

*Proof.* The corollary follows immediately by using the block structure of $\mathcal{P}_p$ described at the beginning of Section 2.1, Proposition 2.2.1 and Corollary 2.2.2. □

**Corollary 2.2.4.** *If $0 \leq m, n \leq p-1$ then*

$$\operatorname{Ext}^q(F_m, F_n) = \begin{cases} \mathbf{k} & \text{if } q = |m-n| + 2r, \text{ where } 0 \leq r \leq p - \max\{m,n\} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* If a highest weight category $\mathcal{C}$ with poset $P$ has a Kazhdan-Lusztig theory, then the following equality holds:

$$\dim \operatorname{Ext}^q(F_\lambda, F_\mu) = \sum_{\tau \in P} \sum_{i+j=q} \dim \operatorname{Ext}^i(F_\lambda, S_\tau) \cdot \dim \operatorname{Ext}^j(W_\tau, F_\mu) \tag{2.9}$$

for $\lambda, \mu \in P$ (cf. [CPS93, Theorem 3.5]). The category $\mathcal{P}_p$ has a Kazhdan-Lusztig theory by Corollary 2.2.3. In view of the block structure of $\mathcal{P}_p$ we restrict the poset of that category to the poset of hooks. By the Kuhn duality $\operatorname{Ext}^q(W_i, F_j) = \operatorname{Ext}^q(F_j, S_i)$ for $0 \leq i, j \leq p-1$. Thus, in the case of our category we can rewrite the equality (2.9) in the following form:

$$\dim \operatorname{Ext}^q(F_m, F_n) = \sum_{0 \leq i \leq p-1} \sum_{q_1+q_2=q} \dim \operatorname{Ext}^{q_1}(F_m, S_i) \cdot \dim \operatorname{Ext}^{q_2}(F_n, S_i).$$

If $i < m$ or $i < n$ then $\dim \operatorname{Ext}^{q_1}(F_m, S_i) \cdot \dim \operatorname{Ext}^{q_2}(F_n, S_i) = 0$ by Proposition 2.2.1. If $i \geq m$ and $i \geq n$, i.e., $i \geq \max\{m,n\}$, then $\dim \operatorname{Ext}^{q_1}(F_m, S_i) \cdot \dim \operatorname{Ext}^{q_2}(F_n, S_i) \neq 0$ if and only if $q_1 = i - m$ and $q_2 = i - n$, by Proposition 2.2.1. By the above observations we obtain

$$\begin{aligned} \dim \operatorname{Ext}^q(F_m, F_n) &= \sum_{0 \leq i \leq p-1} \sum_{q_1+q_2=q} \dim \operatorname{Ext}^{q_1}(F_m, S_i) \cdot \dim \operatorname{Ext}^{q_2}(F_n, S_i) = \\ &= \begin{cases} 1 & \text{if } q_1 = i-m,\ q_2 = i-n, \text{ i.e., } q = 2i - m - n \text{ for } \max\{m,n\} \leq i \leq p-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.10}$$

We note that $2i - m - n = \max\{m,n\} - \min\{m,n\} + 2(i - \max\{m,n\}) = |m-n| + 2(i - \max\{m,n\})$, hence $q = |m-n| + 2r$ for $0 \leq r \leq p - \max\{m,n\} - 1$. The corollary follows from (2.10) and the last remark. □

**Lemma 2.2.5.** *If $0 \leq m, n \leq p-1$ then*

$$\operatorname{Hom}(S_m, \Omega^n) \simeq \operatorname{Hom}((\Omega^n)^\#, W_m) = \begin{cases} \mathbf{k} & \text{if } n = m \text{ or } n = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We have $S_{p-1} = F_{p-1}$, hence $\dim \operatorname{Hom}(S_{p-1}, \Omega^n) = \dim \operatorname{Hom}(F_{p-1}, \Omega^n) = \delta_{p-1,n}$. In case $m < p-1$ we apply the exact functor $\operatorname{Hom}(-, \Omega^n)$ to the short exact sequence (2.3) for $j = m$. Then we have

$$\dim \operatorname{Hom}(S_m, \Omega^n) = \dim \operatorname{Hom}(F_{m+1}, \Omega^n) + \dim \operatorname{Hom}(F_m, \Omega^n) = \begin{cases} 1 & \text{if } n = m \text{ or } n = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

The equivalence $\operatorname{Hom}(S_m, \Omega^n) \simeq \operatorname{Hom}((\Omega^n)^\#, W_m)$ is immediate by the Kuhn duality. □

**Proposition 2.2.6.** *If* $0 \le m, n \le p-1$ *then*

$$\operatorname{Ext}^q(S_m, S_n) = \begin{cases} \mathbf{k} & \textit{if one of the following holds:} \\ & \bullet\ m = n \textit{ and } q = 0, \\ & \bullet\ m < n \textit{ and } q \in \{n-m-1, n-m\}, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* We use the injective resolution of $S_n$ (2.7), hence $\operatorname{Ext}^q(S_m, S_n) = H^q(\operatorname{Hom}(S_m, \Omega^\bullet))$, where $\operatorname{Hom}(S_m, \Omega^\bullet)$ is the complex obtained by applying the functor $\operatorname{Hom}(S_m, -)$ to the resolution of $S_n$ (2.7) with the removed first term. We will now show that in case $m < n$ the differential $\operatorname{Hom}(S_m, \Omega^{m+1}) = \mathbf{k} \xrightarrow{\kappa_{m+1}\circ -} \operatorname{Hom}(S_m, \Omega^m) = \mathbf{k}$ is the zero map. By the acyclicity of the Koszul complex $\kappa_{m+1} = \imath_m s_{m+1}$, where $s_{m+1}$ and $\imath_m$ are the the projection $\Omega^{m+1} \twoheadrightarrow S_m$ and the inclusion $S_m \hookrightarrow \Omega^m$, respecitively. Then it follows from (1.7) that $d_m \imath_m \neq 0$, hence it generates $\operatorname{Hom}(S_m, \Omega^{m+1})$. The considered differential sends an element $c \cdot d_m \imath_m$ with $c \in \mathbf{k}$ to the element $c \cdot \kappa_{m+1} d_m \imath_m = -c \cdot d_{m-1} \kappa_m \imath_m = 0$, i.e., this differential is indeed the zero map. We used here (1.5) in the first equality. Then the proposition follows from Lemma 2.2.5 and the above observation. □

**Corollary 2.2.7.** *If* $0 \le m, n \le p-1$ *then*

$$\operatorname{Ext}^q(W_m, W_n) = \begin{cases} \mathbf{k} & \textit{if one of the following holds:} \\ & \bullet\ m = n \textit{ and } q = 0, \\ & \bullet\ m > n \textit{ and } q \in \{m-n-1, m-n\}, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* The corollary follows from Proposition 2.2.6 by the Kuhn duality. □

As a corollary of the last proposition, we claim that the long exact sequence (2.6) is that induced by the Koszul kernel complex, possibly truncated. Indeed, we saw in Section 1.1 that $S_j = \ker \kappa_j$ for $0 \le j \le p-1$. Letting $q = 0$ in Proposition 2.2.6, we deduce that the long exact sequence (2.6) for $0 \le i \le p-1$ has the following form:

$$0 \to F_i \to S_i \xrightarrow{d_i} S_{i+1} \to \ldots \to S_{p-2} \xrightarrow{d_{p-2}} S_{p-1} \to 0,$$

i.e., this is indeed the long exact sequence induced by the Koszul kernel complex. We also obtain

$$F_i = \ker(S_i \xrightarrow{d_i} S_{i+1}) \tag{2.11}$$

for $0 \le i \le p-1$.

**Proposition 2.2.8.** *If* $0 \leq m, n \leq p-1$ *then*

$$\operatorname{Ext}^q(S_m, F_n) = \begin{cases} \mathbf{k} & \textit{if one of the following holds}: \\ & \quad \bullet\ m < n \textit{ and } q \in \{n-m-1, 2p-m-n-2\}, \\ & \quad \bullet\ m \geq n \textit{ and } q = 2p-m-n-2, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* We use the minimal projective resolution of $S_m$ (2.8), hence

$$\operatorname{Ext}^q(S_m, F_n) = H^q(\operatorname{Hom}(P_\bullet, F_n)),$$

where $\operatorname{Hom}(P_\bullet, F_n)$ is the complex obtained by applying the functor $\operatorname{Hom}(-, F_n)$ to the resolution of $S_m$ (2.8) with the removed first term. Then the proposition follows from Lemma 2.2.5 and the fact that the differentials of the complex $\operatorname{Hom}(P_\bullet, F_n)$ are the zero maps since the resolution (2.8) is minimal. □

**Corollary 2.2.9.** *If* $0 \leq m, n \leq p-1$ *then*

$$\operatorname{Ext}^q(F_m, W_n) = \begin{cases} \mathbf{k} & \textit{if one of the following holds}: \\ & \quad \bullet\ m > n \textit{ and } q \in \{m-n-1, 2p-m-n-2\}, \\ & \quad \bullet\ m \leq n \textit{ and } q = 2p-m-n-2, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* The corollary follows from Proposition 2.2.8 by the Kuhn duality. □

**Proposition 2.2.10.** *If* $0 \leq m, n \leq p-1$ *then*

$$\operatorname{Ext}^q(S_m, W_n) = \begin{cases} \mathbf{k} & \textit{if one of the following holds}: \\ & \quad \bullet\ m = n \textit{ and } q \in \{0, 2p-m-n-3, 2p-m-n-2\}, \\ & \quad \bullet\ m \neq n \textit{ and } q \in \{2p-m-n-3, 2p-m-n-2\}, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* We use the minimal projective resolution of $S_m$ (2.8), hence

$$\operatorname{Ext}^q(S_m, W_n) = H^q(\operatorname{Hom}(P_\bullet, W_n)),$$

where $\operatorname{Hom}(P_\bullet, W_n)$ is the complex obtained by applying the functor $\operatorname{Hom}(-, W_n)$ to the resolution of $S_m$ (2.8) with the removed first term. It was shown in the proof of Proposition 2.2.6 that the differential $\operatorname{Hom}(S_n, \Omega^{n+1}) \to \operatorname{Hom}(S_n, \Omega^n)$ in the complex $\operatorname{Hom}(S_m, \Omega^\bullet)$ considered there is the zero map, hence the differential $\operatorname{Hom}((\Omega^{n+1})^\#, W_n) \xrightarrow{-\circ\kappa_{n+1}^\#} \operatorname{Hom}((\Omega^n)^\#, W_n)$ is the zero map by the Kuhn duality. Now we claim that in case $m < n < p-1$ the differential $\operatorname{Hom}(\Omega^n, W_n) = \mathbf{k} \xrightarrow{-\circ\kappa_{n+1}} \operatorname{Hom}(\Omega^{n+1}, W_n) = \mathbf{k}$ is an isomorphism. Indeed, the projection $s_n : \Omega^n \twoheadrightarrow W_n$ generates $\operatorname{Hom}(\Omega^n, W_n)$. The above differential sends an element $c{\cdot}s_n$ with $c \in \mathbf{k}$ to the element $c{\cdot}s_n\kappa_{n+1}$. Suppose that $s_n\kappa_{n+1} = 0$, i.e., $\operatorname{im}\kappa_{n+1} \subset \ker s_n$. We see that $\operatorname{im}\kappa_{n+1} = S_n$ by the acyclicity of the Koszul complex and $\ker s_n = W_{n-1}$ by the Kuhn dual of (1.8), hence $S_n \subset W_{n-1}$. On the other hand, the composition series of $S_n$ is not contained in the composition series of $W_{n-1}$, contrary to the last inclusion. In consequence, $s_n\kappa_{n+1} \neq 0$, hence the differential under consideration is nonzero, and the claim follows. In case $n = p-1$ we have $W_{p-1} = F_{p-1}$, hence the differential $\operatorname{Hom}(\Omega^{p-1}, W_{p-1}) \to \operatorname{Hom}((\Omega^{p-1})^{\#}, W_{p-1})$ is the zero map since this is a differential in the complex $\operatorname{Hom}(P_\bullet, F_{p-1})$ considered in Proposition 2.2.8. We also observe that in case $n = p-2$ the differential $\operatorname{Hom}(\Omega^{p-1}, W_{p-2}) = \mathbf{k} \xrightarrow{-\circ \kappa_p \kappa_p^{\#}} \operatorname{Hom}((\Omega^{p-1})^{\#}, W_{p-2}) = \mathbf{k}$ is the zero map. Indeed, the map $s_{p-2}\kappa_{p-1}$ generates $\operatorname{Hom}(\Omega^{p-1}, W_{p-2})$. The considered differential sends an element $c \cdot s_{p-2}\kappa_{p-1}$ with $c \in \mathbf{k}$ to the element $c \cdot s_{p-2}\kappa_{p-1}\kappa_p\kappa_p^{\#} = 0$, i.e., this differential is the zero map, as required. Finally, the assertion of the proposition follows from Lemma 2.2.5 and all the above observations. ☐

## 2.3 Yoneda algebras

Let us introduce the notation used in this section. For a complex $C = (C^\bullet, d_C)$ and $t \in \mathbb{Z}$, $C[t] = (C[t]^\bullet, d_{C[t]})$ denotes the complex obtained from $C$ by shifting degrees by $t$, i.e, $C[t]^i = C^{t+i}$ and $d^i_{C[t]} = (-1)^t d^i_C$. For complexes $C, D$ we set $\operatorname{Hom}^t(C, D) = \operatorname{Hom}(C, D[t])$.

### 2.3.1 The Yoneda algebra of Schur functors

The main aim of this subsection is to describe the structure of the Yoneda algebra of Schur functors and to prove that the endomorphism algebra whose cohomology algebra is the Yoneda algebra of Schur functors is a formal DG algebra.

Let $\mathcal{T}_i$ denote the complex obtained by removing $S_i$ in the injective resolution of $S_i$ (2.7). We will now determine a basis of the Yoneda algebra of Schur functors whose elements are induced by the morphisms of complexes defined in the following way. For given $0 \le i \le j \le p-1$ we define the morphism of complexes $\gamma_{ji} \in \operatorname{Hom}^{j-i}(\mathcal{T}_i, \mathcal{T}_j)$ as the natural inclusion. For $0 \le i \le p-1$ let $\widetilde{d}_i : \mathcal{T}_i \to \mathcal{T}_{i+1}$ be the map whose components are given by $\widetilde{d}_i^m = (-1)^{i-m} d_{i-m}$ for $0 \le m \le i$. By (1.5) we see that $\widetilde{d}_i$ is indeed a morphism of complexes. Then for given $0 \le i < j \le p-1$ let $\overline{\gamma}_{ji} \in \operatorname{Hom}^{j-i-1}(\mathcal{T}_i, \mathcal{T}_j)$ be the morphism of complexes given by $\overline{\gamma}_{ji} = \gamma_{j,i+1}\widetilde{d}_i$. We now show that $\gamma_{ji}$ and $\overline{\gamma}_{ji}$ are not null-homotopic.

**Lemma 2.3.1.** *The morphism of complexes $\gamma_{ji}$ (resp. $\overline{\gamma}_{ji}$) is not null-homotopic for any $0 \le i \le j \le p-1$ (resp. $0 \le i < j \le p-1$).*

*Proof.* We prove first the assertion of the lemma for $\gamma_{ji}$. Fix $0 \le i \le j \le p-1$. We suppose that $\gamma_{ji}$ is null-homotopic. Let $h_{ji} = h^*_{ji}$ be a homotopy between $\gamma_{ji}$ and the zero map. We consider the following diagram, where the top row is the complex $\mathcal{T}_i$ and the bottom row is the complex $\mathcal{T}_j[j-i]$.

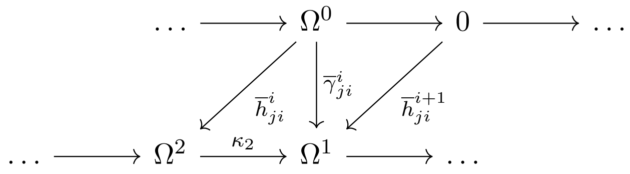
$$\begin{array}{ccccccccc} & & \cdots & \longrightarrow & \Omega^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \swarrow^{h^i_{ji}} & \downarrow{\scriptstyle \gamma^i_{ji}} & \swarrow^{h^{i+1}_{ji}} & & & \\ \cdots & \longrightarrow & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 & \longrightarrow & \cdots & & \end{array}$$

Clearly $h^{i+1}_{ji} = 0$ and, by definition, $\gamma^i_{ji} = \operatorname{id}$. Since $h_{ji}$ is a homotopy between $\gamma_{ji}$ and the zero map, we obtain $\kappa_1 h^i_{ji} = \operatorname{id}$. By Lemma 2.4 we have $\operatorname{Hom}(\Omega^0, \Omega^1) = \mathbf{k}$. It implies that $h^i_{ji} = a \cdot d_0$ for some $a \in \mathbf{k}$, hence

$$\kappa_1 \circ (a \cdot d_0) = \operatorname{id}. \tag{2.12}$$

On the other hand, $\kappa_1 \circ (a \cdot d_0) = a \cdot \kappa_1 d_0 = 0$ by (1.5), contrary to (2.12). Therefore, $\gamma_{ji}$ is not null-homotopic.

Now we turn to the case of the map $\overline{\gamma}_{ji}$. Fix $0 \le i < j \le p-1$. We suppose that $\overline{\gamma}_{ji}$ is null-homotopic. Let $\overline{h}_{ji} = \overline{h}^*_{ji}$ be a homotopy between $\overline{\gamma}_{ji}$ and the zero map. Then there is the following diagram, where the top row is the complex $\mathcal{T}_i$ and the bottom row is the complex $\mathcal{T}_j[j-i-1]$.



Clearly $\overline{h}_{ji}^{i+1} = 0$ and, by definition, $\overline{\gamma}_{ji}^{i} = d_0$. Since $\overline{h}_{ji}$ is a homotopy between $\overline{\gamma}_{ji}$ and the zero map, the following equality holds:

$$\kappa_2 \overline{h}_{ji}^{i} = d_0. \tag{2.13}$$

By Lemma 2.4 we have $\mathrm{Hom}(\Omega^0, \Omega^2) = 0$. It implies that $\overline{h}_{ji}^{i} = 0$, hence $\kappa_2 \overline{h}_{ji}^{i} = 0$, contradicting (2.13). Thus, $\overline{\gamma}_{ji}$ is not null-homotopic. □

The class of the map $\gamma_{ji}$ (resp. $\overline{\gamma}_{ji}$) in $\mathrm{Ext}^{j-i}(S_i, S_j)$ (resp. $\mathrm{Ext}^{j-i-1}(S_i, S_j)$) will be denoted by $[\gamma_{ji}]$ (resp. $[\overline{\gamma}_{ji}]$). Set $S = \bigoplus_{0 \le i \le p-1} S_i$. By Proposition 2.2.6 and Lemma 2.3.1 we obtain

$$\mathrm{Ext}^*(S, S) = \mathrm{span}(\{[\gamma_{ji}] : 0 \le i \le j \le p-1\} \cup \{[\overline{\gamma}_{ji}] : 0 \le i < j \le p-1\}) \tag{2.14}$$

as the graded vector space. By the definition of the maps $\gamma_{ji}$ we have

$$\gamma_{ml} \cdot \gamma_{ji} = \begin{cases} \gamma_{mi} & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}$$

We also find that

$$\gamma_{ml} \cdot \overline{\gamma}_{ji} = \begin{cases} \overline{\gamma}_{mi} & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Indeed, if $j = l$ then

$$\gamma_{mj} \cdot \overline{\gamma}_{ji} = \gamma_{mj}\gamma_{j,i+1}\widetilde{d}_i = \gamma_{m,i+1}\widetilde{d}_i = \overline{\gamma}_{mi}.$$

We note that $\gamma_{j,i+1}\widetilde{d}_i = \widetilde{d}_{j-1}\gamma_{j-1,i}$, hence, similarly to the above, we obtain

$$\overline{\gamma}_{ml} \cdot \gamma_{ji} = \begin{cases} \overline{\gamma}_{mi} & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

We also observe that

$$\overline{\gamma}_{ml} \cdot \overline{\gamma}_{ji} = 0. \tag{2.18}$$

Indeed, in case $j = l$ we have

$$\overline{\gamma}_{ml} \cdot \overline{\gamma_{ji}} = \gamma_{m,j+1}\widetilde{d}_j\gamma_{j,i+1}\widetilde{d}_i = \gamma_{m,j+1}\gamma_{j+1,i+2}\widetilde{d}_{i+1}\widetilde{d}_i = 0.$$

The equalities (2.15) – (2.18) imply the equalities in the algebra $\mathrm{Ext}^*(S, S)$:

$$[\gamma_{ml}] \cdot [\gamma_{ji}] = \begin{cases} [\gamma_{mi}] & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases} \qquad [\gamma_{ml}] \cdot [\overline{\gamma}_{ji}] = \begin{cases} [\overline{\gamma}_{mi}] & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

$$[\overline{\gamma}_{ml}] \cdot [\gamma_{ji}] = \begin{cases} [\overline{\gamma}_{mi}] & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases} \qquad [\overline{\gamma}_{ml}] \cdot [\overline{\gamma}_{ji}] = 0$$

with the appropriate inequality between the indices for a given equality. These equalities describe the multiplication on the Yoneda algebra of Schur functors $\mathrm{Ext}^*(S, S)$.

Let $\mathcal{T} = \bigoplus_{0 \le i \le p-1} \mathcal{T}_i$. We now show the DG formality of the DG algebra $\mathrm{End}^*(\mathcal{T})$ with the components $\mathrm{End}^i(\mathcal{T}) = \prod_{n-m=i} \mathrm{Hom}_{\mathcal{P}_p^{\varnothing}}(\mathcal{T}^m, \mathcal{T}^n)$ and the differential $\mathfrak{d}$ given by $\mathfrak{d}(f^j) = (\kappa f^j - (-1)^i f^{j+1}\kappa)$ for $(f^j) \in \mathrm{End}^i(\mathcal{T})$. We also describe the structure of the Yoneda algebra of Schur functors.

**Theorem 2.3.2.**

(a) *The algebra* $\mathrm{End}^*(\mathcal{T})$ *is a formal DG algebra, i.e., there exists a quasi-isomorphism of DG algebras* $\phi : \mathrm{Ext}^*(S,S) \to \mathrm{End}^*(\mathcal{T})$, *where* $\mathrm{Ext}^*(S,S)$ *is a DG algebra with zero differential.*

(b) *The Yoneda algebra of Schur functors* $\mathrm{Ext}^*(S,S)$ *is isomorphic to the square-zero extension of the algebra* $U_p(\mathbf{k})$ *of* $p \times p$ *upper triangular matrices over* $\mathbf{k}$ *by the* $U_p(\mathbf{k})$*-bimodule* $U_p^+(\mathbf{k})$ *of* $p \times p$ *strictly upper triangular matrices over* $\mathbf{k}$.

*Proof.* Let $A = \bigoplus_{t\in\mathbb{N}} A_t$ be the graded vector space with the components

$$A_t = \mathrm{span}(\{a_{ji} : 0 \le i \le j \le p-1 \text{ and } j-i=t\} \cup \{\overline{a}_{ji} : 0 \le i < j \le p-1 \text{ and } j-i-1=t\})$$

for $0 \le t \le p-1$ and $A_t = 0$ for $t \ge p$, where $a_{ji}, \overline{a}_{ji}$ are the formal symbols for $i,j$ satisfying the conditions given above. We define the multiplication on $A$ as follows:

$$a_{ml}\cdot a_{ji} = \begin{cases} a_{mi} & \text{if } j=l,\\ 0 & \text{otherwise,}\end{cases} \qquad a_{ml}\cdot \overline{a}_{ji} = \begin{cases} \overline{a}_{mi} & \text{if } j=l,\\ 0 & \text{otherwise,}\end{cases}$$
$$\overline{a}_{ml}\cdot a_{ji} = \begin{cases} \overline{a}_{mi} & \text{if } j=l,\\ 0 & \text{otherwise,}\end{cases} \qquad \overline{a}_{ml}\cdot \overline{a}_{ji} = 0$$

with the appropriate inequality between the indices for a given equality. The differential on $A$ is defined to be the zero differential.

Let us first prove (a). Let $\phi : A \to \mathrm{End}^*(\mathcal{T})$ be the linear map such that $\phi(a_{ji}) = \gamma_{ji}$ for $0 \le i \le j \le p-1$ and $\phi(\overline{a}_{ji}) = \overline{\gamma}_{ji}$ for $0 \le i < j \le p-1$. By (2.15)-(2.18) we see that $\phi$ is a graded homomorphism and $\phi$ is a chain map, since $\phi(a_{ji})$ and $\phi(\overline{a}_{ji})$ are morphisms of complexes. In other words, $\phi$ is a DG homomorphism. It suffices to show that $\phi$ is a quasi-isomorphism. It follows from the definition of the differential on $A$ that $H^*(A) \simeq A$. Then it is evident that the induced graded algebra homomorphism $\phi_* : H^*(A) \simeq A \to H^*(\mathrm{End}^*(\mathcal{T})) \simeq \mathrm{Ext}^*(S,S)$ is given by $\phi_*(a_{ji}) = [\gamma_{ji}]$ and $\phi_*(\overline{a}_{ji}) = [\overline{\gamma}_{ji}]$. By (2.14) $\phi_*$ maps a basis of $A$ onto a basis of $\mathrm{Ext}^*(S,S)$. Hence $\phi_*$ is a graded algebra isomorphism. In particular, $\phi : A \simeq \mathrm{Ext}^*(S,S) \to \mathrm{End}^*(\mathcal{T})$ is a quasi-isomorphism, as required.

It remains to prove (b). Let $\{e_{ij} : 1 \le i \le j \le p\}$ (resp. $\{\overline{e}_{ij} : 1 \le i < j \le p\}$) be the standard basis of $U_p(\mathbf{k})$ (resp. $U_p^+(\mathbf{k})$). We will denote by $U_p(\mathbf{k}) \oplus U_p^+(\mathbf{k})$ the square-zero extension of the algebra $U_p(\mathbf{k})$ by the $U_p(\mathbf{k})$-bimodule $U_p^+(\mathbf{k})$. We recall that the multiplication on $U_p(\mathbf{k}) \oplus U_p^+(\mathbf{k})$ is given by $(u_1, \overline{u}_1)\cdot(u_2,\overline{u}_2) = (u_1u_2, u_1\overline{u}_2 + \overline{u}_1u_2)$. Then it is immediate that

$$(e_{ij},0)\cdot(e_{lm},0) = \begin{cases} (e_{im},0) & \text{if } j=l,\\ (0,0) & \text{otherwise,}\end{cases} \qquad (e_{ij},0)\cdot(0,\overline{e}_{lm}) = \begin{cases} (0,\overline{e}_{im}) & \text{if } j=l,\\ (0,0) & \text{otherwise,}\end{cases}$$
$$(0,\overline{e}_{ij})\cdot(e_{lm},0) = \begin{cases} (0,\overline{e}_{ik}) & \text{if } j=l,\\ (0,0) & \text{otherwise,}\end{cases} \qquad (0,\overline{e}_{ij})\cdot(0,\overline{e}_{lm}) = (0,0) \tag{2.19}$$

with the appropriate inequality between the indices for a given equality. Let $\psi : A \to U_p(\mathbf{k}) \oplus U_p^+(\mathbf{k})$ be the linear map given by $\psi(a_{ji}) = (e_{i+1,j+1}, 0)$ for $0 \le i \le j \le p-1$ and $\psi(\overline{a}_{ji}) = (0, \overline{e}_{i+1,j+1})$ for $0 \le i < j \le p-1$. By (2.19) we conclude that $\psi$ is an algebra homomorphism. Since $\psi$ maps a basis of $A$ onto a basis of $U_p(\mathbf{k}) \oplus U_p^+(\mathbf{k})$, $\psi$ is an algebra isomorphism. Thus, we obtain $\mathrm{Ext}^*(S,S) \simeq A \simeq U_p(\mathbf{k}) \oplus U_p^+(\mathbf{k})$, and the proof is complete. □

**Corollary 2.3.3.** *There is an equivalence of triangulated categories*

$$\mathcal{D}^b\mathcal{P}_p^{\varnothing} \simeq \mathcal{D}^b(\mathrm{Ext}^*(S,S) - mod^{gr}).$$

*Proof.* Let $\mathcal{T}^{\#}$ be the resolution obtained from $\mathcal{T}$ by using the Kuhn duality. Clearly $\mathcal{T}^{\#}$ is a cofibrant object in the category of complexes of $\mathcal{P}_p^{\varnothing}$. It is known that $\mathcal{T}^{\#}$ is a small generator of $\mathcal{D}^b\mathcal{P}_p^{\varnothing}$ (cf. [Kra17, Lemma 5.5]). We consider the category $\mathcal{P}_p^{\varnothing}$ to be the DG category concentrated in degree 0. Then the equivalence

$$\mathcal{D}^b\mathcal{P}_p^{\varnothing} \simeq \mathcal{D}^b(\mathrm{End}^*(\mathcal{T}^{\#}) - \mathrm{mod}^{\mathrm{gr}}) \simeq \mathcal{D}^b(\mathrm{End}^*(\mathcal{T})^{\mathrm{op}} - \mathrm{mod}^{\mathrm{gr}})$$

follows from [Kel94a, Theorem 8.2]. By Theorem 2.3.2(a) we obtain

$$\mathcal{D}^b(\mathrm{End}^*(\mathcal{T})^{\mathrm{op}} - \mathrm{mod}^{\mathrm{gr}}) \simeq \mathcal{D}^b(\mathrm{Ext}^*(S,S)^{\mathrm{op}} - \mathrm{mod}^{\mathrm{gr}})$$

(cf. [Kel94b, Example in Section 1.5]). Since $\mathcal{D}^b(\mathcal{C}) \simeq \mathcal{D}^b(\mathcal{C}^{\mathrm{op}})$ for a highest weight category $\mathcal{C}$ with duality (cf. [CPS93, (1.5)]), the following equivalence holds:

$$\mathcal{D}^b(\mathrm{Ext}^*(S,S)^{\mathrm{op}} - \mathrm{mod}^{\mathrm{gr}}) \simeq \mathcal{D}^b(\mathrm{Ext}^*(S,S) - \mathrm{mod}^{\mathrm{gr}}).$$

Using all the above equivalences, we obtain the assertion of the corollary. □

### 2.3.2 The Yoneda algebra of simple functors

The results of this subsection are similar to those in the previous subsection. Our main goal is to describe the structure of the Yoneda algebra of simple functors and to prove the DG formality of the endomorphism algebra whose cohomology algebra is the Yoneda algebra of simple functors.

Fix $0 \le i \le p-1$. We now define a double complex, which provides an injective resolution of the simple functor $F_i$. Set

$$\mathcal{R}_i^{r,s} = \Omega^{i+s-r} \quad \text{if} \quad 0 \le r \le p-1,\ 0 \le s \le p-i-1 \text{ and } r-s \le i.$$

Let $(d_i^{r,s})_v : \mathcal{R}_i^{r,s} \to \mathcal{R}_i^{r,s+1}$ (resp. $(d_i^{r,s})_h : \mathcal{R}_i^{r,s} \to \mathcal{R}_i^{r+1,s}$) be the de Rham differential $d_{i+s-r}$ (resp. the Koszul differential $\kappa_{i+s-r}$) for $r, s$ such that $\mathcal{R}_i^{r,s}, \mathcal{R}_i^{r,s+1} \neq 0$ (resp. $\mathcal{R}_i^{r,s}, \mathcal{R}_i^{r+1,s} \neq 0$). By (1.5) it is easily seen that $(\mathcal{R}_i^{r,s}, (d_i^{r,s})_h, (d_i^{r,s})_v)_{r,s\in\mathbb{Z}}$ is a double complex. We denote it briefly by $\mathcal{R}_i$. We note that in $\mathcal{R}_i$ the rows and columns are, respectively, the truncated Koszul and de Rham complexes. For example, the diagram below is $\mathcal{R}_2$ for $p = 5$.

$$\begin{array}{ccccccccc}
\Omega^4 & \xrightarrow{\kappa_4} & \Omega^3 & \xrightarrow{\kappa_3} & \Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 \\
{\scriptstyle d_3}\uparrow & & {\scriptstyle d_2}\uparrow & & {\scriptstyle d_1}\uparrow & & {\scriptstyle d_0}\uparrow & & \\
\Omega^3 & \xrightarrow{\kappa_3} & \Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 & & \\
{\scriptstyle d_2}\uparrow & & {\scriptstyle d_1}\uparrow & & {\scriptstyle d_0}\uparrow & & & & \\
\Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 & & & &
\end{array}$$

Let us consider the spectral sequence of $\mathcal{R}_i$ with respect to the vertical filtration. By the acyclicity of the Koszul complex we see that

$$E_1^{s,t} = \begin{cases} \ker \kappa_{t+i} = S_{t+i} & \text{if } s = 0 \text{ and } 0 \le t \le p-i-1, \\ 0 & \text{otherwise.} \end{cases}$$

By (1.6) we have $H^0(K_p^{\bullet}) = I^{(1)} = F_0$ and $H^j(K_p^{\bullet}) = 0$ for $j > 0$. Then by the formula for $E_1^{s,t}$ and (2.11) we obtain

$$E_2^{s,t} = \begin{cases} F_i & \text{if } s = 0,\ t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this spectral sequence degenerates at $E_2$ and, in consequence, it follows that

$$H^n(\mathrm{Tot}(\mathcal{R}_i)) = \begin{cases} F_i & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

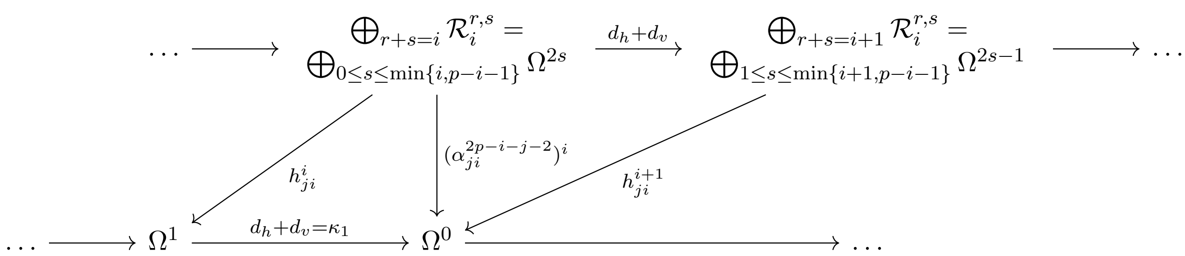
Since each functor $\mathcal{R}_i^{r,s}$ is injective and a direct sum of injectives is injective, the last equality implies that $\mathrm{Tot}(\mathcal{R}_i)$ is an injective resolution of $F_i$. By abuse of notation we use the same symbol $\mathcal{R}_i$ for $\mathrm{Tot}(\mathcal{R}_i)$.

Our next objective is to determine a basis of the Yoneda algebra of simple functors, whose elements are induced by the morphisms of total complexes from $\mathcal{R}_i$ to suitably shifted $\mathcal{R}_j$ defined as follows. For $0 \le i \le j \le p-1$ and $t$ such that $\mathrm{Ext}^t(F_i, F_j) \neq 0$ (cf. Corollary 2.2.4) let $\alpha_{ji}^t$ be the map given by $(\alpha_{ji}^t)^{*,s} = \gamma_{s+\frac{1}{2}(t+i+j),i+s}$ for $0 \le s \le p-1-\frac{1}{2}(t+j+i)$. Assume now that $0 \le j < i \le p-1$. Regarding the double complexes $\mathcal{R}_i$ and $\mathcal{R}_j$ as the complexes of vertical complexes, let $\alpha_{ji}^{i-j}$ be the natural inclusion and define $\alpha_{ji}^t = \alpha_{jj}^{t-(i-j)} \circ \alpha_{ji}^{i-j}$ for $t > i-j$ given above. Less formally, if $i \le j$ then $\alpha_{ji}^t$ embeds the $s$-th row of $\mathcal{R}_i$ into the $s + \frac{1}{2}(t-(j-i))$-th row of $\mathcal{R}_j$, assuming the latter is nonzero. In case $i > j$ we compose $\alpha_{ji}^{j-i}$, which embeds $\mathcal{R}_i$ into $\mathcal{R}_j$, with $\alpha_{jj}^{t-(i-j)}$. It is easy to check that the maps of total complexes induced by $\alpha_{ij}^t$ are morphisms of total complexes. We use for the simplicity the same symbol $\alpha_{ji}^t$ for that. We note that $\alpha_{ji}^t \in \mathrm{Hom}^t(\mathcal{R}_i, \mathcal{R}_j)$ as the morphism of complexes. Let us now show that these morphisms are not null-homotopic.

**Lemma 2.3.4.** *The morphism of total complexes $\alpha_{ij}^t$ is not null-homotopic for any $0 \le i, j \le p-1$ and $t$ such that $\mathrm{Ext}^t(F_i, F_j) \neq 0$.*

*Proof.* Fix $0 \le i, j \le p-1$. By Corollary 2.2.4 the maximal $t$ such that $\mathrm{Ext}^t(F_i, F_j) \neq 0$ is $t = 2p-i-j-2$. We see at once that $\alpha_{ji}^{2p-i-j-2} = \alpha_{jj}^{2p-i-j-2-t} \circ \alpha_{ji}^t$ for any $t < 2p-i-j-2$ satisfying $\mathrm{Ext}^t(F_i, F_j) \neq 0$. Thus, if $\alpha_{ji}^{2p-i-j-2}$ is not null-homotopic, then so is $\alpha_{ji}^t$. Hence it suffices to show that $\alpha_{ji}^{2p-i-j-2}$ is not null-homotopic.

We suppose that $\alpha_{ji}^{2p-i-j-2}$ is null-homotopic. Let $h_{ji} = h_{ji}^*$ be a homotopy between $\alpha_{ji}^{2p-i-j-2}$ and the zero map. Then there is the following diagram, where the top row is the total complex $\mathcal{R}_i$ and the bottom row is the total complex $\mathcal{R}_j[2p-i-j-2]$.




By definition, $(\alpha_{ji}^{2p-i-j-2})^i = \mathrm{id}_{\Omega^0}$. By Lemma 2.4 we have $\mathrm{Hom}(\Omega^m, \Omega^0) = 0$ for $1 < m \le p-1$ and $\mathrm{Hom}(\Omega^1, \Omega^0) = \mathbf{k}$. It implies that $h_{ji}^{i+1} = a \cdot \kappa_1$ for some $a \in \mathbf{k}$. Using Lemma 2.4 again, we have $\mathrm{Hom}(\Omega^m, \Omega^1) = 0$ for $m > 2$, $\mathrm{Hom}(\Omega^2, \Omega^1) = \mathbf{k}$ and $\mathrm{Hom}(\Omega^0, \Omega^1) = \mathbf{k}$. Thus, we conclude that $h_{ji}^i = b \cdot \kappa_2 + c \cdot d_1$ for some $b, c \in \mathbf{k}$. Since $h_{ji}$ is a homotopy between $\alpha_{ji}^{2p-i-j-2}$ and the zero map, the above observations give us the equality

$$\kappa_1 \circ (b \cdot \kappa_2 + c \cdot d_0) + (a \cdot \kappa_1) \circ (\kappa_2 + d_0) = \mathrm{id}_{\Omega^0}, \quad \text{that is,} \quad (a+c) \cdot \kappa_1 d_0 = \mathrm{id}_{\Omega^0}.$$

On the other hand, $(a+c) \cdot \kappa_1 d_0 = 0$ by (1.5), contrary to the equality in the preceding sentence. In particular, $\alpha_{ji}^{2p-i-j-2}$ is not null-homotopic. □

The class of the map $\alpha^t_{ji}$ in $\mathrm{Ext}^t(F_i, F_j)$ will be denoted by $[\alpha^t_{ji}]$. Set $F = \bigoplus_{0\le i\le p-1} F_i$. By Corollary 2.2.4 and Lemma 2.3.4 we have

$$\mathrm{Ext}^*(F,F) = \mathrm{span}\{[\alpha^t_{ji}] : 0 \le i,j \le p-1 \text{ and } \ t = |i-j|+2r, \text{ where } 0 \le r \le p-\max\{i,j\}-1\} \tag{2.20}$$

as the graded vector space. It follows from the definition of the maps $\alpha^t_{ij}$ that

$$\alpha^t_{ml}\cdot\alpha^u_{ji} = \begin{cases}\alpha^{u+t}_{mi} & \text{if } j=l \text{ and } u+t\le 2p-i-m-2,\\ 0 & \text{otherwise.}\end{cases} \tag{2.21}$$

This equality implies the equality in the algebra $\mathrm{Ext}^*(F,F)$:

$$[\alpha^t_{ml}]\cdot[\alpha^u_{ji}] = \begin{cases}[\alpha^{u+t}_{mi}] & \text{if } j=l \text{ and } u+t\le 2p-i-m-2,\\ 0 & \text{otherwise.}\end{cases} \tag{2.22}$$

In the next theorem we use the last equality to describe the structure of the Yoneda algebra of simple functors $\mathrm{Ext}^*(F,F)$. We also prove the DG formality of the DG algebra $\mathrm{End}^*(\mathcal{R})$ with $\mathcal{R} = \bigoplus_{0\le i\le p-1}\mathcal{R}_i$ defined in the similar way as $\mathrm{End}^*(\mathcal{T})$.

Let $B = \bigoplus_{t\in\mathbb{N}} B_t$ be the graded vector space with the components

$$B_t = \mathrm{span}\{b^t_{ji} : 0\le i,j\le p-1 \text{ such that } t=|i-j|+2r, \text{ where } 0\le r\le p-\max\{i,j\}-1\}$$

for $0\le t\le 2p-2$ and $B_t = 0$ for $t\ge 2p-1$, where $b^t_{ij}$ are the formal symbols for $i,j,t$ satisfying the conditions given above. We define the multiplication on $B$ as follows:

$$b^t_{lm}\cdot b^u_{ji} = \begin{cases} b^{t+u}_{mi} & \text{if } j=l \text{ and } u+t\le 2p-i-m-2,\\ 0 & \text{otherwise.}\end{cases}$$

**Theorem 2.3.5.**

(a) *There is a graded algebra isomorphism* $\mathrm{Ext}^*(F,F)\simeq B$.

(b) *There is a graded algebra isomorphism* $\mathrm{Ext}^*(F_i,F_i)\simeq \mathbf{k}[x]/(x^{p-i})$ *for* $0\le i\le p-1$ *and* $x$ *of degree* 2. *In particular,* $\mathrm{Ext}^*(F_i,F_i)$ *is a commutative algebra.*

(c) *The algebra* $\mathrm{End}^*(\mathcal{R})$ *is a formal DG algebra, i.e., there exists a quasi-isomorphism of DG algebras* $\eta : \mathrm{Ext}^*(F,F)\to\mathrm{End}^*(\mathcal{R})$, *where* $\mathrm{Ext}^*(F,F)$ *is a DG algebra with zero differential.*

*Proof.* Let us first prove (a). Let $\eta : B\to\mathrm{Ext}^*(F,F)$ be the linear map given by $\eta(b^t_{ji}) = [\alpha^t_{ji}]$. It follows from (2.22) that $\eta$ is a graded algebra homomorphism. By (2.20) $\eta$ maps a basis of $B$ onto a basis of $\mathrm{Ext}^*(F,F)$. Therefore, $\eta$ is a graded algebra isomorphism, as required.

Now we show (b). Fix $0\le i\le p-1$. We see that the isomorphism $\eta$ defined above restricts to the graded algebra isomorphism

$$\mathrm{Ext}^*(F_i,F_i)\simeq \mathrm{span}\{b^{2r}_{ii} : 0\le r\le p-i-1\} =: B'.$$

Consider $\mathbf{k}[x]$ as the graded algebra with $x$ of degree 2. We define $f:\mathbf{k}[x]\to B'$ to be the linear map given on the basis by

$$f(1) = b^0_{ii}, \qquad f(x^r) = \begin{cases} b^{2r}_{ii} & \text{if } 1\le r\le p-i-1,\\ 0 & \text{if } r>p-i-1.\end{cases}$$

By the definition of $B'$, $b_{ii}^0$ is the identity of $B'$ and $b_{ii}^{2r} = (b_{ii}^2)^r$ for $1 \leq r \leq p-i-1$. Thus, $f$ is a graded algebra homomorphism. It is a simple matter to check that $\ker f = (x^{p-i})$, hence, by the first isomorphism theorem, we obtain $\mathbf{k}[x]/(x^{p-i}) \simeq B' \simeq \mathrm{Ext}^*(F_i, F_i)$ as graded algebras. The second assertion of (b) is the immediate consequence of the first one.

It remains to prove (c). The differential on $B$ is defined to be the zero differential. Let $\xi : B \to \mathrm{End}^*(\mathcal{R})$ be the linear map given by $\xi(b_{ji}^t) = \alpha_{ji}^t$ for $0 \leq i, j \leq p-1$ and $t = |i-j| + 2r$ where $0 \leq r \leq p - \max\{i, j\} - 1$. By (2.21) we see that $\xi$ is a graded algebra homomorphism and $\xi$ is a chain map, since $\xi(b_{ji}^t)$ are morphisms of complexes. In particular, $\xi$ is a DG homomorphism. It is sufficient to show that $\xi$ is a quasi-isomorphism. By the definition of the differential on $B$ we have $H^*(B) \simeq B$. We see at once that the induced homomorphism $\xi_* : H^*(B) \simeq B \to H^*(\mathrm{End}^*(\mathcal{R})) \simeq \mathrm{Ext}^*(F, F)$ is given by $\xi_*(b_{ji}^t) = [\alpha_{ji}^t]$, i.e., $\xi_* = \eta$, where $\eta$ was defined above. We showed that $\eta$ is a graded algebra isomorphism. Hence $\xi : B \simeq \mathrm{Ext}^*(F, F) \to \mathrm{End}^*(\mathcal{R})$ is a quasi-isomorphism. □

**Corollary 2.3.6.** *There is an equivalence of triangulated categories*

$$\mathcal{D}^b\mathcal{P}_p^{\varnothing} \simeq \mathcal{D}^b(\mathrm{Ext}^*(F, F) - mod^{gr}).$$

*Proof.* Let $\mathcal{R}^\#$ be the resolution obtained from $\mathcal{R}$ by using the Kuhn duality. Clearly $\mathcal{R}^\#$ is a cofibrant object in the category of complexes of $\mathcal{P}_p^{\varnothing}$. Since $\mathcal{R}_i^\# \simeq F_i^\# \simeq F_i$ in $\mathcal{D}^b\mathcal{P}_p^{\varnothing}$ for $0 \leq i \leq p-1$, we conclude, by induction on the length of the composition series, that $\mathcal{R}^\#$ is a small generator of $\mathcal{D}^b\mathcal{P}_p^{\varnothing}$. Then the proof is analogous to that of Corollary 2.3.3. □

We also obtain the following corollary, which is by no means obvious from the explicit descriptions of the involved graded algebras.

**Corollary 2.3.7.**
*There is an equivalence of triangulated categories*

$$\mathcal{D}^b(\mathrm{Ext}^*(S, S) - mod^{gr}) \simeq \mathcal{D}^b(\mathrm{Ext}^*(F, F) - mod^{gr}).$$

*Proof.* The corollary follows immediately from Corollaries 2.3.3 and 2.3.6. □

## 2.4 The blocks of p-weight 1

Now we consider the case of the category $\mathcal{P}_d$ for $d > p$. Let $\mathcal{P}_{|\lambda|+p}^{\lambda}$ be the block of $\mathcal{P}_{|\lambda|+p}$ corresponding to a $p$-core $\lambda$. Our objective is to prove the following main theorem in this section. The principal significance of this theorem is that it provides a generalization of the results obtained in the previous sections to the case of $\mathcal{P}_d$ for $p < d < 2p$. It will be instrumental in the computations in $\mathcal{P}_{2p}$ performed in Chapter 3.

**Theorem 2.4.1.** *Fix a nonempty p-core $\lambda$. Let $\theta : \mathcal{P}_p^{\varnothing} \to \mathcal{P}_{|\lambda|+p}^{\lambda}$ be the functor given by $\theta(G) = \pi(S_\lambda \otimes G)$, where $\pi$ is the canonical projection onto $\mathcal{P}_{|\lambda|+p}^{\lambda}$. Then $\theta$ is an equivalence of abelian categories.*

We first prove the following lemma, which will be useful in the proof of Theorem 2.4.1.

**Lemma 2.4.2.** *For any $0 \leq i \leq p-1$ there is exactly one Young diagram $\mu_i$ with p-core $\lambda$ such that $\lambda$ is obtained from $\mu_i$ by removing a skew p-hook corresponding to the hook $(i+1, 1^{p-i-1})$. Moreover, the map $(i+1, 1^{p-i-1}) \mapsto \mu_i$ is an order isomorphism between the poset of hooks of weight $p$ and the poset of Young diagrams of p-weight 1 and with p-core $\lambda$.*

*Proof.* Since a Young diagram is uniquely determined by its $p$-core and $p$-quotient, it is clear that each diagram of $p$-weight 1 is determined by sliding a bead one space down in the abacus configuration of its $p$-core. Fix $0 \le i \le p-1$. Let us consider the abacus configuration of $\lambda$ and let $\mu_i$ be the Young diagram corresponding to the abacus configuration obtained by sliding the $(i+1)$-th bead with the empty space under it (counting from the bead with the lowest label). We observe that there are $p-i-1$ beads between the bead under consideration and the empty space below, since each gap between them corresponds to a bead with an empty space under it, which has been counted before the bead under consideration, and there are exactly $i$ such beads. Then it follows from Lemma 1.4.2 that $\lambda$ arises from the Young diagram $\mu_i$ by removing a skew $p$-hook corresponding to $(i+1, 1^{p-i-1})$. Obviously, if $i \neq j$ then $\mu_i \neq \mu_j$. Thus, the map $(i+1, 1^{p-i-1}) \mapsto \mu_i$ is a bijection.

It remains to prove that the map $(i+1, 1^{p-i-1}) \mapsto \mu_i$ preserves the reversed dominance order. It suffices to show that $\mu_i$ dominates $\mu_{i+1}$ for any $0 \le i \le p-2$. Let $l$ (resp. $m$) be the number of the row of $\mu_{i+1}$ (resp. $\mu_i$) containing the hand of the skew $p$-hook, which has to be removed to obtain $\lambda$. It follows from the construction of the map $(i+1, 1^{p-i-1}) \mapsto \mu_i$ that $l \le m$. Let $(\beta_j)$ (resp. $(\beta'_j)$) be the $\beta$-sequence of $\mu_{i+1}$ (resp. $\mu_i$). It follows from the definition of $\beta$-sequence that $\mu_i$ dominates $\mu_{i+1}$ if and only if $\sum_{r\le s} \beta'_r \le \sum_{r \le s} \beta_r$ for any $s \ge 1$. We see that $(\beta_1, \ldots, \beta_{l-1}, \beta_{l+1}, \ldots, \beta_{l+p-i-1}, \beta_l - p, \beta_{l+p-i}, \ldots)$ and $(\beta'_1, \ldots, \beta'_{m-1}, \beta'_{m+1}, \ldots, \beta'_{m+p-i}, \beta'_m - p, \beta'_{m+p-i+1}, \ldots)$ are both the $\beta$-sequence of $\lambda$ obtained by removing the skew $p$-hook in, respectively, $\mu_{i+1}$ and $\mu_i$. Hence we have

$$\begin{aligned} &\beta_r = \beta'_r \ \text{ for } 1 \le r \le l-1 \text{ and } m+1 \le r \le l+p-i-1 \text{ and } r \ge m+p-i+1, \\ &\beta_l = \beta'_{l+p-i} + p, \qquad \beta_r = \beta'_{r-1} \ \text{ for } l+1 \le r \le m, \\ &\beta_{m+p-i} = \beta'_m - p, \qquad \beta_r = \beta'_{r+1} \ \text{ for } l+p-i \le r \le m+p-i-1. \end{aligned}$$

Then it is a simple matter to check that $\sum_{r \le s} \beta'_r \le \sum_{r\le s} \beta_r$ for any $s \ge 1$. Thus, $\mu_i$ dominates $\mu_{i+1}$, and the proof is complete. □

*Proof of Theorem 2.4.1.* It is clear that $\theta$ is an exact and additive functor. Now we show that $\theta(S_i) = S_{\mu_i}$ and $\theta(W_i) = W_{\mu_i}$ for any $0 \le i \le p-1$, where $\mu_i$ is the Young diagram as given in the proof of Lemma 2.4.2. Fix $0 \le i \le p-1$. By the Littlewood-Richardson rule we obtain

$$\theta(S_i) = \pi(S_\lambda \otimes S_i) = \sum_{\mu \in P_{\lambda,1}} c(i, \lambda; \mu) S_\mu,$$

where $P_{\lambda,1}$ is the set of Young diagrams of $p$-weight 1 with $p$-core $\lambda$ and $c(i,\lambda;\mu)$ is the number of semistandard skew tableaux of shape $\mu/\lambda$ and of the content $(p-i, 1^i)$ that satisfy the Yamanouchi condition. We observe that a skew tableau as given in the definition of $c(i,\lambda;\mu)$ is of shape of a skew $p$-hook corresponding to the hook $(i+1, 1^{p-i-1})$ and there is only one filling such that the corresponding word is a Yamanouchi word. Indeed, it is obvious that $\mu/\lambda$ is a skew $p$-hook and we see that if the $j$-th column of such a skew tableau has only one box, then this column is filled by the number $j$ and, otherwise, this column is filled by the sequence $(j, 1, \ldots, 1)$, reading from the bottom up. The picture below provides an example of such a skew tableau for $p = 7$, $\mu/\lambda = (4,3,3,1)/(2,2)$.

| | | | |
|---|---|---|---|
| | | 1 | 4 |
| | | 1 | |
| 1 | 2 | 3 | |
| 1 | | | |

Therefore, if $\lambda$ arises from $\mu$ by removing a skew $p$-hook corresponding to the hook $(i+1, 1^{p-i-1})$, then $c(i,\lambda;\mu) = 1$ and, otherwise, $c(i,\lambda;\mu) = 0$. By Lemma 2.4.2 there is exactly one Young diagram $\mu = \mu_i$ such that $c(i, \lambda; \mu) = 1$. Hence $\theta(S_i) = S_{\mu_i}$. Clearly $S_\lambda \simeq W_\lambda$, since $\lambda$ is a $p$-core. Then we have

$$\theta(W_i) = S_\lambda \otimes W_i \simeq W_\lambda \otimes W_i = (S_i \otimes S_\lambda)^\# = S_{\mu_i}^\# = W_{\mu_i}.$$

The theorem follows from [PS88, Theorem (5.8)]). □

By the last theorem it is clear that we obtain Ext-computations from those in Section 2.2 by replacing $m$ by $\mu_m$ and $n$ by $\mu_n$. We record in the following corollary the other important and immediate consequences of Theorem 2.4.1 and the results of the previous sections. The last statement of the corollary follows from the easy observation that each diagram of weight $p < d < 2p$ is of $p$-weight at most 1.

**Corollary 2.4.3.** *Let $\lambda$ be a $p$-core.*

*(a) The decomposition matrix $D = (d_{lm})$ of the category $\mathcal{P}^\lambda_{|\lambda|+p}$ is given by*

$$d_{lm} := [W_{\mu_{l-1}} : F_{\mu_{m-1}}] = \begin{cases} 1 & \text{if } m = l \quad \text{or } m = l+1, \\ 0 & \text{otherwise.} \end{cases}$$

*(b) The category $\mathcal{P}^\lambda_{|\lambda|+p}$ has a Kazhdan-Lusztig theory.*

*(c) The endomorphism algebra whose cohomology algebra is the Yoneda algebra of Schur functors in $\mathcal{P}^\lambda_{|\lambda|+p}$, $\mathrm{Ext}^*(S, S)$ where $S = \bigoplus_{0 \le i \le p-1} S_{\mu_i}$, is DG formal.*

*(d) The endomorphism algebra whose cohomology algebra is the Yoneda algebra of simple functors in $\mathcal{P}^\lambda_{|\lambda|+p}$, $\mathrm{Ext}^*(F, F)$ where $F = \bigoplus_{0 \le i \le p-1} F_{\mu_i}$, is DG formal.*

*(e) There are the following equivalences of triangulated categories:*

$$\mathcal{D}^b\mathcal{P}^\lambda_{|\lambda|+p} \simeq \mathcal{D}^b(\mathrm{Ext}^*(S,S) - mod^{gr}), \quad \mathcal{D}^b\mathcal{P}^\lambda_{|\lambda|+p} \simeq \mathcal{D}^b(\mathrm{Ext}^*(F,F) - mod^{gr}),$$
$$\mathcal{D}^b(\mathrm{Ext}^*(S,S) - mod^{gr}) \simeq \mathcal{D}^b(\mathrm{Ext}^*(F,F) - mod^{gr}).$$

*(f) The above statements hold for any block of nonzero $p$-weight of the category $\mathcal{P}_d$ for $p < d < 2p$.*

It was shown in [HTY14] that if two blocks of $\mathcal{P}$ have the same $p$-weight, then they are derived equivalent (cf. [HTY14, Theorem 5]). Theorem 2.4.1 establishes an equivalence of abelian categories (even highest weight categories) in the case of $p$-weight 1. It turns out that the approach used in the proof of this theorem does not work in the case of $p$-weight greater than 1.

**Example.** Let $p = 3$. Then $\mathcal{P}_6^\varnothing$ and $\mathcal{P}_7^{(1)}$ are blocks of $p$-weight 2. Let us consider the functor $\theta : \mathcal{P}_6^\varnothing \to \mathcal{P}_7^{(1)}$ given by $\theta(G) = \pi(S_{(1)} \otimes G)$, where $\pi$ is the canonical projection onto $\mathcal{P}_7^{(1)}$. By the Littlewood-Richardson rule we see that $\theta(S_{(3,2,1)})$ is filtered by $S_{(4,2,1)}$ and $S_{(3,2,1^2)}$. In particular, the image of $S_{(3,2,1)}$ under $\theta$ is not a Schur functor.

# Chapter 3

# Ext-groups between Schur functors in $\mathcal{P}_{2p}$

## 3.1 The category $\mathcal{P}_{2p}$ : notation and basic constructions

Let us recall that if $S_\lambda$ and $S_\eta$ lie in different blocks, then $\mathrm{Ext}^*(S_\lambda, S_\eta) = 0$. By Theorem 1.4.1 the functors $F_\lambda$ and $F_\eta$ (and thus $S_\lambda$ and $S_\eta$) lie in the same block if and only if they have the same $p$-core. If $\lambda$ and $\eta$ have the same $p$-core and the $p$-weight equal to 1, then the Ext-groups between the corresponding Schur functors are completely determined by Theorem 2.4.1 and Proposition 2.2.6. Therefore, all the considerations in this chapter are restricted to the block $\mathcal{P}_{2p}^{\varnothing}$. Obviously, all hooks have the $p$-weight 2. In the next lemma we determine the Young diagrams, whose $p$-quotient consists of the nonempty Young diagrams $\lambda_1 = \lambda_2 = (1)$ corresponding to two runners of the abacus. Since there are $2p$ hooks, $\frac{p(p-1)}{2}$ Young diagrams as described in Lemma 3.1.1 and the total number of Young diagrams of $p$-weight 2 is $\frac{p(p+1)}{2}$ (see Section 1.4), there are no other Young diagrams of $p$-weight 2. We denote the Schur functor corresponding to the Young diagram whose $p$-quotient consists of the nonempty diagrams on the runners $i$ and $j$ by $S_{i,j}$.

**Lemma 3.1.1.** *Let $0 \leq i < j \leq p-1$ and let $\lambda$ be the Young diagram of weight $2p$, whose $p$-quotient consists of two diagrams $(1)$ corresponding to the $i$-th and $j$-th runners of the abacus. Then*

$$\lambda = (j+1, i+2, 2^{p-j-1}, 1^{j-i-1}).$$

*Proof.* Obviously, the $\beta$-sequence of the empty diagram is the sequence $\beta_\varnothing = (-1, -2, -3, \ldots)$. The abacus configuration of $\lambda$ arises from that of the $p$-core of $\lambda$, which is the empty diagram, by sliding the beads on the $i$-th and $j$-th runners one space down. Then we conclude that the $\beta$-sequence of $\lambda$ is

$$\beta_\lambda = (j, i, -1, -2, \ldots, -(p-j-1), -(p-j+1), -(p-j+2), \ldots, -(p-i-1), -(p-i+1), -(p-i+2), \ldots).$$

By definition of the $\beta$-sequence we obtain

$$\lambda_k = \begin{cases} j+1 & \text{for } k = 1, \\ i+2, & \text{for } k = 2, \\ 2 & \text{for } 3 \leq k \leq p-j+1, \\ 1 & \text{for } p-j+2 \leq k \leq p-i, \\ 0 & \text{for } k \geq p-i+1. \end{cases}$$

□

In the next lemma we explain in which way two non-hooks of $p$-weight 2 are related to each other with respect to the dominance order.

**Lemma 3.1.2.** *Let $\lambda$ and $\mu$ be the Young diagrams corresponding to $S_{i_1,j_1}$ and $S_{i_2,j_2}$, respectively. Then $\lambda$ dominates $\mu$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$*

*Proof.* If $\lambda$ dominates $\mu$, then we have immediately, by the definition of the dominance order and Lemma 3.1.1, that $j_1 \leq j_2$. By this assumption, we also know that $\widetilde{\mu}$ dominates $\widetilde{\lambda}$, hence, again by Lemma 3.1.1, $p - i_2 \leq p - i_1$, which is equivalent to $i_1 \leq i_2$. For the converse, it is a simple matter to check by using the definition of the dominance order that if $i_1 \leq i_2$ and $j_1 \leq j_2$, then $\lambda$ dominates $\mu$. □

We will often need to compute Ext-groups between Schur functors in $\mathcal{P}_d$ for $p < d < 2p$. In this case each Young diagram of weight $d$ is a $p$-core or is of $p$-weight 1. By Lemma 2.4.2 each diagram of $p$-weight 1 is the Young diagram $\mu_i$ for some $0 \leq i \leq d-1$. By Theorem 2.4.1 and Proposition 2.2.6 one has

$$\mathrm{Ext}_{\mathcal{P}_d}(S_{\mu_i}, S_{\mu_j}) = \mathrm{Ext}_{\mathcal{P}_p}(S_i, S_j) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \quad \bullet\ i = j \text{ and } q = 0, \\ & \quad \bullet\ i < j \text{ and } q \in \{j-i-1, j-i\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

In the next two section we will compute Ext-groups involving $\Omega^i_{2p}$ for $1 \leq i \leq 2p-1$ (from now on we will omit the subscript, since we work in the category $\mathcal{P}_{2p}$). We will use there broadly the following formula, which follows from the decomposition formula (1.9):

$$E_1^{**} = \sum_{\alpha \subset \lambda} \mathrm{Ext}^*_{\mathcal{P}_i}(\Lambda^i, S_\alpha) \otimes \mathrm{Ext}^*_{\mathcal{P}_{2p-i}}(S^{2p-i}, S_{\lambda/\alpha}) \Rightarrow \mathrm{Ext}^*_{\mathcal{P}_{2p}}(\Omega^i, S_\lambda). \tag{3.2}$$

To simplify notation, we omit subscripts in Ext-groups whenever the context makes them clear.

## 3.2 Computations of the dimensions of $\mathrm{Ext}^*(S_i, S_j)$

In the following lemma we record the result on $\mathrm{Ext}^*(S^d, \Lambda^d)$ for $d < 2p$ for the convenience, since we will use it often in the following section.

**Lemma 3.2.1.** *Let $1 \leq d \leq 2p-1$. Then $\mathrm{Ext}^*(S^d, \Lambda^d) \neq 0$ if and only if $d \in \{1, p, p+1\}$. In these cases one has*

$$\mathrm{Ext}^q(S^1, \Lambda^1) = \begin{cases} \mathbf{k} & \text{if } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{Ext}^q(S^p, \Lambda^p) = \mathrm{Ext}^q(S^{p+1}, \Lambda^{p+1}) = \begin{cases} \mathbf{k} & \text{if } q \in \{p-2, p-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let us recall that $S^d = S_0$ and $\Lambda^d = S_{d-1}$ in $\mathcal{P}_d$. If $d = 1$ then $S^1 = \Lambda^1$ and the lemma follows from (1.2). For $2 \leq d \leq p-1$ or $p+2 \leq d \leq 2p-1$ we observe that $(d)$ and $(1^d)$ have the different $p$-cores, hence $\mathrm{Ext}^*(S^d, \Lambda^d) = 0$. For $d = p$ we apply Proposition 2.2.6. In case $d = p+1$ we use (3.1). It is worth noting that this lemma also follows from [Aki89, Corollary §5]. □

Now we turn to two lemmas that are necessary for the proof of Theorem 3.2.4.

**Lemma 3.2.2.** *Let* $1 \le i \le 2p-1$ *and* $0 \le j \le 2p-1$. *Then*

$$\mathrm{Ext}^q(\Omega^i, S_j) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i \in \{j, j+1\} \text{ and } q = 0, \\ & \bullet\ i \in \{j-p, j-p+1\} \text{ and } q \in \{p-1, p-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let $\lambda$ be the Young diagram $(j+1, 1^{2p-j-1})$. If $i > j+1$ then $\lambda$ does not contain the Young diagram $(i)$, hence $\mathrm{Ext}^*(\Omega^i, S_j) = \mathrm{Ext}^*(\Lambda^i \otimes S^{2p-i}, S_j) = 0$ by (3.2) and (1.2). If $i = j+1$ then $\lambda/(i)$ is the Young diagram $(1^{2p-j-1})$. Thus, again by (3.2) and (1.2),

$$\mathrm{Ext}^q(\Omega^{j+1}, S_j) = \mathrm{Ext}^q(\Lambda^{j+1} \otimes S^{2p-j-1}, S_j) = \mathrm{Ext}^q(S^{2p-j-1}, S_{\lambda/(j+1)}) =$$
$$\mathrm{Ext}^q(S^{2p-j-1}, S^{2p-j-1}) = \begin{cases} \mathbf{k} & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider $i \le j$. Then $S_{\lambda/(i)} \simeq S^{2p-j-1} \otimes \Lambda^{j-i+1}$. In consequence, by (3.2), (1.2) and the exponential formula (1.10) we obtain

$$\mathrm{Ext}^q(\Omega^i, S_j) = \mathrm{Ext}^q(\Lambda^i \otimes S^{2p-i}, S_j) = \mathrm{Ext}^q(S^{2p-i}, S_{\lambda/(i)}) = \mathrm{Ext}^q(S^{2p-i}, S^{2p-j-1} \otimes \Lambda^{j-i+1}) =$$
$$\bigoplus_{r=0}^{q} \mathrm{Ext}^r(S^{2p-i-1}, S^{2p-i-1}) \otimes \mathrm{Ext}^{q-r}(S^{j-i+1}, \Lambda^{j-i+1}) = \mathrm{Ext}^q(S^{j-i+1}, \Lambda^{j-i+1}).$$

To end the proof, it suffices to apply Lemma 3.2.1. □

**Lemma 3.2.3.** *Let* $1 \le i \le 2p-1$. *Then*

$$\mathrm{Ext}^q(S_0, \Omega^i) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i = 1 \text{ and } q = 0, \\ & \bullet\ i \in \{p, p+1\} \text{ and } q \in \{p-2, p-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By the exponential formula (1.10) we have

$$\mathrm{Ext}^q(S_0, \Omega^i) = \mathrm{Ext}^q(S^{2p}, S^{2p-i} \otimes \Lambda^i) = \bigoplus_{m=0}^{q} \mathrm{Ext}^m(S^{2p-i}, S^{2p-i}) \otimes \mathrm{Ext}^{q-m}(S^i, \Lambda^i) =$$
$$\mathrm{Ext}^q(S^i, \Lambda^i).$$

In the last equality we use (1.2). The lemma follows from Lemma 3.2.1. □

**Theorem 3.2.4.** *Let* $0 \le i, j \le 2p-1$ *and* $r = j - i$. *Then*

$$\mathrm{Ext}^q(S_i, S_j) \simeq \mathrm{Ext}^q(W_j, W_i) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ 0 \le r \le p-1 \text{ and } q \in \{r-1, r\}, \\ & \bullet\ p+1 \le r \le 2p-1 \text{ and } q \in \{r-3, r-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The equivalence $\mathrm{Ext}^*(S_i, S_j) \simeq \mathrm{Ext}^*(W_j, W_i)$ follows by the Kuhn duality. By (1.2) we have $\mathrm{Ext}^q(S_j, S_j) = \mathbf{k}$ for $q = 0$ and the trivial group otherwise. Moreover, if $j < i$ then the Young diagram corresponding to $S_j$ strictly dominates that corresponding to $S_i$. In consequence, $\mathrm{Ext}^*(S_i, S_j) = 0$ by Proposition 1.1.1(d). In the sequel we assume that $j > i$. We now divide the proof into several cases.

*Case 1.* $0 < j - i < p$

We apply the long exact sequence for $\mathrm{Ext}(-, S_j)$ to the short exact sequence (1.8)

$$0 \to S_j \to \Omega^j \to S_{j-1} \to 0.$$

We have recalled above that $\mathrm{Ext}^*(S_j, S_j) = \mathrm{Hom}(S_j, S_j) = \mathbf{k}$. By Lemma 3.2.2 we have $\mathrm{Ext}^*(\Omega^j, S_j) = \mathrm{Hom}(\Omega^j, S_j) = \mathbf{k}$. It follows from the long exact sequence under consideration that

$$\dim \mathrm{Hom}(S_{j-1}, S_j) = \dim \mathrm{Ext}^1(S_{j-1}, S_j), \quad \mathrm{Ext}^q(S_{j-1}, S_j) = 0 \text{ for } q \geq 2.$$

Suppose that $\mathrm{Hom}(S_{j-1}, S_j) = 0$. Then $\mathrm{Ext}^1(S_{j-1}, S_j) = 0$, hence $\Omega^j = S_{j-1} \oplus S_j$. Since the Young diagram corresponding to $S_{j-1}$ dominates that corresponding to $S_j$, we have $\mathrm{Hom}(S_j, S_{j-1}) = 0$. By (1.2), $\mathrm{Hom}(S_{j-1}, S_{j-1}) = \mathbf{k}$. Then we have isomorphisms of $\mathbf{k}$-algebras:

$$\mathrm{End}(\Omega^j) = \mathrm{End}(S_{j-1} \oplus S_j) \simeq \begin{bmatrix} \mathrm{Hom}(S_{j-1}, S_{j-1}) & \mathrm{Hom}(S_{j-1}, S_j) \\ \mathrm{Hom}(S_j, S_{j-1}) & \mathrm{Hom}(S_j, S_j) \end{bmatrix} = \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix} \simeq \mathbf{k} \times \mathbf{k}.$$

We have seen in (1.7) that $d_{j-1}\kappa_j \neq 0$. By (1.5) we get

$$(d_{j-1}\kappa_j)^2 = d_{j-1}(\kappa_j d_{j-1})\kappa_j = d_{j-1}(-d_{j-2}\kappa_{j-1})\kappa_j = 0,$$

hence we have a nonzero nilpotent in $\mathrm{End}(\Omega^j)$. This leads to the contradiction, since there are no nonzero nilpotents in a product of fields. Thus,

$$\mathrm{Ext}^q(S_{j-1}, S_j) = \begin{cases} \mathbf{k} & \text{if } q \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Take $i + 1 \leq m \leq j - 1$. By Lemma 3.2.2, $\mathrm{Ext}^q(\Omega^m, S_j) = 0$. By that and the long exact sequence for $\mathrm{Ext}^*(-, S_j)$ applied to the short exact sequence (1.8)

$$0 \to S_m \to \Omega^m \to S_{m-1}$$

we obtain $\mathrm{Ext}^q(S_{m-1}, S_j) = \mathrm{Ext}^{q-1}(S_m, S_j)$. Using all these equalities for $i + 1 \leq m \leq j - 1$ and the above result for $\mathrm{Ext}^*(S_{j-1}, S_j)$, we get

$$\mathrm{Ext}^q(S_i, S_j) = \mathrm{Ext}^{q-(j-i-1)}(S_{j-1}, S_j) = \begin{cases} \mathbf{k} & \text{if } q - (j - i - 1) \in \{0, 1\}, \\ & \text{i.e., } q \in \{j - i - 1, j - i\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Case 2.* $i = 0, j = p$

By the long exact sequence for $\mathrm{Ext}(S_0, -)$ applied to the short exact sequence (1.8)

$$0 \to S_p \to \Omega^p \to S_{p-1} \to 0$$

and by Lemma 3.2.3 we obtain $\mathrm{Ext}^q(S_0, S_p) = 0$ for $q \notin \{p-2, p-1, p\}$. Now we will show that $\mathrm{Ext}^{p-2}(S_0, S_p)$ and $\mathrm{Ext}^{p-1}(S_0, S_p)$ are also the zero space.

Let us consider the resolution of $S_{p-1}$

$$0 \to S_{p-1} \to \Omega^{p-1} \xrightarrow{\kappa_{p-1}} \Omega^{p-2} \xrightarrow{\kappa_{p-2}} \dots \xrightarrow{\kappa_2} \Omega^1 \xrightarrow{\kappa_1} \Omega^0 \to 0$$

and the injective resolution of $S_p \otimes \Lambda^p$ obtained from the injective resolution of $\Lambda^p$ (2.7) by tensoring by $S^p$. We note that the above resolution of $S_{p-1}$ is injective, since $\Omega^i$ for $i \leq p-1$ is injective as the tensor product of injectives (cf. [FS97, Corollary 2.11]). Then there exists vertical maps $f_i$ that each square in the following diagram commutes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \Omega^p & \xrightarrow{\mathrm{id}\otimes\kappa_p} & S^p \otimes S^1 \otimes \Lambda^{p-1} & \longrightarrow & \dots & \longrightarrow & S^p \otimes S^{p-1} \otimes \Lambda^1 & \xrightarrow{\mathrm{id}\otimes\kappa_1} & S^p \otimes S^p & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle f_1} & & & & \big\downarrow{\scriptstyle f_{p-1}} & & \big\downarrow{\scriptstyle f_p} & & \\
0 & \longrightarrow & S_{p-1} & \hookrightarrow & S^{p+1} \otimes \Lambda^{p-1} & \longrightarrow & \dots & \longrightarrow & S^{2p-1} \otimes \Lambda^1 & \xrightarrow{\kappa_1} & S^{2p} & \longrightarrow & 0
\end{array}.$$

We will show that we can set $f_i = m_i \otimes \mathrm{id}$ for $1 \leq i \leq p-1$ and $f_p = m_p$, where $m_i : S^p \otimes S^i \to S^{p+i}$ is the multiplication map. The leftmost square commutes by the definition of $\kappa_p : \Omega^p \to \Omega^{p-1}$, the acyclicity of the Koszul differential and the fact that $S_{p-1}$ is the kernel of $\kappa_{p-1}$. We now will show that the following square commutes for $1 \leq i \leq p-1$ (in case $i = p-1$ there is no exterior power in the tensor products at the right side of the diagram and the vertical map is $m_p$, but we show the commutativity in the same way as below):

$$\begin{array}{ccc}
S^p \otimes S^{p-i} \otimes \Lambda^i & \xrightarrow{\mathrm{id}\otimes\kappa_i} & S^p \otimes S^{p-i+1} \otimes \Lambda^{i-1} \\
\big\downarrow{\scriptstyle m_{p-i}\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle m_{p-i+1}\otimes\mathrm{id}} \\
S^{2p-i} \otimes \Lambda^i & \xrightarrow{\kappa_i} & S^{2p-i+1} \otimes \Lambda^{i-1}
\end{array}$$

Let us evaluate all the functors in this diagram at the arbitrary vector space $V$ with a basis $\{v_1, \dots, v_n\}$ and take a basis element $v_{i_1} \dots v_{i_p} \otimes v_{j_1} \dots v_{j_{p-i}} \otimes v_{l_1} \wedge \dots \wedge v_{l_i} \in S^p \otimes S^{p-i} \otimes \Lambda^i$. Then

$$v_{i_1} \dots v_{i_p} \otimes v_{j_1} \dots v_{j_{p-i}} \otimes v_{l_1} \wedge \dots \wedge v_{l_i} \xmapsto{m_{p-i}\otimes\mathrm{id}} v_{i_1} \dots v_{i_p} v_{j_1} \dots v_{j_{p-i}} \otimes v_{l_1} \wedge \dots \wedge v_{l_i} \xmapsto{\kappa_i}$$
$$\sum_{\sigma \in \Sigma_i'} \mathrm{sgn}(\sigma) v_{i_1} \dots v_{i_p} v_{j_1} \dots v_{j_{p-i}} v_{l_{\sigma(1)}} \otimes v_{l_{\sigma(2)}} \wedge \dots \wedge v_{l_{\sigma(i)}}$$

and

$$v_{i_1} \dots v_{i_p} \otimes v_{j_1} \dots v_{j_{p-i}} \otimes v_{l_1} \wedge \dots \wedge v_{l_i} \xmapsto{\mathrm{id}\otimes\kappa_i}$$
$$v_{i_1} \dots v_{i_p} \otimes \sum_{\sigma \in \Sigma_i'} \mathrm{sgn}(\sigma) v_{j_1} \dots v_{j_{p-i}} v_{l_{\sigma(1)}} \otimes v_{l_{\sigma(2)}} \wedge \dots \wedge v_{l_{\sigma(i)}} \xmapsto{m_{p-i+1}\otimes\mathrm{id}}$$
$$\sum_{\sigma \in \Sigma_i'} \mathrm{sgn}(\sigma) v_{i_1} \dots v_{i_p} v_{j_1} \dots v_{j_{p-i}} v_{l_{\sigma(1)}} \otimes v_{l_{\sigma(2)}} \wedge \dots \wedge v_{l_{\sigma(i)}},$$

where $\Sigma_i' = \{\sigma \in \Sigma_i : \sigma(2) < \dots < \sigma(i)\}$. Thus, we see that the diagram is commutative. Let $c_p : S^{2p} \to S^p \otimes S^p$ be the comultiplication map. We observe that $m_p \circ c_p = \binom{2p}{p} \cdot \mathrm{id}$. Indeed, we consider, similarly as above, a vector space $V$ with a basis $\{v_1, \dots, v_n\}$ and we take a basis element $v_{i_1} \dots v_{i_{2p}} \in S^{2p}$. Then

$$v_{i_1} \dots v_{i_{2p}} \xmapsto{c_p} \sum_{\sigma \in \Sigma_{2p}^p} v_{i_{\sigma(1)}} \dots v_{i_{\sigma(p)}} \otimes v_{i_{\sigma(p+1)}} \dots v_{i_{\sigma(2p)}} \xmapsto{m_p} \sum_{\sigma \in \Sigma_{2p}^p} v_{i_1} \dots v_{i_{2p}} = \binom{2p}{p} \cdot (v_{i_1} \dots v_{i_{2p}}),$$

where $\Sigma_{2p}^p = \{\sigma \in \Sigma_{2p} : \sigma(1) < \dots < \sigma(p) \text{ and } \sigma(p+1) < \dots < \sigma(2p)\}$. Hence $m_p \circ c_p = \binom{2p}{p} \cdot \mathrm{id}$. $\binom{2p}{p}$ is not divisible by $p$, hence it is invertible in the field $\mathbf{k}$. In consequence, $m_p \circ \left( \frac{1}{\binom{2p}{p}} \cdot c_p \right) = \mathrm{id}\,.$

In other words, $m_p$ is a retraction. In the similar way we argue that $m_{p-1} \otimes \mathrm{id}$ is a retraction – one shows in the same manner that $(m_{p-1} \otimes \mathrm{id}) \circ (c_{p-1} \otimes \mathrm{id}) = \binom{2p-1}{p} \cdot \mathrm{id}$ and, clearly, $\binom{2p-1}{p}$ is not divisible by $p$. Now we apply the functor $\mathrm{Hom}(S^{2p}, -)$ to both resolutions and we consider the right part of the Hom-complexes

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \mathrm{Hom}(S^{2p}, S^p \otimes \Omega^2_p) \simeq 0 & \longrightarrow & \mathrm{Hom}(S^{2p}, S^p \otimes \Omega^1_p) \simeq \mathbf{k} & \xrightarrow{0} & \mathrm{Hom}(S^{2p}, S^p \otimes S^p) \simeq \mathbf{k} & \longrightarrow 0 \\
& \downarrow & & \downarrow{\scriptstyle m_{p-1}\circ -} & & \downarrow{\scriptstyle m_p \circ -} & \\
\dots \longrightarrow & \mathrm{Hom}(S^{2p}, \Omega^2_{2p}) \simeq 0 & \longrightarrow & \mathrm{Hom}(S^{2p}, \Omega^1_{2p}) \simeq \mathbf{k} & \xrightarrow{0} & \mathrm{Hom}(S^{2p}, S^{2p}) \simeq \mathbf{k} & \longrightarrow 0
\end{array}.$$

We use here the exponential formula (1.10) to deduce that only two last Hom-spaces in both Hom-complexes are nonzero and one-dimensional. The homologies of the first and second complex are given by $\mathrm{Ext}^*(S_0, \Omega^p)$ and $\mathrm{Ext}^*(S_0, S_{p-1})$, which were computed in Lemma 3.2.3 and in Case 1., respectively. By that $(\mathrm{id} \otimes \kappa_1) \circ -$ (in the first Hom-complex) and $\kappa_1 \circ -$ (in the second Hom-complex) are the zero maps. Since $m_{p-1} \otimes \mathrm{id}$ and $m_p$ are retractions, $m_{p-1} \otimes \mathrm{id} \circ -$ and $m_p \circ -$ are also rectractions between one-dimensional spaces. In consequence, these maps are isomorphisms. Since all the maps in the Hom-complexes are the zero map, the maps on the homologies, i.e., on Ext's, induced by these isomorphims are also isomorphisms. Therefore, $\mathrm{Ext}^q(S_0, \Omega^p) \simeq \mathrm{Ext}^q(S_0, S_{p-1})$ for $q = p-2$ or $q = p-1$. Then, by the long exact sequence considered at the beginning of this case, we have $\mathrm{Ext}^q(S_0, S_p) = 0$ for $q \in \{p-2, p-1, p\}$. Thus, we have $\mathrm{Ext}^*(S_0, S_p) = 0$.

*Case 3. $j - i = p$ and $i > 0$*

It follows from the long exact sequence for $\mathrm{Ext}(S_i, -)$ applied to the short exact sequence (1.8)

$$0 \to S_{p+i} \to \Omega^{p+i} \to S_{p+i-1} \to 0,$$

by using Lemma 3.2.3 and results derived in Case 1., that $\mathrm{Ext}^q(S_i, S_{p+i}) = 0$ for $q \notin \{p-1, p-2, p\}$. We also conclude, by the long exact sequence for $\mathrm{Ext}(-, S_{i+p})$ applied to the same short exact sequence, that $\mathrm{Ext}^p(S_i, S_{i+p}) = 0$. Now we turn to consider the degrees $q = p-2$ and $q = p-1$. We will prove by induction on $0 \le i \le p-1$ that $\mathrm{Ext}^q(S_i, S_{i+p}) = 0$ for $q \in \{p-2, p-1\}$. The base of induction is the Case 2. Now we assume that the equality holds for some $i < p-1$ and we prove that it holds also for $i+1$.

Let us consider the following commutative diagram (except the anticommutative squares involving the connecting homomorphisms) for $q \in \{p-2, p-1\}$ whose columns and rows are the long exact sequences for Ext's applied to the short exact sequences (1.8) for $m = p+i+1$ and $m = i+1$, respectively:

$$\begin{array}{ccccccc}
& \dots & & \dots & & \dots & \\
& \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & \mathrm{Ext}^q(S_i, S_{p+i+1}) & \longrightarrow & \mathrm{Ext}^q(\Omega^{i+1}, S_{p+i+1}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^q(S_{i+1}, S_{p+i+1}) & \longrightarrow \dots \\
& \downarrow & & \downarrow{\scriptstyle \alpha_q} & & \downarrow & \\
\dots \longrightarrow & \mathrm{Ext}^q(S_i, \Omega^{p+i+1}) & \longrightarrow & \mathrm{Ext}^q(\Omega^{i+1}, \Omega^{p+i+1}) \simeq \mathbf{k}^2 & \longrightarrow & \mathrm{Ext}^q(S_{i+1}, \Omega^{p+i+1}) \simeq \mathbf{k} & \longrightarrow \dots \\
& \downarrow & & \downarrow{\scriptstyle \beta_q} & & \downarrow{\scriptstyle \gamma_q} & \\
\dots \longrightarrow & \mathrm{Ext}^q(S_i, S_{p+i}) = 0 & \longrightarrow & \mathrm{Ext}^q(\Omega^{i+1}, S_{p+i}) \simeq \mathbf{k} & \xrightarrow{\simeq} & \mathrm{Ext}^q(S_{i+1}, S_{p+i}) \simeq \mathbf{k} & \longrightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & \\
& \dots & & \dots & & \dots &
\end{array}$$

We have applied above the result of Case 1., the inductive hypothesis and Lemma 3.2.2. We also use the following computations, which are similar to those in Lemma 3.2.2 and in (1.9):

$$\mathrm{Ext}^q(S_{i+1}, \Omega^{p+i+1}) = \mathrm{Ext}^q_{\mathcal{P}(2)_{2p}}\left(\sum_{\xi \subset (i+2,1^{2p-i-2})} S_\xi \boxtimes S_{(i+2,1^{2p-i-2})/\xi}, S^{p-i-1} \boxtimes \Lambda^{p+i+1}\right) =$$
$$\bigoplus_{m=0}^{q} \mathrm{Ext}^m(S^{p-i-1}, S^{p-i-1}) \otimes \mathrm{Ext}^{q-m}(S^p \otimes \Lambda^{i+1}, \Lambda^{p+i+1})) =$$
$$\mathrm{Hom}(S^{p-i-1}, S^{p-i-1}) \otimes \mathrm{Ext}^q(S^p \otimes \Lambda^{i+1}, \Lambda^{p+i+1}) = \mathrm{Ext}^q(S^p, \Lambda^p) \otimes \mathrm{Hom}(\Lambda^{i+1}, \Lambda^{i+1}) = \mathbf{k}$$

and

$$\mathrm{Ext}^q(\Omega^{i+1}, \Omega^{p+i+1}) = \mathrm{Ext}^q_{\mathcal{P}(2)_{2p}}(\Omega^{i+1}(- \oplus -), S^{p-i-1} \boxtimes \Lambda^{p+i+1}) =$$
$$\mathrm{Ext}^q_{\mathcal{P}(2)_{2p}}\left(\bigoplus_{\substack{0 \le l \le 2p-i-1 \\ 0 \le j \le i+1 \\ l+j=p-i-1}} (S^l \otimes \Lambda^j) \boxtimes (S^{2p-l-i-1} \otimes \Lambda^{i-j+1}), S^{p-i-1} \boxtimes \Lambda^{p+i+1}\right) =$$
$$\bigoplus_{\substack{0 \le l \le 2p-i-1 \\ 0 \le j \le i+1 \\ l+j=p-i-1}} \bigoplus_{m=0}^{q} \mathrm{Ext}^m(S^l \otimes \Lambda^j, S^{p-i-1}) \otimes \mathrm{Ext}^{q-m}(S^{2p-l-i-1} \otimes \Lambda^{i-j+1}, \Lambda^{p+i+1}) =$$
$$(\mathrm{Hom}(S^{p-i-1}, S^{p-i-1}) \otimes \mathrm{Ext}^q(S^p, \Lambda^p) \otimes \mathrm{Hom}(\Lambda^{i+1}, \Lambda^{i+1})) \oplus$$
$$(\mathrm{Hom}(S^{p-i-2}, S^{p-i-2}) \otimes \mathrm{Hom}(\Lambda^1, S^1) \otimes \mathrm{Ext}^q(S^{p+1}, \Lambda^{p+1}) \otimes \mathrm{Hom}(\Lambda^i, \Lambda^i)) = \mathbf{k}^2.$$

Let us come back to the diagram given above. Suppose that $\gamma_q = 0$. Then, by the commutativity of the right bottom square in the diagram and the fact that the bottom arrow in this square is an isomorphism, we obtain $\beta_q = 0$. In consequence, $\operatorname{im}\alpha_q = \ker\beta_q = \mathrm{Ext}^q(\Omega^{i+1}, \Omega^{p+i+1}) \simeq \mathbf{k}^2$, hence $\dim\operatorname{im}\alpha_q = 2$. On the other hand, $\dim\operatorname{im}\alpha_q \le \dim\mathrm{Ext}^q(\Omega^{i+1}, S_{p+i+1}) = 1$ by the rank-nullity theorem. Thus, $\gamma_q \neq 0$, hence $\gamma_q$ is an isomorphism since the domain and codomain are one-dimensional. Then $\mathrm{Ext}^q(S_{i+1}, S_{p+i+1}) = 0$, as desired, and the statement for any $1 \le i \le p-1$ follows from the induction principle.

*Case 4.* $j - i > p$

We apply the long exact sequence for $\mathrm{Ext}^*(S_0, -)$ to the short exact sequence

$$0 \to S_{p+1} \to \Omega^{p+1} \to S_p \to 0.$$

By the result of Case 2. and Lemma 3.2.3 we obtain

$$\mathrm{Ext}^q(S_0, S_{p+1}) = \mathrm{Ext}^q(S_0, \Omega^{p+1}) = \begin{cases} \mathbf{k} & \text{if } q \in \{p-2, p-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we assume that $j > p+1$. Take $p+2 \le m \le j$. We apply the long exact sequences for $\mathrm{Ext}(S_0, -)$ to the short exact sequences (1.8)

$$0 \to S_m \to \Omega^m \to S_{m-1} \to 0.$$

By Lemma 3.2.3 we have $\mathrm{Ext}^*(S_0, \Omega^m) = 0$. Then we have $\mathrm{Ext}^q(S_0, S_{m-1}) = \mathrm{Ext}^{q+1}(S_0, S_m)$. Using all these equalities and the result for $\mathrm{Ext}^*(S_0, S_{p+1})$, we obtain

$$\mathrm{Ext}^q(S_0, S_j) = \mathrm{Ext}^{q-(j-p-1)}(S_0, S_{p+1}) = \begin{cases} \mathbf{k} & \text{if } q-(j-p-1) \in \{p-2, p-1\}, \\ & \text{i.e., } q \in \{j-3, j-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now take $1 \leq m \leq i$. We apply the long exact sequence for $\mathrm{Ext}(-, S_j)$ to the short exact sequences (1.8)

$$0 \to S_m \to \Omega^m \to S_{m-1} \to 0.$$

Since $\mathrm{Ext}^*(\Omega^m, S_j) = 0$ by Lemma 3.2.3, we get $\mathrm{Ext}^q(S_{m-1}, S_j) = \mathrm{Ext}^{q-1}(S_m, S_j)$. Using all these equalities, we have

$$\mathrm{Ext}^q(S_i, S_j) = \mathrm{Ext}^{q+i}(S_0, S_j) = \begin{cases} \mathbf{k} & \text{if } q+i \in \{j-3, j-2\}, \text{ i.e., } q \in \{j-i-3, j-i-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

We have considered all the possible cases. Therefore, the proof is complete. □

## 3.3 Computations of the dimensions of $\mathrm{Ext}^*(S_i, S_{j,k})$ and $\mathrm{Ext}^*(S_{j,k}, S_i)$

**Lemma 3.3.1.** *Fix $1 \leq i \leq p-1$. Let $\lambda = (1^{p-i})$ and let $\mu_l$ be the unique Young diagram of $p$-weight 1 whose $p$-core $\lambda$ is obtained by the removal of the skew hook corresponding to the hook $(l+1, 1^{p-l-1})$ (see Lemma 2.4.2). Then we have*

$$\mu_l = \begin{cases} (1^{2p-i}) & \text{if } k = 0, \\ (l+1, 2^{p-i}, 1^{i-l-1}) & \text{if } 1 \leq l \leq i-1, \\ (l+2, 2^{p-l-1}, 1^{l-i}) & \text{if } i \leq l \leq p-1. \end{cases}$$

*Proof.* We will say that the box of the Young diagram, which lies in the $i$-th row and $j$-th column, has the coordinates $(i, j)$. To prove the lemma, we describe in each case how to obtain the $p$-core $\lambda$ from the given Young diagram by removing the skew hook.

In $(1^{2p-i})$ we remove the skew $p$-hook whose hand and foot have the coordinates, respectively, $(p-i+1, 1)$ and $(2p-i, 1)$. The removed skew $p$-hook is the Young diagram $(1^p)$, hence $\mu_0 = (1^{2p-i})$.

Now we consider $(l+1, 2^{p-i}, 1^{i-l-1})$ with $1 \leq l \leq i-1$. We remove the skew $p$-hook whose hand and foot have the coordinates, respectively, $(1, l+1)$ and $(p-l, 1)$. The removed skew $p$-hook corresponds to the hook $(l+1, 1^{p-l-1})$, hence $\mu_l = (l+1, 2^{p-i}, 1^{i-l-1})$.

Finally, let us consider $(l+2, 2^{p-l-1}, 1^{l-i})$ with $i \leq l \leq p-1$. We remove the skew $p$-hook whose hand and foot have the coordinates, respectively, $(1, l+2)$ and $(p-l, 2)$. In this case the removed skew hook is the hook $(l+1, 1^{p-l-1})$, hence $\mu_l = (l+2, 2^{p-l-1}, 1^{l-i})$ for $i \leq l \leq p-1$. □

**Lemma 3.3.2.** *Let $1 \leq i \leq 2p$ and $0 \leq j < k \leq p-1$. Then*

$$\mathrm{Ext}^q(\Omega^i, S_{j,k}) = \begin{cases} \mathbf{k} & \textit{if one of the following holds:} \\ & \bullet\ k > j+1,\ i \in \{k, k+1\} \textit{ and } q \in \{j, j+1\}, \\ & \bullet\ j > 0,\ k > j+1,\ i \in \{j, j+1\} \textit{ and } q \in \{k-2, k-1\}, \\ & \bullet\ j = 0,\ k > 1,\ i = 1 \textit{ and } q \in \{k-2, k-1\}, \\ & \bullet\ j > 0,\ k = j+1,\ i = j \textit{ and } q \in \{j-1, j\}, \\ & \bullet\ k = j+1,\ i = j+2 \textit{ and } q \in \{j, j+1\}, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* Let $\lambda$ be the Young diagram corresponding to $S_{j,k}$. We refer to (3.2) and the notation introduced there. Since $\Lambda^i = S_{(i)}$ is the Schur functor corresponding to the minimal Young diagram, $\mathrm{Ext}^q(\Lambda^i, S_\alpha) \neq 0$ if and only if $\alpha = (i)$. Thus, we can assume that $\alpha = (i)$ and, by (1.2),

$$\mathrm{Ext}^*(\Omega^i, S_\lambda) = \mathrm{Ext}(S^{2p-i}, S_{\lambda/\alpha}). \tag{3.3}$$

In view of Lemma 3.1.1 we have $i \leq k+1$. We note that the $p$-core of $(1^{2p-i})$ is $(1^{p-i})$. By the Littlewood-Richardson rule there is a filtration of $S_{\lambda/\alpha}$ by Schur functors (see Theorem 1.2.1). Since $\mathrm{Ext}^*(S^{2p-i}, S_\xi) = 0$ for $\xi$ not belonging to the block corresponding to the $p$-core $(1^{p-i})$, it is sufficient to restrict our attention to the multiplicities of $S_{\mu_l}$ in the filtration under consideration.

Assume that $i < j$. Then we claim that for any $0 \leq l \leq p-1$ the multiplicity of $S_{\mu_l}$ in the filtration of $S_{\lambda/\alpha}$ is 0. We have the following filling of $\lambda/\alpha$ satisfying the Yamanouchi condition, whose shape is known by Lemma 3.1.1 ($\star$ stay for some numbers being irrelevant for our discussion):

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\star$ | $\star$ | $n$ | $\cdots$ | $\star$ | $\cdots$ | $k{+}1$ |
| 1 | 2 | 3 | $\cdots$ | $i$ | $i{+}1$ | $i{+}2$ | $i{+}3$ | $\cdots$ | $j{+}2$ | | |
| 1 | 2 | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | |
| $\vdots$ | $\vdots$ | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | |
| 1 | | | | | | | | | | | |
| $\vdots$ | | | | | | | | | | | |
| 1 | | | | | | | | | | | |

We have $3 \leq n \leq i+3$, because rows are increasing and columns are non-decreasing. In particular, $\lambda/\alpha$ is filled with two numbers $n$. On the other hand, it follows from Lemma 3.3.1 that each number greater than 3 appears in the content of $\widetilde{\mu_l}$ at most once. Hence, we have reached the contradiction, and the claim is proved. In consequence, by (3.3), $\mathrm{Ext}^*(\Omega^i, S_{j,k}) = \mathrm{Ext}^*(S^{2p-i}, S_{\lambda/\alpha}) = 0$.

Now we consider the case $i = j$. We have $j > 0$, since $i \geq 1$. We see that there is exactly one filling of $\lambda/\alpha$ satisfying the Yamanouchi condition:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1 | 2 | $j{+}3$ | $\cdots$ | $k{+}1$ |
| 1 | 2 | 3 | $\cdots$ | $j$ | $j{+}1$ | $j{+}2$ | | | |
| 1 | 2 | | | | | | | | |
| 1 | 2 | | | | | | | | |
| $\vdots$ | $\vdots$ | | | | | | | | |
| 1 | 2 | | | | | | | | |
| 1 | | | | | | | | | |
| $\vdots$ | | | | | | | | | |
| 1 | | | | | | | | | |

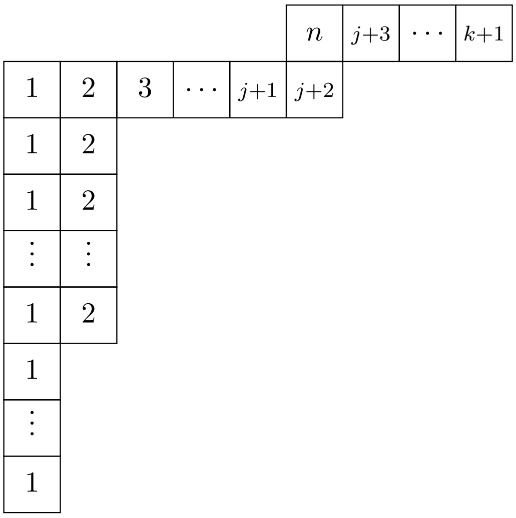
In particular, there is only one diagram $\mu_l$ such that the multiplicity of $S_{\mu_l}$ in the filtration of $S_{\lambda/\alpha}$ is nonzero. By the filling given above and Lemma 3.1.1 the content of $\lambda/\alpha$ is

$$\widetilde{\mu_l} = (p-j, p-k+1, 1^{k-1}), \text{ hence } \mu_l = (k+1, 2^{p-k}, 1^{k-j-1}).$$

By Lemma 3.3.1 we have $l = k-1$. Then, by using (3.3) and (3.1), it follows that

$$\mathrm{Ext}^q(\Omega^j, S_{j,k}) = \mathrm{Ext}^q(S^{2p-j}, S_{\mu_{k-1}}) = \mathrm{Ext}^q_{\mathcal{P}_{2p-j}}(S_{\mu_0}, S_{\mu_{k-1}}) \begin{cases} \mathbf{k} & \text{if } q \in \{k-2, k-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we turn to the case $i = j+1$. We see that we have the following filling of $\lambda/\alpha$ satisfying the Yamanouchi condition:



where $n \in \{1, 2\}$. Suppose that $n = 1$. Then it follows from Lemma 3.1.1 that we filled diagram by $p-j$ ones and $p-k$ twos. By the Littlewood-Richardson rule these numbers are the lengths of the first two columns in $\mu_l$. Since $j \neq i$, we deduce, by using Lemma 3.3.1 and by inspection of the lengths of the first two columns, that $p-l = p-j$ and $p-i+1 = p-k$. Equivalently, $l = j$ and $k = i-1$. However, we have $k > j$, i.e., $k > i-1$. Thus, $n = 2$. If $k = j+1$ then we have $\lambda/\alpha$ filled by $p-j-1$ ones and $p-j$ twos. It means that $\mu_l$ has more boxes in the second column than in the first one, which is impossible. In this case there is no filling of $\lambda/\alpha$ with content $\widetilde{\mu_l}$ for any $0 \leq l \leq p-i$, hence

$$\mathrm{Ext}^*(\Omega^{j+1}, S_{j,j+1}) = 0.$$

Now we assume $k > j+1$. The content of $\lambda/\alpha$ is $\widetilde{\mu_l} = (p-j-1, p-k+1, 1^{k-1})$, hence $\mu_l = (k+1, 2^{p-k}, 1^{k-j-2})$. By Lemma 3.3.1 we have $l = k-1$. Thus, by using (3.3) and (3.1), we conclude that

$$\mathrm{Ext}^q(\Omega^{j+1}, S_{j,k}) = \mathrm{Ext}^q(S^{2p-j-1}, S_{\mu_{k-1}}) = \mathrm{Ext}^q_{\mathcal{P}_{2p-j-1}}(S_{\mu_0}, S_{\mu_{k-1}}) = \begin{cases} \mathbf{k} & \text{if } q \in \{k-2, k-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider $i > j+1$. If $i = k+1$ then $S_{\lambda/\alpha} = S_\xi$ for $\xi = (j+2, 2^{p-k-1}, 1^{k-j-1})$. By Lemma 3.3.1 we see that $\xi = \mu_{j+1}$, hence, by (3.3) and (3.1),

$$\mathrm{Ext}^q(\Omega^{k+1}, S_{j,k}) = \mathrm{Ext}^q(S^{2p-k-1}, S_{\mu_{j+1}}) = \mathrm{Ext}^q_{\mathcal{P}_{2p-k-1}}(S_{\mu_0}, S_{\mu_{j+1}}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j, j+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that $j+1 < i < k+1$. In particular, $k > j+1$. Then $\lambda/\alpha$ is disconnected, hence $S_{\lambda/\alpha} \simeq S_\xi \otimes \Lambda^{k-i+1}$ where $\xi = (j+2, 2^{p-k-1}, 1^{k-j-1})$. It follows from the exponential formula (1.10) that

$$\mathrm{Ext}^q(S^{2p-i}, S_{\lambda/\alpha}) = \mathrm{Ext}^q(S^{2p-i}, S_\xi \otimes \Lambda^{k-i+1}) =$$
$$\bigoplus_{0\le l\le q} \mathrm{Ext}^{q-l}(S^{2p-k-1}, S_\xi) \otimes \mathrm{Ext}^{q-l}(S^{k-i+1}, \Lambda^{k-i+1}) = \mathrm{Ext}^q(S^{2p-i-1}, S_\xi). \tag{3.4}$$

The last equality follows from the observation that $k-i-1 \le p-i < p$ and from Lemma 3.2.1. We now observe that if $k = i$ then the $p$-core of $\xi$ is $(1^{p-i-1})$ and it is simple matter to check that $\xi = \mu_{j+1}$. Then, by (3.3), (3.4) and (3.1), we get

$$\mathrm{Ext}^q(\Omega^i, S_{j,k}) = \mathrm{Ext}^q(S^{2p-i}, S_{\lambda/\alpha}) = \mathrm{Ext}^q(S^{2p-i-1}, S_{\mu_{j+1}}) = \mathrm{Ext}^q_{\mathcal{P}_{2p-i-1}}(S_{\mu_0}, S_{\mu_{j+1}}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j, j+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

□

We now establish two short exact sequences, which will be used in later arguments. Let $0 \le m_1 < m_2 \le p-1$ such that $m_2 - m_1 > 1$ and let $\lambda$ be the Young diagram corresponding to $S_{m_1,m_2}$. Then it is possible to remove the last box from the first row of $\lambda$ preserving the Young diagram condition. Denote by $S_{\widehat{m_1,m_2}}$ the Schur functor corresponding to the diagram obtained by deleting that box. By Lemma 3.1.1 this diagram is $\eta = (m_2, m_1+2, 2^{p-m_2-1}, 1^{m_2-m_1-1})$. By the Littlewood-Richardson rule, $S_{\widehat{m_1,m_2}} \otimes I = S_{\widehat{m_1,m_2}} \otimes S_{(1)}$ has a filtration by Schur functors corresponding to the Young diagrams obtained by adding a single box to $\eta$, each appearing with multiplicity 1. Let $\alpha$ be any such diagram. If the box is added to the first row of $\eta$, then $\alpha = \lambda$. Otherwise, the first row of $\alpha$ remains of length $m_2$. Now we assume that $S_\alpha \in \mathcal{P}^\varnothing_{2p}$. Then by Lemma 3.1.1 the second column of $\alpha$ has length $p - m_2 + 2$ whereas $\eta$ has $p - m_2 + 1$ boxes in its second column. Hence, to satisfy the assumption, a box has to be added to the second column of $\eta$. In this case, again by Lemma 3.1.1, we have $S_\alpha = S_{m_1,m_2-1}$. This argument shows that all quotients in the filtration under consideration, other than $S_{m_1,m_2}$ and $S_{m_1,m_2-1}$, lie outside the block $\mathcal{P}^\varnothing_{2p}$. In particular, in $\mathcal{P}^\varnothing_{2p}$ there is a short exact sequence

$$0 \to S_{m_1,m_2} \to S_{\widehat{m_1,m_2}} \otimes I \to S_{m_1,m_2-1} \to 0; \tag{3.5}$$

where, by abuse of notation, we denote both $S_{\widehat{m_1,m_2}} \otimes I$ and its image under the canonical projection onto $\mathcal{P}^\varnothing_{2p}$ by the same symbol. Now fix $1 \le m_1 < m_2 \le p-1$ and let $\lambda$ be the Young diagram corresponding to $S_{m_1,m_2}$. This time we can remove the last box in the second row of $\lambda$. We denote the Schur functor corresponding to the Young diagram obtained by removing this box by $S_{\overline{m_1,m_2}}$. Using the Littlewood-Richardson formula and the similar reasoning to the above, $S_{\overline{m_1,m_2}} \otimes I = S_{\overline{m_1,m_2}} \otimes S_{(1)}$ has a filtration with quotients $S_{m_1,m_2}$, $S_{m_1-1,m_2}$ and some Schur functors lying outside the block $\mathcal{P}^\varnothing_{2p}$. In particular, this gives the following short exact sequence in $\mathcal{P}^\varnothing_{2p}$ (with the same abuse of notation as above):

$$0 \to S_{m_1,m_2} \to S_{\overline{m_1,m_2}} \otimes I \to S_{m_1-1,m_2} \to 0. \tag{3.6}$$

**Lemma 3.3.3.** *Let $0 \le i \le 2p-1$ and $0 \le i_1 < j_1 \le p-1$ such that $j_1 - i_1 > 1$. Then*

$$\mathrm{Ext}^q(S_i, S_{\widehat{i_1,j_1}} \otimes I) = \begin{cases} \mathbf{k} & \text{if } i \in \{j_1-1, j_1\} \text{ and } q \in \{i_1, i_1+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Now, let $0 \le i \le 2p-1$ and $1 \le i_1 < j_1 \le p-1$. Then*

$$\mathrm{Ext}^q(S_i, S_{\overline{i_1,j_1}} \otimes I) = \begin{cases} \mathbf{k} & \text{if } i \in \{i_1-1, i_1\} \text{ and } q \in \{j_1-2, j_1-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We begin by proving the first formula. Let $\lambda$ and $\eta$ be the Young diagrams corresponding, respectively, to $S_i$ and $S_{\widehat{i_1,j_1}}$. By using Lemma 3.1.1 we observe that the $p$-core of $\eta$ is $(j_1, 1^{p-j_1-1})$: the foot is the $(p-i_1)$-th row and the hand is in the second row. We also note that $\eta = \mu_{i_1+1}$ (in the sense of Lemma 2.4.2). By the decomposition formula (1.9) we have

$$E_1^{**} = \sum_{\alpha\subset\lambda} \mathrm{Ext}^*(S_\alpha, S_{\widehat{i_1,j_1}}) \otimes \mathrm{Ext}^*(S_{\lambda/\alpha}, I) = \sum_{\alpha\subset\lambda} \mathrm{Ext}^*_{\mathcal{P}_{2p-1}}(S_\alpha, S_{\mu_{i_1+1}}) \Rightarrow \mathrm{Ext}^*(S_i, S_{\widehat{i_1,j_1}} \otimes I). \tag{3.7}$$

If all Young diagrams of weight $2p-1$ contained in $\lambda$ have the different $p$-cores than that of $\eta$, then we have $\mathrm{Ext}^*(S_i, S_{\widehat{i_1,j_1}}) = 0$ by (3.7). Hence assume that $\alpha$ is a Young diagram of weight $2p-1$ with $p$-core $(j_1, 1^{p-j_1-1})$ and contained in $\lambda$. Since $\alpha \subset \lambda$, $\alpha$ is a hook and the length of its first row is not greater than $p$. Then we conclude $\alpha = (j_1, 1^{2p-j_1-1}) = \mu_0$. The diagram $\alpha$ is obtained from $\lambda$ by removing a single box, hence we have $\lambda = (j_1+1, 1^{2p-j_1-1})$ or $\lambda = (j_1, 1^{2p-j_1})$, i.e., $i \in \{j_1, j_1+1\}$. Therefore, by the above discussion, if $i \notin \{j_1, j_1+1\}$ then $\mathrm{Ext}^*(S_i, S_{i_1,j_1}) = 0$. Otherwise, by (3.7), we have

$$\mathrm{Ext}^q(S_i, S_{\widehat{i_1,j_1}} \otimes I) = \mathrm{Ext}^q(S_{\mu_0}, S_{\mu_{i_1+1}}) = \begin{cases} \mathbf{k} & \text{if } q \in \{i_1, i_1+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

The second formula follows by the same argument as presented above; the only substantive change is that the $p$-core of the Young diagram corresponding to $S_{\overline{i_1,j_1}}$ is $(i_1, 1^{p-i_1-1})$. □

**Theorem 3.3.4.** *Let $0 \le i \le 2p-1$ and $0 \le j < k \le p-1$. Then*

$$\mathrm{Ext}^q(S_i, S_{j,k}) = \mathrm{Ext}^q(W_{j,k}, W_i) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i = k,\ q \in \{j, j+1\}, \\ & \bullet\ i = j,\ q \in \{k, k+1\}, \\ & \bullet\ i < j,\ q \in \{k+j-i-3, k+j-i-2, \\ & \quad k+j-i, k+j-i+1\}, \\ & \bullet\ j < i < k,\ q \in \{k+j-i-1, k+j-i+1\}, \\ \mathbf{k}^2 & \text{if } j < i < k,\ q = k+j-i, \\ 0 & \text{otherwise.} \end{cases}$$

*Moreover,* $\mathrm{Ext}^*(S_{j,k}, S_i) = \mathrm{Ext}^*(W_i, W_{j,k}) = \mathrm{Ext}^*(S_{2p-i-1}, S_{p-k-1,p-j-1})$. *In consequence,*

$$\mathrm{Ext}^q(S_{j,k}, S_i) = \mathrm{Ext}^q(W_i, W_{j,k}) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i = p+j,\ q \in \{p-k-1, p-k\}, \\ & \bullet\ i = p+k,\ q \in \{p-j-1, p-j\}, \\ & \bullet\ i > p+k,\ q \in \{i-j-k-4, \\ & \quad i-j-k-3, i-j-k-1, i-j-k\}, \\ & \bullet\ p+j < i < p+k,\ q \in \{i-j-k-2, i-j-k\}, \\ \mathbf{k}^2 & \text{if } p+j < i < p+k,\ q = i-j-k-1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The equalities $\mathrm{Ext}^*(S_i, S_{j,k}) = \mathrm{Ext}^*(W_{j,k}, W_i)$ and $\mathrm{Ext}^*(S_{j,k}, S_i) = \mathrm{Ext}^*(W_i, W_{j,k})$ are the immediate consequence of the Kuhn duality. The conjugate diagram to the hook $(i+1, 1^{2p-i-1})$

is $(2p-i, 1^i)$. If $\lambda$ is the Young diagram corresponding to $S_{j,k}$ then by Lemma 3.1.1 we have $\widetilde{\lambda} = (p-j, p-k+1, 2^k, 1^{k-j-1})$. This diagram corresponds to $S_{p-k-1,p-j-1}$. Then by the isomorphism (1.3) we obtain $\mathrm{Ext}^*(S_{j,k}, S_i) = \mathrm{Ext}^*(S_{2p-i-1}, S_{p-k-1,p-j-1})$. The last formula in the lemma is the consequence of the rest of the equalities in the assertion. We now turn to the computation of $\mathrm{Ext}^*(S_i, S_{j,k})$.

Since for $i \geq k+1$ the Young diagram corresponding to $S_{j,k}$ is greater than that corresponding to $S_i$ (in the dominance order), we have $\mathrm{Ext}^*(S_i, S_{j,k}) = 0$ in this case. Hence we assume that $i \leq k$. Now we divide the proof into several parts.

*Case 1.* $i = k$

We have mentioned in the previous paragraph that $\mathrm{Ext}^*(S_{k+1}, S_{j,k}) = 0$. Then, by the long exact sequence for $\mathrm{Ext}(-, S_{j,k})$ applied to the short exact sequence (1.8)

$$0 \to S_{k+1} \to \Omega^{k+1} \to S_k \to 0$$

and by Lemma 3.3.2, we obtain

$$\mathrm{Ext}^q(S_k, S_{j,k}) = \mathrm{Ext}^q(\Omega^{k+1}, S_{j,k}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j, j+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Case 2.* $i = j$

We begin with the special case $k = j+1$. We apply the long exact sequence for $\mathrm{Ext}(-, S_{j,j+1})$ to the short exact sequence (1.8)

$$0 \to S_{j+1} \to \Omega^{j+1} \to S_j \to 0.$$

Since $\mathrm{Ext}^*(\Omega^{j+1}, S_{j,j+1}) = 0$ by Lemma 3.3.2, we obtain

$$\mathrm{Ext}^q(S_j, S_{j,j+1}) = \mathrm{Ext}^{q-1}(S_{j+1}, S_{j,j+1}) = \begin{cases} \mathbf{k} & \text{if } q-1 \in \{j, j+1\}, \text{ i.e., } q \in \{j+1, j+2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now assume $k > j+1$. Take $j+2 \leq m \leq k$. We apply the long exact sequence for $\mathrm{Ext}(S_j, -)$ to the short exact sequence (3.5)

$$0 \to S_{j,m} \to S_{\widehat{j,m}} \otimes I \to S_{j,m-1} \to 0.$$

By Lemma 3.3.3 we have $\mathrm{Ext}^*(S_j, S_{\widehat{j,m}}) = 0$. Hence we have $\mathrm{Ext}^q(S_j, S_{j,m}) = \mathrm{Ext}^{q-1}(S_j, S_{j,m-1})$. Using all these equalities and the result for $\mathrm{Ext}^*(S_j, S_{j,j+1})$, we obtain

$$\mathrm{Ext}^q(S_j, S_{j,k}) = \mathrm{Ext}^{q-(k-j-1)}(S_j, S_{j,j+1}) = \begin{cases} \mathbf{k} & \text{if } q-(k-j-1) \in \{j+1, j+2\}, \\ & \text{i.e., } q \in \{k, k+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Case 3.* $i < j$

We begin with the special case $i = j-1$ and $k = j+1$. Applying the long exact sequence for $\mathrm{Ext}(-, S_{j,j+1})$ to the short exact sequence (1.8)

$$0 \to S_j \to \Omega^j \to S_{j-1}$$

and using Lemma 3.3.2 together with the result of Case 2., we get the following exact sequence:

$$
\begin{array}{llll}
\ldots \longrightarrow \operatorname{Ext}^{j-2}(S_{j-1}, S_{j,j+1}) & \longrightarrow \operatorname{Ext}^{j-2}(\Omega^j, S_{j,j+1}) \simeq 0 & \longrightarrow \operatorname{Ext}^{j-2}(S_j, S_{j,j+1}) \simeq 0 & \\
\operatorname{Ext}^{j-1}(S_{j-1}, S_{j,j+1}) & \longrightarrow \operatorname{Ext}^{j-1}(\Omega^j, S_{j,j+1}) \simeq \mathbf{k} & \longrightarrow \operatorname{Ext}^{j-1}(S_j, S_{j,j+1}) \simeq 0 & \\
\operatorname{Ext}^{j}(S_{j-1}, S_{j,j+1}) & \longrightarrow \operatorname{Ext}^{j}(\Omega^j, S_{j,j+1}) \simeq \mathbf{k} & \longrightarrow \operatorname{Ext}^{j}(S_j, S_{j,j+1}) \simeq 0 & \\
\operatorname{Ext}^{j+1}(S_{j-1}, S_{j,j+1}) & \longrightarrow \operatorname{Ext}^{j+1}(\Omega^j, S_{j,j+1}) \simeq 0 & \longrightarrow \operatorname{Ext}^{j+1}(S_j, S_{j,j+1}) \simeq \mathbf{k} & \\
\operatorname{Ext}^{j+2}(S_{j-1}, S_{j,j+1}) & \longrightarrow \operatorname{Ext}^{j+2}(\Omega^j, S_{j,j+1}) \simeq 0 & \longrightarrow \operatorname{Ext}^{j+2}(S_j, S_{j,j+1}) \simeq \mathbf{k} & \\
\operatorname{Ext}^{j+3}(S_{j-1}, S_{j,j+1}) & \longrightarrow \operatorname{Ext}^{j+3}(\Omega^j, S_{j,j+1}) \simeq 0 & \longrightarrow \operatorname{Ext}^{j+3}(S_j, S_{j,j+1}) \simeq 0 & \longrightarrow \ldots
\end{array}
$$

Therefore, it follows that

$$\operatorname{Ext}^q(S_{j-1}, S_{j,j+1}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j-1, j, j+2, j+3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that $i < j-1$ and $k = j+1$. Let $i+1 \leq m \leq j-1$. By Lemma 3.3.2, $\operatorname{Ext}^*(\Omega^m, S_{j,j+1}) = 0$. Then we apply the long exact sequence for $\operatorname{Ext}(-, S_{j,j+1})$ to the short exact sequence (1.8)

$$0 \to S_m \to \Omega^m \to S_{m-1} \to 0$$

and we get $\operatorname{Ext}^q(S_{m-1}, S_{j,j+1}) = \operatorname{Ext}^{q-1}(S_m, S_{j,j+1})$. Using all these equalities, we obtain

$$\operatorname{Ext}^q(S_i, S_{j,j+1}) = \operatorname{Ext}^{q-(j-1-i)}(S_{j-1}, S_{j,j+1}) = \begin{cases} \mathbf{k} & \text{if } q-(j-i-1) \in \{j-1, j, j+2, j+3\}, \\ & \text{i.e., } q \in \{2j-i-2, 2j-i-1, \\ & \qquad 2j-i+1, 2j-i+2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $k > j+1$. Take $j+2 \leq m \leq k$. By Lemma 3.3.3 we have $\operatorname{Ext}^*(S_i, S_{\widehat{j,m}} \otimes I) = 0$. Then, by the long exact sequence for $\operatorname{Ext}(S_i, -)$ applied to short exact sequence (3.5)

$$0 \to S_{j,m} \to S_{\widehat{j,m}} \otimes I \to S_{j,m-1} \to 0,$$

we have $\operatorname{Ext}^q(S_i, S_{j,m}) = \operatorname{Ext}^{q-1}(S_i, S_{j,m-1})$. Thus, using all these equalities and the result for $\operatorname{Ext}^*(S_i, S_{j,j+1})$, we obtain

$$\operatorname{Ext}^q(S_i, S_{j,k}) = \operatorname{Ext}^{q-(k-j-1)}(S_i, S_{j,j+1}) = \begin{cases} \mathbf{k} & \text{if } q-(k-j-1) \in \{2j-i-2, 2j-i-1, \\ & \quad 2j-i+1, 2j-i+2\}, \\ & \text{i.e., } q \in \{k+j-i-3, k+j-i-2, \\ & \qquad k+j-i, k+j-i+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Case 4.* $j < i < k$

We begin with the special case $i = 1$ and $j = 0$. Applying the long exact sequence for $\mathrm{Ext}(-, S_{0,k})$ to the short exact sequence (1.8)

$$0 \to S_1 \to \Omega^1 \to S_0 \to 0$$

and using Lemma 3.3.2 together with the result of Case 3., we get the following exact sequence:

$$\begin{array}{rcccl}
\ldots \longrightarrow \mathrm{Ext}^{k-3}(S_0, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k-3}(\Omega^1, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k-3}(S_1, S_{0,k}) \\
\mathrm{Ext}^{k-2}(S_0, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k-2}(\Omega^1, S_{0,k}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{k-2}(S_1, S_{0,k}) \\
\mathrm{Ext}^{k-1}(S_0, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k-1}(\Omega^1, S_{0,k}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{k-1}(S_1, S_{0,k}) \\
\mathrm{Ext}^{k}(S_0, S_{0,k}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{k}(\Omega^1, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k}(S_1, S_{0,k}) \\
\mathrm{Ext}^{k+1}(S_0, S_{0,k}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{k+1}(\Omega^1, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k+1}(S_1, S_{0,k}) \\
\mathrm{Ext}^{k+2}(S_0, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k+2}(\Omega^1, S_{0,k}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{k+2}(S_1, S_{0,k}) \longrightarrow \ldots
\end{array}$$

Hence, we conlude that

$$\mathrm{Ext}^q(S_1, S_{0,k}) = \begin{cases} \mathbf{k} & \text{if } q \in \{k-2, k\}, \\ \mathbf{k}^2 & \text{if } q = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now assume $i > 1$ and $j = 0$. Take $2 \leq m \leq i$. By Lemma 3.3.2, $\mathrm{Ext}^*(\Omega^m, S_{0,k}) = 0$. Then, by the long exact sequences for $\mathrm{Ext}(-, S_{0,k})$ applied to the short exact sequence (1.8)

$$0 \to S_m \to \Omega^m \to S_{m-1} \to 0,$$

we have $\mathrm{Ext}^q(S_m, S_{0,k}) = \mathrm{Ext}^{q+1}(S_{m-1}, S_{0,k})$. Using all these equalities, we obtain

$$\mathrm{Ext}^q(S_i, S_{0,k}) = \mathrm{Ext}^{q+i-1}(S_1, S_{0,k}) = \begin{cases} \mathbf{k} & \text{if } q+i-1 \in \{k-2, k\}, \\ & \text{i.e } q \in \{k-i-1, k-i+1\}, \\ \mathbf{k}^2 & \text{if } q+i-1 = k-1, \text{ i.e } q = k-i, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, assume that $j > 0$ and take $1 \leq m \leq j$. In particular, $m < i$. By Lemma 3.3.3, $\mathrm{Ext}^*(S_i, S_{\overline{m,k}} \otimes I) = 0$. Then, by the long exact sequence for $\mathrm{Ext}(S_i, -)$ applied to the short exact sequence (3.6)

$$0 \to S_{m,k} \to S_{\overline{m,k}} \otimes I \to S_{m-1,k} \to 0,$$

we have $\mathrm{Ext}^q(S_i, S_{m,k}) = \mathrm{Ext}^{q-1}(S_i, S_{m-1,k})$. Using all these equalities and the result for $\mathrm{Ext}^*(S_i, S_{0,k})$, we get

$$\mathrm{Ext}^q(S_i, S_{j,k}) = \mathrm{Ext}^{q-j}(S_i, S_{0,k}) = \begin{cases} \mathbf{k} & \text{if } q-j \in \{k-i-1, k-i+1\}, \\ & \text{i.e } q \in \{k+j-i-1, k+j-i+1\}, \\ \mathbf{k}^2 & \text{if } q-j = k-i, \text{ i.e } q = k+j-i, \\ 0 & \text{otherwise.} \end{cases}$$

All the possible cases were considered, hence the proof is complete. □

## 3.4 Computations of the dimensions of $\mathrm{Ext}^*(S_{i_1,j_1}, S_{i_2,j_2})$

**Lemma 3.4.1.** *Let $0 \le i_1, j_1 \le p-1$ such that $j_1 - i_1 > 1$ and $0 \le i_2 < j_2 \le p-1$. Then*

$$\mathrm{Ext}^q(S_{\widehat{i_1,j_1}} \otimes I, S_{i_2,j_2}) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ j_2 - i_2 > 1,\ j_1 \in \{j_2, j_2+1\},\ i_1 = i_2 \text{ and } q = 0, \\ & \bullet\ j_2 - i_2 > 1,\ j_1 \in \{j_2, j_2+1\},\ i_1 < i_2 \text{ and } \\ & \quad q \in \{i_2 - i_1 - 1, i_2 - i_1\}, \\ & \bullet\ j_2 - i_2 > 1,\ j_1 \in \{i_2, i_2+1\} \text{ and } q \in \{j_2 - i_1 - 3, j_2 - i_1 - 2\}, \\ & \bullet\ j_2 = i_2 + 1,\ j_1 = j_2 + 1,\ i_1 = i_2 \text{ and } q = 0, \\ & \bullet\ j_2 = i_2 + 1,\ j_1 = j_2 + 1,\ i_1 < i_2 \text{ and } q \in \{i_2 - i_1 - 1, i_2 - i_1\}, \\ & \bullet\ j_2 = i_2 + 1,\ j_1 = i_2 \text{ and } q \in \{j_2 - i_1 - 3, j_2 - i_1 - 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

*We now assume that $1 \le i_1 < j_1 \le p-1$ and $0 \le i_2 < j_2 \le p-1$. Then*

$$\mathrm{Ext}^q(S_{\overline{i_1,j_1}} \otimes I, S_{i_2,j_2}) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ j_2 - i_2 > 1,\ i_1 \in \{i_2, i_2+1\},\ j_1 = j_2 \text{ and } q = 0, \\ & \bullet\ j_2 - i_2 > 1,\ i_1 \in \{i_2, i_2+1\},\ j_1 < j_2 \text{ and } \\ & \quad q \in \{j_2 - j_1 - 1, j_2 - j_1\}, \\ & \bullet\ j_2 = i_2 + 1,\ i_1 = i_2,\ j_1 = j_2 \text{ and } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We begin by proving the first formula. Let $\lambda$ and $\eta$ be the Young diagrams such that $S_\lambda = S_{i_2,j_2}$ and $S_\eta = S_{\widehat{i_1,j_1}}$. We observed in the proof of Lemma 3.3.3 that the $p$-core of $\eta$ is $(j_1, 1^{p-j_1-1})$ and $\eta = \mu_{i_1+1}$ (in the sense of Lemma 2.4.2). By the decomposition formula (1.9) we have

$$E_1^{**} = \sum_{\alpha \subset \lambda} \mathrm{Ext}^*(S_{\widehat{i_1,j_1}}, S_\alpha) \otimes \mathrm{Ext}^*(I, S_{\lambda/\alpha}) = \sum_{\alpha \subset \lambda} \mathrm{Ext}^*_{\mathcal{P}_{2p-1}}(S_{\mu_{i+1}}, S_\alpha) \Rightarrow \mathrm{Ext}^*(S_{\widehat{i_1,j_1}} \otimes I, S_{i_2,j_2}). \tag{3.8}$$

If all the diagrams of weight $2p-1$ contained in $\lambda$ have the different $p$-cores than that of $\eta$, then by (3.8) we have $\mathrm{Ext}^*(S_{\widehat{i_1,j_1}} \otimes I, S_{i_2,j_2}) = 0$. Hence assume that there exists a Young diagram $\alpha$ of weight $2p-1$ with the $p$-core $(j_1, 1^{p-j_1-1})$ and contained in $\lambda$. In particular, this is not a hook. If one can remove from $\alpha$ a skew $p$-hook with the foot in the first column, then by Lemma 3.1.1 its hand is in the second row and $\alpha = \mu_{i_2+1}$. In this case we see that the length of the first row of $\alpha$ is $j_1$. Then $\lambda$ has the length $j_1 + 1$ if $\alpha$ is obtained from $\lambda$ by removing a box from the first row and $j_1$ otherwise. We note that it is not possible to remove a box from the first row if $j_2 = i_2 + 1$. Hence, by Lemma 3.1.1,

- if $j_2 > i_2 + 1$ then $j_1 + 1 = j_2 + 1$ or $j_1 = j_2 + 1$, i.e., $j_1 \in \{j_2, j_2 + 1\}$;
- if $j_2 = i_2 + 1$ then $j_1 = j_2 + 1$.

Now we consider the case if one can remove from $\alpha$ a skew $p$-hook with the foot in the second column. Then by Lemma 3.1.1 its hand is in the first row and $\alpha = \mu_{j_2-1}$. In this situation the length of the second row is $j_1+1$. Therefore, the length of the second row of $\lambda$ is $j_1+2$ if one obtains $\alpha$ by removing a box from the second row of $\lambda$ and $j_1+1$ otherwise. Let us note that this is not possible to remove a box from the second row if $i_2 = 0$. Hence, by Lemma 3.1.1,

- if $i_2 > 0$ then $j_1+1 = i_2+2$ or $j_1+2 = i_2+2$, i.e., $j_1 \in \{i_2, i_2+1\}$;
- if $i_2 = 0$ then $j_1+1 = i_2+2$, i.e., $j_1 = i_2+1$.

By these observation we note that it is not possible to hold simultaneously $j_1 \in \{i_2, i_2+1\}$ and $j_1 \in \{j_2, j_2+1\}$. In particular there is the only one Schur functor with the nonzero multiplicity in the filtration (3.8). Therefore, by the above discussion, (3.8) and (3.1), we have

- if $j_1 \notin \{i_2, i_2+1, j_2, j_2+1\}$, then
$$\mathrm{Ext}^*(S_{\widehat{i_1,j_1}} \otimes I, S_{i_2,j_2}) = 0 \quad ;$$
- if $j_2 - i_2 > 1,\ j_1 \in \{j_2, j_2+1\}$ or $j_2 - i_2 = 1,\ j_1 = j_2+1$, then
$$\mathrm{Ext}^q(S_{\widehat{i_1,j_1}} \otimes I, S_{i_2,j_2}) = \mathrm{Ext}^q_{\mathcal{P}_{2p-1}}(S_{\mu_{i_1+1}}, S_{\mu_{i_2+1}}) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i_1 = i_2 \text{ and } q = 0, \\ & \bullet\ i_1 < i_2 \text{ and } q \in \{i_2 - i_1 - 1, i_2 - i_1\}, \\ 0 & \text{otherwise} \end{cases} ;$$
- if $j_2 - i_2 > 1,\ j_1 \in \{i_2, i_2+1\}$ or $j_2 - i_2 = 1,\ j_1 = i_2$, then
$$\mathrm{Ext}^q(S_{\widehat{i_1,j_1}} \otimes I, S_{i_2,j_2}) = \mathrm{Ext}^q_{\mathcal{P}_{2p-1}}(S_{\mu_{i_1+1}}, S_{\mu_{j_2-1}}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j_2 - i_1 - 3, j_2 - i_1 - 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

The second formula follows by the same argument as presented above; the only substantive change is that the $p$-core of the Young diagram corresponding to $S_{\overline{i_1,j_1}}$ is $(i_1, 1^{p-i_1-1})$. □

**Theorem 3.4.2.** *Let $0 \le i_1 < j_1 \le p-1$, $0 \le i_2 < j_2 \le p-1$ and $i_1 < j_1, i_2 < j_2$. Then*
$$\mathrm{Ext}^q(S_{i_1,j_1}, S_{i_2,j_2}) = \begin{cases} \mathbf{k} & \text{if } i_1 \le i_2,\ j_1 \le j_2 \text{ and one of the following holds:} \\ & \bullet\ i_1 = i_2,\ q \in \{j_2 - j_1 - 1, j_2 - j_1\}, \\ & \bullet\ j_1 = j_2,\ q \in \{i_2 - i_1 - 1, i_2 - i_1\}, \\ & \bullet\ j_1 = i_2,\ q \in \{j_2 - i_1 - 1, j_2 - i_1\}, \\ & \bullet\ j_1 < i_2 \text{ and } q \in \{i_2 + j_2 - i_1 - i_2 - 4, i_2 + j_2 - i_1 - i_2 - 3, \\ & \quad i_2 + j_2 - i_1 - i_2 - 1, i_2 + j_2 - i_1 - i_2\}, \\ & \bullet\ i_1 < i_2 < j_1 < j_2 \text{ and } q \in \{i_2 + j_2 - i_1 - i_2 - 2, \\ & \quad i_2 + j_2 - i_1 - i_2\}, \\ \mathbf{k}^2 & \text{if } i_1 < i_2 < j_1 < j_2 \text{ and } q = i_2 + j_2 - i_1 - j_1 - 1, \\ 0 & \text{otherwise.} \end{cases}$$
*Moreover,* $\mathrm{Ext}^*(W_{i_1,j_1}, W_{i_2,j_2}) = \mathrm{Ext}^*(S_{i_2,j_2}, S_{i_1,j_1})$.

*Proof.* The last equality in the assertion follows immediately from the first part of the theorem by the Kuhn duality. Now we turn to the computation of the Ext-groups between Schur functors. In view of Proposition 1.1.1(d) and Lemma 3.1.2 it is sufficient to assume that $i_1 \leq i_2$ and $j_1 \leq j_2$. We divide the proof into four cases depending on indices $i_1, i_2, j_1, j_2$.

*Case 1.* $i_1 = i_2$ *or* $j_1 = j_2$

First, we consider the case $i_1 = i_2$. Let $i_1 = p-2$. By (1.2) we have

$$\mathrm{Ext}^*(S_{p-2,p-1}, S_{p-2,p-1}) = \mathrm{Hom}(S_{p-2,p-1}, S_{p-2,p-1}) = \mathbf{k}.$$

Now we assume that $i_1 < p-2$. Since the Young diagrams corresponding to $S_{i_1+1,j_1}$ and $S_{i_1,j_2}$ are noncomparable (with respect to the dominance order), we have $\mathrm{Ext}^*(S_{i_1+1,j_1}, S_{i_1,j_2}) = 0$ by Proposition 1.1.1(d). Then, by the long exact sequence for $\mathrm{Ext}(-, S_{i_1,j_2})$ applied to the short exact sequence (3.6)

$$0 \to S_{i_1+1,j_1} \to S_{\overline{i_1+1,j_1}} \otimes I \to S_{i_1,j_1} \to 0$$

and by Lemma 3.4.1, we obtain

$$\mathrm{Ext}^q(S_{i_1,j_1}, S_{i_1,j_2}) = \mathrm{Ext}^q(S_{\overline{i_1+1,j_1}} \otimes I, S_{i_1,j_2}) = \begin{cases} \mathbf{k} & \text{if } j_1 = j_2,\ q = 0 \text{ or } j_1 < j_2, \\ & \quad q \in \{j_2 - j_1, j_2 - j_1 - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

The case $j_1 = j_2$ follows from the above result. Indeed, we have observed at the beginning of the proof of Theorem 3.3.4 that the conjugate of the Young diagram corresponding to $S_{u,v}$ corresponds to $S_{p-v-1,p-u-1}$. Then by the isomorphism (1.3) we have

$$\mathrm{Ext}^*(S_{i_1,j_1}, S_{i_2,j_1}) = \mathrm{Ext}^*(S_{p-j_1-1,p-i_2-1}, S_{p-j_1-1,p-i_1-1}) = \begin{cases} \mathbf{k} & \text{if } i_1 = i_2,\ q = 0 \text{ or } i_1 < i_2, \\ & \quad q \in \{i_2 - i_1 - 1, i_2 - i_1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Case 2.* $j_1 = i_2$

Assume $i_1 = j_1 - 1$. We apply the long exact sequence for $\mathrm{Ext}(S_{j_1-1,j_1}, -)$ to the short exact sequence (3.6)

$$0 \to S_{j_1,j_2} \to S_{\overline{j_1,j_2}} \otimes I \to S_{j_1-1,j_2} \to 0.$$

Now we compute $\mathrm{Ext}^*(S_{j_1-1,j_1}, S_{\overline{j_1,j_2}} \otimes I)$. Let $\lambda$ and $\eta$ be the Young diagrams corresponding to $S_{j_1-1,j_1}$ and $S_{\overline{j_1,j_2}}$, respectively. By Lemma 3.1.1, we have $\eta = (j_2+1, j_1+1, 2^{p-j_2-1}, 1^{j_2-j_1-1})$. Hence $\widetilde{\eta} = (p-j_1, p-j_2+1, 2^{j_1-1}, 1^{j_2-j_1})$. This is obtained from $(p-j_1+1, p-j_2+1, 2^{j_1-1}, 1^{j_2-j_1})$ by removing a box from the first row. By Lemma 3.1.1 again, it follows that $\widetilde{\eta}$ corresponds to $S_{\widehat{p-j_2-1,p-j_1}}$. Then by the isomorphism (1.4) and Lemma 3.4.1 we obtain

$$\mathrm{Ext}^*(S_{j_1-1,j_1}, S_{\overline{j_1,j_2}} \otimes I) = \mathrm{Ext}^*(S_{\widetilde{\eta}} \otimes I, S_{\widetilde{\lambda}}) = \mathrm{Ext}^*(S_{\widehat{p-j_2-1,p-j_1}} \otimes I, S_{p-j_1-1,p-j_1}) = 0$$

Therefore, it follows from the long exact sequence under consideration that

$$\mathrm{Ext}^q(S_{j_1-1,j_1}, S_{j_1,j_2}) = \mathrm{Ext}^{q-1}(S_{j_1-1,j_1}, S_{j_1-1,j_2}) = \begin{cases} \mathbf{k} & \text{if } q-1 \in \{j_2 - j_1 - 1, j_2 - j_1\}, \\ & \text{i.e., } q \in \{j_2 - j_1, j_2 - j_1 + 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $i_1 < j_1 - 1$ and take $i_1 + 1 \leq m \leq j_1 - 1$. We apply the long exact sequence for $\mathrm{Ext}(-, S_{j_1,j_2})$ to the short exact sequence (3.6)

$$0 \to S_{m,j_1} \to S_{\overline{m,j_1}} \otimes I \to S_{m-1,j_1} \to 0.$$

We have $\mathrm{Ext}^q(S_{m-1,j_1}, S_{j_1,j_2}) = \mathrm{Ext}^{q-1}(S_{m,j_1}, S_{j_1,j_2})$, since $\mathrm{Ext}^*(S_{\widehat{m,j_1}} \otimes I, S_{j_1,j_2}) = 0$ by Lemma 3.4.1. Using all these equalities and the result for $\mathrm{Ext}^*(S_{j_1-1,j_1}, S_{j_1,j_2})$, we conclude that

$$\mathrm{Ext}^q(S_{i_1,j_1}, S_{j_1,j_2}) = \mathrm{Ext}^{q-(j_1-i_1-1)}(S_{j_1-1,j_1}, S_{j_1,j_2}) = \begin{cases} \mathbf{k} & \text{if } q-(j_1-i_1-1) \in \{j_2-j_1,\ j_2-j_1+1\}, \\ & \text{i.e., } q \in \{j_2-i_1-1,\ j_2-i_1\}, \\ 0 & \text{otherwise.} \end{cases}.$$

*Case 3.* $j_1 < i_2$

Consider the case $j_1 = i_2 - 1$. Applying the long exact sequence for $\mathrm{Ext}(-, S_{i_2,j_2})$ to the short exact sequence (3.5)

$$0 \to S_{i_1,i_2} \to S_{\widehat{i_1,i_2}} \otimes I \to S_{i_1,i_2-1} \to 0$$

and using Lemma 3.4.1 together with the result of Case 2., we get the following exact sequence:

$$\begin{array}{ccccc}
\cdots & & & & \\
\downarrow & & & & \\
\mathrm{Ext}^{j_2-i_1-4}(S_{i_1,i_2-1}, S_{i_2,j_2}) & \longrightarrow & \mathrm{Ext}^{j_2-i_1-4}(S_{\widehat{i_1,i_2}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-4}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 \\
\mathrm{Ext}^{j_2-i_1-3}(S_{i_1,i_2-1}, S_{i_2,j_2}) & \longrightarrow & \mathrm{Ext}^{j_2-i_1-3}(S_{\widehat{i_1,i_2}} \otimes I, S_{i_2,j_2}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{j_2-i_1-3}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 \\
\mathrm{Ext}^{j_2-i_1-2}(S_{i_1,i_2-1}, S_{i_2,j_2}) & \longrightarrow & \mathrm{Ext}^{j_2-i_1-2}(S_{\widehat{i_1,i_2}} \otimes I, S_{i_2,j_2}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{j_2-i_1-2}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 \\
\mathrm{Ext}^{j_2-i_1-1}(S_{i_1,i_2-1}, S_{i_2,j_2}) & \longrightarrow & \mathrm{Ext}^{j_2-i_1-1}(S_{\widehat{i_1,i_2}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-1}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq \mathbf{k} \\
\mathrm{Ext}^{j_2-i_1}(S_{i_1,i_2-1}, S_{i_2,j_2}) & \longrightarrow & \mathrm{Ext}^{j_2-i_1}(S_{\widehat{i_1,i_2}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq \mathbf{k} \\
\mathrm{Ext}^{j_2-i_1+1}(S_{i_1,i_2-1}, S_{i_2,j_2}) & \longrightarrow & \mathrm{Ext}^{j_2-i_1+1}(S_{\widehat{i_1,i_2}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1+1}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 \\
& & & & \downarrow \\
& & & & \cdots
\end{array}$$

Therefore, we obtain

$$\mathrm{Ext}^q(S_{i_1,i_2-1}, S_{i_2,j_2}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j_2-i_1-3, j_2-i_1-2, j_2-i_1, j_2-i_1+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now assume $j_1 < i_2 - 1$ and take $j_1 + 1 \le m \le i_2 - 1$. We apply the long exact sequence for $\mathrm{Ext}(-, S_{i_2,j_2})$ to the short exact sequence (3.5)

$$0 \to S_{i_1,m} \to S_{\widehat{i_1,m}} \otimes I \to S_{i_1,m-1} \to 0.$$

Since $\mathrm{Ext}^*(S_{\widehat{i_1,m}} \otimes I, S_{i_2,j_2}) = 0$ by Lemma 3.4.1, $\mathrm{Ext}^q(S_{i_1,m-1}, S_{i_2,j_2}) = \mathrm{Ext}^{q-1}(S_{i_1,m}, S_{i_2,j_2})$.

Using all these equalities and the result for $\mathrm{Ext}^*(S_{i_1,i_2-1}, S_{i_2,j_2})$, we obtain

$$\mathrm{Ext}^q(S_{i_1,j_1}, S_{i_2,j_2}) = \mathrm{Ext}^{q-(i_2-j_1-1)}(S_{i_1,i_2-1}, S_{i_2,j_2}) =$$
$$\begin{cases} \mathbf{k} & \text{if } q-(i_2-j_1-1) \in \{j_2-i_1-3, j_2-i_1-2, j_2-i_1, j_2-i_1+1\}, \\ & \text{i.e., } q \in \{i_2+j_2-i_1-j_1-4, i_2+j_2-i_1-j_1-3, i_2+j_2-i_1-j_1-1, \\ & \qquad i_2+j_2-i_1-j_1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Case 4.* $i_1 < i_2 < j_1 < j_2$

We begin with the special case $j_1 = i_2+1$. Applying the long exact sequence for $\mathrm{Ext}(-, S_{i_2,j_2})$ to the short exact sequence (3.5)

$$0 \to S_{i_1,i_2+1} \to S_{\widehat{i_1,i_2+1}} \otimes I \to S_{i_1,i_2} \to 0$$

and using Lemma 3.4.1 together with the result of Case 2., we get the following exact sequence:

$$\begin{array}{lllll}
\cdots \downarrow & & & & \\
\mathrm{Ext}^{j_2-i_1-4}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-4}(S_{\widehat{i_1,i_2+1}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-4}(S_{i_1,i_2+1}, S_{i_2,j_2}) \\
\mathrm{Ext}^{j_2-i_1-3}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-3}(S_{\widehat{i_1,i_2+1}} \otimes I, S_{i_2,j_2}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{j_2-i_1-3}(S_{i_1,i_2+1}, S_{i_2,j_2}) \\
\mathrm{Ext}^{j_2-i_1-2}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-2}(S_{\widehat{i_1,i_2+1}} \otimes I, S_{i_2,j_2}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{j_2-i_1-2}(S_{i_1,i_2+1}, S_{i_2,j_2}) \\
\mathrm{Ext}^{j_2-i_1-1}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{j_2-i_1-1}(S_{\widehat{i_1,i_2+1}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1-1}(S_{i_1,i_2+1}, S_{i_2,j_2}) \\
\mathrm{Ext}^{j_2-i_1}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq \mathbf{k} & \longrightarrow & \mathrm{Ext}^{j_2-i_1}(S_{\widehat{i_1,i_2+1}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1}(S_{i_1,i_2+1}, S_{i_2,j_2}) \\
\mathrm{Ext}^{j_2-i_1+1}(S_{i_1,i_2}, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1+1}(S_{\widehat{i_1,i_2+1}} \otimes I, S_{i_2,j_2}) \simeq 0 & \longrightarrow & \mathrm{Ext}^{j_2-i_1+1}(S_{i_1,i_2+1}, S_{i_2,j_2}) \\
& & & & \downarrow \cdots
\end{array}$$

Therefore, we conclude that

$$\mathrm{Ext}^q(S_{i_1,i_2+1}, S_{i_2,j_2}) = \begin{cases} \mathbf{k} & \text{if } q \in \{j_2-i_1-3, j_2-i_1-1\}, \\ \mathbf{k}^2 & \text{if } q = j_2-i_1-2, \\ 0 & \text{otherwise.} \end{cases}$$

Now assume $j_1 > j_2+1$ and take $i_2+2 \leq m \leq j_1$. We apply the long exact sequence for $\mathrm{Ext}(-, S_{i_2,j_2})$ to the short exact sequence (3.5)

$$0 \to S_{i_1,m} \to S_{\widehat{i_1,m}} \otimes I \to S_{i_1,m-1} \to 0.$$

Since $\mathrm{Ext}^*(S_{\widehat{i_1,m}} \otimes I, S_{i_2,j_2}) = 0$ by Lemma 3.4.1, $\mathrm{Ext}^q(S_{i_1,m}, S_{i_2,j_2}) = \mathrm{Ext}^{q+1}(S_{i_1,m-1}, S_{i_2,j_2})$.

Using all these equalities and the result for $\mathrm{Ext}^*(S_{i_1,i_2+1}, S_{i_2,j_2})$, we obtain

$$\mathrm{Ext}^q(S_{i_1,j_1}, S_{i_2,j_2}) = \mathrm{Ext}^{q+(j_1-i_2-1)}(S_{i_1,i_2+1}, S_{i_2,j_2}) =$$
$$\begin{cases} \mathbf{k} & \text{if } q+(j_1-i_2-1) \in \{j_2-i_1-3, j_2-i_1-1\}, \\ & \text{i.e., } q \in \{i_2+j_2-i_1-j_1-2, i_2+j_2-i_1-j_1\}, \\ \mathbf{k}^2 & \text{if } q+(j_1-i_2-1) = j_2-i_1-2, \text{ i.e., } q = i_2+j_2-i_1-j_1-1, \\ 0 & \text{otherwise.} \end{cases}$$

We have considered all the possible cases, hence the proof is complete. □

# Chapter 4

# Strict polynomial functors with bounded domain

## 4.1 The category $\mathcal{P}_{d,n}$: definition and basic properties

In this section we define the category of homogeneous strict polynomial functors of degree $d$ with bounded domain in the spirit of the definition of strict polynomial functors recalled in Section 1.1 and we discuss properties of that category.

Let $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}$ be the full subcategory of $\Gamma^d\mathcal{V}ect_{\mathbf{k}}$ consisting of vector spaces over $\mathbf{k}$ with dimension at most $n$. The category $\mathcal{P}_{d,n}$ of homogeneous strict polynomial functors of degree $d$ with bounded domain $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}$ is defined as the category of $\mathbf{k}$-linear functors $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n} \to \mathcal{V}ect_{\mathbf{k}}$. It is an abelian category. The Kuhn duality $(-)^{\#}$ in $\mathcal{P}_{d,n}$ is defined as the restriction of the Kuhn duality in $\mathcal{P}_d$ to the category $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}$ and we define the tensor product $\otimes : \mathcal{P}_{d,n} \times \mathcal{P}_{e,n} \to \mathcal{P}_{d+e,n}$ in the same manner. Let $\Gamma^{d,n} = \mathrm{Hom}_{\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}}(\mathbf{k}^n, -)$ and $S^{d,n} = (\Gamma^{d,n})^{\#}$.

**Proposition 4.1.1.** *$\Gamma^{d,n}$ is a compact projective generator of $\mathcal{P}_{d,n}$ and $S^{d,n}$ is a compact injective cogenerator of $\mathcal{P}_{d,n}$. In particular, $\mathcal{P}_{d,n}$ has enough injective and projective objects.*

*Proof.* By the Yoneda lemma we have

$$\mathrm{Hom}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}, F) \simeq F(\mathbf{k}^n) \tag{4.1}$$

for any $F \in \mathcal{P}_{d,n}$, hence $\Gamma^{d,n}$ is a projective functor. We also see from (4.1) that $\Gamma^{d,n}$ is compact. It remains to prove that $\Gamma^{d,n}$ is a generator of $\mathcal{P}_{d,n}$, i.e., $\mathrm{Hom}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}, -)$ is a faithful functor. By the Yoneda lemma it is sufficient to show that for any $F, G \in \mathcal{P}_{d,n}$, if $\eta : F \to G$ is a natural transformation such that $\eta(\mathbf{k}^n) = 0$, then $\eta = 0$, but it follows from the fact that for any $m < n$, $\mathbf{k}^m$ is a retract of $\mathbf{k}^n$ in $\mathcal{V}ect_{\mathbf{k}}^{\leq n}$, hence also in $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}$.

By the Kuhn duality $S^{d,n}$ is an injective cogenerator of $\mathcal{P}_{d,n}$. The last statement is evident from the first part of the proposition. □

Let us recall that in Section 1.1 we denoted the category of finite dimensional homogeneous polynomial $\mathrm{GL}_n$-modules of degree $d$ by $\mathrm{GL}_n^{\mathrm{Pol},d}$ −mod.

**Theorem 4.1.2.** *There are equivalences of abelian categories $\mathcal{P}_{d,n} \simeq S(n,d)$-mod and $\mathcal{P}_{d,n} \simeq \mathrm{GL}_n^{Pol,d}$-mod. In particular, $\mathcal{P}_{d,n} \simeq \mathcal{P}_d$ for $n \geq d$.*

*Proof.* It follows from (4.1) that there are equivalences of $\mathbf{k}$-algebras

$$\mathrm{End}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}) \simeq (\Gamma^{d,n}(\mathbf{k}^n))^{\mathrm{op}} = (\Gamma^d(\mathrm{End}_{\mathbf{k}}(\mathbf{k}^n)))^{\mathrm{op}} \simeq S(n,d)^{\mathrm{op}}.$$

Let $\mathsf{P}_{\mathsf{d,n}}$ be the category of $\mathbf{k}$-linear functors from $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}$ to the category of all vector spaces over $\mathbf{k}$. Then by Proposition 4.1.1 and the Gabriel theorem (cf. [Mit65, Theorem 4.1]) we have $\mathsf{P}_{\mathsf{d,n}} \simeq \mathrm{Mod} - \mathrm{End}_{\mathsf{P}_{\mathsf{d,n}}}(\Gamma^{d,n})$. Since this equivalence given by $\mathrm{Hom}_{\mathsf{P}_{\mathsf{d,n}}}(\Gamma^{d,n}, -)$ preserves finite-dimensionality, it restricts to the equivalence

$$\mathcal{P}_{d,n} \simeq \mathrm{mod} - \mathrm{End}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}) \simeq \mathrm{mod} - S(n,d)^{\mathrm{op}} \simeq S(n,d) - \mathrm{mod}.$$

It immediately implies the last statement, because $\mathcal{P}_d \simeq S(n,d)$-mod for $n \geq d$. The second equivalence in Theorem follows from the first one and the equivalence $S(n,d) - \mathrm{mod} \simeq \mathrm{GL}_n^{\mathrm{Pol},d} - \mathrm{mod}$ (cf. [Mar93, Theorem 2.2.7]). □

**Corollary 4.1.3.** *$\mathcal{P}_{d,n}$ is a highest weight category with poset $\Lambda(d,n)$ of Young diagrams of weight $d$ and with at most $n$ columns, with the partial order on $\Lambda(d,n)$ being the reversed dominance order.*

*Proof.* The assertion follows immediately from Theorem 4.1.2 and the fact that $S(n,d)$-mod is a highest weight category (cf. [Par89, Theorem 4.1]). □

In the following theorem we describe the categories $\mathcal{P}_{d,n}$ and $\mathcal{D}^b\mathcal{P}_{d,n}$ in terms of recollements of abelian and triangulated categories (see, e.g., [BBD82] and [Psa14]) and we characterize costandard and standard objects in $\mathcal{P}_{d,n}$.

**Theorem 4.1.4.** *Denote the full subcategory of $\mathcal{P}_d$ consisting of strict polynomial functors of degree $d$, which assign the zero space to vector spaces with dimension at most $n$ by $\mathcal{P}_d^{>n}$. Let $i_* : \mathcal{P}_d^{>n} \to \mathcal{P}_d$ be the inclusion functor and let $j^* : \mathcal{P}_d \to \mathcal{P}_{d,n}$ be the exact functor, which restricts a given strict polynomial functor of degree $d$ to the category $\Gamma^d\mathcal{V}ect_{\mathbf{k}}^{\leq n}$.*

*(a) There is a recollement of abelian categories*

$$\mathcal{P}_d^{>n} \underset{\overset{\longleftarrow}{i^!}}{\overset{\overset{i^*}{\longleftarrow}}{\xrightarrow{\ i_*\ }}} \mathcal{P}_d \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{\ j^*\ }}} \mathcal{P}_{d,n}$$

*and the recollement of triangulated categories induced by the recollement given above on the level of bounded derived categories*

$$\mathcal{D}^b\mathcal{P}_d^{>n} \underset{\overset{\longleftarrow}{Ri^!}}{\overset{\overset{Li^*}{\longleftarrow}}{\xrightarrow{\ i_*\ }}} \mathcal{D}^b\mathcal{P}_d \underset{\overset{\longleftarrow}{Rj_*}}{\overset{\overset{Lj_!}{\longleftarrow}}{\xrightarrow{\ j^*\ }}} \mathcal{D}^b\mathcal{P}_{d,n}.$$

*(b) Let $\lambda \in \Lambda(d,n)$. The costandard (resp. standard, simple) object corresponding to $\lambda$ in $\mathcal{P}_{d,n}$ is the functor $j^*(S_\lambda)$ (resp. $j^*(W_\lambda)$, $j^*(F_\lambda)$). Thus, costandard and standard functors in $\mathcal{P}_{d,n}$ will be also called, respectively, Schur and Weyl functors and will be denoted by $S_\lambda$ and $W_\lambda$. The simple functor in $\mathcal{P}_{d,n}$ corresponding to $\lambda \in \Lambda(d,n)$ will be denoted by $F_\lambda$.*

*(c) Let $\lambda \in \Lambda(d,n)$. Then $j_*(S_\lambda) \simeq Rj_*(S_\lambda) \simeq S_\lambda$ and $j_!(W_\lambda) \simeq Lj_!(W_\lambda) \simeq W_\lambda$.*

*The functor $j_*$ (resp. $j_!$) preserves injective envelopes (resp. projective covers).*

*Proof.* We observe that $J = \Lambda(d) \setminus \Lambda(d,n)$ is an order ideal of $\Lambda(d)$ and $\mathcal{P}_d^{>n}$ consists of strict polynomial functors of degree $d$ with composition factors $F_\lambda$, where $\lambda \in J$. Then the part (a) follows from [CPS88, Theorem 3.9(b)]. (In [CPS88, Thm 3.9(b)] only the triangulated recollement was established, however, the abelian case is proved in the similar manner by using the recollement ($R/ReR$-mod, $R$-mod, $eRe$-mod) where $R$ is a ring and $e \in R$ is an idempotent (see, e.g., [Psa14, Example 2.7]).) The part (b) follows from (a) and [CPS89, Lemma 1.4(b)].

Now we turn to the proof of (c). Fix $\mu \in \Lambda(d,n)$. By the right adjointness of $Rj_*$ to $j^*$ and part (b) we have

$$\mathrm{Hom}_{\mathcal{DP}_d}(W_\lambda, Rj_*(S_\mu)[t]) \simeq \mathrm{Ext}^t_{\mathcal{P}_{d,n}}(j^*W_\lambda, S_\mu) \simeq \mathrm{Ext}^t_{\mathcal{P}_{d,n}}(W_\lambda, S_\mu) = 0$$

for any $t \neq 0$, $\lambda \in \Lambda(d,n)$. Therefore $Rj_*(S_\lambda) \simeq j_*S_\mu$ and, by Proposition 1.1.1(h), it is a good object (i.e., admits a good filtration). Let us recall that for a good $X \in \mathcal{P}_d$ we have $\ell(X) = \dim \mathrm{Hom}_{\mathcal{P}_d}(W, X)$, where $\ell(X)$ is the length of a corresponding good filtration of $X$ and $W = \bigoplus_{\lambda\in\Lambda(d,n)} W_\lambda$, also by Proposition 1.1.1(h) and the fact that $\mathrm{End}(F_\lambda) = \mathbf{k}$ (cf. [Jan03, Proposition 2.8]). By the adjointness, Proposition 1.1.1(f) and the part (b) again, we obtain

$$\ell(j_*(S_\mu)) = \dim \mathrm{Hom}_{\mathcal{P}_d}(W, j_*(S_\mu)) = \dim \mathrm{Hom}_{\mathcal{P}_{d,n}}(j^*(W), S_\mu) =$$
$$\dim \mathrm{Hom}_{\mathcal{P}_{d,n}}(W, S_\mu) = \dim \mathrm{Hom}_{\mathcal{P}_{d,n}}(W_\mu, S_\mu) = 1,$$

In consequence, $j_*(S_\mu) \simeq S_\mu$. The functor $j_*$ preserves injectives, because it has an exact left adjoint. Let $G \in \mathcal{P}_{d,n}$ be the injective envelope of $F \in \mathcal{P}_{d,n}$. Then, since $j_*$ preserves monomorphisms, we have an embedding $j_*(F) \subset j_*(G)$. Suppose that $j_*(G)$ is not the injective envelope of $j_*(F)$, i.e., there exists an injective $H \in \mathcal{P}_d$ such that $j_*(F) \subset H \subset j_*(G)$. Since $H$ is injective, it is a direct summand of $j_*(G)$. By applying $j^*$ we get $F \subset j^*(H) \subset G$, since $j^*j_* \simeq \mathrm{id}$. This gives a contradiction with the minimal property of envelope, because $j^*(H)$ is a direct summand of $G$, hence it is injective. This shows that $j_*$ preserves injective envelopes.

The proofs of the respective facts for $j_!(W_\lambda)$ are analogous. □

Now we recall the block structure of the Schur algebra $S(n,d)$ (cf. [Don94]). By Theorem 4.1.2 this is also the block structure of $\mathcal{P}_{d,n}$. For $\lambda \in \Lambda(d,n)$ with the conjugate Young diagram $\widetilde{\lambda}$ of the form $\widetilde{\lambda} = (\lambda'_1, \ldots, \lambda'_n)$ we define

$$\alpha(\lambda) = \max\left\{r \geq 0 : \forall_{1\leq i\leq n-1}\ \lambda'_i - \lambda'_{i+1} \equiv -1 \pmod{p^r}\right\}.$$

Then, identifying a block with a subset of $\Lambda(d,n)$, $\lambda$ and $\mu$ are in the same block of $\mathcal{P}_{d,n}$ if and only if $\lambda$ and $\mu$ have the same $p$-core and $\alpha(\lambda) = \alpha(\mu)$.

## 4.2 The homological algebra in $\mathcal{P}_{p,n}$

In the present section we study the homological algebra in the category $\mathcal{P}_{p,n}$. Since for $d < p$, $S(n,d)$ is a semisimple algebra (cf. [Mar93, Theorem 2.2.8]), $\mathcal{P}_{d,n}$ is a semisimple category for $d < p$. Thus, the first non-trivial case from the point of view of homological algebra is $d = p$. Moreover, if $n \geq p$ then $\mathcal{P}_{p,n} \simeq \mathcal{P}_p$ by Theorem 4.1.2. Therefore we assume in the sequel that $n < p$ and we call this situation *unstable*. Let us observe that $\alpha(\lambda) = 0$ for any $\lambda \in \Lambda(p,n)$. It is also easily seen that each diagram in $\Lambda(p,n)$ not being a $p$-hook is a $p$-core and all $p$-hooks in $\Lambda(p,n)$ have the same $p$-core, namely the empty set. As the consequence, the only non-trivial homological computations appear in the block of $\mathcal{P}_{p,n}$ corresponding to the subset of $\Lambda(p,n)$ consisting of $p$-hooks, which will be denoted by $\mathcal{P}^{\varnothing}_{p,n}$. Set $S_i = S_{(i+1,1^{p-i-1})}, W_i = W_{(i+1,1^{p-i-1})}$ and $F_i = F_{(i+1,1^{p-i-1})}$ for $0 \leq i \leq n-1$.

We start with collecting the facts which are formal consequences of the recollement between $\mathcal{P}_p$ and $\mathcal{P}_{p,n}$ described in the previous section and the corresponding properties of $\mathcal{P}_p$ established in Section 2.2.

**Theorem 4.2.1.** *(a) The decomposition matrix $D = (d_{ij}) \in M_{n\times n}(\mathbb{Z})$ of $\mathcal{P}^{\varnothing}_{p,n}$ is given by*

$$d_{ij} = [W_{i-1} : F_{j-1}] = \begin{cases} 1 & \text{if } j = i \text{ or } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Let* $0 \leq i, j \leq n-1$. *Then:*

$$\mathrm{Ext}^q(F_i, S_j) = \begin{cases} \mathbf{k} & \text{if } j \geq i \text{ and } q = j-i, \\ 0 & \text{otherwise.} \end{cases}$$

(c) *The category* $\mathcal{P}_{p,n}$ *has a Kazhdan-Lusztig theory relative to the function* $l : \Lambda(p,n) \to \mathbb{Z}$ *given by* $l((i+1, 1^{p-i-1})) = i$ *for* $0 \leq i \leq n-1$ *and* $l(\lambda) = 0$ *for* $\lambda$ *not being a p-hook.*

(d)

$$\mathrm{Ext}^q(F_i, F_j) = \begin{cases} \mathbf{k} & \text{if } q = |i-j| + 2r, \text{ where } 0 \leq r \leq n - \max\{i,j\} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(e)

$$\mathrm{Ext}^q(S_i, S_j) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i = j \text{ and } q = 0, \\ & \bullet\ j > i \text{ and } q \in \{j-i-1, j-i\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The statement (a) follows from the fact that $j^*$ is exact and preserves $F_i$ and $S_i$ for $0 \leq i \leq n-1$. The computation in (b) follows from Proposition 2.2.1, since the groups under consideration are the same in $\mathcal{P}_{p,n}$ and $\mathcal{P}_p$ because of the equality $j_*(S_j) = S_j$. The statement (c) follows from (b) (although it is also a formal consequence of the existence of the recollement setup). The computation of the Ext-groups in (d) is in turn a formal consequence of the existence of a Kazhdan-Lusztig theory in $\mathcal{P}_{p,n}$. The computation in (e) follows from Proposition 2.2.6 and the fact that $j_*(S_i) = S_i$. □

Now we investigate multiplicative structures and formality phenomena in $\mathcal{P}_{p,n}$. Analogous questions in $\mathcal{P}_p$ were studied in Section 2.3. We will again use the close relation between $\mathcal{P}_p$ and $\mathcal{P}_{p,n}$. A general picture is quite similar in both cases, but the crucial constructions of Section 2.3.2 need to be aptly adopted to $\mathcal{P}_{p,n}$. Namely, we constructed there an explicit injective resolution $\mathcal{R}_i$ of $F_i$ in $\mathcal{P}_p$ consisting of the parts of the de Rham and Koszul complexes spliced together. Unfortunately, $j^*(\mathcal{R}_i)$ will not be suitable for us, since it is not injective anymore. Nevertheless, we shall obtain an injective resolution $\mathcal{R}_{i,n}$ of $F_i$ in $\mathcal{P}_{p,n}$ by a similar construction. Namely, set

$$\mathcal{R}_{i,n}^{r,s} = \Omega^{i+s-r} \qquad \text{if } 0 \leq r \leq n-1,\ 0 \leq s \leq n-i-1 \text{ and } r-s \leq i.$$

The horizontal and vertical differentials are, respectively, the de Rham and Koszul differentials. Since $\kappa d + d\kappa = 0$, $\mathcal{R}_{i,n}^{**}$ is a double complex. We observe that $\mathcal{R}_{i,n}$ is the truncation of the double complex providing $\mathcal{R}_i$ at the vertical degree $n-i-1$. For instance, $\mathcal{R}_{3,5}$ for $p = 7$ is of the following form:

$$\begin{array}{ccccccccc} \Omega^4 & \xrightarrow{\kappa_4} & \Omega^3 & \xrightarrow{\kappa_3} & \Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 \\ {\scriptstyle d_3}\uparrow & & {\scriptstyle d_2}\uparrow & & {\scriptstyle d_1}\uparrow & & {\scriptstyle d_0}\uparrow & & \\ \Omega^3 & \xrightarrow{\kappa_3} & \Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 & & \end{array} .$$

Since $\Omega^k$ is injective for $0 \leq k \leq n-1$ (as being a summand in the injective cogenerator $S^{p,n}$) and $H^*(\mathrm{Tot}(\mathcal{R}_{i,n})) = H^0(\mathrm{Tot}(\mathcal{R}_{i,n})) = F_i$ (see p. 34), $\mathrm{Tot}(\mathcal{R}_{i,n})$ is an injective resolution of $F_i$. We use for the simplicity the same symbol $\mathcal{R}_{i,n}$ for that. Set $\mathcal{R} = \bigoplus_{0 \leq i \leq n-1} \mathcal{R}_{i,n}$ and $F = \bigoplus_{0 \leq i \leq n-1} F_i$. We now observe that by truncation we get a quasi-isomorphism $j^*(\mathcal{R}_i) \simeq \mathcal{R}_{i,n}$, which, in particular, shows that $j^*$ induces an epimorphism $\mathrm{Ext}^*_{\mathcal{P}_p}(F,F) \to \mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F,F)$. Also, since the construction of $\mathcal{R}_{i,n}$ is analogous to that of $\mathcal{R}_i$, the arguments of Section 2.3.2 apply to the current situation and we obtain:

**Theorem 4.2.2.**

*(a)* *Let* $B = \bigoplus_{t \in \mathbb{N}} B_t$ *be the graded algebra with grading*

$$B_t = \mathrm{span}\{b^t_{ji} : 0 \leq i, j \leq n-1 \textit{ and } t = |i-j| + 2r, \textit{ where } 0 \leq r \leq n - \max\{i,j\} - 1\}$$

*for* $0 \leq t \leq 2n-2$ *and* $B_t = 0$ *for* $t \geq 2n-1$*, where* $b^t_{ji}$ *are the formal symbols for* $i, j, t$ *satisfying the above conditions. The multiplication on* $B$ *is given by*

$$b^t_{lm} \cdot b^u_{ji} = \begin{cases} b^{t+u}_{mi} & \textit{if } j = l \textit{ and } u + t \leq 2n - i - m - 2, \\ 0 & \textit{otherwise.} \end{cases}$$

*Then there is a graded algebra isomorphism* $\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F, F) \simeq B$.

*(b)* *There is a graded algebra isomorphism* $\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F_i, F_i) \simeq \mathbf{k}[x]/(x^{n-i})$ *for* $0 \leq i \leq n-1$ *and* $x$ *of degree* 2*. In particular,* $\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F_i, F_i)$ *is a commutative algebra.*

*(c)* *The algebra* $\mathrm{End}^*(\mathcal{R})$ *is a formal DG algebra, i.e., there exists a quasi-isomorphism of DG algebras* $\eta : \mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F, F) \to \mathrm{End}^*(\mathcal{R})$*, where* $\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F, F)$ *is regarded as a DG algebra with zero differential.*

*(d)* *There is an equivalence of triangulated categories* $\mathcal{D}^b\mathcal{P}^{\varnothing}_{p,n} \simeq \mathcal{D}^b(\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F, F) - mod^{gr})$.

We remark that similar results concerning the Yoneda algebra of Schur functors, analogous to Theorem 2.3.2 and Corollary 2.3.3, also hold — $j^*(\mathcal{T}_i)$ is an injective resolution of $S_i$ — though we leave their formulation to the interested reader.

The computations in $\mathcal{P}_{p,n}$ obtained so far could be hardly called surprising. However, when we try to compute the groups $\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(F_i, W_j)$ and $\mathrm{Ext}^*_{\mathcal{P}_{p,n}}(S_i, W_j)$, we encounter some interesting phenomena. Analogously to the proofs of Propositions 2.2.8 and 2.2.10, we will use the minimal injective resolution of $W_i$, let us call it $L_{i,n}$ (in fact (2.8) gives the minimal projective resolution of $S_i$ in $\mathcal{P}_p$, hence we consider the Kuhn dual case). Again we cannot just apply $j^*$ to the dual of (2.8), since the resulting complex is not injective. Instead, we need to adapt the idea of its construction to our context. Namely, this time we concatenate the truncated dual Koszul and Koszul complexes, but we cut them off earlier than it was done in (2.8) in order to get an injective complex. Thus, we put as $L_{i,n}$ the following:

$$(\Omega^{i+1})^\# \longrightarrow (\Omega^{i+2})^\# \longrightarrow \ldots \longrightarrow (\Omega^{n-1})^\# \xrightarrow{\alpha} \Omega^{n-1} \longrightarrow \Omega^{n-2} \longrightarrow \ldots \longrightarrow \Omega^0,$$

where $\alpha = (\Omega^{n-1})^\# \xrightarrow{\kappa_n^\#} (\Omega^n)^\# \simeq \Omega^n \xrightarrow{\kappa_n} \Omega^{n-1}$. We remark that $L_{i,n}$ is the minimal injective resolution of $W_i$ in $\mathcal{P}_{p,n}$, since $\Omega^q$ is the injective envelope of $F_q$ for $0 \leq q \leq n-1$ by Proposition 2.1.1 and [CPS89, Lemma 1.2].

Then by repeating the arguments from the proofs of Proposition 2.2.8, Corollary 2.2.9 and Proposition 2.2.10 we obtain:

**Proposition 4.2.3.** *We have the following computations in* $\mathcal{P}_{p,n}$*:*

*(a)* *For* $0 \leq i, j \leq n-1$

$$\mathrm{Ext}^q(S_i, F_j) = \begin{cases} \mathbf{k} & \textit{if one of the following holds:} \\ & \bullet\ i < j \textit{ and } q \in \{j-i-1, 2n-i-j-2\}, \\ & \bullet\ i \geq j \textit{ and } q = 2n-i-j-2, \\ 0 & \textit{otherwise} \end{cases}$$

*and*

$$\operatorname{Ext}^q(F_i, W_j) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i > j \text{ and } q \in \{i-j-1, 2n-i-j-2\}, \\ & \bullet\ i \leq j \text{ and } q = 2n-i-j-2, \\ 0 & \text{otherwise.} \end{cases}$$

*(b) For* $0 \leq i, j \leq n-1$

$$\operatorname{Ext}^q(S_i, W_j) = \begin{cases} \mathbf{k} & \text{if one of the following holds:} \\ & \bullet\ i = j \text{ and } q \in \{0, 2p-i-j-3, 2p-i-j-2\}, \\ & \bullet\ i \neq j \text{ and } q \in \{2p-i-j-3, 2p-i-j-2\}, \\ 0 & \text{otherwise.} \end{cases}$$

We point out here that although our construction is similar to that in $\mathcal{P}_p$, we cannot say that the result is just the truncation of the corresponding computation in $\mathcal{P}_p$, since the Ext-groups are shifted. In particular, in this case $j^*$ is not epimorphic on the Ext-groups. Some even more striking examples of non-obvious behavior of $j^*$ will be provided in the next paragraph.

It was observed in [AB88, pp. 183-184] that $I^{\otimes d}$ is not projective if $d > n$. The conceptual reason for this is that it is not a direct summand in a projective generator of $\mathcal{P}_{d,n}$ anymore. Hence, let us compute some Ext-groups in $\mathcal{P}_{p,n}$ involving $I^{\otimes p}$. To this end, we shall use the adjunction $\{j^*, Rj_*\}$ between the categories $\mathcal{D}^b\mathcal{P}_d$ and $\mathcal{D}^b\mathcal{P}_{d,n}$. Let us start with an easy general observation:

**Proposition 4.2.4.** *For any* $F \in \mathcal{P}_{d,n}$ *and projective* $P \in \mathcal{P}_d$ *we have:*

$$\operatorname{Ext}^q_{\mathcal{P}_{d,n}}(j^*(P), F) \simeq \operatorname{Hom}_{\mathcal{P}_d}(P, H^q(Rj_*(F))).$$

*Proof.* It follows from the aforementioned adjunction and the fact that the functor $\operatorname{Hom}_{\mathcal{P}_d}(P, -)$ is exact, hence it commutes with cohomology. □

Thus, we see that the task of computing $\operatorname{Ext}^*_{\mathcal{P}_{p,n}}(I^{\otimes p}, F)$ is essentially reduced to that of computing $H^*(Rj_*(F))$. For this we shall use for various $F \in \mathcal{P}_{p,n}$ their $j_*$-acyclic resolutions, which are often simpler than the injective ones. One of these is $K_{i,n}$, the truncated complex of "Koszul kernels" in $\mathcal{P}_{p,n}$ given by:

$$S_i \xrightarrow{d} S_{i+1} \xrightarrow{d} \dots \xrightarrow{d} S_{n-1},$$

which is a resolution of $F_i$ by (2.6) and the exactness of $j^*$. The second is the truncated de Rham complex $M_{i,n}$:

$$\Omega^{i+1} \xrightarrow{d_{i+1}} \Omega^{i+2} \xrightarrow{d_{i+2}} \dots \xrightarrow{d_{n-1}} \Omega^n,$$

which is a resolution of $W_i$. In fact, $M_{i,n}$ is the first part of the complex $L_{i,n}$, because $(\Omega^j)^\# \simeq \Omega^j$ for $j > 0$. It is not injective, since $\Omega^n$ is not injective, but it is, as we will see, $j_*$-acyclic. We shall also need the complex $M'_{i,n}$ in $\mathcal{P}_p$ given by:

$$\Omega^{i+1} \xrightarrow{d_{i+1}} \Omega^{i+2} \xrightarrow{d_{i+2}} \dots \xrightarrow{d_{n-2}} \Omega^{n-1} \xrightarrow{\beta} S_{n-1},$$

where the map $\beta$ is the composition of the canonical projection and the de Rham differential:

$$\Omega^{n-1} \longrightarrow S_{n-2} \xrightarrow{d_{n-2}} S_{n-1}.$$

In the next theorem we still use the convention introduced in Theorem 4.1.4 that for "generally known functors" we denote their restriction $j^*$ by the same symbol. Thus, formula like $j_*(F) \simeq F$ actually means that for some $F \in \mathcal{P}_{p,n}$ we have $j^*(j_*(F)) \simeq F$.

Now we have:

**Theorem 4.2.5.** *For any* $0 \leq i \leq n-1$ *we have:*

*(a)* $Rj_*(F_i) = j_*(K_{i,n}) \simeq K_{i,n}$. *Hence* $j_*(F_{n-1}) \simeq Rj_*(F_{n-1}) \simeq S_{n-1}$ *and for* $0 \leq i < n-1$, *we have:*

$$H^q(Rj_*(F_i)) \simeq \begin{cases} F_i & \text{for } q = 0, \\ F_n & \text{for } q = n-i-1, \\ 0 & \text{otherwise.} \end{cases}$$

*(b)* $Rj_*(W_i) = j_*(M_{i,n}) \simeq M'_{i,n}$. *Hence* $j_*(W_{n-1}) \simeq Rj_*(W_{n-1}) \simeq S_{n-1}$ *and for* $0 \leq i < n-1$, *we have:*

$$H^q(Rj_*(W_i)) = \begin{cases} W_i & \text{for } q = 0, \\ F_n & \text{for } q = n-i-2,\ q = n-i-1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We first observe that, by Theorem 4.1.4(c), the complex $K_{i,n}$ is $j_*$-acyclic, hence it may be used for computing $Rj_*(F_i)$. Since $j_*(S_q) \simeq S_q$ for $q \leq n-1$ (Theorem 4.1.4(c) again), we see that the complexes $j_*(K_{i,n})$ and $K_{i,n}$ consist of isomorphic objects. Then by the faithfulness of $j_*$, $j_*(d) \neq 0$, but since $\dim(\mathrm{Hom}_{\mathcal{P}_p}(S_q, S_{q+1})) = 1$ by Proposition 2.2.6, we conclude that $j_*(d) = d$ up to a nonzero scalar. This shows that $Rj_*(K_{i,n}) \simeq K_{i,n}$ in $\mathcal{D}^b\mathcal{P}_p$. The formula for cohomology of $Rj_*(F_i)$ follows from the fact that the kernel (resp. cokernel) of the map $d : S_q \to S_{q+1}$ in $\mathcal{P}_p$ for $q < p$ is isomorphic to $F_q$ (resp. $F_{q+1}$) (see (2.11) and the preceding discussion).

For the second part, we start by invoking Theorem 4.1.4(c) again to show that $M_{i,n}$ is $j_*$-acyclic, since all its terms except $\Omega^n$ are injective while $\Omega^n \simeq S_{n-1}$ in $\mathcal{P}_{p,n}$. More precisely, by the fact mentioned before Proposition 4.2.3 that for $q < n$, $\Omega^q$ is the injective envelope of $S_q$ and Theorem 4.1.4(c), we obtain $j_*(\Omega^q) \simeq \Omega^q$ for $0 \leq q < n$ and $j_*(\Omega^n) \simeq S_{n-1}$. We conclude that $j_*(M_{i,n}) \simeq M'_{i,n}$ by the similar argument to that used in the proof of the first part, since the Hom-spaces between the functors under inspection are one-dimensional by, e.g., an elementary computation using the sum-diagonal adjunction. Then the fact that $H^0(M'_{i,n}) \simeq W_i$ is the main point of (2.8). The rest of the formula for $H^*(Rj_*(W_i))$ could be derived from the description of the kernel/cokernel of $\beta$, but it also follows immediately from the first part of the Theorem and the long exact sequence of cohomology applied to the distinguished triangle

$$Rj_*(F_{i+1}) \to Rj_*(W_i) \to Rj_*(F_i)$$

for $i < n-1$ coming from the short exact sequence in $\mathcal{P}_{p,n}$:

$$0 \to F_{i+1} \to W_i \to F_i \to 0,$$

which is a consequence of Theorem 4.2.1(a). □

Now we can apply our computation and Proposition 4.2.4 to $P = I^{\otimes p}$ and compute the corresponding Ext-groups. In order to describe an extra structure carried by these Exts, which comes from the action of the symmetric group $\Sigma_p$ on $I^{\otimes p}$, we shall formulate our results in terms of "the derived Schur functor" $Rs$. Namely, classically (see, e.g., [Mar93, Section 4]) one considers "the Schur functor" $s : \mathcal{P}_d \longrightarrow \mathbf{k}[\Sigma_d]$-mod given by the formula:

$$s(F) := \mathrm{Hom}_{\mathcal{P}_d}(I^{\otimes d}, F).$$

In fact, over a field of characteristic zero $s$ establishes an equivalence between $\mathcal{P}_d$ and the category $\mathbf{k}[\Sigma_d]-$mod of finite dimensional $\mathbf{k}$-representations of $\Sigma_d$. Over a field of positive characteristic $s$ is not an equivalence anymore, but it still preserves an important information. For example, for a $p$-regular $\lambda$ (i.e., a Young diagram with no $p$ or more rows of the same length), $s$ takes the simple functor $F_\lambda$ to the simple $\mathbf{k}[\Sigma_d]$-module associated to $\lambda$, which we shall denote by $G_\lambda$ (see, e.g., [JK81, Chapter 8.4] where an explicit construction of simple $\Sigma_d$-modules is given, or [Kuh02] where the relevant recollement diagram is studied). In particular, we denote by $G_i$ the simple $\mathbf{k}[\Sigma_p]$-module associated to the corresponding hook diagram. Also, $s(S_\lambda)$ and $s(W_\lambda)$ can be explicitly described. They are called, respectively, the Specht and dual Specht modules and denoted by $Sp_\lambda$ and $Sp'_\lambda$ [JK81, Chapter 7] (again, we will also use the notations: $Sp_i$ and $Sp'_i$).

Now, since in our situation $I^{\otimes p}$ is not projective, it is natural to consider its derived functor operating on the bounded derived categories:

$$Rs : \mathcal{D}^b\mathcal{P}_{p,n} \longrightarrow \mathcal{D}^b\mathbf{k}[\Sigma_p]-\mathrm{mod},$$

given by the formula:

$$Rs(F) := \mathrm{RHom}_{\mathcal{D}^b\mathcal{P}_{p,n}}(I^{\otimes p}, F),$$

which translates to Ext-computations via the formula:

$$H^t(Rs(F)) \simeq \mathrm{Ext}^t_{\mathcal{P}_{p,n}}(I^{\otimes p}, F).$$

Thus, within the formalism of total derived functors, we can formulate our Ext-computations as follows (for completeness, we also add the obvious computation of $Rs(S_\lambda)$:

**Corollary 4.2.6.** *There are the following isomorphisms in $\mathcal{D}^b\mathbf{k}[\Sigma]_p-\mathrm{mod}$:*

*(a) For any $0 \leq i \leq n-1$, $Rs(S_i) = Sp_i$.*

*(b) $Rs(F_0) = G_n[-(n-1)]$, $Rs(F_{n-1}) = Sp_{n-1}$, and for $0 < i < n-1$:*

$$\mathrm{Ext}^q_{\mathcal{P}_{p,n}}(I^{\otimes p}, F_i) \simeq H^q(Rs(F_i)) = \begin{cases} G_i & \text{for } q = 0, \\ G_n & \text{for } q = n-1-i, \\ 0 & \text{otherwise.} \end{cases}$$

*(c) $Rs(W_{n-1}) = Sp_{n-1}$, and for $0 \leq i < n-1$:*

$$\mathrm{Ext}^q_{\mathcal{P}_{p,n}}(I^{\otimes p}, W_i) \simeq H^q(Rs(W_i)) = \begin{cases} Sp'_i & \text{for } q = 0, \\ G_n & \text{for } q = n-i-2,\ q = n-i-1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The statements (b) and (c) follow from the discussion in the two paragraphs preceding the corollary, Proposition 4.2.4 and Theorem 4.2.5. In (a) we use Theorem 4.1.4 instead of Theorem 4.2.5. □

**Remark:** Let us point out for some interesting phenomena here. Firstly, our formulas are overlapping at the end of the poset, since in $\mathcal{P}_{p,n}$ we have $S_{n-1} \simeq F_{n-1} \simeq W_{n-1}$ (because $(n, 1^{p-n})$ is minimal in its block). This, for example, shows that $j^*$ is not full, since $\mathrm{Hom}_{\mathcal{P}_p}(I^{\otimes p}, F_{n-1}) = G_{n-1}$ while $\mathrm{Hom}_{\mathcal{P}_{p,n}}(I^{\otimes p}, F_{n-1}) = Sp_{n-1}$. At the other end of the scale something interesting also happens. Namely, since $(1^p)$ is $p$-singular (i.e., not $p$-regular) we have $s(F_0) = 0$, but, as we see, $Rs(F_0)$ is not only nonzero but it is the shifted $G_n$, which could not be hit classically, since we do not have $F_n$ in $\mathcal{P}_{p,n}$.

These phenomena have quite a surprising consequence for $n = p - 1$. We recall that for an abelian category $\mathcal{A}$, $K_0(\mathcal{D}^b\mathcal{A})$ denotes the abelian group generated by objects of the bounded derived category of $\mathcal{A}$ with the relations coming from distinguished triangles (see, e.g., [CPS93, Section 2]). Since we have the relation $X[1] = -X$, $K_0(\mathcal{D}^b\mathcal{A})$ is isomorphic to the usual $K_0(\mathcal{A})$, hence, for an artinian abelian category $\mathcal{A}$, it is a free abelian group of rank equal to the number of non-isomorphic simples in $\mathcal{A}$. Now we have:

**Corollary 4.2.7.** *$Rs$ induces an isomorphism:*

$$K_0(Rs) : K_0(\mathcal{D}^b\mathcal{P}_{p,p-1}) \xrightarrow{\simeq} K_0(\mathcal{D}^b\mathbf{k}[\Sigma_p]{-}\mathrm{mod}).$$

*Proof.* It suffices to show that $K_0(Rs)$ is onto. To this end, we first observe that $\pm[G_{p-1}]$ lies in the image, since $Rs(F_0) = G_{p-1}[-(p-2)]$. Then, by the rest of computations in Corollary 4.2.6, we see that all $[G_i]$ for $0 \leq i < p-1$ lie in the image of $Rs$ too. □

This can be contrasted with the fact that $Rs$ is extremely far from being an isomorphism. It is not only because $\mathbf{k}[\Sigma_p]{-}\mathrm{mod}$ has infinite homological dimension, but often $s$ acts trivially on Ext-groups just for a dimension reason. For example, we have $\mathrm{Ext}^*_{\mathcal{P}_{p,p-1}}(F_0, F_0) = \mathbf{k}[x]/(x^{p-1})$ with $|x| = 2$ while $\mathrm{Ext}^*_{\mathbf{k}[\Sigma_p]-\mathrm{mod}}(G_{p-1}, G_{p-1}) \simeq H^*(\Sigma_p, \mathbf{k})$, which is a subring of $\mathbf{k}[x] \otimes \Lambda^*(y)$ with $|x| = 2, |y| = 1$ consisting of elements of degree congruent to 0 or $-1$ modulo $2p$.

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