University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Olga Ziemiańska

Complexes of groups on the categories with loops

Doctoral dissertation

Supervisor

Professor Stefan Jackowski

Institute of Mathematics
University of Warsaw

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Author's declaration:
Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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date *Olga Ziemiańska*

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The dissertation is ready to be refereed.

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date *Professor Stefan Jackowski*

Streszczenie

Niniejsza rozprawa poświęcona jest uogólnieniu klasycznej teorii kompleksów grup wprowadzonej A. Haefliegera. Pokazujemy, że kompleksy grup Haefliegera są szczególnym przypadkiem słabych funktorów zdefiniowanych przez W. Thomasona. Pozwala to zaprezentować wyniki Haefliegera w znacznie szerszym kategoryjnym kontekście.

Głównym wynikiem rozprawy jest klasyfikacja epimorfizmów kompleksów grup. Jej szczególnym przypadkiem jest klasyfikacja epimorfizmów kompleksów grup z abelowym jadrem i epimorfizmów z lokalnie stałym jądrem zaprezentowana przez Haefliegera. Dowodzimy, że istnieje wzajemnie jednoznaczna odpowiedniość pomiędzy klasami równoważności epimorfizmów kompleksów grup a elementami drugiej grupy kohomologii pewnej małej kategorii. Jeśli kategoria ta jest zdefiniowana przez pewną dyskretną grupę, wówczas otrzymujemy dobrze znaną klasyfikację rozszerzeń grup.

Ponadto dla każdego epimorfizmu kompleksów grup konstruujemy odpowiednik kategoryjnego jądra tego epimorfizmu. Jest to kompleks grup wraz z homomorfizmem spełniającym pewne uniwersalne własności.

W ostanim rozdziale pokazujemy, że epimorfizm kompleksów grup indukuje epimorfizm ich grup podstawowych. Co więcej, dla każdego epimorfizmu kompleksów grup $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ konstruujemy homomorfizm kompleksów grup $\mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$, który jest nakryciem. Ma ono te własność, że ciąg $\mathcal{K} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ homomorfizmów kompleksów grup indukuje rozszerzenie grup podstawowych.

Słowa kluczowe

kompleks grup, epimorfizm kompleksów grup, rozszerzenie kompleksów grup, kategoria małych kategorii, op-lax funktor, kohomologie małych kategorii, grupa podstawowa kompleksu grup, grupoid, nakrycie małej kategorii, nakrycie kompleksu grup

Klasyfikacja tematyczna według AMS

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Abstract

The thesis is devoted to a generalization of the classical theory of complexes of groups introduced by A. Haefliеger. We show that a complex of groups defined by Haefliger is a special case of an op-lax functor defined by R.W.Thomason. This allows us to present Haefliger's results in a much more general categorical context.

The main result of the thesis is the classification of epimorphisms of complexes of groups. A special case of this is the classification of epimorphisms of complexes of groups with abelian or locally constant kernel given by Haefliger. We prove that there exists a natural bijective corrspondence between equivalence classes of epimorphisms of complexes of groups and elements of the second cohomology group of a certain small category. If this category is defined by a discrete group, then we obtain the well known classification of extensions of groups.

In addition, for each epimorphism of complexes of groups we construct an anlogue of the categorical kernel of the given epimorphism. It is a complex of groups and a homomorphism which satisfy a certain universal property.

In the last Chapter we prove that each epimorphism of complexes of groups yields an epimorphism of the fundamental groups. Moreover, for a given epimorphism of complexes of groups $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ we consturct a homomorphism of complexes of groups $\mathcal{K} \longrightarrow \mathcal{G}$ which is a covering. The sequence of homomorphisms $\mathcal{K} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ yields an extension of the fundamental groups.

Keywords and phrases

complex of groups, epimorphism of complexes of groups, extension of complexes of groups, category of small categories, op-lax functor, cohomology of small categories, fundamental group of a complex of groups, grupoid, covering of a small category, covering of a complex of groups

AMS Mathematics Subject Classification

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Contents

Introduction

We generalize the notion of a complex of groups defined by A. Haefliger ([H1], [B-H]) in the following way:

Definition *Let $\mathcal{C}$ be a small category and let* Gr *denote the category of groups. A twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is given by*

1. *for each object $c \in \mathrm{Ob}\,\mathcal{C}$ a group $\mathcal{G}(c)$*
2. *for each morphism $l : c' \longrightarrow c \in \mathrm{Mor}\,\mathcal{C}$ a homomorphism of groups $\mathcal{G}(l) : \mathcal{G}(c') \longrightarrow \mathcal{G}(c)$*
3. *for two composable morphisms $c'' \xrightarrow{l'} c' \xrightarrow{l} c \in \mathrm{Mor}\,\mathcal{C}$ an element $g_{l,l'} \in \mathcal{G}(c)$, called the twisting element, such that*
   i) $\mathrm{Ad}(g_{l,l'})\mathcal{G}(ll') = \mathcal{G}(l)\mathcal{G}(l')$, where $\mathrm{Ad}(g_{l,l'})$ is the conjugation by $g_{l,l'}$
   ii) $\mathcal{G}(l)(g_{l',l''})g_{l,l'l''} = g_{l,l'}g_{ll',l''}$ for each triple $\cdot \xrightarrow{l''} \cdot \xrightarrow{l'} \cdot \xrightarrow{l} \cdot \in \mathrm{Mor}\,\mathcal{C}$ of composable morphisms (cocycle condition)

As will turn out later, we can assume that a twisted diagram of groups satisfies the normalizing condition, that is $\mathcal{G}(\mathrm{id}_c) = \mathrm{id}_{\mathcal{G}(c)}$ for each $c \in \mathrm{Ob}\,\mathcal{C}$.

"Sheafs" of groups A twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ assings to every object c of an indexing category $\mathcal{C}$ a group $\mathcal{G}(c)$ and to every morphism $c' \longrightarrow c$ a homomorphism $\mathcal{G}(c') \longrightarrow \mathcal{G}(c)$, however it does not have to be completely functorial - it preserves composition only up to a compatible family of inner automorphisms. A discrete group is a special case of a twisted diagram of groups, we assume that $\mathcal{C} = *$ is the category which consists of one object. We consider twisted diagrams of groups as generalizations of groups or "sheafs" of groups modelled on $\mathcal{C}$. Many concepts associated with groups carry over to twisted diagrams of groups. A homomorphism $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ of twisted diagrams of groups over a functor $F : \mathcal{C}' \longrightarrow \mathcal{C}$ consists of local homomorphisms of local groups $\{\phi_{c'} : \mathcal{G}'(c') \longrightarrow \mathcal{G}(F(c'))\}_{c' \in \mathrm{Ob}\,\mathcal{C}'}$ subject to some relations (2.1.5).

Group complexes defined by group actions The starting point of Haefliger's work was the following example. Assume that a group G acts on a simplicial complex $\widetilde{X}$ in such a way that the orbit space $X := \widetilde{X}/G$ has a natural simplicial structure and the quotient map $q : \widetilde{X} \to X$ is simplicial. Simplices of X are partially ordered by (reverse) inclusion; thus they form a category $\mathcal{C}$. We define a weak functor from $\mathcal{C}$ to the category of groups $\mathcal{G} : \mathcal{C} \to \mathrm{Gr}$ by assigning to every simplex $c \in \mathcal{C}$ first a simplex $\tilde{c} \in q^{-1}(c)$ and then its stabilizer (isotropy subgroup) $G_{\tilde{c}}$. If $c' \subset c$ then we pick up an element $g \in G$ such that $\tilde{c'}$ is a face of the simplex $g\tilde{c}$. We define a monomorphism $\psi_{c'c} : G_{\tilde{c}} \to G_{\tilde{c'}}$ as the composition of the conjugation by $\mathrm{Ad}(g) : G_{\tilde{c}} \to G_{g\tilde{c}}$ and the inclusion $G_{g\tilde{c}} \subset G_{\tilde{c'}}$. Thus we obtain a "weak" functor from the category of simplices to the category of groups and monomorphisms. Because of these choices, if we consider the composition $G_{\tilde{c}} \longrightarrow G_{\widetilde{c'}} \longrightarrow G_{\widetilde{c''}}$ then the monomorphism $\psi_{c''c} \neq \psi_{c''c'}\psi_{c'c}$ and differs from it by the conjugation with an element of the group $G_{\widetilde{c''}}$ called the twisting element. These twisting elements satisfy the cocycle condition. Note that $\mathcal{C}$ is a small category such that the only endomorphisms of objects are identities. Such a category is called a small category without loops or *scwol* for short.

These considerations led Haefliger in 1990 to his definition of complexes of groups: i.e. "weak" functors defined on categories related to simplicial complexes with values in the category of groups and monomorphisms. A complex of groups associated to an action of a group in a way described above is called *developable*.

Haefliger and Thomason Much earlier in 1979 Bob Thomason considered - for a homotopy theoretical purpose - similar ideas in a much more general categorical context. He considered "weak" functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ (he called them "op-lax functors") from an arbitrary small category to the category of small categories. We note that the definition of Haefliger is a special case of Thomason's when we assume that $\mathcal{C}$ has no loops and the functor takes values in the category of groups and monomorphisms. This is because every group G can be considered as a small category $\mathcal{B}G$ with a single object and the group G as its morphisms.

Twisted diagram of groups associated to an extension of groups We will present an example of a twisted diagram of groups on a small category associated to a group G. Note that $\mathcal{B}G$ is a category with loops.

Let $N \rightarrowtail \widetilde{G} \overset{\eta}{\twoheadrightarrow} G$ be an extension of groups. Any set-theoretical cross-section of η yields a twisted diagram of groups $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ such that $\mathcal{F}(*) = N$. For details cf. Example 2.1.10. Let $\mathcal{E}G$ be a category whose objects correspond to elements of G and for each pair of objects g_1, g_2 there exist unique morphism $g_1 \overset{g_1^{-1}g_2}{\longrightarrow} g_2$. The group G acts on $\mathcal{E}G$ in the obvious way with a quotient $\mathcal{B}G$. Hence the group $\widetilde{G}$ acts on $\mathcal{E}G$ via the epimorphism η, namely $\widetilde{g}.g = \eta(\widetilde{g})g$. Clearly the isotropy subgroup of each object is isomorphic to N. It turns out that the associated twisted diagram of groups is isomorphic to $\mathcal{F}$. Therefore, twisted diagram of groups $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ is developable.

Graphs of groups and complexes of groups The Bass-Serre theory of graphs of groups analyzes the algebraic structure of groups acting by automorphisms on simplicial trees. It was formalized by J.P.Serre in [S]. The theory relates group actions on trees with decomposing groups as iterated applications of the operations of free product with amalgamation and HNN extension, via the notion of the fundamental group of a graph of groups. To every graph of groups $\mathcal{G}$, one can associate a Bass-Serre covering tree $\widetilde{X}$, which is a tree that comes equipped with a natural group action of the fundamental group. Moreover, the quotient graph of groups is isomorphic to $\mathcal{G}$. The fundamental theorem of this theory says that if G acts on a tree $\widetilde{X}$ and $\mathcal{G}$ is the associated graph of groups then G is isomorphic to the fundamental group of $\mathcal{G}$.

The theory of complexes of groups provides a higher-dimensional generalization of Bass-Serre theory. One can define an analogue of the fundamental group of a graph of groups for a complex of groups. However, in order for this notion to have good algebraic properties (such as embeddability of the local groups in it) and in order for a good analogue of the notion of the Bass-Serre covering tree to exist in this context, one needs to require some sort of "non-positive curvature" condition. For details cf. Corson [C] and Stallings [St].

The classifying category of a twisted diagram of groups Each twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ yields a small category $\mathcal{B}\mathcal{G}$ called the classifying category of $\mathcal{G}$. It is a special case of the Grothendieck construction defined by Thomason [T]. Roughly speaking it is a small category "generated" by $\mathcal{C}$ and the local groups $\{\mathcal{G}(c)\}_{c\in \mathrm{Ob}\,\mathcal{C}}$ as automorphisms of objects. In particular there exists a projection $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ which is a bijection on objects set. If $\mathcal{G}$ is a complex of groups, then $\mathcal{B}\mathcal{G}$ is the classifying category defined by Haefliger ([H1], [B-H]). If a twisted diagram of groups is a group G then its classifying category is $\mathcal{B}G$. Assume that $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ is a twisted diagram associated to an extension $N \rightarrowtail \widetilde{G} \overset{\eta}{\twoheadrightarrow} G$. Then the classifying category of $\mathcal{F}$ is isomorphic to $\mathcal{B}\widetilde{G}$.

Assume that a twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a complex of groups associated to an action of a group G on a simply connected simplicial complex $\widetilde{X}$. Then the geometric realization of the classifying category $\mathcal{B}\mathcal{G}$ is homotopy equivalent to the Borel construction $\mathrm{E}\,G \times_G \widetilde{X}$ where $\mathrm{E}\,G$ is the universal covering of the Eilenberg-MacLane space $\mathrm{B}\,G$.

Fundamental group According to Haefliger ([H1], [B-H]) the fundamental group of a complex of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is the group generated by the local groups and the fundamental group of the small category $\mathcal{C}$. The main theorem of the theory of complexes of groups says that a complex of groups is developable if and only if the local groups inject into the fundamental group. This theorem carries over to twisted diagrams of groups.

It turns out that the fundamental group of a complex of groups $\mathcal{G}$ is isomorphic to the fundamental group of the classifying category of $\mathcal{G}$. Hence we define the fundamental group of a twisted diagram of groups to be the fundamental group of its classifying category. Let $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups associated to an extension $N \rightarrowtail \widetilde{G} \overset{\eta}{\twoheadrightarrow} G$. Then the fundamental group of $\mathcal{F}$ is isomorphic to $\widetilde{G}$. In case of a diagram of groups of the form $G_1 \longleftarrow H \longrightarrow G_2$ its fundamental group is isomorphic to the push-out of the diagram, i.e. its direct limit. We prove in Theorem 4.2.13 that the fundamental group of a diagram of groups $F : \mathcal{C} \longrightarrow \mathrm{Gr}$ is isomorphic to its direct limit if and only if $\mathcal{C}$ is simply connected.

Therefore the notion of the fundamental group of a twisted diagram of groups provides a unified approach to direct limit and extension of groups.

Classification of epimorphisms of twisted diagrams of groups One of the main results of the thesis is the classification of epimorphisms of twisted diagrams of groups. We say that a homomorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ over $\mathcal{C}$ is an epimorphism if each local homomorphism $\varphi_c : \widetilde{\mathcal{G}}(c) \longrightarrow \mathcal{G}(c)$ is surjective.

We will extend the classical relation between group cohomology and extensions of groups from single groups to twisted diagrams of groups; in particular complexes of groups introduced by Haefliger [H1]. We begin with a description of the classical situation in the way suitable for generalizations. For detailed discussion cf. [B2], [R].

Assume G, N are discrete groups and let $\phi : G \longrightarrow \mathrm{Out}(N) := \mathrm{Aut}(N, N)/\,\mathrm{Inn}(N)$ be a homomorphism. One asks whether ϕ comes from an extension $N \rightarrowtail \widetilde{G} \twoheadrightarrow G$. It is the case if certain obstruction element $o(\phi) \in H^3(G; Z(N))$ vanishes, where $Z(N)$ is the center of N. Then equivalence classes of extensions are in bijective correspondence with elements of $H^2(G; Z(N))$ or equivalently with twisted actions of G on N, where the twisting is defined by the corresponding cocycle.

Let Rep denote the category whose objects are groups but morphisms are reprezentations i.e. $\mathrm{Mor}_{\mathrm{Rep}}(G,H) := \mathrm{Hom}(G,H)/\operatorname{Inn}(H)$. Then any twisted diagram of groups composed with projection $\mathrm{Gr} \longrightarrow \mathrm{Rep}$ gives a strict functor to the category Rep. Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $F : \mathcal{BG} \longrightarrow \mathrm{Rep}$ be a functor. Then there exists a certain abelian module $Z_F : \mathcal{BG} \longrightarrow \mathcal{A}b$ and the classification theorem takes the following form:

Theorem 3.4.6 *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and let $F : \mathcal{BG} \longrightarrow \mathrm{Rep}$ be a functor. If an obstruction element $o(F) \in H^3(\mathcal{BG}; Z_F)$ vanishes then there is an epimorphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that the corresponding twisted diagram $\mathcal{BG} \longrightarrow \mathrm{Gr}$ is a lifting of F. Moreover, set of equivalence classes of such liftings is in a natural bijective correspondence with the elements of $H^2(\mathcal{BG}; Z_F)$.*

Observe that if $\mathcal{G}$ is a group then the Theorem reduces to the classical case described above.

Extension of twisted diagrams of groups If $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is an epimorphism of twisted diagrams of groups then for each object $c \in \mathrm{Ob}\,\mathcal{C}$ we obtain an extension $N_c \rightarrowtail \widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c)$. A natural question arises; can we define a "kernel" twisted diagram of groups $\mathcal{N} : \mathcal{C} \longrightarrow \mathrm{Gr}$ such that $\mathcal{N}(c) = N_c$ for each $c \in \mathrm{Ob}\,\mathcal{C}$? The answer turns out to be a bit complicated, in particular it may happen that $\mathcal{N}$ does not exist. In this case we obtain only a "presheaf" of groups on the small category $\mathcal{C}$.

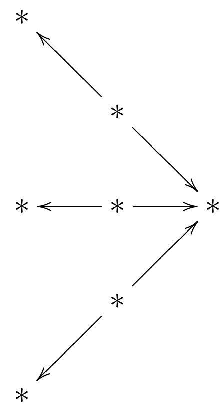
Let $\mathcal{N}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and let $\phi : \mathcal{N}' \longrightarrow \widetilde{\mathcal{G}}$ over $F : \mathcal{C}' \longrightarrow \mathcal{C}$ be a homomorphism of twisted diagrams of groups. What does it mean that the composition $\varphi \circ \phi$ is trivial? Two interpretations are possible. We say that $\varphi \circ \phi$ is trivial on the local groups if for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ the local homomorphism $(\varphi \circ \phi)_{c'} : \mathcal{N}(c') \longrightarrow \mathcal{G}(F(c'))$ is trivial. We prove that an epimorphism of twisted diagrams of groups yields an epimorphism of fundamental groups of these twisted diagrams. This justifies second interpretation, we say that $\varphi \circ \phi$ is trivial on the fundamental groups if the induced homomorphism $(\varphi \circ \phi)_* : \pi_1(\mathcal{N}, c') \longrightarrow \pi_1(\mathcal{G}, F(c'))$ is trivial.

"Categorical" kernel of an epimorphism of twisted diagrams of groups Let $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ be an epimorphism of twisted diagrams of groups. In order to construct the "kernel" twisted diagram we prove that there is one to one correspondece between the equivalence classes of extentions $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ and equivalence classes of twisted diagrams of groups on $\mathcal{BG}$ (Theorem 3.4.4). In particular for each epimorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ there exists a twisted diagram $\mathcal{F}_\varphi : \mathcal{BG} \longrightarrow \mathrm{Gr}$ such that $\mathcal{F}_\varphi(c) = \ker\big(\widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c)\big)$. Moreover, there exists a homomorphism of twisted diagrams of groups $\psi : \mathcal{F}_\varphi \longrightarrow \widetilde{\mathcal{G}}$ such that the composition $\varphi \circ \psi$ is trivial on the local groups. It turns out that $\mathcal{F}_\varphi$ satisfies the following universal property; for each twisted diagram of groups $\mathcal{F}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$ and a homomorphism $\psi' : \mathcal{F}' \longrightarrow \widetilde{\mathcal{G}}$ such that $\varphi \circ \psi'$ is trivial on the local groups, there exists a unique homomorphism $\bar{\psi}' : \mathcal{F}' \longrightarrow \mathcal{F}_\varphi$ such that $\psi \circ \bar{\psi}' = \psi'$.

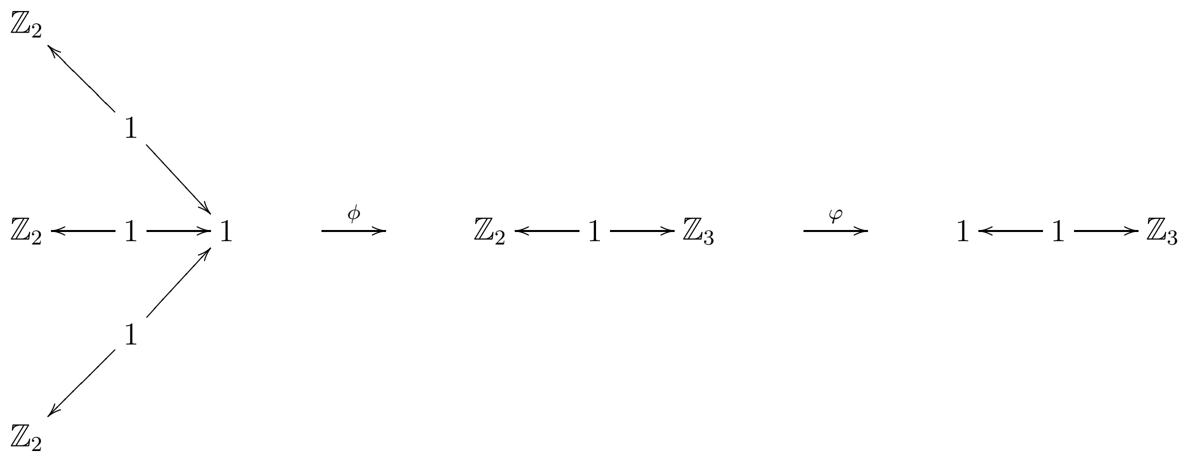
Therefore $\mathcal{F}_\varphi : \mathcal{BG} \longrightarrow \mathrm{Gr}$ is an analogue of the categorical kernel of the epimorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$.

Kernel of an epimorphism of twisted diagrams of groups - second approach
Let $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ be an epimorphism of twisted diagrams of groups. Our goal is to construct a twisted diagram of groups $\mathcal{K} : \mathcal{D} \longrightarrow \mathrm{Gr}$ and a homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ such that $\varphi \circ \phi$ is trival on the local groups and on the fundamental groups.

Consider the following example. Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a diagram of groups $1 \longleftarrow 1 \longrightarrow \mathbb{Z}_3$ defined on a small category $\mathcal{C} = * \longleftarrow * \longrightarrow *$. It is a graph of groups and its Bass-Serre covering tree is the geometric realization of a small category $\mathcal{D}$ given by



Clearly $\mathbb{Z}_3$ acts on $\mathcal{D}$ with a quotient $\mathcal{C}$. Consider the following sequence of graphs of groups:



We obtain an extension of the fundamental groups

$$\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrowtail \mathbb{Z}_2 * \mathbb{Z}_3 \twoheadrightarrow \mathbb{Z}_3$$

and for each $d \in \mathrm{Ob}\,\mathcal{D}$ we obtain an extension of the local groups.

Assume $\widetilde{\mathcal{G}}, \mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ are twisted diagrams of groups. We prove (Theorem 6.3.7) that for each epimorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ there exists a twisted diagram of groups $\mathcal{K} : \mathcal{D} \longrightarrow \mathrm{Gr}$ and a homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ such that $\varphi \circ \phi$ is trivial on the local groups and on the fundamental groups. Moreover ϕ is final for homomorphisms $\phi' : \mathcal{K}' \longrightarrow \widetilde{\mathcal{G}}$ satisfying these properties, i.e. for each twisted diagram of groups $\mathcal{K}' : \mathcal{D}' \longrightarrow \mathrm{Gr}$ and a homomorphism $\phi' : \mathcal{K}' \longrightarrow \widetilde{\mathcal{G}}$ such that $\varphi \circ \phi'$ is trivial on the local and fundamental groups there exists a unique homomorphism $\bar{\phi}' : \mathcal{K}' \longrightarrow \mathcal{K}$ such that $\phi\bar{\phi}' = \phi'$.

Furthermore, the sequence of homomorphisms

$$\mathcal{K} \rightarrowtail \widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$$

yields an extension of the fundamental groups and for each object $d \in \mathrm{Ob}\,\mathcal{D}$ an extension of the local groups.

It turns out that the small category $\mathcal{D}$, on which $\mathcal{K}$ is defined, is strictly related to the twisted diagram of groups $\mathcal{G}$. In particular, if $\mathcal{G}$ is a graph of groups then the geometric realization of $\mathcal{D}$ is the Bass-Serre tree of $\mathcal{G}$.

Structure of the thesis Chapter 1 collects some of the important notions and constructions concerning weak functors. The main source for this chapter is Thomason's paper [T].

In Chapter 2 some of the basics of the theory of twisted diagrams of groups are presented. Section 2.1 provides the definitions of twisted diagrams of groups, homomorphism and equivalence of twisted diagrams of groups.

Section 2.2. introduces the classifying category of a twisted diagram of groups. We present the properties of the classifying category which will be usefull later. The classifying category of a twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is the Grothendieck construction of the corresponding weak functor. Therefore it comes with the projection $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ on the small category $\mathcal{C}$. Observe that for a single group G this projection is simply $\mathcal{B}G \longrightarrow *$. Theorem 2.2.9 says, that a small category $\mathcal{D}$ and a functor $p : \mathcal{D} \longrightarrow \mathcal{C}$ are associated to a twisted diagram of groups as its classifying category and the corresponding projection, if and only if p satisfies certain properties. In particular if p is associated to a twisted diagram of groups then the preimage $p^{-1}(c) \subset \mathcal{D}$ of each object c of $\mathcal{C}$ is isomorphic to a group. Moreover the small category $\mathcal{D}$ is "generated" by $\mathcal{C}$ and these groups.

We prove in Theorem 2.2.13 that the category of twisted diagrams of groups is equivalent to the category of functors satisfying assertions of Theorem 2.2.9.

Chapter 3 presents the classification of epimorphisms of twisted diagrams of groups. It starts with a definition of the cohomology groups of a small category $\mathcal{C}$ with coefficients in an abelian module $M : \mathcal{C} \longrightarrow \mathcal{A}b$. If $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ is a twisted diagram of groups with values in the category of abelian groups then we can forget about the twisting elemetns and obtain an abelian module $|\mathcal{F}| : \mathcal{C} \longrightarrow \mathcal{A}b$. We prove in Section 3.1 that the elements of the second cohomology group $H^2(\mathcal{C}; M)$ are in one to one correspondence with equivalence classes of twisted diagrams of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ such that $|\mathcal{F}| = M$.

As we have already observed, any twisted diagram of group composed with the projection $\mathrm{Gr} \longrightarrow \mathrm{Rep}$ gives a strict functor $\mathcal{C} \longrightarrow \mathrm{Rep}$. The natural question is when a functor $F : \mathcal{C} \longrightarrow \mathrm{Rep}$ lifts to a twisted diagram of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and how many such liftings exist? Section 3.2 answers this question and the answer is given in terms of cohomology of the small category $\mathcal{C}$ with coefficients in a certain abelian module $Z_F : \mathcal{C} \longrightarrow \mathcal{A}b$ (3.2.4) associated to F. This abelian module generalizes the notion of the center of a group, in particular for each $c \in \mathrm{Ob}\,\mathcal{C}$ we have $Z_F(c) \subset Z(F(c))$.

Theorem 3.2.5 *To every functor $F : \mathcal{C} \longrightarrow \mathrm{Rep}$ one assignes in a natural way an obstruction element $o(F) \in H^3(\mathcal{C}; Z_F)$ such that $o(F)$ vanishes if and only if the functor F has a lifting to a twisted diagram of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$. Moreover the equivalence classes of such liftings are in bijective correspondence with elements of the group $H^2(\mathcal{C}; Z_F)$.*

Section 3.3 concerns the case when $\mathcal{C}$ is a category defined by a group G. Then Theorem 3.2.5 reduces to the classical case.

The folowing theorem establishes the relation between the surjective homomorphisms of twisted diagrams of groups and twisted diagrams of groups defined on the classifying category of a twisted diagram of groups $\mathcal{BG}$.

Theorem 3.4.4 *There is a natural bijective correspondence between equivalence classes of epimorphisms $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ and equivalence classes of twisted diagrams defined on the category $\mathcal{BG}$.*

Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $F : \mathcal{BG} \longrightarrow \mathrm{Rep}$ be a functor. Does there exist an epimorphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that the corresponding twisted diagram $\mathcal{BG} \longrightarrow \mathrm{Gr}$ is a lifting of F? Theorem 3.4.6 is a straightforward corollary from Theorems 3.2.5 and 3.4.4.

Theorem 3.4.6 *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and let $F : \mathcal{BG} \longrightarrow \mathrm{Rep}$ be a functor. If an obstruction element $o(F) \in H^3(\mathcal{BG}; Z_F)$ vanishes then there is an epimorphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that the corresponding twisted diagram $\mathcal{BG} \longrightarrow \mathrm{Gr}$ is a lifting of F. Moreover, set of equivalence classes of such liftings is in a natural bijective correspondence with the elements of $H^2(\mathcal{BG}; Z_F)$.*

Note that Theorem 3.4.6 contains as special cases theorems of Haefliger concerning extensions of complexes of groups with abelian kernels [Thm. 5.2. H2] and with locally constant (not necessary abelian) kernels [Thm. 6.3. H2]. The reason that we can provide a unified approach to those result, and prove a more general theorem is that we consider twisted diagrams over arbitrary small categories, also with loops, wheras Haefliger works with complexes defined on small categories without loops.

Chapter 4 concerns the notion of the fundamental group of a twisted diagram of groups.

Section 4.1 is devoted to introductory material and basic definitions concerning fundamental group of a small category. The fundamental group of a small category is defined as the fundamental group of the geometric realization of the given category. To each category $\mathcal{C}$ one can assign a certain grupoid $\pi\mathcal{C}$ called the fundamental grupoid of $\mathcal{C}$. If the geometric realization of $\mathcal{C}$ is connected then the fundamental grupoid and the fundamental group are equivalent small categories.

We define the fundamental group of a twisted diagram of groups as the fundamental group of its classifying category. In Section 4.2 we prove that the fundamental group of a twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is generated by the local groups and the fundamental group of $\mathcal{C}$ (Theorem 4.2.7). We also prove (Proposition 4.2.6) that the natural projection $\mathcal{BG} \longrightarrow \mathcal{C}$ yields an epimorphism of fundamental groups.

Assume that a twisted diagram of groups is a functor $F : \mathcal{C} \longrightarrow \mathrm{Gr}$. Therefore there exists its direct limit $\operatorname{colim} F$. The natural question is how is $\operatorname{colim} F$ related to the fundamental group of F. Theorem 4.2.13 motivated by E.D. Farjoun [Fa] proves that the direct limit is the push-out of the following diagram

$$\begin{array}{ccc} \pi_1(\mathcal{C}, c_0) & \longrightarrow & \pi_1(F, c_0) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \operatorname{colim} F \end{array}$$

In particular the direct limit of $F : \mathcal{C} \longrightarrow \mathrm{Gr}$ is isomorphic to the fundamental group of F if and only if the geometric realization of $\mathcal{C}$ is simply connected.

Chapter 5 starts with the theory of coverings of small categories. A functor $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ is said to be a covering if its geometric realization $\mathrm{B}\,\phi : \mathrm{B}\,\mathcal{C} \longrightarrow \mathrm{B}\,\mathcal{C}'$ is a topological covering. D. Quillen in [Q] has proved that the category of topological coverings of $\mathrm{B}\,\mathcal{C}$ is equivalent to the category of morphism inverting functors $\mathcal{C} \longrightarrow \mathbf{Sets}$. In Theorem 5.1.7 we present a similar result, namely we prove that the category of coverings of the small category $\mathcal{C}$ is equivalent to the category of functors $\pi\mathcal{C} \longrightarrow \mathbf{Sets}$, where $\pi\mathcal{C}$ is the fundamental grupoid associated to $\mathcal{C}$. This implies that the category of coverings of $\mathcal{C}$ is equivalent to the category of topological coverings of $\mathrm{B}\,\mathcal{C}$.

We say that G acts on a small category $\mathcal{D}$ without inversion if for each $d \in \mathrm{Ob}\,\mathcal{D}$, $gd = d$ implies that g fixes each morphism $l : d \longrightarrow d'$ of $\mathcal{D}$. If the geometric realization of $\mathcal{D}$ is a Bass-Serre tree then this condition means that G does not inverse cells, hence the terminology. The projection $\mathcal{D} \longrightarrow \mathcal{D}/G$ induced by the action of G is so called right-covering. Note that if the action of G is free then clearly it is without inversion and the geometric realization of $\mathcal{D} \longrightarrow \mathcal{D}/G$ is a topological G-covering.

As we have observed an action without inversion of G on $\mathcal{D}$ yields a (developable) twisted diagram of groups $\mathcal{G} : \mathcal{D}/G \longrightarrow \mathrm{Gr}$. The main result of Section 5.3 says that there exists a functor $\mathcal{D} \longrightarrow \mathcal{B}\mathcal{G}$ and it is equivalent to a covering, i.e. there exist a small category $\mathcal{E}$ and a covering $\mathcal{E} \longrightarrow \mathcal{B}\mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D} & \overset{\approx}{\hookrightarrow} & \mathcal{E} \\ & \searrow & \downarrow \\ & & \mathcal{B}\mathcal{G} \end{array}$$

The action of G on $\mathcal{D}$ yields a functor $S_{\mathcal{D}} : G \longrightarrow \mathrm{Cat}$. The Grothendieck construction $\mathcal{B}S_{\mathcal{D}}$ of this functor is equivalent to the classifying category $\mathcal{B}\mathcal{G}$. Moreover the following diagram commutes

$$\begin{array}{ccccc} \mathcal{D} & \overset{\approx}{\hookrightarrow} & \mathcal{E} & \overset{\approx}{\hookrightarrow} & \mathcal{E}G \times \mathcal{D} \\ & \searrow & \downarrow{\scriptstyle /G} & & \downarrow{\scriptstyle /G} \\ & & \mathcal{B}\mathcal{G} & \overset{\approx}{\hookrightarrow} & \mathcal{B}S_{\mathcal{D}} \\ & & & & \downarrow \\ & & & & \mathcal{B}G \end{array}$$

Section 5.4 is devoted to generalization of the developability criterion given by Bridson and Haeflieger in [B-H]. They have proved that a complex of groups $\mathcal{G}$ is developable if and only if there exists a group G and a homomorphism $\Phi : \mathcal{G} \longrightarrow G$ which is injective on the local groups. This theorem carries over to twisted diagrams of groups.

In Chapter 6 we develope the theory of coverings of complexes of groups.

We say that a homomorphism $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ of twisted diagrams of groups is a covering if the associated functor $\mathcal{B}\phi : \mathcal{B}\mathcal{G}' \longrightarrow \mathcal{B}\phi$ is equal to the composition

$$\begin{array}{ccc} \mathcal{B}\mathcal{G}' & \overset{\approx}{\hookrightarrow} & \mathcal{E} \\ & \overset{\mathcal{B}\phi}{\searrow} & \downarrow \\ & & \mathcal{B}\mathcal{G} \end{array}$$

where $\mathcal{E} \longrightarrow \mathcal{BG}$ is a covering of small categories and $\mathcal{BG}' \overset{\approx}{\hookrightarrow} \mathcal{E}$ is an inclusion and an equivalence of small categories. For precise definition and examples see Section 6.1.

Section 6.2 concerns a special case of a covering of twisted diagrams of groups which is a covering of (Heaflieger's) complexes of groups. Each complex of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is locally developable, i.e. for each $c \in \mathrm{Ob}\,\mathcal{C}$ there exists a small category $\mathcal{D}_c$ with an action of the local group $\mathcal{G}(c)$ such that $\mathcal{D}_c/\mathcal{G}(c) \simeq \mathcal{C}/c$. We prove an analgoue of Theorem given in [B-H]. Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ over $F : \mathcal{C}' \longrightarrow \mathcal{C}$ be a homomorphism of complexes of groups. Theorem 6.2.9 proves that ϕ is a covering if and only if it is injective on the local groups and for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ the induced functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is an isomorphism.

For each twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ we define the universal covering of $\mathcal{G}$. It is a covering of twisted diagrams of groups $\hat{\phi} : \hat{\mathcal{G}} \longrightarrow \mathcal{G}$, such that $\mathcal{B}\hat{\mathcal{G}}$ is equivalent to the universal covering of the small category $\mathcal{BG}$. The main Theorem of Chapter 6 takes the following form

Theorem 6.3.7 *Let $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ over $\mathcal{C}$ be an epimorphism of twisted diagrams of groups. Let $\hat{\mathcal{G}} : \hat{\mathcal{D}} \longrightarrow \mathrm{Gr}$ over $\hat{p} : \hat{\mathcal{D}} \longrightarrow \mathcal{C}$ be the universal covering of $\mathcal{G}$. Then there exists a twisted diagram of groups $\mathcal{K} : \mathcal{B}\hat{\mathcal{G}} \longrightarrow \mathrm{Gr}$ and a homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ over $F : \mathcal{B}\hat{\mathcal{G}} \longrightarrow \mathcal{C}$ satisfying*

1. *$\varphi \circ \phi : \mathcal{K} \longrightarrow \mathcal{G}$ is trivial on the local groups*
2. *$(\varphi \circ \phi)_* : \pi_1(\mathcal{K}, d_0) \longrightarrow \pi_1(\mathcal{G}, F(d_0))$ is trivial*
3. *the homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ is a covering and it is final for homomorphisms $\phi' : \mathcal{K}' \longrightarrow \widetilde{\mathcal{G}}$ satisfying 1. and 2.*

It turns out that the sequence of homomorphisms

$$\mathcal{K} \rightarrowtail \widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$$

yields an extension of the fundamental groups and for each object $d \in \mathrm{Ob}\,\mathcal{B}\hat{\mathcal{G}}$ an extension of the local groups.

Chapter 1

Weak functors *à* la Thomason

This chapter is devoted to introductory material, basic definitions and some standard results.

Small categories – Let us recall that a small category $\mathcal{C}$ is a category whose morphisms form a set; if c and c' are objects of $\mathcal{C}$ and if l is a morphisms of c in c', namely belongs to $\mathrm{Mor}_{\mathcal{C}}(c, c')$, then c is denoted by $i(l)$ and c' by $t(l)$. Two morphisms l and l' are composable iff $t(l') = i(l)$. We shall often identify an object c of $\mathcal{C}$ with the identity morphism id_c of this object.

The idea of weak functors was firstly introduced by R.W.Thomason [T]. He considered "weak" functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ (he called them "op-lax functors") from an arbitrary small category to the category of small categories. Roughly speaking a weak functor or op-lax functor is something less then a functor. It has many properties similar to the properties of functors like for example the notion of a weak natural transformation or the Grothendieck construction.

1.1 The geometric realization of a small category

To every small category one assignes in a natural way a certain topological space. This space is called the geometric realization of this category and one constructs it by assigning a certain simplicial space. For a defintion of a smiplicial space cf. [D].

Definition 1.1.1. *Suppose that $\mathcal{C}$ is a small category. Consider the poset $\mathbf{n}$ as a category with one morphism $i \longrightarrow j$ is $i \leq j$, and no other morphisms. The singular complex or nerve $\mathrm{N}\,\mathcal{C}$ of $\mathcal{C}$ is the simplicial space given by*

$$(\mathrm{N}\,\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}(\mathbf{n}, \mathcal{C})$$

More explicitely, an n-simplex σ of $\mathrm{N}\,\mathcal{C}$ is just a composable sequence of n morphisms in $\mathcal{C}$

$$\sigma(0) \longrightarrow \sigma(1) \longrightarrow \ldots \longrightarrow \sigma(n)$$

Definition 1.1.2. *The geometric realization $\mathrm{B}\,\mathcal{C}$ of a small category $\mathcal{C}$ is the geometric realization (3.15 [D]) of the simplicial space $\mathrm{N}\,\mathcal{C}$.*

Remark 1.1.3. The topological space $\mathrm{B}\,\mathcal{C}$ is a CW complex whose p-cells are in one to one correspondence with the p-simplices of the nerve which are nondegenrate, i.e. such that none of the arrows is an identity map.

1.2 Weak functors

Definition 1.2.1. *An op-lax functor or a weak functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ consists of functions assigning:*

1. *to each object c of $\mathcal{C}$, a category $\mathcal{F}(c)$*
2. *to each morphism $l : c_1 \longrightarrow c_0$, a functor $\mathcal{F}(l) : \mathcal{F}(c_1) \longrightarrow \mathcal{F}(c_2)$*
3. *to each composable pair of morphisms $c_2 \xrightarrow{l_2} c_1 \xrightarrow{l_1} c_0$ in $\mathcal{C}$, a natural transformation $f_{l_1,l_2} : \mathcal{F}(l_1 l_2) \implies \mathcal{F}(l_1)\mathcal{F}(l_2)$*
4. *to each object c of $\mathcal{C}$ a natural transformation $f(c) : \mathcal{F}(\mathrm{id}_c) \implies \mathrm{id}_{\mathcal{F}(c)}$*

These must satisfy the conditions that for

$$c_3 \xrightarrow{l_3} c_2 \xrightarrow{l_2} c_1 \xrightarrow{l_1} c_0$$

the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{F}(l_1 l_2 l_3) & \xRightarrow{f_{l_1,l_2 l_3}} & \mathcal{F}(l_1)\mathcal{F}(l_2 l_3) \\
\Big\Downarrow f_{l_1 l_2, l_3} & & \Big\Downarrow \mathcal{F}(l_1) f_{l_2,l_3} \\
\mathcal{F}(l_1 l_2)\mathcal{F}(l_3) & \xRightarrow[f_{l_1,l_2}\mathcal{F}(l_3)]{} & \mathcal{F}(l_1)\mathcal{F}(l_2)\mathcal{F}(l_3)
\end{array}$$

and that for $l_1 : c_1 \longrightarrow c_0$ the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{F}(l_1) & \xRightarrow{f_{\mathrm{id}_{c_0}, l_1}} & \mathcal{F}(\mathrm{id}_{c_0})\mathcal{F}(l_1) \\
\Big\Downarrow f_{l_1, \mathrm{id}_{c_1}} & \searrow^{\mathrm{id}_{\mathcal{F}(l_1)}} & \Big\Downarrow f(c_0) \\
\mathcal{F}(l_1)\mathcal{F}(\mathrm{id}_{c_1}) & \xRightarrow[f(c_1)]{} & \mathcal{F}(l_1)
\end{array}$$

Remark 1.2.2. Note that a functor $F : \mathcal{C} \longrightarrow \mathrm{Cat}$ is a weak functor with $f_{l_1,l_2} = \mathrm{id}$, $f(c) = \mathrm{id}$.

Definition 1.2.3. *For $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ a weak functor. The Grothendieck construction $\mathcal{BF}$ of $\mathcal{F}$ is a small category with objects the pairs (c, x) with c an object of $\mathcal{C}$ and x an object of $\mathcal{F}(c)$, and with morphisms $(l, f) : (c_1, x_1) \longrightarrow (c_0, x_0)$ given by a morphism $l : c_1 \longrightarrow c_0$ in $\mathcal{C}$ and $f : \mathcal{F}(l)(x_1) \longrightarrow x_0$ in $\mathcal{F}(c_0)$. Composition is defined by*

$$(l_1, f_1)(l_2, f_2) = (l_1 l_2, f_1 \mathcal{F}(l_1)(f_2) f_{l_1,l_2})$$

Remark 1.2.4. There is an obvious projection at the first coordinate functor $\pi : \mathcal{BF} \longrightarrow \mathcal{C}$.

Definition 1.2.5. *Given $\mathcal{F}, \mathcal{G}$, weak functors $\mathcal{C} \longrightarrow \mathrm{Cat}$, a weak natural transformation $\eta : \mathcal{F} \implies \mathcal{G}$ consist of functions assigning*

1. *to each object c in $\mathcal{C}$, a functor $\eta(c) : \mathcal{F}(c) \longrightarrow \mathcal{G}(c)$*

2. *to each $c_1 \xrightarrow{l_1} c_0$ in $\mathcal{C}$, a natural transformation $\eta(l_1) : \mathcal{G}(l_1)\eta(c_1) \implies \eta(c_0)\mathcal{F}(l_1)$ such that for $c_2 \xrightarrow{l_2} c_1 \xrightarrow{l_1} c_0$ we have $\eta(c_0)(f_{l_1,l_2})\eta(l_1 l_2) = \eta(l_1)\mathcal{G}(l_1)(\eta(l_2))g_{l_1,l_2}$ i.e. the following diagram commutes*

$$\begin{array}{ccccc} \mathcal{G}(l_1 l_2)\eta(c_2) & \xRightarrow{\eta(l_1 l_2)} & \eta(c_0)\mathcal{F}(l_1 l_2) & \xRightarrow{\eta(c_0)f_{l_1,l_2}} & \eta(c_0)\mathcal{F}(l_1)\mathcal{F}(l_2) \\ {\scriptstyle g_{l_1,l_2}}\Downarrow & & & & \Uparrow{\scriptstyle \eta(l_1)} \\ \mathcal{G}(l_1)\mathcal{G}(l_2)\eta(c_2) & & \xRightarrow[\mathcal{G}(l_1)\eta(l_2)]{} & & \mathcal{G}(l_1)\eta(c_1)\mathcal{F}(l_2) \end{array}$$

and for $c \in \operatorname{Ob}\mathcal{C}$ we have $\eta(c)f(c)\eta(\mathrm{id}_c) = g(c)\eta(c)$ i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{G}(\mathrm{id}_c)\eta(c) & \xRightarrow{\eta(\mathrm{id}_c)} & \eta(c)\mathcal{F}(\mathrm{id}_c) \\ & {\scriptstyle g(c)\eta(c)}\searrow & \Downarrow{\scriptstyle \eta(c)f(c)} \\ & & \eta(c) \end{array}$$

Remark 1.2.6. Note that a natural transformation $\eta : \mathcal{F} \longrightarrow \mathcal{G}$ of functors is a weak natural transformation with $\eta(l) = \mathrm{id}$. Given weak natural transformations $\mathcal{F} \implies \mathcal{G}$, $\mathcal{G} \implies \mathcal{H}$ there is an obvious composite weak natural transformation $\mathcal{F} \implies \mathcal{H}$. Thus for a fixed $\mathcal{C}$ we have a category of weak functors $\mathcal{C} \longrightarrow \mathrm{Cat}$ and weak natural transformations between such, $\text{Op-lax}(\mathcal{C}, \mathrm{Cat})$.

Definition 1.2.7. *Assume $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Cat}$ are op-lax functors and $\eta : \mathcal{F} \implies \mathcal{G}$ is a weak natural transformation. We define a functor $\mathcal{B}\eta : \mathcal{BF} \longrightarrow \mathcal{BG}$ on objects by $\mathcal{B}\eta(c, x) = (c, \eta(c)(x))$. For a morphism in $\mathcal{BF}$, $(l, f) : (c_1, x_1) \longrightarrow (c_0, x_0)$, we have a morphism in $\mathcal{G}(c_0)$,*

$$\mathcal{G}(l)\eta(c_1)(x_1) \overset{\eta(l)(x_1)}{\longrightarrow} \eta(c_0)\mathcal{F}(l)(x_1) \overset{\eta(c_0)(f)}{\longrightarrow} \eta(c_0)(x_0)$$

We set $\mathcal{B}\eta(l, f) = (l, \eta(c_0)(f)\eta(l)(x_1))$.

Remark 1.2.8. One notes that 1.2.3 and 1.2.7 determine a functor $\mathcal{B} : \text{Op-lax}(\mathcal{C}, \mathrm{Cat}) \longrightarrow \mathrm{Cat}$.

The following proposition will be useful later. It says that the local equivalence of functors yields an eqivalence of the Grothendieck constructions.

Proposition 1.2.9. *Assume $F, G : \mathcal{C} \longrightarrow \mathrm{Cat}$ are functors and $\eta : F \implies G$ is a natural transformation such that for each object $c \in \operatorname{Ob}\mathcal{C}$ the corresponding functor $\eta(c) : F(c) \longrightarrow G(c)$ is an equivalence of categories. Then $\mathcal{B}\eta : \mathcal{B}F \longrightarrow \mathcal{B}G$ is an equivalence of categories.*

Proof. Let $(c, y) \in \operatorname{Ob}\mathcal{B}G$ where $c \in \operatorname{Ob}\mathcal{C}$ and $y \in \operatorname{Ob}G(c)$. There exists an object $y' \in \operatorname{Ob}G(c)$ such that $y' = \eta(c)(x)$ and y' is isomorphic to y in $G(c)$. Then $\mathcal{B}\eta(c, x) = (c, y')$ is isomorphic to (c, y) in $\mathcal{B}G$.

For a morphism in $\mathcal{B}F$, $(l, f) : (c_1, x_1) \longrightarrow (c_0, x_0)$, we have a morphism in $G(c_0)$,

$$G(l)\eta(c_1)(x_1) = \eta(c_0)F(l)(x_1) \overset{\eta(c_0)(f)}{\longrightarrow} \eta(c_0)(x_0)$$

We have $\mathcal{B}\eta(l, f) = (l, \eta(c_0)(f))$, hence $\mathcal{B}\eta$ yields an isomorphism $\operatorname{Mor}_{\mathcal{B}F}\big((c_1, x_1), (c_0, x_0)\big) \simeq \operatorname{Mor}_{\mathcal{B}G}\big(\mathcal{B}\eta(c_1, x_1), \mathcal{B}\eta(c_0, x_0)\big)$. Therefore $\mathcal{B}\eta : \mathcal{B}F \longrightarrow \mathcal{B}G$ is an equivalence of categories. ∎

Definition 1.2.10. *Assume $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor, and $\mathcal{F} : \mathcal{C}' \longrightarrow \mathrm{Cat}$ a weak functor. The pull-back weak functor $F^*\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ is a composition $\mathcal{F} \circ F : \mathcal{C} \longrightarrow \mathrm{Cat}$.*

Definition 1.2.11. *Assume $F : \mathcal{C} \to \mathcal{C}'$ is a functor. A homomorphism $\Phi : \mathcal{F} \to \mathcal{G}$ (over F) of weak functors $\mathcal{F} : \mathcal{C} \to \mathrm{Cat}$, $\mathcal{G} : \mathcal{C}' \to \mathrm{Cat}$ is a weak natural transformation $\eta : \mathcal{F} \implies F^*\mathcal{G}$.*

Proposition 1.2.12. *Let $\Phi : \mathcal{F} \to \mathcal{G}$ (over $F : \mathcal{C} \longrightarrow \mathcal{C}'$) be a homomorphism of weak functors. Then Φ yields a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{BF} & \xrightarrow{\Phi} & \mathcal{BG} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' \end{array}$$

Proof. The homomorphism $\Phi : \mathcal{F} \to \mathcal{G}$ (over F) of weak functors $\mathcal{F} : \mathcal{C} \to \mathrm{Cat}$, $\mathcal{G} : \mathcal{C}' \to \mathrm{Cat}$ is a weak natural transformation $\eta : \mathcal{F} \implies F^*\mathcal{G}$. This natural transformation yields a functor $\mathcal{B}\eta : \mathcal{BF} \longrightarrow \mathcal{B}F^*\mathcal{G}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{BF} & \xrightarrow{\mathcal{B}\eta} & \mathcal{B}F^*\mathcal{G} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

There exists a diagram

$$\begin{array}{ccc} \mathcal{B}F^*\mathcal{G} & \xrightarrow{F^*} & \mathcal{BG} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' \end{array}$$

given by $(l, f) \longrightarrow (F(l), f)$. Therefore we obtain

$$\begin{array}{ccc} \mathcal{BF} & \xrightarrow{\Phi} & \mathcal{BG} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' \end{array}$$

□

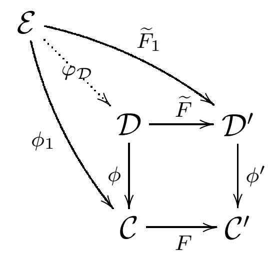
Proposition 1.2.13. *Consider functors $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and $\phi' : \mathcal{D}' \longrightarrow \mathcal{C}'$ of small categories. There exists a category $\mathcal{D}$ and a commutative diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\widetilde{F}} & \mathcal{D}' \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' \end{array}$$

such that for each commutative diagram of the form

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\widetilde{F}_1} & \mathcal{D}' \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi'} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{C}' \end{array}$$

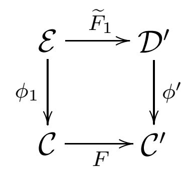
there exists exactly one functor $\varphi_{\mathcal{D}} : \mathcal{E} \longrightarrow \mathcal{D}$ such that the following diagram commutes



We say that $\mathcal{D}$ is the pull-back category of the given diagram.

Proof. We define objects of $\mathcal{D}$ to be the set of pairs (c, d') where $c \in \mathrm{Ob}\,\mathcal{C}$ and $d' \in \mathrm{Ob}\,\mathcal{D}'$ such that $F(c) = \phi'(d')$ and morphisms to be the set $\{(l, k') \;\; l \in \mathrm{Mor}\,\mathcal{C},\; k' \in \mathrm{Mor}\,\mathcal{D}', F(l) = \phi'(k')\}$. The functors $\widetilde{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ and $\phi : \mathcal{D} \longrightarrow \mathcal{C}$ are given by the natural projections.

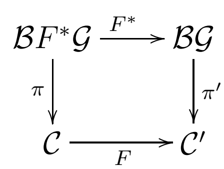
Assume that we have



For each $\widetilde{l} \in \mathrm{Mor}\,\mathcal{E}$ we define $\varphi_{\mathcal{D}} : \mathcal{E} \longrightarrow \mathcal{D}$ to be $\varphi_{\mathcal{D}}(\widetilde{l}) = (\phi_1(\widetilde{l}), \widetilde{F}_1(\widetilde{l}))$. Then $\phi\varphi_{\mathcal{D}} = \phi_1$ and $\widetilde{F}\varphi_{\mathcal{D}} = \widetilde{F}_1$. If there exists a functor $\varphi : \mathcal{E} \longrightarrow \mathcal{D}$ such that $\phi\varphi = \phi_1$ and $\widetilde{F}\varphi = \widetilde{F}_1$ then $\varphi = \varphi_{\mathcal{D}}$ on the set of morphisms thus $\varphi = \varphi_{\mathcal{D}}$.

□

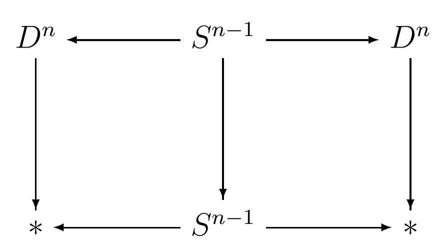
Proposition 1.2.14. *The category $\mathcal{B}F^*\mathcal{G}$ is the pull-back category of the diagram*



Proof. The set of morphisms of the category $\mathcal{B}F^*\mathcal{G}$ consists of pairs (l, f) where $l \in \mathrm{Mor}\,\mathcal{C}$ and $f \in \mathrm{Mor}\,\mathcal{G}(t(l'))$ such that $F(l) = l'$. Let $\mathcal{D}$ be the pull-back of the latter diagram. The isomorphism $\mathcal{B}F^*\mathcal{G} \longrightarrow \mathcal{D}$ is given by $(l, f) \longrightarrow (l, (F(l), f))$. □

1.3 Homotopy colimit and the functor associated to a weak functor

The construction of the homotopy colimit is motivated by the fact that ordinary colimits are not well-behaved with respect to weak equivalences. For instance, consider the following commutative diagram of topological spaces (wher D^n is the n-disk and S^{n-1} its boundary sphere).



All three vertical arrows are weak equivalences (even homotopy equivalences) but the colimit of the top row is homeomorphic to S^n, the colimit of the bottom row is a one-point space $*$, and the map $S^n \longrightarrow *$ induced by the diagram is not a weak equivalence.

Homotopy colimits on the other hand have a strong invariance property.

Definition 1.3.1. *For a (weak) functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ let $\mathrm{N}\,\mathcal{F}$ denote the nerve of a Grothendieck construction $\mathcal{B}\mathcal{F}$ and $\mathrm{B}\,\mathcal{F}$ its geometric realization.*

Assume that we have functors $F, F' : \mathcal{C} \longrightarrow \mathrm{Cat}$ and a natural transformation $\eta : F \implies F'$ such that it induces a homotopy equivalence $\mathrm{B}\,\eta(c) : \mathrm{B}\,F(c) \longrightarrow \mathrm{B}\,F'(c)$ for each object c of $\mathcal{C}$, then the geometric realization of the simplicial map $\mathrm{hocolim}\,\mathrm{N}\,\eta : \mathrm{hocolim}\,\mathrm{N}\,F \longrightarrow \mathrm{hocolim}\,\mathrm{N}\,F'$ is a homotopy equivalence.

The homotopy colimit construction is functorial, in the sense that a natural transformation $\eta : F \implies F'$ of functors $\mathcal{C} \longrightarrow \mathrm{Cat}$ induces a map

$$\mathrm{hocolim}\,\mathrm{N}\,\eta : \mathrm{hocolim}\,\mathrm{N}\,F \longrightarrow \mathrm{hocolim}\,\mathrm{N}\,F'$$

The homotopy colimit construction is also functorial in $\mathcal{C}$, in the sense that if $j : \mathcal{D} \longrightarrow \mathcal{C}$ is a functor and j^*F denotes the composite Fj, then there is a natural map $\mathrm{hocolim}\,\mathrm{N}(j^*F) \longrightarrow \mathrm{hocolim}\,\mathrm{N}\,F$.

The following theorems were proved by R.W. Thomason in [T].

Theorem 1.3.2. *(Homotopy colimit theorem). Let $F : \mathcal{C} \longrightarrow \mathrm{Cat}$ be a functor. The geometric realization of*

$$\phi : \mathrm{hocolim}\,\mathrm{N}\,F \longrightarrow \mathrm{N}\,F$$

is a homotopy equivalence.

We do not have a $\mathrm{hocolim}\,\mathrm{N}\,\mathcal{F}$ defined for a weak functor $\mathcal{F}$, so we cannot compare it to $\mathrm{N}\,\mathcal{F}$. Instead, we will naturaly associate a functor $\overline{\mathcal{F}} : \mathcal{C} \longrightarrow \mathrm{Cat}$ to the weak functor $\mathcal{F}$, and compare $\mathrm{N}\,\mathcal{F}$ to $\mathrm{N}\,\overline{\mathcal{F}}$ and $\mathrm{hocolim}\,\mathrm{N}\,\overline{\mathcal{F}}$.

Definition 1.3.3. *For a weak functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ we define a functor $\overline{\mathcal{F}} : \mathcal{C} \longrightarrow \mathrm{Cat}$. For each $c \in \mathrm{Ob}\,\mathcal{C}$, $\overline{\mathcal{F}}(c)$ is a category whose objects are (l, c', x'), $l : c' \longrightarrow c$ a morphism in $\mathcal{C}$ and x' an object of $F(c')$. A morphism $(l_1, f_1) : (l, c', x') \longrightarrow (l', c'', x'')$ is a $l_1 : c' \longrightarrow c''$ such that $l' l_1 = l$ and $f_1 : \mathcal{F}(l_1)(x') \longrightarrow x''$. For each morphism $l_1 : c_1 \longrightarrow c_0$ in $\mathcal{C}$ there is a natural transformation $\overline{\mathcal{F}} : \overline{\mathcal{F}}(c_1) \longrightarrow \overline{\mathcal{F}}(c_0)$ which assign $(l, c', x') \longrightarrow (l_1 l, c', x')$ and is the identity on morphisms.*

For a weak natural transformation $\eta : \mathcal{F} \implies \mathcal{G}$, $\overline{\eta} : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}$ is the natural transformation of functors induced by $\mathcal{B}\eta : \mathcal{B}\mathcal{F} \longrightarrow \mathcal{B}\mathcal{G}$.

Definition 1.3.4. *Let $\mathcal{C}$ be a small category. For any object $c \in \mathcal{C}$ one defines the left fibre category over c, denoted by $\mathcal{C}/c$, whose objects are pairs $(i(l), i(l) \xrightarrow{l} c)$. A morphism $(i(l_1), i(l_1) \xrightarrow{l_1} c) \longrightarrow (i(l_2), i(l_2) \xrightarrow{l_2} c)$ in $\mathcal{C}/c$ is a morphism $k : t(l_1) \longrightarrow t(l_2)$ for which the corresponding triangle over c commutes.*

For each $c \in \mathrm{Ob}\,\mathcal{C}$ there exists a natural projection $l_c : \mathcal{C}/c \longrightarrow \mathcal{C}$.

For $(c \xrightarrow{k} c') \in \mathrm{Mor}\,\mathcal{C}$ there is a natural functor $\mathcal{C}/k : \mathcal{C}/c \longrightarrow \mathcal{C}/c'$ given by $(c'', c'' \xrightarrow{l} c) \longrightarrow (c'', c'' \xrightarrow{kl} c')$.

Remark 1.3.5. Let $\mathcal{C}$ be a small category and $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$ a weak functor. Let $\overline{\mathcal{F}} : \mathcal{C} \longrightarrow \mathrm{Cat}$ be an associated functor defined in 1.3.3.

Let $l_c^*\mathcal{F} : \mathcal{C}/c \longrightarrow \mathrm{Cat}$ be a weak functor induced by $l_c : \mathcal{C}/c \longrightarrow \mathcal{C}$. Then the value of a functor $\overline{\mathcal{F}}(c)$ is isomorphic to a small category $\mathcal{B}l_c^*\mathcal{F}$.

Definition 1.3.6. *A weak natural transformation* $j : \mathcal{F} \longrightarrow \overline{\mathcal{F}}$ *is determined by the formula*

$$j(c) : \mathcal{F}(c) \longrightarrow \overline{\mathcal{F}}(c)$$

and is the functor sending x *to* (id_c, c, x) *and* $f_1 : x \longrightarrow x'$ *to* $(\mathrm{id}_c, f(c)(x')\mathcal{F}(\mathrm{id}_c)(f_1))$. *For* $l : c_1 \longrightarrow c_0$, $j(l) : \overline{\mathcal{F}}(l)j(c_1) \implies j(c_0)\mathcal{F}(l)$ *is the natural transformation with components given at* x *by* $(l, \mathrm{id}) : (l, c_1, x) \longrightarrow (\mathrm{id}, c_0, \mathcal{F}(l)(x))$.

These $j : \mathcal{F} \longrightarrow \overline{\mathcal{F}}$ *are such that for any weak natural transformation* $\eta : \mathcal{F} \longrightarrow \mathcal{G}$ *the diagram*

$$\begin{array}{ccc} \mathcal{F} & \overset{j}{\Longrightarrow} & \overline{\mathcal{F}} \\ {\scriptstyle \eta}\Downarrow & & \Downarrow{\scriptstyle \overline{\eta}} \\ \mathcal{G} & \underset{j}{\Longrightarrow} & \overline{\mathcal{G}} \end{array}$$

commutes.

Lemma 1.3.7. *For each* $c \in \mathrm{Ob}\,\mathcal{C}$, *the functor* $j(c) : \mathcal{F}(c) \longrightarrow \overline{\mathcal{F}}(c)$ *has a left adjoint* $i(c) : \overline{\mathcal{F}}(c) \longrightarrow \mathcal{F}(c)$ *sending* (l, c', x') *to* $\mathcal{F}(l)(x')$ *and* $(l_1, f) : (l, c', x') \longrightarrow (l', c'', x'')$ *to*

$$\mathcal{F}(l)(x') = \mathcal{F}(l'l_1)(x') \xrightarrow{f_{l',l_1}} \mathcal{F}(l')\mathcal{F}(l_1)(x') \xrightarrow{\mathcal{F}(l')(f)} \mathcal{F}(l')(x'')$$

Theorem 1.3.8. *For a weak functor* $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$, *we have a diagram*

$$\mathrm{N}\,\mathcal{F} \xrightarrow{\mathrm{N}_{\mathcal{C}}\, j} \mathrm{N}\,\overline{\mathcal{F}} \longrightarrow \mathrm{hocolim}\,\mathrm{N}\,\overline{\mathcal{F}}$$

Its geometric realization is a diagram of natural homotopy equivalnces.

Corollary 1.3.9. *Assume* $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Cat}$ *are weak functors and* $\eta : \mathcal{F} \implies \mathcal{G}$ *is a weak natural transformation. If for each* $c \in \mathrm{Ob}\,\mathcal{C}$, *the geometric realization of* $\eta(c)$ *is a homotopy equivalence, then* $\mathrm{B}\,\eta : \mathrm{B}\,\mathcal{F} \longrightarrow \mathrm{B}\,\mathcal{G}$ *is a homotopy equivalence.*

Chapter 2

Twisted diagrams of groups

Section 2.1 of this chapter describes weak functors which take values in the category of groups. We will call them twisted diagrams of groups, because of the twisting elements which corresponds to the natural transformations associated to the given weak functor. We will also explain when a given twisted diagram of groups determines a complex of groups defined by Bridson and Haeflieger in [B-H](Proposition 2.1.4).

The Grothendieck category of a twisted diagram of groups is a generalization of the classifying category of a complex of groups defined in [B-H]. Some important properties of this category will be described in Section 2.2. We will prove that for a given small category $\mathcal{C}$, the category of twisted diagrams of groups defined on $\mathcal{C}$ is equivalent to the category of functors $p : \mathcal{D} \longrightarrow \mathcal{C}$ satisfying certain properties (Theorems 2.2.9 and 2.2.13).

2.1 Twisted diagrams of groups

For a given subcategory of the category of small categories Cat, like for example groups, grupoids or EI-categories, we can consider weak functors which take values in this subcategory.

Every group G can be considered as a small category $\mathcal{B}G$ with a single object and morphisms corresponding to G.

From now on, a weak functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a weak functor such that for each object c of $\mathcal{C}$

$$\mathcal{F}(c) = \mathcal{B}G = \ast \circlearrowleft_{g \in G}$$

is a category defined by a group.

Definition 2.1.1. *A weak functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$ to the category of groups will be called a twisted diagram of groups.*

Proposition 2.1.2. *Assume $\Phi, \Psi : G \to H$ are homomorphisms of groups. We can consider these homomorphisms as functors $\mathcal{B}\Psi : \mathcal{B}G \to \mathcal{B}H$ and $\mathcal{B}\Phi : \mathcal{B}G \to \mathcal{B}H$ in the category of groups $\mathrm{Gr} \subset \mathrm{Cat}$. Let $\alpha : \mathcal{B}\Psi \implies \mathcal{B}\Phi$ be a natural transformation. Then α is a conjugation by some element of the group H.*

Proof. For each element $g \in G$ there is a commutative diagram

$$\begin{array}{ccc} *_H & \xrightarrow{\mathcal{B}\Psi(g)} & *_H \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ *_H & \xrightarrow[\mathcal{B}\Phi(g)]{} & *_H \end{array}$$

This implies $\Psi(g) = h\Phi(g)h^{-1}$ and $\alpha = \mathrm{Ad}(h^{-1})$. □

Proposition 2.1.3. *A twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is given by*

1. *for each object $c \in \mathrm{Ob}\,\mathcal{C}$ a group $\mathcal{G}(c)$*
2. *for each morphism $l : c \longrightarrow c' \in \mathrm{Mor}\,\mathcal{C}$ a homomorphism of groups $\mathcal{G}(l) : \mathcal{G}(c) \longrightarrow \mathcal{G}(c')$*
3. *for two composable morphisms $l, l' \in \mathrm{Mor}\,\mathcal{C}$ an element $g_{l,l'} \in \mathcal{G}(t(ll')) = \mathcal{G}(t(l))$, called the twisting element, such that*
   i) $\mathrm{Ad}(g_{l,l'})\mathcal{G}(ll') = \mathcal{G}(l)\mathcal{G}(l')$
   ii) $\mathcal{G}(l)(g_{l',l''})g_{l,l'l''} = g_{l,l'}g_{ll',l''}$ *for each triple* $\cdot \xrightarrow{l''} \cdot \xrightarrow{l'} \cdot \xrightarrow{l} \cdot \in \mathrm{Mor}\,\mathcal{C}$ *of composable morphisms (cocycle condition)*
4. *for each object $c \in \mathrm{Ob}\,\mathcal{C}$ an element $g(c) \in \mathcal{G}(c)$ such that $\mathcal{G}(\mathrm{id}_c) = \mathrm{Ad}(g(c))$ and for $l \in \mathrm{Mor}\,\mathcal{C}$, $i(l) = c$, $t(l) = c'$*
   i) $g(c') = g^{-1}_{\mathrm{id}_{c'},l}$
   ii) $\mathcal{G}(l)(g(c)) = g^{-1}_{l,\mathrm{id}_c}$

Proof. Follows directly from the defintion of a weak functor and Proposition 2.1.2. □

Proposition 2.1.4. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups. Assume that for each morphism $l \in \mathrm{Mor}\,\mathcal{C}$ the given homomorphism of groups is a monomorphism, for each object $c \in \mathrm{Ob}\,\mathcal{C}$ an element $g(c)$ is trivial and $\mathcal{C}$ is a small category without loops, that is such a category whose endomorphisms are identities of objects. Then $\mathcal{G}$ determines a complex of groups defined by Haeflieger ([H1], [B-H]).*

Proof. The proof is straightforward. Let $\mathcal{C}$ be a small category without loops (*scwol*) and $\{G_\sigma, \psi_l, g_{l,l'}\}$ a complex of groups defined on it. We put $\mathcal{G}(c) = G_c$, $\mathcal{G}(l) = \psi_l$ and the twisting elements of this twisted diagram are the twisting elements of the complex of groups. □

We define a homomorphism of twisted diagrams of groups as a weak natural transformation of the corresponding weak functors. Therefore

Proposition 2.1.5. *A homomorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ of twisted diagrams of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and $\mathcal{G}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$ over a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is given by the following data:*

1. *for each $c \in \mathrm{Ob}\,\mathcal{C}$ a homomorphism $\phi_c : \mathcal{G}(c) \longrightarrow \mathcal{G}'(F(c))$ called the local homomorphism*

2. *for each* $l \in \operatorname{Mor}\mathcal{C}$ *an element* $\phi(l) \in \mathcal{G}'(F(t(l)))$ *such that*

   (i) $\operatorname{Ad}(\phi(l))\mathcal{G}'(F(l))\phi_{i(l)} = \phi_{t(l)}\mathcal{G}(l)$

   (ii) $\phi_{t(l)}(g_{l,l'})\phi(ll') = \phi(l)\mathcal{G}'(F(l))(\phi(l'))g'_{F(l),F(l')}$ *for two composable* $l, l' \in \operatorname{Mor}\mathcal{C}$

Proof. A homomorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ of twisted diagrams of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and $\mathcal{G}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$ over $F : \mathcal{C} \longrightarrow \mathcal{C}'$ considered as a weak natural transformation of weak functors is given by:

1. for each $c \in \operatorname{Ob}\mathcal{C}$ a homomorphism $\phi_c : \mathcal{G}(c) \longrightarrow \mathcal{G}'(F(c))$

2. for each $l \in \operatorname{Mor}\mathcal{C}$ a natural transformation $\mathcal{G}'(F(l))\phi_{i(l)} \Longrightarrow \phi_{t(l)}\mathcal{G}(l)$

satisfying properties from 1.2.11. Then using 2.1.2 we obtain the latter equations. □

We say that homomorphism ϕ is simple if for each $l \in \operatorname{Mor}\mathcal{C}$ the element $\phi(l)$ is trivial.

We will often denote the homomoprhism ϕ as a pair $\phi = (\phi_c, \phi(l))$.

Remark 2.1.6. If F is an isomorphism and ϕ_c is an isomorphism for every $c \in \operatorname{Ob}\mathcal{C}$, then ϕ is called an isomorphism.

Definition 2.1.7. *Assume that* $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ *and* $\mathcal{G}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ *are twisted diagrams of groups. We say that* $\mathcal{G}$ *and* $\mathcal{G}'$ *are equivalent if there exists an isomorphism over the identity of* $\mathcal{C}$

$$\phi : \mathcal{G} \longrightarrow \mathcal{G}'$$

Proposition 2.1.8. *We say that twisted diagrams* $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ *and* $\mathcal{G}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ *differ by a coboundary* $\{g_l\}_{l \in \operatorname{Mor}\mathcal{C}}$ *if for each* $c \in \operatorname{Ob}\mathcal{C}$ *the corresponding groups are equal*

$$\mathcal{G}'(c) = \mathcal{G}(c)$$

and for each $l \in \operatorname{Mor}\mathcal{C}$ *there exists an element* $g_l \in \mathcal{G}(t(l))$ *such that*

$$\mathcal{G}'(l) = \operatorname{Ad}(g_l) \circ \mathcal{G}(l)$$

and the twisting elelements satisfy

$$g'_{l,l'} = g_l\mathcal{G}(l)(g_{l'})g_{l,l'}g_{ll'}^{-1}$$

Assume that $\mathcal{G}$ *and* $\mathcal{G}'$ *differ by a coboundary. Then* $\mathcal{G}$ *and* $\mathcal{G}'$ *are equivalent.*

Proof. Assume that $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and $\mathcal{G}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ are twisted diagrams of groups which differ by a coboundary. Then for each $c \in \operatorname{Ob}\mathcal{C}$ we have $\mathcal{G}(c) = \mathcal{G}'(c)$ and for each $l \in \operatorname{Mor}\mathcal{C}$ there exists an element g_l of the group $\mathcal{G}(t(l))$ such that $\mathcal{G}'(l) = \operatorname{Ad}(g_l) \circ \mathcal{G}(l)$. The twisting elements satisfy $g'_{l,l'} = g_l\mathcal{G}(g_{l'})g_{l,l'}g_{ll'}^{-1}$. Then we can define $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ as follows

$$\phi_c = \mathrm{id}_{\mathcal{G}(c)}$$

and

$$\phi(l) = g_l^{-1}$$

According to 2.1.5 ϕ is a well defined isomorphism of twisted diagrams of groups, over the identity of $\mathcal{C}$. Thus $\mathcal{G}$ and $\mathcal{G}'$ are equivalent. □

Corollary 2.1.9. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups. Then $\mathcal{G}$ is equivalent to a twisted diagram $\mathcal{G}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ such that for each object $c \in \mathcal{C}$ the corresponding element $g(c) \in \mathcal{G}(c)$ is trivial.*

Proof. The proof is straightforward. For each $c \in \mathrm{Ob}\,\mathcal{C}$ choose $g_{\mathrm{id}_c} = g(c)^{-1}$. □

From now on we will assume that a given twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ satisfies a normalizing condition, i.e.

$$1 = g(c) \in \mathcal{G}(c)$$

We will end this section with an example of a twisted diagram of groups on a small category $\mathcal{B}G$.

Example 2.1.10. *Extension of groups*
Let $N \overset{\iota}{\rightarrowtail} \widetilde{G} \overset{\eta}{\twoheadrightarrow} G$ be an extension of groups and let $\mathcal{B}G$ be a category defined by a group G. Choose any set theoretical section $s : G \longrightarrow \widetilde{G}$ (not necessarily a homomorphism). We define a twisted diagram of groups

$$\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$$

as follows; Assign to the single object of the category $\mathcal{B}G$ a group N:

$$\mathcal{F}(*) := N$$

To each morphism $g \in \mathrm{Mor}\,\mathcal{B}G = G$ assign an automorphism of the group N given by

$$\mathcal{F}(g) : N \longrightarrow N = N \simeq \iota(N) \overset{\mathrm{Ad}(s(g))}{\longrightarrow} \iota(N) \simeq N$$

Note that for two elements $g_1, g_2 \in G$

$$\mathcal{F}(g_1 g_2) \neq \mathcal{F}(g_1)\mathcal{F}(g_2)$$

and differs by the conjugation with an element $s(g_1)s(g_2)s(g_1g_2)^{-1} \in \iota(N)$. We define

$$n_{g_1,g_2} = \iota^{-1}_{|N}\big(s(g_1)s(g_2)s(g_1g_2)^{-1}\big) \in N$$

and

$$n(*) = \iota^{-1}_{|N}\big(s(e)^{-1}\big) \in N$$

where $e \in G$ is the trivial element of the group G.

It is straightforward to check that for $g_1, g_2, g_3 \in G = \mathrm{Mor}\,\mathcal{B}G$ the corresponding twisting elements satisfy the cocycle condition. Moreover, for any $g \in G = \mathrm{Mor}\,\mathcal{B}G$ we have

$$n(*) = \iota^{-1}_{|N}\big(s(e)^{-1}\big) = \iota^{-1}_{|N}\big(s(g)s(g)^{-1}s(e)^{-1}\big) = n^{-1}_{\mathrm{id}_*,g}$$

and

$$\mathcal{F}(g)(n(*)) = \iota^{-1}_{|N}\big(s(g)s(e)^{-1}s(g)^{-1}\big) = n^{-1}_{g,\mathrm{id}_*}$$

Note that the twisting elements measure to what extend our section fails to be a homomorphism. Moreover the twisted diagram of groups $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ is equivalent to a diagram of groups if and only if the group $\widetilde{G}$ is isomorphic to a semidirect product of the groups N and G.

2.2 The classifying category of a twisted diagram of groups

The classifying category of a twisted diagram of groups is a generalization of the category defined by a group and on the other hand the classifying category of a complex of groups ([H1],[B-H]).

Definition 2.2.1. *The classifying category of a twisted diagram of groups is the Grothendieck construction of the corresponding weak functor.*

Remark 2.2.2. Let $\mathcal{C} = *$ be the category with one object and $\mathcal{G} : * \longrightarrow \mathrm{Gr}$ a twisted diagram of groups. Then the classifying category $\mathcal{B}\mathcal{G}$ is the classifying category of the group $\mathcal{G}(*)$. Let $\mathcal{I}_{\mathcal{C}} : \mathcal{C} \longrightarrow * \in \mathrm{Gr}$ be a diagram of groups such that $\mathcal{I}_{\mathcal{C}}(c) = 1$. Then $\mathcal{B}\mathcal{I}_{\mathcal{C}} \simeq \mathcal{C}$.

Remark 2.2.3. Let $\mathcal{C}$ be a category without loops and $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ a complex of groups. Then the Grothedieck construction $\mathcal{B}\mathcal{G}$ is the classifying category of a complex of groups defined by in [H1] and [B-H].

Example 2.2.4. Consider an extension of groups $N \rightarrowtail \widetilde{G} \overset{\eta}{\twoheadrightarrow} G$ and the twisted diagram of groups associated to it $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ described in Example 2.1.10. Then the classifying category of $\mathcal{F}$ is isomorphic to the category defined by the group $\widetilde{G}$

$$\mathcal{B}\mathcal{F} \simeq \mathcal{B}\widetilde{G}$$

The isomorphism is given by $(g, n) \longrightarrow ns(g)$. Moreover, the associated projection (1.2.4)

$$p : \mathcal{B}\mathcal{F} \longrightarrow \mathcal{B}G$$

equals

$$\mathcal{B}\eta : \mathcal{B}\widetilde{G} \longrightarrow \mathcal{B}G$$

The definition of a homomorphism of twisted diagrams of groups given in 2.1.5 is quite complicated. The notion of a classifying category will simplify it, namely

Remark 2.2.5. Assume that $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and $\mathcal{G}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$ are twisted diagrams of groups and $\phi = (\phi_c, \phi(l)) : \mathcal{G} \longrightarrow \mathcal{G}'$ is a homomorphism over $F : \mathcal{C} \longrightarrow \mathcal{C}'$. Then due to 1.2.12 we obtain a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{B}\mathcal{G} & \xrightarrow{\Phi} & \mathcal{B}\mathcal{G}' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \end{array}$$

given by $\Phi(l, g) = (F(l), \phi_{t(l)}(g)\phi(l))$.

Moreover each diagram of this form defines a homomorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ of twisted diagrams of groups given by $\phi_c = \Phi_{|\mathcal{G}(c)} : \mathcal{G}(c) \longrightarrow \mathcal{G}'(F(c))$. If $\Phi(l, 1) = (F(l), g')$ then we define $\phi(l) := g'$.

This implies

Corollary 2.2.6. *Assume that* $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ *and* $\mathcal{G}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ *are two twisted diagrams of groups. Then* $\mathcal{G}$ *and* $\mathcal{G}'$ *are equivalent if and only if there exists a homomorphism*

$$\phi : \mathcal{G} \longrightarrow \mathcal{G}'$$

such that the associated diagram is of the form

$$\begin{array}{ccc} \mathcal{B}\mathcal{G} & \xrightarrow{\Phi} & \mathcal{B}\mathcal{G}' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \mathcal{C} & \end{array}$$

and Φ *is an isomorphism.*

Definition 2.2.7. *We say that extensions* $N \rightarrowtail \widetilde{G} \twoheadrightarrow G$ *and* $N \rightarrowtail \widetilde{G}' \twoheadrightarrow G$ *are equivalent if the following diagram*

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\simeq} & \widetilde{G}' \\ & \searrow \quad \swarrow & \\ & G & \end{array}$$

commutes.

As a collorary we obtain the following:

Proposition 2.2.8. *Assume that* $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ *and* $\mathcal{F}' : \mathcal{B}G \longrightarrow \mathrm{Gr}$ *are twisted diagrams of groups associated to extensions of groups (2.1.10). Then* $\mathcal{F}$ *and* $\mathcal{F}'$ *are equivalent if and only if the corresponding extensions are equivalent.*

Proof. Let $\mathcal{F}$ be associated to $N \rightarrowtail \widetilde{G} \overset{s}{\leftrightarrows} G$ and $\mathcal{F}'$ to $N \rightarrowtail \widetilde{G}' \overset{s'}{\leftrightarrows} G$. Then the following diagrams commute

$$\begin{array}{ccc} \mathcal{B}\mathcal{F} & \xrightarrow{\simeq} & \mathcal{B}\widetilde{G} \\ & \searrow \quad \swarrow & \\ & \mathcal{B}G & \end{array} \qquad \begin{array}{ccc} \mathcal{B}\mathcal{F}' & \xrightarrow{\simeq} & \mathcal{B}\widetilde{G}' \\ & \searrow \quad \swarrow & \\ & \mathcal{B}G & \end{array}$$

Therefore, $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if and only if the corresponding extensions are equivalent. □

As we have observed, the twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ yields a projection $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$. Assume that for a small category $\mathcal{C}$ we are given a functor $p : \mathcal{D} \longrightarrow \mathcal{C}$. The natural question is, when the functor p is associated to a twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$? This question is answered in:

Theorem 2.2.9. *Let* $\mathcal{C}$ *be a small category. A category* $\mathcal{D}$ *and a functor* $p : \mathcal{D} \longrightarrow \mathcal{C}$ *is associated to a twisted diagram of groups as the classifying category of this twisted diagram of groups if and only if it satisfies the following conditions:*

1. $p : \mathcal{D} \longrightarrow \mathcal{C}$ *is a bijection on object sets and* p *is onto*
2. *The subcategory* $G_c^p := \{g \in \mathrm{Mor}_{\mathcal{D}}(c, c) : p(g) = \mathrm{id}_c\}$ *of* $\mathcal{D}$ *is a group.*

3. *For each $l \in \operatorname{Mor}\mathcal{C}$ let X_l be a subset of morphisms $\operatorname{Mor}_{\mathcal{D}}(c, c')$ such that for each $x \in X_l$ $p(x) = l$. Then the groups G^p_c, $G^p_{c'}$ acts on this set in the natural way*

$$G^p_{c'} \circlearrowright X_l \circlearrowleft G^p_c$$

such that $g'x = g' \circ x \in \operatorname{Mor}_{\mathcal{D}}(c, c')$ and $xg = x \circ g \in \operatorname{Mor}_{\mathcal{D}}(c, c')$. These actions satisfy

   - *the action of the group $G^p_{c'}$ is transitive and free*
   - *for each $x \in X_l$ there exists a homomorphism $\psi^p_x : G^p_c \longrightarrow G^p_{c'}$ given by $x \circ h = \psi^p_x(h) \circ x$*

Proof. These properties are clearely satisfied if $\mathcal{D}$ is the classifying category of a twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and p is an associated projection $\mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$. Conversely assume that $p : \mathcal{D} \longrightarrow \mathcal{C}$ satisfies the above properites. Let $s : \mathcal{C} \longrightarrow \mathcal{D}$ be any section of p, i.e. a map such that $ps = \mathrm{id}_{\mathcal{C}}$ (s does not have to be a functor). We will choose s such that for each object $s(\mathrm{id}_c) = \mathrm{id}_c$. Then we define a twisted diagram of groups on $\mathcal{C}$ as follows:

1. $\mathcal{G}(c) = G^p_c$ for each $c \in \operatorname{Ob}\mathcal{C}$
2. $\mathcal{G}(l) = \psi^p_{s(l)}$ for each $l \in \operatorname{Mor}\mathcal{C}$
3. elements $g_{l,l'}$ are uniquely defined by the equality $g_{l,l'}s(ll') = s(l)s(l')$ in $\mathcal{D}$

Note that $\mathcal{G}$ satisfy the normalizing condition $g(c) = 1$ for each object $c \in \mathcal{C}$.

Then $\mathcal{D}$ is clearly isomorphic to the classifying category $\mathcal{B}\mathcal{G}$: the isomorphism sends (l, g) to $gs(l)$. Another choice of section would give a twisted diagram of groups equivalent to $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$.

□

Corollary 2.2.10. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ the associated projection. Then p splits, i.e. there exists a functor $s : \mathcal{C} \longrightarrow \mathcal{B}\mathcal{G}$ such that $ps = \mathrm{id}_{\mathcal{C}}$, if and only if $\mathcal{G}$ is equivalent to a diagram of groups (functor).*

Proof. Assume that there exists a functor $s : \mathcal{C} \longrightarrow \mathcal{B}\mathcal{G}$. Then twisted diagram of groups $\mathcal{G}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ defined as $\mathcal{G}'(c) = G^p_c$ and $\mathcal{G}'(l) = \psi^p_{s(l)}$ is a diagram of groups. Thus $\mathcal{G}$ is equivalent to a diagram of groups.

Assume that $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a functor and $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ the associated projection. Then $s : \mathcal{C} \longrightarrow \mathcal{B}\mathcal{G}$ defined as $s(l) = (l, 1)$ is a functor. □

The following observation will be usefull later

Corollary 2.2.11. *Assume that $p : \mathcal{D} \longrightarrow \mathcal{C}$ satisfies the assertions of 2.2.9 and let $p' : \mathcal{D} \longrightarrow \mathcal{E}$ be a functor such that for each $g \in G^p_c \subset \operatorname{Aut}_{\mathcal{D}}(c)$ we have $p'(g) = \mathrm{id}_{p'(c)}$. Then there exists a uniqe functor $\widetilde{p'} : \mathcal{C} \longrightarrow \mathcal{E}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{p'} & \mathcal{E} \\ {\scriptstyle p}\downarrow & \nearrow_{\widetilde{p'}} & \\ \mathcal{C} & & \end{array}$$

Definition 2.2.12. *Let $\mathcal{C}$ be a small category. The category of twisted diagrams of groups on $\mathcal{C}$ is defined as a category whose objects are twisted diagrams of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and morphisms are homomorphisms of twisted diagrams of groups over the identity of the category $\mathcal{C}$. We will denote it $\mathrm{CGr}_{\mathcal{C}}$.*

The following theorem is a collorary from the Theorem 2.2.9 and Remark 2.2.5.

Theorem 2.2.13. *Let $\mathcal{C}$ be a small category. The category $\mathrm{CGr}_{\mathcal{C}}$ of twisted diagrams of groups on $\mathcal{C}$ is equivalent to the category $\downarrow \mathcal{C}$ whose objects are functors $\mathcal{D} \xrightarrow{p} \mathcal{C}$ satisfying assertions of 2.2.9 and morphisms are given by the commutative diagrams*

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{D} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

This equivalence is given by the natural functor $\mathcal{B}_{\mathcal{C}} : \mathrm{CGr}_{\mathcal{C}} \longrightarrow \downarrow \mathcal{C}$ assigning to $\mathcal{G}$ the projection $\mathcal{BG} \longrightarrow \mathcal{C}$.

Proof. The functor $\mathcal{B}_{\mathcal{C}} : \mathrm{CGr}_{\mathcal{C}} \longrightarrow \downarrow \mathcal{C}$ is given by $\mathcal{B}_{\mathcal{C}}(\mathcal{G}) = (\mathcal{BG} \xrightarrow{p} \mathcal{C})$ on objects and $\mathcal{B}_{\mathcal{C}}(\mathcal{G}' \longrightarrow \mathcal{G}) = \begin{array}{ccc} \mathcal{BG}' & \longrightarrow & \mathcal{BG} \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{C} & \end{array}$ on morphisms. The functor $\Phi' : \downarrow \mathcal{C} \longrightarrow \mathrm{CGr}_{\mathcal{C}}$ is given by $\Phi'(\mathcal{D} \xrightarrow{p} \mathcal{C}) = \mathcal{G}$ where $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a twisted diagram of groups constructed in the proof of Theorem 2.2.9. The commutative diagram

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{D} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

yields a diagram

$$\begin{array}{ccccccc} \mathcal{BG}' & \xrightarrow{\simeq} & \mathcal{D}' & \longrightarrow & \mathcal{D} & \xrightarrow{\simeq} & \mathcal{BG} \\ & & & \searrow \quad \swarrow & & & \\ & & & \mathcal{C} & & & \end{array}$$

which according to 2.2.5 defines a homomorphism of twisted diagrams of groups. Then $\Phi' \circ \mathcal{B}_{\mathcal{C}} \simeq \mathrm{id}_{\mathrm{CGr}_{\mathcal{C}}}$ and $\mathcal{B}_{\mathcal{C}} \circ \Phi' \simeq \mathrm{id}_{\downarrow \mathcal{C}}$. □

The following Proposition will be very usefull in our further considerations. It says roughly that a composition of twisted diagrams of groups is a twisted diagram of groups.

Proposition 2.2.14. *Let $r : \mathcal{E} \longrightarrow \mathcal{D}$ and $p : \mathcal{D} \longrightarrow \mathcal{C}$ satisfy assertions of the Theorem 2.2.9. Then the composition $p \circ r : \mathcal{E} \longrightarrow \mathcal{C}$ satisfies 2.2.9.*

Proof. If $r : \mathcal{E} \longrightarrow \mathcal{D}$ and $p : \mathcal{D} \longrightarrow \mathcal{C}$ satisfy assertions of the Theorem 2.2.9, then

1. $p \circ r : \mathcal{E} \longrightarrow \mathcal{C}$ is onto and is bijection on object sets because p and r are onto and bijections on object sets

2. We will prove that $G_c^{pr} = (pr)^{-1}(\mathrm{id}_c)$ is a group and moreover there exists an extension of the form
$$G_c^r \rightarrowtail G_c^{pr} \overset{r}{\twoheadrightarrow} G_c^p$$
If $y_1,\ y_2 \in \mathrm{End}_{\mathcal{E}}(c)$ such that $pr(y_1) = pr(y_2) = \mathrm{id}_c$ then $pr(y_1 y_2) = \mathrm{id}_c$. Assume that $y \in \mathrm{End}_{\mathcal{E}}(c)$ such that $pr(y) = \mathrm{id}_c$. Then $r(y) \in G_c^p$. The functor r is onto thus there exists $y' \in \mathrm{End}_{\mathcal{E}}(c)$ such that $r(y') = (r(y))^{-1}$. Then $yy',\ y'y \in G_c^r$ and let $yy' := g_1$ and $y'y := g_2$. Thus $y \circ (y' g_1^{-1}) = \mathrm{id}_c = (g_2^{-1} y') \circ y$ and $(g_2^{-1} y') = (g_2^{-1} y') \circ y \circ (y' g_1^{-1}) = (y' g_1^{-1})$. This proves $(g_2^{-1} y') = (y' g_1^{-1}) = y^{-1}$. Thus G_c^{pr} is a group which projects on G_c^p and the kernel of this epimorphism is G_c^r.

3. Let $l \in \mathrm{Mor}\,\mathcal{C},\ i(l) = c,\ t(l) = c'$ and denote Y_l the subset of $\mathrm{Mor}_{\mathcal{E}}(c, c')$ such that $pr(y) = l$ for each $y \in Y_l$. Then $Y_l = \coprod_{x \in X_l} Y_x$, where $X_l \subset \mathrm{Mor}_{\mathcal{D}}(c, c')$ such that $p(x) = l$ for each $x \in X_l$, and $Y_x \subset \mathrm{Mor}_{\mathcal{E}}(c, c')$ such that $r(y) = x$ for each $y \in Y_x$.

   - Let $G' = G_{c'}^{pr}$. We will prove that G' acts freely and transitively on Y_l.

     Assume that $g'y = y$ for some $y \in Y_l$ and $g' \in G'$. Then there exists $x \in X_l$ such that $y \in Y_x$. Thus $r(g')x = x$. This implies $r(g') = 1$ and then $g' \in G_{c'}^r$. Then $g' = 1$.

     Let $y_1,\ y_2 \in Y_l$. Then $y_1 \in Y_{x_1}$ and $y_2 \in Y_{x_2}$. We pick $g \in G_{c'}^p$ such that $gx_1 = x_2$ and we pick $g' \in G_{c'}^{pr}$ such that $r(g') = g$. Then $g'y_1 \in Y_{x_2}$. The group $G_{c'}^r$ acts transitively on Y_{x_2}. This proves that $G_{c'}^{pr}$ acts transitively on Y_l.
   - A homomorphism $\psi_y^{pr} : G_c^{pr} \longrightarrow G_{c'}^{pr}$ is induced by the following diagram

$$\begin{array}{ccccc} G_c^r & \rightarrowtail & G_c^{pr} & \twoheadrightarrow & G_c^p \\ \downarrow{\scriptstyle \psi_y^r} & & \vdots{\scriptstyle \psi_y^{pr}} & & \downarrow{\scriptstyle \psi_{r(y)}^p} \\ G_{c'}^r & \rightarrowtail & G_{c'}^{pr} & \twoheadrightarrow & G_{c'}^p \end{array}$$

□

Remark 2.2.15. Let $\widetilde{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$, $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$, $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ denote the twisted diagrams of groups associated respectively to $p \circ r : \mathcal{E} \longrightarrow \mathcal{C}$, $p : \mathcal{D} \longrightarrow \mathcal{C}$, $r : \mathcal{E} \longrightarrow \mathcal{D}$. Then we have the following diagram of homomorphisms of twisted diagrams of groups

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{=} & \mathcal{E} & \xrightarrow{r} & \mathcal{D} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle p} \\ \mathcal{D} & \xrightarrow{p} & \mathcal{C} & \xrightarrow{=} & \mathcal{C} \end{array}$$

corresponding to $\mathcal{F} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$,

Remark 2.2.16. Note that for each $c \in \mathrm{Ob}\,\mathcal{C}$ there exists an extension
$$\mathcal{F}(c) \rightarrowtail \widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c)$$
which is equal to
$$G_c^r \rightarrowtail G_c^{pr} \overset{r}{\twoheadrightarrow} G_c^p$$

This observation implies:

Proposition 2.2.17. *Let* $\mathcal{F} : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$ *be a twisted diagram of groups defined on the classifying category of a twisted diagram of groups* $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$*. For each* $c \in \mathrm{Ob}\,\mathcal{C}$ *let* $\mathcal{F}_c : \mathcal{B}\mathcal{G}(c) \longrightarrow \mathrm{Gr}$ *be a restriction of the weak functor* $\mathcal{F}$ *to the subcategory* $\mathcal{B}\mathcal{G}(c) \subset \mathcal{B}\mathcal{G}$*. This restriction defines a weak functor* $\widetilde{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr} \subset \mathrm{Cat}$ *given by*

$$\widetilde{\mathcal{G}}(c) = \mathcal{B}\mathcal{F}_c$$

Note $\mathcal{B}\mathcal{F}_c$ *is a small category associated to a group. Moreover*

$$\mathcal{B}\widetilde{\mathcal{G}} \simeq \mathcal{B}\mathcal{F}$$

Proof. Let $\mathcal{E}$ be the classifying category of $\mathcal{F}$ and $r : \mathcal{E} \longrightarrow \mathcal{B}\mathcal{G}$ the associated projection, let $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ be the projection associated to $\mathcal{G}$. Then $\widetilde{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is equivalent to a twisted diagram of groups associated to $p \circ r : \mathcal{E} \longrightarrow \mathcal{C}$ and $\mathcal{F}_c : \mathcal{B}\mathcal{G}(c) \longrightarrow \mathrm{Gr}$ is a twisted diagram of groups associated to the extension $\mathcal{F}(c) \rightarrowtail \widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c)$ (the construction described in 2.1.10). Obviousely $\mathcal{B}\widetilde{\mathcal{G}} \simeq \mathcal{B}\mathcal{F} = \mathcal{E}$. $\square$

Chapter 3

Cohomology of small categories and extensions of twisted diagrams of groups

Section 3.1 presents the definition and examples of cohomology of small categories. Given a twisted diagram of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ we can forget about the twisting elements and consider it as a functor $|\mathcal{F}| : \mathcal{C} \longrightarrow \mathcal{A}b$. We will prove in Proposition 3.1.7 that for a given functor $F : \mathcal{C} \longrightarrow \mathcal{A}b$ there is one to one correpsondence between the elemetns of the group $H^2(\mathcal{C}; F)$ and the equivalence classes of the twisted diagrams of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ such that $|\mathcal{F}| = F$.

One can generalize the above observation as follows. We define a category of representations Rep to be a category whose objects are groups and

$$\mathrm{Mor}_{\mathrm{Rep}}(H, G) := \mathrm{Mor}_{\mathrm{Gr}}(H, G)/\,\mathrm{Inn}(G)$$

For a given twisted diagram of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$ the composition with the natural projection $\mathrm{Gr} \longrightarrow \mathrm{Rep}$ gives a diagram of representations $F : \mathcal{C} \longrightarrow \mathrm{Rep}$. Section 3.2 answers the question: when does a diagram of representations lift to a twisted diagram of groups and how many such liftings exist? An answer will be given in terms of cohomology of small categories (Theorem 3.2.5).

Section 3.3 considers the case when the small category $\mathcal{C}$ is a category associated to a group G. We will explain the relation between the epimorphisms of groups and the twisted diagrams of groups defined on the category associated to a group.

A. Haeflieger in [H2] has classified extensions of complexes of groups with abelian kernel and extensions with locally constant kernel. We will extend this classification on twisted diagrams of groups. Section 3.4 describes the relation between the surjective homomorphisms of twisted diagram of groups and twisted diagrams of groups defined on the classifying category of a certain twisted diagram of groups (Theorem 3.4.4). Then as a collorary from 3.2.5 and 3.4.4 we obtain the classification Theorem (3.4.6).

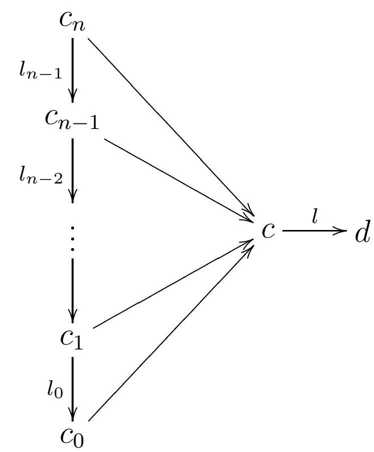
3.1 Cohomology of small categories

Definition 3.1.1. *For a small category $\mathcal{C}$ we denote by $\mathcal{C}$-mod* $:= \mathrm{Hom}(\mathcal{C}, \mathcal{A}b)$ *the category of (covariant) functors $\mathcal{C} \longrightarrow \mathcal{A}b$ and call its elements $\mathcal{C}$-modules. This agrees with the notion of G-module which is now called $\mathcal{B}G$-module. For $M, M' \in \mathcal{C}$-mod we* *denote by* $\mathrm{Hom}_{\mathcal{C}}(M, M')$ *the set of all morphisms* $M \longrightarrow M'$ *in* $\mathcal{C}$*-mod (i.e. natural transformations).*

Let $\underline{\mathbb{Z}} : \mathcal{C} \longrightarrow \mathcal{A}b$ be a constant $\mathcal{C}$-module given by $\underline{\mathbb{Z}}(c) = \mathbb{Z}$ for each object of $\mathcal{C}$ and the identity for each morphism of $\mathcal{C}$. The functor $\lim_{\mathcal{C}} = \mathrm{Hom}(\underline{\mathbb{Z}}, -) : \mathcal{C} - mod \longrightarrow \mathcal{A}b$ is left exact which implies that one can define the derived functors. Their value on the $\mathcal{C}$-module M will be denoted $H^*(\mathcal{C}; M)$ and called the n-th cohomology of $\mathcal{C}$ with coefficients in the module M.

From the defintion of the right derived functor one has to construct an injective resolution I^* of the $\mathcal{C}$-module M. $H^n(\mathcal{C}; M)$ is then the n-th cohomology of the cochain complex $\mathrm{Hom}(\underline{\mathbb{Z}}, I^*)$. These cohomology groups can be computed also as the cohomology groups of the cochain complex $\mathrm{Hom}(P_*, M)$ where P_* is a projective resolution of the $\mathcal{C}$-module $\underline{\mathbb{Z}}$.

We define the chain complex functor which assigns to an object $c \in \mathcal{C}$ the chain complex $C_*(\mathcal{C}/c)$. The generator of $C_n(\mathcal{C}/c)$ is an n-chain of objects of $\mathcal{C}/c$:



Since the arrows $c_i \longrightarrow c$ for $i < n$ are determined by the others we may think of this generator as of the $(n+1)$-chain $[c_n \longrightarrow c_{n-1} \longrightarrow \ldots \longrightarrow c_0 \longrightarrow c]$ ending in c. Thus $C_n(\mathcal{C}/c)$ can be thought of as the free abelian group over the set of chains $[c_n \longrightarrow \ldots \longrightarrow c_0 \longrightarrow c]$. For $l : c \longrightarrow d$ the morphism $C_*(\mathcal{C}/l)$ just composes the last morphism of the chain $[c_n \longrightarrow \ldots \longrightarrow c_0 \xrightarrow{k} c]$ with l yielding $[c_n \longrightarrow \ldots \longrightarrow c_0 \xrightarrow{lk} d]$.

Proposition 3.1.2. *For every* $n \geq 0$ *the functor* $C_n(\mathcal{C}/-)$ *is a projective* $\mathcal{C}$*-module. The chain complex* $C_*(\mathcal{C}/-)$ *is a projective resolution in* $\mathcal{C}$*-mod of the constant functor* $\underline{\mathbb{Z}}$.

Proof. We will denote $C_n(\mathcal{C}/-)$ as C_n. We shall prove that for an arbitrary epimorphism $M' \longrightarrow M$ of $\mathcal{C}$-modules the induced homomorphism $\mathrm{Hom}_{\mathcal{C}}(C_n, M') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_n, M)$ is also an epimorphism. For every $\mathcal{C}$-module M we have a bijection

$$\mathrm{Hom}_{\mathcal{C}}(C_n, M) \simeq \prod_{[c_n \longrightarrow \ldots \longrightarrow c_0]} M(c_0)$$

Before we give the formula for this map we will introduce the notation $(m_{[c_n \to \ldots \to c_0]})$ for the element of the product. Here the element $m_{[c_n \to \ldots \to c_0]} \in M(c_0)$ is the component corresponding to $[c_n \longrightarrow \ldots \longrightarrow c_0]$. The map sends each natural transformation τ to the collection $(\tau([c_n \longrightarrow \ldots \longrightarrow c_0 \xrightarrow{\mathrm{id}} c_0]))$. We will construct the inverse map in order to prove bijectivity. Let $(m_{[c_n\to\ldots\to c_0]})$ be an element in the right hand side group. Its inverse image is the transformation $C_n(c) \longrightarrow M(c)$ sending $[c_n \longrightarrow \ldots \longrightarrow c_0 \xrightarrow{k} c]$ to $M(k)(m_{[c_n\to\ldots\to c_0]})$. For naturality let us consider a morphism $l : c \longrightarrow d$ and the diagram

$$\begin{array}{ccccc} [c_n \longrightarrow \ldots \longrightarrow c_0 \xrightarrow{k} c] \in & C_n(c) & \xrightarrow{C_n(l)} & C_n(d) & \ni [c_n \longrightarrow \ldots \longrightarrow c_0 \xrightarrow{lk} d] \\ & \downarrow & & \downarrow & \\ M(k)(m_{[c_n\to\ldots\to c_0]}) \in & M(c) & \xrightarrow{M(l)} & M(d) & \ni M(lk)(m_{[c_n\to\ldots\to c_0]}) \end{array}$$

It is easy to observe that these maps are inverse to each other. The cartesian product preserves epimorphisms. As the map corresponding to $\mathrm{Hom}_{\mathcal{C}}(C_n, M') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_n, M)$ on left is just the product of epimorphisms on the right it is epi. This completes the proof of projectivity of C_*.

To prove that the complex $C_*(\mathcal{C}/-)$ is acyclic (i.e. has zero homology) it is enough to note that the category $\mathcal{C}/c$ has a final object $(c,\ c \xrightarrow{\mathrm{id}} c)$ thus the chain complex $C_*(\mathcal{C}/c)$ is exact for every $c \in \mathrm{Ob}\,\mathcal{C}$ and then $C_*(\mathcal{C}/-)$ is an exact sequence of $\mathcal{C}$-modules. □

This implies the explicite computation of the cohomology groups:

Corollary 3.1.3. *Given a $\mathcal{C}$-module $M : \mathcal{C} \longrightarrow \mathcal{A}b$ we consider the cochain complex $C^*(\mathcal{C}; M)$ on $\mathcal{C}$ with coefficients in M. A n-cochain $f \in C^n(\mathcal{C}; M)$ is a map associating to a sequence $c_n \xrightarrow{l_{n-1}} c_{n-1} \longrightarrow, \ldots, \longrightarrow c_1 \xrightarrow{l_0} c_0$ an element $f(l_0, \ldots, l_{n-1}) \in M(c_0)$. The coboundary $\delta f \in C^{n+1}(\mathcal{C}; M)$ is defined by*

$$\delta f(l_0, \ldots l_n) = M(l_0)(f(l_1, \ldots, l_n)) - \sum_{i=0}^{n-1} (-1)^i f(l_0, \ldots, l_i l_{i+1}, \ldots, l_n) - (-1)^n f(l_0, \ldots, l_{n-1})$$

Then $H^n(\mathcal{C}; M)$ is the n-th cohomology group of this cochain complex.

Remark 3.1.4. A homomorphism of $\mathcal{C}$-module M in a $\mathcal{C}$-module M' induces a homomorphism of $H^k(\mathcal{C}; M)$ in $H^k(\mathcal{C}; M')$. If $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor, then it induces a homomorphism F^* of $H^k(\mathcal{C}'; M)$ in $H^k(\mathcal{C}; F^*M)$. Those homomorphisms are obtained from the natural associated homomorphisms of the cochain complexes.

Remark 3.1.5. Assume that $\mathcal{C} = \mathcal{B}G$. Then $\mathcal{C}$-module M is a G-module and

$$H^k(\mathcal{C}; M) = H^k(G; M(*))$$

Remark 3.1.6. Let $M : \mathcal{C} \longrightarrow \mathcal{A}b$ be a $\mathcal{C}$-module. The first cohomology group $H^1(\mathcal{C}; M)$ is given by the set of equivalence classes $\{[f] \mid f \in C^1(\mathcal{C}; M),\ \delta f = 0\}$. Thus for $c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0$ in $\mathcal{C}$ we have $M(l_0)(f(l_1)) - f(l_0 l_1) + f(l_0) = 0$.

Assume that $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ is a twisted diagram of groups. Then for each pair of morphisms $c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0$ the composition $\mathcal{F}(l_0)\mathcal{F}(l_1)$ differs from $\mathcal{F}(l_0 l_1)$ by a conjugation with some element of the abelian group $\mathcal{F}(c_0)$. Thus $\mathcal{F}(l_0)\mathcal{F}(l_1) = \mathcal{F}(l_0 l_1)$ and $\mathcal{F}$ defines a functor $F : \mathcal{C} \longrightarrow \mathcal{A}b$ given by $F(c) = \mathcal{F}(c)$, $F(l) = \mathcal{F}(l)$. We will denote the functor $F := |\mathcal{F}|$.

The following proposition gives an explicit description of the second cohomology group of $\mathcal{C}$.

Proposition 3.1.7. *Let $F : \mathcal{C} \longrightarrow \mathcal{A}b$ be a functor. Then the group $H^2(\mathcal{C}; F)$ acts freely and transitively on the set of equivalence classes of twisted diagrams of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ such that $|\mathcal{F}| = F$.*

Proof. Let $g \in H^2(\mathcal{C}; F)$ be an element of the second cohomology group of $\mathcal{C}$. Then according to 3.1.3 g is given as an equivalence class of cocycles

$$\big\{\{f(l_0, l_1)\}_{[c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0] \in \mathrm{Mor}\,\mathcal{C}},\ f(l_0, l_1) \in F(c_0)\big\}$$

such that for $c_3 \xrightarrow{l_2} c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0$ these elements satisfy the (multiplicative) cocycle formula

$$F(l_0)(f(l_1, l_2)) f(l_0, l_1 l_2)^{-1} f(l_0 l_1, l_2) f(l_0, l_1)^{-1} = 0$$

which is the cocycle condition defined in 2.1.3. Two cocycles $\{f\}$ and $\{f'\}$ are equivalent if there exists a cochain $b \in C^1(\mathcal{C}; F)$ such that $\{f'\} = \{(\delta b) f\}$.

Each cocycle f defines a twisted diagram of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{A}b$ given by $\mathcal{F}(c) = F(c)$, $\mathcal{F}(l) = F(l)$, $f_{l_1, l_2} = f(l_1, l_2)$. If $[\{f\}] = [\{f'\}]$ then $\mathcal{F}$ and $\mathcal{F}'$ differ by a coboundary δb and then $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.

Assume that $\mathcal{F}, \mathcal{F}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ are equivalent twisted diagrams of groups. Then there exists a homomorphism $\phi = (\mathrm{id}_{\mathcal{F}(c)}, \phi(l)) : \mathcal{F} \longrightarrow \mathcal{F}'$ and $\mathcal{F}, \mathcal{F}'$ differ by a coboundary $\{\phi(l)\}_{l \in \mathrm{Mor}\,\mathcal{C}}$.

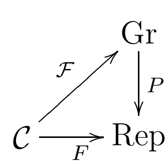
Thus an action of $H^2(\mathcal{C}; F)$ on the set of equivalence classes of twisted diagrams of groups given by: $[\{f\}][\mathcal{F}'] = [\mathcal{F}'']$ such that $[\{f''\}] = [\{ff'\}]$ is well defined, free and transitive. □

3.2 Lifting of diagrams of representations to twisted diagrams of groups

Proposition 3.2.1. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups. Then the composition of $\mathcal{F}$ with the functor $P : \mathrm{Gr} \longrightarrow \mathrm{Rep}$ is a functor.*

Proof. $\mathrm{Rep}(G, H) = \mathrm{Hom}(G, H)/\,\mathrm{Inn}(H)$, then for two composable morphisms l, l' in $\mathcal{C}$ we have $P\mathcal{F}(l) P\mathcal{F}(l') = P\mathcal{F}(ll')$. □

Definition 3.2.2. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $F \longrightarrow \mathrm{Rep}$ an associated functor. We will call the twisted diagram of groups $\mathcal{F}$ a lifting of the functor F.*



Assume that we are given a functor $F : \mathcal{C} \longrightarrow \mathrm{Rep}$. Does this functor lift to any twisted diagram of groups $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Gr}$? Is there any classification of such liftings? We will answer both these questions in the proceding Section. In order to do this we will define a certain abelian module $Z_F : \mathcal{C} \longrightarrow \mathcal{A}b$ associated to F.

Definition 3.2.3. *Let $\mathcal{C}$ be a small category. For any object $c \in \mathcal{C}$ one defines the right fibre category under c, denoted by $c\backslash\mathcal{C}$, whose objects are pairs $(t(l), c \xrightarrow{l} t(l))$. A morphism $(t(l_1), c \xrightarrow{l_1} t(l_1)) \longrightarrow (t(l_2), c \xrightarrow{l_2} t(l_2))$ in $c\backslash\mathcal{C}$ is a morphism $k : t(l_1) \longrightarrow t(l_2)$ for which the corresponding triangle under c commutes.*

For $(c \xrightarrow{l'} c') \in \operatorname{Mor}\mathcal{C}$ there is a natural inclusion $l'\backslash\mathcal{C} : c'\backslash\mathcal{C} \hookrightarrow c\backslash\mathcal{C}$ given by $(c'', c' \xrightarrow{l} c'') \longrightarrow (c'', c \xrightarrow{ll'} c'')$.

For a given group N, let $Z(N)$ denote its center.

Definition 3.2.4. *Let $F : \mathcal{C} \longrightarrow \operatorname{Rep}$ be a functor. One can assign to this functor a diagram of groups $Z_F : \mathcal{C} \longrightarrow \mathcal{A}b$, called obstruction functor defined as follows: for each $c \in \operatorname{Ob}\mathcal{C}$*

$$Z_F(c) = \bigcap_{l \in c\backslash\mathcal{C}} F(l)^{-1}(Z(F(t(l))))$$

and for each $(c \xrightarrow{l'} c') \in \operatorname{Mor}\mathcal{C}$

$$Z_F(l) : Z_F(c) \longrightarrow Z_F(c')$$

is given by

$$\bigcap_{l \in c\backslash\mathcal{C}} F(l)^{-1}(Z(F(t(l)))) < \bigcap_{k \in c'\backslash\mathcal{C}} F(kl')^{-1}(Z(F(t(kl')))) =$$

$$= \bigcap_{k \in c'\backslash\mathcal{C}} F(l')^{-1}F(k)^{-1}(Z(F(t(k)))) \xrightarrow{F(l')} \bigcap_{k \in c'\backslash\mathcal{C}} F(k)^{-1}(Z(F(t(k))))$$

An inner automorphism of a group is the identity on its center. Then

$$Z_F : \mathcal{C} \longrightarrow \mathcal{A}b$$

is a well defined functor. If $\mathcal{C} = *$ then $Z_F(*) = Z(F(*))$. Therefore we can roughly say that Z_F is a center of the functor F.

Let $F : \mathcal{C} \longrightarrow \operatorname{Rep}$ be a functor and let $Z_F : \mathcal{C} \longrightarrow \mathcal{A}b$ be an obstruction functor. Then

Theorem 3.2.5. *To every functor $F : \mathcal{C} \longrightarrow \operatorname{Rep}$ one assignes in a natural way an obstruction element $o(F) \in H^3(\mathcal{C}; Z_F)$ such that $o(F)$ vanishes if and only if the functor F has a lifting to a twisted diagram $\mathcal{F} : \mathcal{C} \longrightarrow \operatorname{Gr}$. Moreover equivalence classes of such liftings are in bijective corrspondence with elements of the group $H^2(\mathcal{C}; Z_F)$.*

Proof. Given the functor $F : \mathcal{C} \longrightarrow \operatorname{Rep}$, we try to construct a "2-cocycle" $\{f\}$ verifying the cocycle condition 2.1.3. We choose a map $\widetilde{F} : \mathcal{C} \longrightarrow \operatorname{Gr}$ such that $P \circ \widetilde{F} = F$ and a cochain $\big\{\{f(l_0, l_1)\}_{[c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0] \in \operatorname{Mor}\mathcal{C}},\ f(l_0, l_1) \in F(c_0)\big\}$ such that

$$\operatorname{Ad}(f(l_0, l_1))\widetilde{F}(l_0 l_1) = \widetilde{F}(l_0)\widetilde{F}(l_1)$$

One can define a unique 3-cochain $\{d_{[c_3 \xrightarrow{l_2} c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0]}\}$, assininig to each triple of composable morphisms an element of the center $Z(F(c_0))$,

$$d(l_0, l_1, l_2) \in Z(F(c_0))$$

satisfying

$$\widetilde{F}(l_0)(f(l_1,l_2))f(l_0,l_1l_2)=d(l_0,l_1,l_2)f(l_0,l_1)f(l_0l_1,l_2)$$

We will prove that d is a 3-cocycle with coefficients in $Z_F:\mathcal{C}\longrightarrow\mathcal{A}b$. We have
$\widetilde{F}(l_0)(f(l_1,l_2))f(l_0,l_1l_2)f(l_0l_1,l_2)^{-1}f(l_0,l_1)^{-1}=d(l_0,l_1,l_2)$
$\widetilde{F}(l_0)\widetilde{F}(l_1)(f(l_2,l_3))\widetilde{F}(l_0)(f(l_1,l_2l_3))\big(\widetilde{F}(l_0)(f(l_1l_2,l_3))\big)^{-1}\big(\widetilde{F}(l_0)(f(l_1,l_2))\big)^{-1}=\widetilde{F}(l_0)(d(l_1,l_2,l_3))$
$\widetilde{F}(l_0l_1)(f(l_2,l_3))f(l_0l_1,l_2l_3)f(l_0l_1l_2,l_3)^{-1}f(l_0l_1,l_2)^{-1}=d(l_0l_1,l_2,l_3)$
$\widetilde{F}(l_0)(f(l_1l_2,l_3))f(l_0,l_1l_2l_3)f(l_0l_1l_2,l_3)^{-1}f(l_0,l_1l_2)^{-1}=d(l_0,l_1l_2,l_3)$
$\widetilde{F}(l_0)(f(l_1,l_2l_3))f(l_0,l_1l_2l_3)f(l_0l_1,l_2l_3)^{-1}f(l_0,l_1)^{-1}=d(l_0,l_1,l_2l_3)$
We will prove that

$$d(l_0,l_1,l_2)^{-1}d(l_0,l_1l_2,l_3)^{-1}d(l_0l_1,l_2,l_3)d(l_0,l_1,l_2l_3)=\widetilde{F}(l_0)(d(l_1,l_2,l_3))$$

First note
$d(l_0,l_1,l_2)^{-1}d(l_0l_1,l_2,l_3)=$
$=f(l_0l_1,l_2)f(l_0,l_1l_2)^{-1}\big(\widetilde{F}(l_0)(l_1,l_2)\big)^{-1}f(l_0,l_1)\widetilde{F}(l_0l_1)(f(l_2,l_3))f(l_0l_1,l_2l_3)f(l_0l_1l_2,l_3)^{-1}f(l_0l_1,l_2)^{-1}$
Then
$d(l_0,l_1,l_2)^{-1}d(l_0l_1,l_2,l_3)d(l_0,l_1,l_2l_3)=$
$=f(l_0l_1,l_2l_3)f(l_0l_1l_2,l_3)^{-1}f(l_0,l_1l_2)^{-1}\big(\widetilde{F}(l_0)(l_1,l_2)\big)^{-1}f(l_0,l_1)\widetilde{F}(l_0l_1)(f(l_2,l_3))f(l_0,l_1)^{-1}$
$\widetilde{F}(l_0)(f(l_1,l_2l_3))f(l_0,l_1l_2l_3)f(l_0l_1,l_2l_3)^{-1}=$
$=f(l_0l_1l_2,l_3)^{-1}f(l_0,l_1l_2)^{-1}\big(\widetilde{F}(l_0)(l_1,l_2)\big)^{-1}\widetilde{F}(l_0)\widetilde{F}(l_1)(f(l_2,l_3))\widetilde{F}(l_0)(f(l_1,l_2l_3))f(l_0,l_1l_2l_3)$
and this implies
$d(l_0,l_1,l_2)^{-1}d(l_0l_1,l_2,l_3)d(l_0,l_1,l_2l_3)d(l_0,l_1l_2,l_3)^{-1}=\widetilde{F}(l_0)(d(l_1,l_2,l_3))$. Then $\{d\}$ is a 3-cocycle with coefficients in Z_F.

If $\{f'\}$ is another map satisfying $\mathrm{Ad}(f'(l_0,l_1))\widetilde{F}(l_0l_1)=\widetilde{F}(l_0)\widetilde{F}(l_1)$, then there is unique 2-cochain $\{b_{[c_2\xrightarrow{l_1}c_1\xrightarrow{l_0}c_0]}\}$ such that $f'(l_0,l_1)=b(l_0,l_1)f(l_0,l_1)$. Note $b(l_0,l_1)\in Z(F(c_0))$.‘

Then the 3-cochain $\{d'\}$ associated to $\{f'\}$ is the cochain $\{d\}$ modified by the coboundary of the 2-cochain b, namely

$$d'(l_0,l_1,l_2)=\widetilde{F}(l_0)(b(l_1,l_2))b(l_0,l_1l_2)b(l_0l_1,l_2)^{-1}b(l_0,l_1)^{-1}d(l_0,l_1,l_2)$$

Note that $\widetilde{F}(l_0)(b(l_1,l_2))\in Z(F(c_0))$. This proves that b is a 2-cochain with coefficients in $Z_F:\mathcal{C}\longrightarrow\mathcal{A}b$, i.e. $b\in C^2(\mathcal{C};Z_F)$. Thus $\{d\}$ and $\{d'\}$ give the same element $o(F)\in H^3(\mathcal{C};Z_F)$.

Assume that $\widetilde{F}':\mathcal{C}\longrightarrow$ Gr is another map such that $P\circ\widetilde{F}'=F$. Then for each morphism of $\mathcal{C}$ we have $\widetilde{F}'(l)=\mathrm{Ad}(g_l)\widetilde{F}(l)$, where $g_l\in F(t(l))$. Let
$f'(l_0,l_1):=g_{l_0}\widetilde{F}(l_0)(g_{l_1})f(l_0,l_1)g_{l_0l_1}^{-1}$ for each pair $.\xrightarrow{l_1}.\xrightarrow{l_0}.$ of morphisms in $\mathcal{C}$. Then

$$\mathrm{Ad}(f'(l_0,l_1))\widetilde{F}'(l_0l_1)=\widetilde{F}'(l_0)\widetilde{F}'(l_1)$$

and if $\{d'\}$ is the unique cochain defined by the equation
$\widetilde{F}'(l_0)(f'(l_1,l_2))f'(l_0,l_1l_2)=d'(l_0,l_1,l_2)f'(l_0,l_1)f'(l_0l_1,l_2)$ then $\{d'\}=\{d\}$.

Thus the cohomology class of $\{d\}$ is independent of the choice of $\widetilde{F}$ and f.

Therefore, the functor $F:\mathcal{C}\longrightarrow$ Rep has a lifting if and only if the element $o(F)\in H^3(\mathcal{C};Z_F)$ vanishes. This proves the first part of the Theorem.

We will prove that the group $H^2(\mathcal{C};Z_F)$ acts freely and transitively on the set of equivalence classes of liftings of F.

Assume that $\mathcal{F},\mathcal{F}':\mathcal{C}\longrightarrow$ Gr are liftings of $F:\mathcal{C}\longrightarrow$ Rep. Then for each morphism $l\in\mathrm{Mor}\,\mathcal{C}$ there exists an element $g_l\in F(t(l))$ such that $\mathcal{F}(l)=\mathrm{Ad}(g_l)\circ\mathcal{F}'(l)$.

Let $\mathcal{F}'' : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted digram of groups such that $\mathcal{F}'$ and $\mathcal{F}''$ differ by a coboundary $\{g_l\}_{l\in \mathrm{Mor}\,\mathcal{C}}$. Thus the twisted diagram of groups $\mathcal{F}' : \mathcal{C} \longrightarrow \mathrm{Gr}$ is equivalent to a twisted diagram of groups $\mathcal{F}'' : \mathcal{C} \longrightarrow \mathrm{Gr}$

$$\mathcal{F}'' \approx \mathcal{F}'$$

such that for each morphism $l \in \mathrm{Mor}\,\mathcal{C}$

$$\mathcal{F}''(l) = \mathcal{F}(l)$$

The twisted diagram $\mathcal{F}''$ is a lifting of the functor F, thus for each pair of composable morphisms of $\mathcal{C}$

$$f''_{l_0,l_1} = d_{l_0,l_1} f_{l_0,l_1}$$

where $d_{l_0,l_1} \in Z(F(t(l_0l_1)))$. The "cocycles" $\{f\}, \{f''\}$ satisfy the cocycle condition, thus for $\cdot \xrightarrow{l_2} \cdot \xrightarrow{l_1} \cdot \xrightarrow{l_0} \cdot$ of morphisms in $\mathcal{C}$

$1 = \mathcal{F}''(l_0)(f''_{l_1,l_2}) f''_{l_0,l_1l_2} (f''_{l_0l_1,l_2})^{-1} (f''_{l_0,l_1})^{-1} =$

$= \mathcal{F}(l_0)(d_{l_1,l_2}) d_{l_0,l_1l_2} (d_{l_0l_1,l_2})^{-1} (d_{l_0,l_1})^{-1} \mathcal{F}(l_0)(f_{l_1,l_2}) f_{l_0,l_1l_2} (f_{l_0l_1,l_2})^{-1} (f_{l_0,l_1})^{-1} =$

$= \mathcal{F}(l_0)(d_{l_1,l_2}) d_{l_0,l_1l_2} (d_{l_0l_1,l_2})^{-1} (d_{l_0,l_1})^{-1}$

This proves $\mathcal{F}(l_0)(d_{l_1,l_2}) \in Z(F(t(l_0)))$ and then d is a cocycle

$$d \in C^2(\mathcal{C}; Z_F)$$

The map d is a 2-cocycle whose cohomology class is independent on the choice of f and f''. Its vanishing implies the existence of an equivalence between the $\mathcal{F}$ and $\mathcal{F}''$.

Conversely given a lifiting $\mathcal{F}$ and a 2-cocycle d_{l_0,l_1}, the formula

$$f''_{l_0,l_1} = d_{l_0,l_1} f_{l_0,l_1}$$

defines a "cocycle" $\{f''\}$ verifying the cocycle condition and this gives a twisted diagram of groups $\mathcal{F}'' : \mathcal{C} \longrightarrow \mathrm{Gr}$. Thus it defines an action

$$[\{d\}][\mathcal{F}] = [\mathcal{F}'']$$

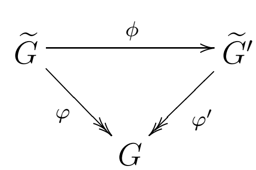
of the group $H^2(\mathcal{C}; Z_F)$ on the set of equivalence classes of liftings of F. If $[\{d\}] = 1$ then $[\mathcal{F}] = [\mathcal{F}'']$.

This proves the second part of the Theorem. □

3.3 Epimorphisms of groups

Let G be a group and $\mathcal{B}G$ the small category defined by G. As we have observed in Proposition 2.2.8 there is a bijective corrspondence between the equivalence classes of extensions over G and equivalence classes of twisted diagrams of groups on $\mathcal{B}G$.

Definition 3.3.1. *Assume $\varphi : \widetilde{G} \twoheadrightarrow G$, $\varphi' : \widetilde{G}' \twoheadrightarrow G$ are epimorphisms of groups. We say that these epimorphisms are equivalent if there exists an isomorphism of groups $\phi : \widetilde{G} \longrightarrow \widetilde{G}'$ such that the following diagram*



commutes.

Then according to 2.2.8

Remark 3.3.2. The equivalence classes of epimorphisms over G are in bijective correspondence with equivalence classes of twisted diagrams of groups over $\mathcal{B}G$.

Remark 3.3.3. Let $F : \mathcal{B}G \longrightarrow \text{Rep}$ be a functor such that $F(*) = N$. Then F is simply a homomorphism of groups $F : G \longrightarrow \text{Out}(N)$. The $\mathcal{B}G$-module $Z_F : \mathcal{B}G \longrightarrow \mathcal{A}b$ defined in 3.2.4 is a homomorphism $Z_F : G \longrightarrow \text{Aut}(Z(N))$.

Then the Theorem 3.2.5 reduces to the classical case ([B2], [R]);

Proposition 3.3.4. *Let $F : G \longrightarrow \text{Out}(N)$ be a homomorphism of groups. Then F comes from an epimorphism $\widetilde{G} \twoheadrightarrow G$ if and only if a certain obstruction element $o(F) \in H^3(G; Z(N))$ vanishes. The equivalence classes of epimorphisms are in bijective correspondence with the elements of $H^2(G; Z(N))$.*

Proof. Use 3.2.5 for $\mathcal{C} = \mathcal{B}G$ and then 3.3.2. □

3.4 Epimorphisms of twisted diagrams of groups

Example 3.4.1. The epimorphism of groups

$$SL_2\mathbb{Z} \twoheadrightarrow PSL_2\mathbb{Z}$$

could be describe as the homomorphism of colimit groups of diagrams of groups

$$(\mathbb{Z}_6 \longleftarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4) \twoheadrightarrow (\mathbb{Z}_3 \longleftarrow 1 \longrightarrow \mathbb{Z}_2)$$

which is an epimorphism on local groups.

This example is a special case of surjective homomorphism of complexes of groups, considered by Haeflieger [H2].

Definition 3.4.2. *Assume that $\mathcal{G} : \mathcal{C} \longrightarrow \text{Gr}$ and $\widetilde{\mathcal{G}} : \mathcal{C} \longrightarrow \text{Gr}$ are twisted diagrams of groups defined on the category $\mathcal{C}$. A surjective homomorphism or epimorphism of twisted diagrams of groups $\varphi : \widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$ is a homomorphism over the identity of $\mathcal{C}$ such that all the local homomorphisms are surjective; i.e. for each $c \in \mathcal{C}$*

$$\varphi_c : \widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c)$$

Definition 3.4.3. *Assume that $\varphi : \widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$, $\varphi' : \widetilde{\mathcal{G}}' \twoheadrightarrow \mathcal{G}$ are epimorphisms of twisted diagrams of groups. We say that φ, φ' are equivalent if there exists an isomorphism $\phi : \widetilde{\mathcal{G}} \longrightarrow \widetilde{\mathcal{G}}'$ over the identity of $\mathcal{C}$ such that the following diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{G}} & \xrightarrow{\ \phi\ } & \widetilde{\mathcal{G}}' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & \mathcal{G} & \end{array}$$

commutes.

The following Theorem is a generalization of the Remark 3.3.2

Theorem 3.4.4. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups. There is one to one correspondence between the equivalence classes of epimorphisms*

$$\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$$

and equivalence classes of twisted diagrams of groups defined on the classifying category of the twisted diagram of groups $\mathcal{G}$;

$$\mathcal{F} : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$$

Proof. Let $\varphi : \widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$ be a surjective homomorphism of twisted diagrams of groups. The homomorphism φ is given by a commutative diagram

$$\begin{array}{ccc} \mathcal{B}\widetilde{\mathcal{G}} & \xrightarrow{\ r\ } & \mathcal{B}\mathcal{G} \\ & {\scriptstyle p\circ r}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{C} & \end{array}$$

We will prove that functor $r : \mathcal{B}\widetilde{\mathcal{G}} \longrightarrow \mathcal{B}\mathcal{G}$ satisfies assertions of the Theorem 2.2.9. First note, that p and $p \circ r$ satisfy these assertions. Then

1. The functor r is onto because the homomorphism φ is locally onto. The functors p and $p \circ r$ are bijections on the object sets thus r is a bijection on the object sets.

2. For each $c \in \mathrm{Ob}\,\mathcal{C}$ the subcategory $G^r_c = r^{-1}(\mathrm{id}_c)$ is the kernel of the local epimorphism $\varphi_c : \widetilde{\mathcal{G}}(c) \longrightarrow \mathcal{G}(c)$, hence it is a group

3.
   - Let $Y_x \subset \mathrm{Mor}_{\mathcal{B}\widetilde{\mathcal{G}}}(c, c')$ such that $r(y) = x$. Let $g' \in G^r_{c'}$ and assume that $gy = y$. $G^r_{c'} \subset \widetilde{\mathcal{G}}(c')$ and then $g'y = y$ implies $g' = 1$. For $y_1,\ y_2 \in Y_x$ there exists $\widetilde{g} \in \widetilde{\mathcal{G}}(c')$ such that $\widetilde{g}y_1 = y_2$. Then $r(\widetilde{g})x = x$ which implies $\widetilde{g} \in \ker \varphi_{c'} = G^r_{c'}$.
   - The homomorphism $\psi^r_y : G^r_c \longrightarrow G^r_{c'}$ is induced by the following diagram

$$\begin{array}{ccccc} G^r_c & \rightarrowtail & \widetilde{\mathcal{G}}(c) & \twoheadrightarrow & \mathcal{G}(c) \\ {\scriptstyle \psi^r_y}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi^{pr}_y} & & \Big\downarrow{\scriptstyle \psi^p_{r(y)}} \\ G^r_{c'} & \rightarrowtail & \widetilde{\mathcal{G}}(c') & \twoheadrightarrow & \mathcal{G}(c') \end{array}$$

Then according to Theorem 2.2.9, the epimorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ yields a twisted diagram of groups $\mathcal{F}_\varphi : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$. Moreover $\mathcal{F}_\varphi(c) = \ker(\widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c))$ for each $c \in \mathrm{Ob}\,\mathcal{B}\mathcal{G} = \mathrm{Ob}\,\mathcal{C}$.

Conversely let $\mathcal{F}_\varphi : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups defined on the category $\mathcal{B}\mathcal{G}$. Let $r : \mathcal{B}\mathcal{F}_\varphi \longrightarrow \mathcal{B}\mathcal{G}$ and $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ be the associated projections. Then, by Propsition 2.2.14, the composition functor $p \circ r : \mathcal{B}\mathcal{F}_\varphi \longrightarrow \mathcal{C}$ defines a twisted diagram of groups

$$\begin{array}{ccccc} \mathcal{B}\widetilde{\mathcal{G}} & \xrightarrow{\ \simeq\ } & \mathcal{B}\mathcal{F}_\varphi & \xrightarrow{\ r\ } & \mathcal{B}\mathcal{G} \\ & {\scriptstyle \widetilde{p}}\searrow & \Big\downarrow{\scriptstyle pr} & \swarrow{\scriptstyle p} & \\ & & \mathcal{C} & & \end{array}$$

Thus we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{B}\widetilde{\mathcal{G}} & \longrightarrow & \mathcal{B}\mathcal{G} \\ & {\scriptstyle\widetilde{p}}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{C} & \end{array}$$

which defines a surjective homomorphism of twisted diagrams of groups $\widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$.

According to Remarks 2.2.15 and 2.2.16, for each object $c \in \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{B}\mathcal{G}$ there exists an extension of groups

$$\mathcal{F}_\varphi(c) \rightarrowtail \widetilde{\mathcal{G}}(c) \twoheadrightarrow \mathcal{G}(c)$$

and the commutative diagram

$$\begin{array}{ccccc} \mathcal{B}\mathcal{F}_\varphi & \xrightarrow{\simeq} & \mathcal{B}\widetilde{\mathcal{G}} & \longrightarrow & \mathcal{B}\mathcal{G} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} & & \downarrow{\scriptstyle p} \\ \mathcal{B}\mathcal{G} & \xrightarrow{p} & \mathcal{C} & \xrightarrow{=} & \mathcal{C} \end{array}$$

which defines the homomorphisms

$$\mathcal{F}_\varphi \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$$

Clearly the surjective homomorphisms $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, $\varphi' : \widetilde{\mathcal{G}}' \longrightarrow \mathcal{G}$ are equivalent if and only if the associated twisted diagrams of groups $\mathcal{F}_\varphi : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$ and $\mathcal{F}'_{\varphi'} : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$ are equivalent.

□

The twisted diagram of groups $\mathcal{F}_\varphi$ satisfies the following universal property:

Proposition 3.4.5. *Let $\mathcal{F}_\varphi : \mathcal{B}\mathcal{G} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups associated to an epimorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ over $\mathrm{id}_\mathcal{C}$ and $\phi : \mathcal{F}_\varphi \longrightarrow \widetilde{\mathcal{G}}$ over $p : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{C}$ the associated homomorphism. Assume that $\phi' : \mathcal{G}' \longrightarrow \widetilde{\mathcal{G}}$ over $s : \mathcal{D} \longrightarrow \mathcal{C}$ is a homomorphism of twisted diagram of groups such that $\varphi \circ \phi'$ is trivial on the local groups. Then there exists a unique homomorphism $\bar{\phi}' : \mathcal{G}' \longrightarrow \mathcal{F}_\varphi$ over a functor $\bar{s} : \mathcal{D} \longrightarrow \mathcal{B}\mathcal{G}$ such that $\phi \circ \bar{\phi}' = \phi'$ and $p \circ \bar{s} = s$.*

Proof. In view of 2.2.11 there exists a unique functor $\bar{s} : \mathcal{D} \longrightarrow \mathcal{B}\mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}\mathcal{G}' & \longrightarrow & \mathcal{B}\widetilde{\mathcal{G}} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{\bar{s}} & \mathcal{B}\mathcal{G} \\ & \searrow & \downarrow \\ & & \mathcal{C} \end{array}$$

This diagram yields a (unique) homomorphism $\bar{\phi}' : \mathcal{G}' \longrightarrow \mathcal{F}_\varphi$ over s such that the following diagram commutes

$$\begin{array}{ccc} & & \mathcal{F}_\varphi \\ & {\scriptstyle\bar{\phi}'}\nearrow & \downarrow{\scriptstyle\phi} \\ \mathcal{G}' & \xrightarrow[\phi']{} & \widetilde{\mathcal{G}} \end{array}$$

□

The following Theoerem is a corollary from 3.2.5 and 3.4.4.

Theorem 3.4.6. *Let* $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ *be a twisted diagram of groups and* $F : \mathcal{BG} \longrightarrow \mathrm{Rep}$ *be any functor. Let* $Z_F : \mathcal{BG} \longrightarrow \mathcal{A}b$ *be an obstruction functor defined in 3.2.4. Then*

1. *there exists an epimorphism* $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ *of twisted diagrams of groups, such that the associated twisted diagram of groups* $\mathcal{F} : \mathcal{BG} \longrightarrow \mathrm{Gr}$ *is a lifting of* F *if and only if a certain element* $o(F) \in H^3(\mathcal{BG}; Z_F)$ *vanishes*

2. *the set of equivalence classes of such epimorphisms is in bijection with* $H^2(\mathcal{BG}; Z_F)$.

Proof. The proof follows directly from 3.4.4 and 3.2.5. □

Remark 3.4.7. Assume that $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is a surjective homomorphism of complexes of groups and for each object c the corresponding epimorphism of local groups has abelian kernel. Then the Theorem 3.4.6 reduces to the Haeflieger's theorem [Thm. 5.2. H2].

If $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is a surjective homomorphism of complexes of groups and for each object c the corresponding epimorphism of local groups has constant (not necessary abelian) kernel, then the Theorem 3.4.6 reduces to the Haeflieger's theorem [Thm. 6.3. H2].

Chapter 4

Fundamental group

The fundamental group of a twisted diagram of groups is a generalization of a direct limit of a diagram of groups. Precisely, for each twisted diagram of groups there exists a "weak" direct limit of the corresponding weak functor. This weak direct limit of a twisted diagram of groups is defined as the fundamental group of its classifying category.

Section 4.1 is devoted to introductory material and basic definitions concerning fundamental group of a small category. This fundamental group is defined as the fundamental group of the geometric realization of the given category. We will prove that to each category $\mathcal{C}$ one can assign a certain grupoid called the fundamental grupoid or the grupoid associated to $\mathcal{C}$. It is constructed by formally inverting all of the morphisms of $\mathcal{C}$. If the geometric realization of $\mathcal{C}$ is connected then the fundamental grupoid and the fundamental group are equivalent small categories.

Section 4.2 starts with the Theorem motivated by E. D. Farjoun [Fa]. It is the reformulation of the Seifert-van Kampen theorem, concerning the fundametal group of a union of spaces. Precisely, we will prove that the fundamental group of a (connected) homotopy colimit is isomorphic to the fundamental group of a certain twisted diagram of groups.

Each twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ yields a projection $p : \mathcal{BG} \longrightarrow \mathcal{C}$ and then a homomorphism of fundamental groups $p_* : \pi_1(\mathcal{G}, c_0) \longrightarrow \pi_1(\mathcal{C}, c_0)$. We will prove that this homomorphism is onto (4.2.6). Assume that $\mathcal{G}$ is a diagram of groups. Then the epimorphism p_* splits. Moreover there exist the direct limit of this diagram. The Theorem 4.2.13 establishes the relation between the fundamental group and the direct limit of $\mathcal{G}$.

4.1 Fundamental groupoid and fundamental group of a small category

The fundamental group of a small category $\mathcal{C}$ is defined as the fundamental group of its geometric realization. We will prove that this group is isomorphic to the group of automorphisms of the grupoid associated to $\mathcal{C}$, defined by P. Gabriel and M. Zisman in [G-Z]. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said make a morphism l of $\mathcal{C}$ invertible if $F(l)$ is invertible. [G-Z] associated with each category $\mathcal{C}$ and each subset $\Sigma \subset \mathrm{Mor}\, \mathcal{C}$ a category of fractions $\mathcal{C}[\Sigma^{-1}]$ and a functor $P_\Sigma : \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}]$ verifying the following conditions:

- P_Σ makes the morphisms of Σ invertible

- If a functor $F : \mathcal{C} \longrightarrow \mathcal{X}$ makes the morphisms of Σ invertible, there exists unique functor $\widetilde{F} : \mathcal{C}[\Sigma^{-1}] \longrightarrow \mathcal{X}$ such that $F = \widetilde{F} \circ P_\Sigma$.

We will describe this construction for $\Sigma = \operatorname{Mor}\mathcal{C}$. In this case the category of fraction $\mathcal{C}[\Sigma^{-1}]$ turns out to be a grupoid. We will denote it $\pi\mathcal{C}$ and call the fundamental grupoid or the grupoid associated to $\mathcal{C}$.

$\mathcal{C}$-paths Let $\mathcal{C}$ be a small category. We will define a combinatorial path in the category $\mathcal{C}$. We associate two symbols l^+ and l^- to each morphism $l \in \operatorname{Mor}\mathcal{C}$. The set of symbols l^+, l^- with $l \in \operatorname{Mor}\mathcal{C}$ is denoted $\operatorname{Mor}^\pm \mathcal{C}$. Given $\alpha \in \operatorname{Mor}^\pm \mathcal{C}$, we define its initial object $i(\alpha)$ and its terminal object $t(\alpha)$ by the formula:

$$i(l^+) = i(l),\ t(l^+) = t(l),\ i(l^-) = t(l),\ t(l^-) = i(l)$$

For $\alpha = l^+$ (resp. l^-), we define $\alpha^{-1} = l^-$ (resp. l^+).

A *path in $\mathcal{C}$ joining* an object c to an object d is a sequence $\gamma = (\alpha_1, \ldots, \alpha_k)$, where each $\alpha \in \operatorname{Mor}^\pm \mathcal{C}$, $t(\alpha_i) = i(\alpha_{i-1})$ for $i = k, \ldots, 2$ and $i(\alpha_k) = c$, $t(\alpha_1) = d$. If $\gamma' = (\alpha'_1, \ldots, \alpha'_{k'})$ is a path in $\mathcal{C}$ joining d to e, then one can compose γ and γ' to obtain the path $\gamma'\gamma = (\alpha'_1, \ldots, \alpha'_{k'}, \alpha_1, \ldots, \alpha_k)$ joining c to e. The inverse of the path γ is the path $\gamma^{-1} = (\alpha_k^{-1}, \ldots, \alpha_1^{-1})$. If $i(\gamma) = t(\gamma)$ then γ is called a loop at c.

Equivalence of paths Let $\gamma = (\alpha_1, \ldots, \alpha_k)$ be a path in $\mathcal{C}$ joining c to d. Consider following three operations on γ:

1. Assume that for some $k \geq j > 2$, we have $\alpha_j = l_j^+$ and $\alpha_{j-1} = l_{j-1}^+$ (resp. $\alpha_j = l_j^-$ and $\alpha_{j-1} = l_{j-1}^-$). Then the composition $l_{j-1}l_j$ is defined (resp. $l_j l_{j-1}$) and we get a new path γ' in $\mathcal{C}$ by replacing the subsequence (α_{j-1}, α_j) of γ by $(l_{j-1}l_j)^+$ (resp. $(l_j l_{j-1})^-$).

2. Assume that for some $k \geq j > 2$, we have $\alpha_{j-1} = \alpha_j^{-1}$. Then we get a new path γ' by deleting from γ the subsequence (α_{j-1}, α_j).

3. Assume that for some j, the morphism α_j is associated to id_c. Then we get a new path by deleting α_j.

If γ and γ' are related in this way, then we say that are obtained from each other by an elementary equivalence. Two paths γ and γ' are defined to be *equivalent* if one can pass from the first to the second by a sequence of elementary equivalences. The set of equivalence classes of paths in $\mathcal{C}$ joinig c to d is denoted $\pi_1(\mathcal{C}, c, d)$. If $[\gamma] \in \pi_1(\mathcal{C}, c, d)$ and $[\gamma'] \in \pi_1(\mathcal{C}, d, e)$ then $[\gamma'][\gamma] = [\gamma'\gamma] \in \pi_1(\mathcal{C}, c, e)$ and $[\gamma]^{-1} = [\gamma^{-1}] \in \pi_1(\mathcal{C}, d, e)$.

Definition 4.1.1. *Let $\mathcal{C}$ be a small category. The grupoid associated to $\mathcal{C}$ is a small category $\pi\mathcal{C}$ such that the set of its objects is equal to the set of objects of $\mathcal{C}$ and the set of morphisms is given by*

$$\operatorname{Mor}_{\pi\mathcal{C}}(c, d) = \pi_1(\mathcal{C}, c, d)$$

Note, $\pi\mathcal{C}$ is well defined small category. Each morphism of $\pi\mathcal{C}$ is invertible thus $\pi\mathcal{C} \in \mathrm{Grp} \subset \mathrm{Cat}$.

Proposition 4.1.2. *The above construction is natural, i.e. there exists a functor $\pi : \mathrm{Cat} \longrightarrow \mathrm{Grp}$ such that for each small category $\mathcal{C}$*

$$\pi(\mathcal{C}) = \pi\mathcal{C}$$

Proof. Let $\mathcal{C}$ be a small category and $\pi\mathcal{C}$ the associated grupoid. There exists a functor $\pi_{\mathcal{C}} : \mathcal{C} \longrightarrow \pi\mathcal{C}$ which is an identity on the set of objects and maps each morphism $l \in \operatorname{Mor}\mathcal{C}$ to the equivalence class $[l^+] \in \pi_1(\mathcal{C}, i(l), t(l))$.

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a morphism in Cat. Then there exists an induced functor $\pi F : \pi\mathcal{C} \longrightarrow \pi\mathcal{D}$ of the associated grupoids given by $F : \operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{D}$ on the set of objects and $\pi F([(l_1^{\pm}, \ldots, l_k^{\pm})]) = [(F(l_1)^{\pm}, \ldots, F(l_k)^{\pm})]$ on morphisms. The functor $\pi F : \operatorname{Mor}\pi\mathcal{C} \longrightarrow \operatorname{Mor}\pi\mathcal{D}$ is well defined because if $[(l_1^{\pm}, \ldots, l_k^{\pm})] = [(f_1^{\pm}, \ldots, f_{k'}^{\pm})]$ then $[(F(l_1)^{\pm}, \ldots, F(l_k)^{\pm})] = [(F(f_1)^{\pm}, \ldots, F(f_{k'})^{\pm})]$

For two composable functors F and F' we have $\pi(F \circ F') = \pi F \circ \pi F'$, thus the map $\pi : \text{Cat} \longrightarrow \text{Grp}$ is a functor.

□

Proposition 4.1.3. *If $F : \mathcal{C} \longrightarrow \mathcal{D}$ makes the morphisms of $\mathcal{C}$ invertible then there exists an extension of F on $\pi\mathcal{C}$, i.e. a functor $\widetilde{F} : \pi\mathcal{C} \longrightarrow \mathcal{D}$ such that the following diagram*

$$\begin{array}{ccc} & & \pi\mathcal{C} \\ & {}^{\pi_{\mathcal{C}}}\nearrow & \downarrow{\scriptstyle \widetilde{F}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

commutes.

Proof. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Assume that for each $l \in \operatorname{Mor}\mathcal{C}$ the image $F(l)$ is an invertible morphism in $\mathcal{D}$. We extend the functor F to a functor $\widetilde{F} : \pi\mathcal{C} \longrightarrow \mathcal{D}$ as follows; $\widetilde{F}([l^+]) = F(l)$ and $\widetilde{F}([l^-]) = F(l)^{-1}$ and then
$\widetilde{F}([(l_1^{\pm}, \ldots, l_k^{\pm})]) = F(l_1)^{\pm} \circ \ldots \circ F(l_k)^{\pm}$. Therefore the following diagram

$$\begin{array}{ccc} & & \pi\mathcal{C} \\ & {}^{\pi_{\mathcal{C}}}\nearrow & \downarrow{\scriptstyle \widetilde{F}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

commutes. □

Remark 4.1.4. Note that the extension $\widetilde{F} : \pi\mathcal{C} \longrightarrow \mathcal{D}$ of F is unique.

Corollary 4.1.5. *Let $\mathcal{C}$ be a small category, $\pi\mathcal{C}$ the associated grupoid and $\pi_{\mathcal{C}} : \mathcal{C} \longrightarrow \pi\mathcal{C}$ the natural functor. Then $\pi_{\mathcal{C}}$ is initial for functors $\mathcal{C} \longrightarrow \mathcal{X}$ of $\mathcal{C}$ to any groupoid $\mathcal{X}$, i.e. for each functor $F : \mathcal{C} \longrightarrow \mathcal{X}$ there exists a unique functor $\widetilde{F} : \pi\mathcal{C} \longrightarrow \mathcal{X}$ such that the following diagram*

$$\begin{array}{ccc} & & \pi\mathcal{C} \\ & {}^{\pi_{\mathcal{C}}}\nearrow & \downarrow{\scriptstyle \widetilde{F}} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{X} \end{array}$$

commutes.

Remark 4.1.6. If $\mathcal{C}$ is a grupoid then $\pi\mathcal{C} = \mathcal{C}$.

Proof. Let γ be a path in $\mathcal{C}$. All morphisms of $\mathcal{C}$ are invertible thus $\gamma = l \in \operatorname{Mor}\mathcal{C}$. Moreover if two paths γ and γ' are homotopic then they are equal morphisms of $\mathcal{C}$. Thus $\operatorname{Mor}\pi\mathcal{C} = \operatorname{Mor}\mathcal{C}$ and then $\pi\mathcal{C} = \mathcal{C}$. □

Corollary 4.1.7. *The functor* $\pi : \mathrm{Cat} \longrightarrow \mathrm{Gr}$ *is left adjoint to the inclusion* $\mathrm{Grp} \subset \mathrm{Cat}$.

Definition 4.1.8. *The fundamental group of* $\mathcal{C}$ *is defined to be the fundamental group of its geometric realization*

$$\pi_1(\mathcal{C}, c) := \pi_1(\mathrm{B}\,\mathcal{C}, c)$$

Definition 4.1.9. *We say that a small category* $\mathcal{C}$ *is connected if the geometric realization of* $\mathcal{C}$ *is a connected topological space.*

Definition 4.1.10. *Let* $\mathcal{C}$ *be a connected small category. Consider a graph* $\mathrm{B}\,\mathcal{C}^{(1)}$ *whose set of vertices is* $\mathrm{Ob}\,\mathcal{C}$ *and whose set of* 1*-cells is* $\mathrm{Mor}\,\mathcal{C}$*; an element* $l \in \mathrm{Mor}\,\mathcal{C}$ *is considered as an edge joining the vertices* $i(l)$ *and* $t(l)$*. Let* T *be any maximal tree in* $\mathrm{B}\,\mathcal{C}^{(1)}$*. Let* $\mathcal{T} \subset \mathrm{Mor}\,\mathcal{C}$ *be the subset of morphisms of the category* $\mathcal{C}$ *associated to the maximal tree* T*. We define a maximal tree of the category* $\mathcal{C}$ *to be* $\mathcal{T} \subset \mathrm{Mor}\,\mathcal{C}$*.*

Remark 4.1.11. The fundamental group of $\mathcal{BC}$ is isomorphic to the fundamentl group of its 2-skeleton, namely $\pi_1(\mathrm{B}\,\mathcal{C}, c_0) = \pi_1(\mathrm{B}\,\mathcal{C}^{(2)}, c_0)$. Moreover there exists a homotopy equivalence $\mathrm{B}\,\mathcal{C} \approx (\mathrm{B}\,\mathcal{C})/T$, which implies $\pi_1(\mathcal{C}, c_0) \simeq \pi_1(\mathrm{B}\,\mathcal{C}^{(2)}/T, *)$.

As a corollary we obtain the following presentation of the fundamental group of the given category $\mathcal{C}$

Corollary 4.1.12. *The fundamental group of a connected category* $\mathcal{C}$ *is isomorphic to the group* $\pi_1(\mathcal{C}, \mathcal{T})$ *given by the following presentation. It is generated by the set*

$$\coprod \mathrm{Mor}\,\mathcal{C}$$

subjected to the relations

1. $(l^+)^{-1} = l^{-1}$ *and* $(l^{-1})^{-1} = l^+$
2. $l^+ l'^+ = (ll')^+$ *for a pair* (l, l') *of composable morphisms*
3. $l = 1 \quad \forall l \in \mathcal{T}$

Proposition 4.1.13. *Let* c_0 *be an object of the small connected category* $\mathcal{C}$*. There exists an isomorphism of groups* $\Theta : \mathrm{Aut}_{\pi\mathcal{C}}(c_0) \longrightarrow \pi_1(\mathcal{C}, \mathcal{T})$*.*

Proof. Each element of the group $\mathrm{Aut}_{\pi\mathcal{C}}(c_0)$ is given by a sequence $(\alpha_1, \ldots, \alpha_k)$ of composable morphisms of $\pi\mathcal{C}$ such that $i(\alpha_k) = t(\alpha_1) = c_0$ and each morphism α_i equals l_i^+ or l_i^- where $l_i \in \mathrm{Mor}\,\mathcal{C}$. We define $\Theta : \mathrm{Aut}_{\pi\mathcal{C}}(c_0) \longrightarrow \pi_1(\mathcal{C}, \mathcal{T})$ to be a map sending

$$\alpha_1 \circ \ldots \circ \alpha_k \longrightarrow \alpha_1 \ldots \alpha_k$$

For each object $c \in \mathrm{Ob}\,\pi\mathcal{C}$, let $\gamma_c = (\alpha_1, \ldots, \alpha_k)$ be the unique sequence of composable maps in $\pi\mathcal{C}$ such that no consecutive elements are inverse to each other, $t(\alpha_1) = c_0$, $i(\alpha_k) = c$ and each α_i is contained in $\mathcal{T}' = \pi_{\mathcal{C}}(\mathcal{T})$. Let α_c denote the composition $\alpha_1 \circ \ldots \circ \alpha_k$ of morphisms from γ_c.

Then we define $\Theta' : \pi_1(\mathcal{C}, \mathcal{T}) \longrightarrow \mathrm{Aut}_{\pi\mathcal{C}}(c_0)$ to be a homomorphism mapping the generator l^+ to the morphism $\alpha_{t(l)} l^+ \alpha_{i(l)}^{-1}$. This homomorphism is well defined because the relations are satisfied; in particular $\Theta'(l^+) = 1$ if $l \in \mathcal{T}$. The group $\mathrm{Aut}_{\pi\mathcal{C}}(c_0)$ is generated by the elements of the form $\alpha_{t(l)} l^+ \alpha_{i(l)}^{-1}$ and the homomorphisms Θ and Θ' are inverse to each other. ∎

Definition 4.1.14. *Let X be a topological space. The grupoid πX associated to X is a small category whose objects are elements $x \in X$ and morphisms are given by homotopy equvalence classes of paths $\omega \subset X$, i.e.*

$$\mathrm{Mor}_{\pi X}(x, y) := \{[\omega] \mid i(\omega) = x,\ t(\omega) = y\}$$

Proposition 4.1.15. *Let $\mathcal{X}$ be a connected grupoid, $x \in \mathrm{Ob}\,\mathcal{X}$. The inclusion of the small category $\mathrm{Aut}_{\mathcal{X}}(x) \hookrightarrow \mathcal{X}$ is an equivalence of small categories.*

Proof. Each object of $\mathcal{X}$ is isomorphic with x. Thus the inclusion $\mathrm{Aut}_{\mathcal{X}}(x) \hookrightarrow \mathcal{X}$ is an equivalence of small categories. □

Remark 4.1.16. Let X be a topological space and πX the associated grupoid. Then the fundamental group $\pi_1(X, x)$ is defined as the group $\mathrm{Aut}_{\pi X}(x)$.

Remark 4.1.17. Let $\mathcal{C}$ be a small category, $\mathrm{B}\,\mathcal{C}$ its geometric realization and $\pi\mathcal{C}$, $\pi\,\mathrm{B}\,\mathcal{C}$ the associated grupoids. There exists a natural functor $I : \pi\mathcal{C} \longrightarrow \pi\,\mathrm{B}\,\mathcal{C}$ which sends an object c to the vertex $c \in \mathrm{B}\,\mathcal{C}$. Each morphism $[\gamma]$ of $\pi\mathcal{C}$ is maped to the equivalence class of the corrsponding edge path $\omega \subset (\mathrm{B}\,\mathcal{C})^{(1)}$ in the 1-skeleton of the geometric realization. If $[\gamma] = [\gamma']$ then the corresponding paths ω and ω' are homotopic in $\mathrm{B}\,\mathcal{C}$. Thus I is well defined functor.

Proposition 4.1.18. *Let $\mathcal{C}$ be a small category and $\mathrm{B}\,\mathcal{C}$ its geometric realization. Then the natural functor $I : \pi\mathcal{C} \longrightarrow \pi\,\mathrm{B}\,\mathcal{C}$ defined in 4.1.17 is an inclusion and an equivalence of categories.*

Proof. Let $c, d \in \mathrm{Ob}\,\mathcal{C}$. According to the cellular aproximation theorem the functor $\mathrm{Mor}_{\pi\mathcal{C}}(c, d) \longrightarrow \mathrm{Mor}_{\pi\,\mathrm{B}\,\mathcal{C}}(c, d)$ induced by I is onto. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a connected component of $\mathcal{C}$. Then due to 4.1.13, 4.1.15 and 4.1.16 for each object c of $\mathcal{C}_0$ the following diagram

$$\begin{array}{ccc} \pi\mathcal{C}_0 & \xrightarrow{\ I\ } & \pi\,\mathrm{B}\,\mathcal{C}_0 \\ \approx\uparrow & & \uparrow\approx \\ \mathrm{Aut}_{\pi\mathcal{C}_0}(c) & \xrightarrow{\ \simeq\ } & \mathrm{Aut}_{\pi\,\mathrm{B}\,\mathcal{C}_0}(c) \end{array}$$

commutes. Therefore the functor $\mathrm{Mor}_{\pi\mathcal{C}_0}(c, d) \longrightarrow \mathrm{Mor}_{\pi\,\mathrm{B}\,\mathcal{C}_0}(c, d)$ induced by I is an inclusion. Hence $\mathrm{Mor}_{\pi\mathcal{C}_0}(c, d) \simeq \mathrm{Mor}_{\pi\,\mathrm{B}\,\mathcal{C}_0}(c, d)$. The small category $\pi\,\mathrm{B}\,\mathcal{C}_0$ is a connected grupoid, thus each two objects of $\pi\,\mathrm{B}\,\mathcal{C}_0$ are isomorphic. Therefore $I_{|\mathcal{C}_0} : \pi\mathcal{C}_0 \longrightarrow \pi\,\mathrm{B}\,\mathcal{C}_0$ is an equivalence of small categories. It is inclusion because it is inclusion on the set of objects. This implies $I : \pi\mathcal{C} \longrightarrow \pi\,\mathrm{B}\,\mathcal{C}$ is inclusion and equivalence of categories. □

Proposition 4.1.19. *For a given category $\mathcal{C}$ the fundamental groups of $\mathcal{C}$ and $\pi\mathcal{C}$ are isomorphic.*

Proof. According to 4.1.6 the small category $\pi\pi\mathcal{C}$ is equal to $\pi\mathcal{C}$. Thus according to 4.1.15 the fundamental groups $\pi_1(\mathcal{C}, c)$ and $\pi_1(\pi\mathcal{C}, c)$ are isomorphic. □

Remark 4.1.20. Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor. The restriction of the induced functor $\pi F : \pi\mathcal{C} \longrightarrow \pi\mathcal{C}'$, $\pi F_{|} : \mathrm{Aut}_{\pi\mathcal{C}}(c) \longrightarrow \mathrm{Aut}_{\pi\mathcal{C}'}(F(c))$ defines a homomorphism of fundamental groups

$$\pi_1 F : \pi_1(\mathcal{C}, c) \longrightarrow \pi_1(\mathcal{C}', F(c))$$

Remark 4.1.21. Let c_1, c_2 be objects of the category $\mathcal{C}$. Let $\gamma \in \operatorname{Mor} \pi\mathcal{C}$ be a morphism such that $i(\gamma) = c_1$ and $t(\gamma) = c_2$. Then there exists a homomorphism of fundamental groups $\phi : \pi_1(\mathcal{C}, c_1) \longrightarrow \pi_1(\mathcal{C}, c_2)$ given by $\phi(g) = \gamma g \gamma^{-1}$ where $g \in \operatorname{Aut}_{\pi\mathcal{C}}(c_1)$.

Theorem 4.1.22. *Let $F : \mathcal{D} \longrightarrow \mathrm{Cat}$ be a functor such that for each each $d \in \operatorname{Ob} \mathcal{D}$ the corresponding category $F(d) = \mathcal{C}_d$ is connected. Consider a map $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ assigning to each object $d \in \operatorname{Ob} \mathcal{D}$ first the base object $c_d \in \mathcal{C}_d$ and then the fundamental group of $\mathcal{C}_d$, i.e.*

$$\mathcal{F}(d) = \operatorname{Aut}_{\pi\mathcal{C}_d}(c_d)$$

For each morphism $l : d \longrightarrow d'$ we define a homomorphisms of groups $\mathcal{F}(l) : \mathcal{F}(d) \longrightarrow \mathcal{F}(d')$ to be the composition $\operatorname{Aut}_{\pi\mathcal{C}_d}(c_d) \longrightarrow \operatorname{Aut}_{\pi\mathcal{C}_{d'}}(F(l)(c_d)) \longrightarrow \operatorname{Aut}_{\pi\mathcal{C}_{d'}}(c_{d'})$ of homomorphisms defined in 4.1.20 and 4.1.21. Then $\mathcal{F}$ is a twisted diagram of groups.

Moreover, there exists a weak natural transformation of weak functors

$$\eta : \mathcal{F} \implies \pi F$$

given by $\eta_d : \operatorname{Aut}_{\mathcal{C}_d}(c_d) \overset{\approx}{\hookrightarrow} \pi\mathcal{C}_d$.

Proof. The homomorphism $\mathcal{F}(l) : \mathcal{F}(d) \longrightarrow \mathcal{F}(d')$ is given by the composition

$$\operatorname{Aut}_{\pi\mathcal{C}_d}(c_d) \xrightarrow{\pi F(l)} \operatorname{Aut}_{\pi\mathcal{C}_{d'}}(F(l)(c_d)) \xrightarrow{\operatorname{Ad}(\gamma_l)} \operatorname{Aut}_{\pi\mathcal{C}_{d'}}(c_{d'})$$

where $\gamma_l \in \operatorname{Mor} \pi\mathcal{C}_{d'}$ such that $i(\gamma_l) = F(l)(c_d)$ and $t(\gamma_l) = c_{d'}$. Let $d_2 \xrightarrow{l_1} d_1 \xrightarrow{l_0} d_0$ be morphisms of $\mathcal{D}$. Then the composition $\mathcal{F}(l_0)\mathcal{F}(l_1)$ differs from the homomorphism $\mathcal{F}(l_0 l_1)$ by the conjugation with an element

$$g_{l_0,l_1} = \gamma_{l_0} \circ \pi F(l_0)(\gamma_{l_1}) \circ \gamma_{l_0 l_1}^{-1} \in \operatorname{Aut}_{\pi\mathcal{C}_{d_0}}(c_{d_0})$$

These elements satisfy the cocycle condition defined in 2.1.3 thus $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ is a twisted diagram of groups. Note that different choice of the objects $\{c_d \in \mathcal{C}_d\}_{d \in \operatorname{Ob} \mathcal{D}}$ gives an isomorphic twisted diagram of groups and different choice of the paths $\{\gamma_l\}_{l \in \operatorname{Mor} \mathcal{D}}$ gives a twisted diagram which differs from $\mathcal{F}$ by a coboundary.

The following diagram

$$\begin{array}{ccc} \mathcal{F}(d) & \xrightarrow{\mathcal{F}(l)} & \mathcal{F}(d') \\ \downarrow & & \downarrow \\ \pi F(d) & \xrightarrow{\pi F(l)} & \pi F(d') \end{array}$$

is commutative up to a natural transformation $\operatorname{Ad}(\gamma_l)$. These diagrams define a weak natural transformation $\eta : \mathcal{F} \implies \pi F$ (1.2.5).

□

4.2 Fundamental group of a twisted diagram of groups

Definition 4.2.1. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $\mathcal{B}\mathcal{G}$ its classifying category. We define a fundamental group of a twisted diagram of groups to be the fundamentl group of its classifying category*

$$\pi_1(\mathcal{G}, c_0) := \pi_1(\mathcal{B}\mathcal{G}, c_0)$$

Fundamental group of a (connected) homotopy colimit Let $F : \mathcal{D} \longrightarrow \mathrm{Cat}$ be any functor and $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ the twisted diagram of groups defined in 4.1.22. The natural question is how the fundamental group of this twisted diagram of groups is related to F. The following Theorem answers this question:

Theorem 4.2.2. *Let $F : \mathcal{D} \longrightarrow \mathrm{Cat}$ be a functor such that each $F(d)$ is a connected category. Then the fundamental group of the Grothendieck construction $\mathcal{B}F$ is isomorphic to the fundamental group of the twisted diagram of groups $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{Gr}$ defined in 4.1.22.*

Proof. According to 4.1.22 there exists a weak natural transformation $\eta : \mathcal{F} \Longrightarrow \pi F$ such that for each object d of $\mathcal{D}$, η_d is an equivalence of small categories. Then due to 1.3.9

$$\pi_1(\mathcal{B}\mathcal{F}, d_0) \simeq \pi_1(\mathcal{B}(\pi F), d_0)$$

The small category $\mathcal{B}F$ is connected (because each $F(d)$ is connected) thus $\pi_1(\mathcal{B}F, d_0) \overset{\approx}{\hookrightarrow} \pi(\mathcal{B}F)$. The map $\pi(\mathcal{B}F) \longrightarrow \pi(\mathcal{B}(\pi F))$ sending $[(l,f)]^+ \longrightarrow [(l,[f]^+)]^+$ is an isomorphism. Therefore,

$$\pi_1(|\operatorname{hocolim} \mathrm{N}\, F|, d_0) \simeq \pi_1(\mathcal{B}F, d_0) \simeq \pi_1(\mathcal{B}\mathcal{F}, d_0)$$

□

Presentations of the fundamental group of a twisted diagram of groups

Proposition 4.2.3. *Let $\mathcal{C}$ be a small category. There exists a functor $\iota : \mathcal{C} \longrightarrow \mathcal{B}\pi_1(\mathcal{C}, c)$.*

Proof. The functor ι is given by the composition $\mathcal{C} \xrightarrow{\pi_{\mathcal{C}}} \pi\mathcal{C} \xrightarrow{j} \pi_1(\mathcal{C}, c)$ where j is an inverse functor of the equivalence $\mathrm{Aut}_{\pi\mathcal{C}}(c) \hookrightarrow \pi\mathcal{C}$.

We define $j : \pi\mathcal{C} \longrightarrow \mathrm{Aut}_{\pi\mathcal{C}}(c)$ as follows. For each $c_1 \in \mathrm{Ob}\,\pi\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$ we choose a morphism $\alpha_{c_1} \in \mathrm{Mor}\,\pi\mathcal{C}$ such that $i(\alpha) = c$ and $t(\alpha) = c_1$. Then for each $\gamma \in \mathrm{Mor}\,\pi\mathcal{C}$ we define $j(\gamma) = \alpha^{-1}_{t(\gamma)}\gamma\alpha_{i(\gamma)}$ □

Proposition 4.2.4. *Let $\iota : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{B}\pi_1(\mathcal{G}, c_0)$ be a functor defined in 4.2.3. The following commutative diagram*

$$\begin{array}{ccc} \mathcal{B}\mathcal{G} & \xrightarrow{\iota} & \mathcal{B}\pi_1(\mathcal{G}, c_0) \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & * \end{array}$$

defines a homomorphism in the category of twisted diagrams of groups

$$\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$$

Remark 4.2.5. If $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a trivial diagram of groups (i.e. all of $\mathcal{G}(c) = 1$), then $\pi_1(\mathcal{G}, c_0) = \pi_1(\mathcal{C}, c_0)$ and $\iota_{\mathcal{G}} = \iota$.

More generally,

Proposition 4.2.6. *Homomorphism of $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ to a trivial diagram of groups $\mathcal{I} : \mathcal{C} \longrightarrow \mathrm{Gr}$ induces a surjective homomorphism $\pi_1(\mathcal{G}, c_0) \longrightarrow \pi_1(\mathcal{C}, c_0)$ which splits if $\mathcal{G}$ is a diagram of groups.*

Proof. The homomorphism $\mathcal{G} \longrightarrow \mathcal{I}$ of twisted diagrams of groups induces a projection $p : \mathcal{BG} \longrightarrow \mathcal{C}$ of its classifying categories. Consider a map associating to each loop $\gamma = (\alpha_1, \dots, \alpha_k)$ in $\mathcal{BG}$ the loop $(p(\alpha_1), \dots, p(\alpha_k))$ in $\mathcal{C}$. This map defines a homomorphism $\pi_1(\mathcal{G}, c_0) \longrightarrow \pi_1(\mathcal{C}, c_0)$. According to 2.2.9 the functor p is onto and it is the identity on the set of objects. Then each loop γ' in $\mathcal{C}$ can be lifted to a loop γ in $\mathcal{BG}$.

If $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a diagram of groups, then according to 2.2.10, the functor p splits thus the epimorphism $\pi_1(\mathcal{G}, c_0) \longrightarrow \pi_1(\mathcal{C}, c_0)$ splits also. □

Theorem 4.2.7. *Let $\mathcal{T}$ be a maximal tree in $\mathcal{C}$. Then the fundamental group $\pi_1(\mathcal{G}, c_0)$ is isomorphic to the group $\pi_1(\mathcal{G}, \mathcal{T})$ which has the following presentation:*

The generators are all elements of $\mathcal{G}(c)$ for each $c \in \mathrm{Ob}\,\mathcal{C}$ and all elements $l \in \mathrm{Mor}\,\mathcal{C}$. The relations are:

1. *the relations in the groups $\mathcal{G}(c)$*
2. *$(l^+)^{-1} = l^{-1}$ and $(l^{-1})^{-1} = l^+$*
3. *for $l \in \mathrm{Mor}\,\mathcal{C}$, $h \in \mathcal{G}(i(l))$, then $\mathcal{G}(l)(h) = l^+ h l^{-1}$*
4. *for a pair (l, l') of composable morphisms $l^+ l'^+ = g_{l,l'}(ll')^+$*
5. *$l = 1$ for $l \in \mathcal{T}$*

Proof. First note that the maximal tree in the category $\mathcal{C}$ yields a maximal tree in $\mathcal{BG}$. Then use 2.2.9 and 4.1.12. □

Remark 4.2.8. There exists a functor $\mathcal{BG} \longrightarrow \pi_1(\mathcal{G}, \mathcal{T})$ which sends (l, g) to gl^+. This functor defines a homomorphism $i_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, \mathcal{T})$.

Fundamental group and colimit of a diagram of groups Assume that we have a diagram of groups that is a functor $F : \mathcal{C} \longrightarrow \mathrm{Gr}$. The diagram of groups is a special case of a twisted diagram of groups thus we have a fundamental group of this diagram. For a given diagram of groups we have also its direct limit. The following section concerns the relation between the colimit and the fundamental group of a diagram of groups.

Remark 4.2.9. A morphism $\Phi = (\Phi_c, \Phi(l))$ from a twisted diagram $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ to a group G, where G is considered as a twisted diagram of groups, consists of a homomorphism $\Phi_c : \mathcal{G}(c) \longrightarrow G$ for each $c \in \mathrm{Ob}\,\mathcal{C}$ and an element $\Phi(l) \in G$ for each $l \in \mathrm{Mor}\,\mathcal{C}$ such that

1. $\Phi_{t(l)} \mathcal{G}(l) = \mathrm{Ad}(\Phi(l)) \Phi_{i(l)}$
2. $\Phi_{t(l)}(g_{l,l'}) \Phi(ll') = \Phi(l) \Phi(l')$

We say that Φ is simple if $\mathcal{G}$ is a diagram of groups and each $\Phi(l)$ is trivial.

Let $\mathcal{T}$ be a maximal tree of the small category $\mathcal{C}$.

Proposition 4.2.10. *Let $F : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a diagram of groups and $I : \mathcal{C} \longrightarrow \mathrm{Gr}$ a trivial diagram of groups. There exists a commutative diagram*

$$\begin{array}{ccc} F & \xrightarrow{\phi} & I \\ \downarrow{\scriptstyle i_F} & & \downarrow{\scriptstyle i_I} \\ \pi_1(F,\mathcal{T}) & \xrightarrow{\phi_*} & \pi_1(\mathcal{C},\mathcal{T}) \end{array}$$

where i_F, i_I are defined as in 4.2.8. Moreover, functors $\mathcal{B}\phi$ and ϕ_* split.

Proof. Due to 2.2.10 the natural projection $\mathcal{B}\phi : \mathcal{B}F \longrightarrow \mathcal{C}$ splits and according to 4.2.6 the homomorphism ϕ_* splits. Therefore the following diagram

$$\begin{array}{ccc} \mathcal{B}F & \underset{\mathcal{B}\phi}{\overset{s}{\rightleftarrows}} & \mathcal{C} \\ \downarrow & & \downarrow \\ \pi_1(F,\mathcal{T}) & \underset{\phi_*}{\overset{\bar{s}}{\rightleftarrows}} & \pi_1(\mathcal{C},\mathcal{T}) \end{array}$$

commutes. Clearly the homomorphism $\bar{s}$ sends the generator $l^+ \in \pi_1(\mathcal{C},\mathcal{T})$ to the generator $l^+ \in \pi_1(\mathcal{G},\mathcal{T})$.

□

Remark 4.2.11. The functor $\Phi_F : F \longrightarrow \operatorname{colim} F$ induces a commutaive diagram

$$\begin{array}{ccc} \mathcal{B}F & \rightleftarrows & \mathcal{C} \\ \downarrow & & \downarrow \\ \operatorname{colim}_{\mathcal{C}} F & \rightleftarrows & 1 \end{array}$$

The morphism (l, g) is mapped to $\mathcal{B}\Phi_F(g) \in \operatorname{colim} F$.

Proposition 4.2.12. *Let $\pi_1(F,\mathcal{T})$ be a fundamental group of a given diagram of groups $F : \mathcal{C} \longrightarrow \mathrm{Gr}$ and $i_F : F \longrightarrow \pi_1(F,\mathcal{T})$ the induced homomorphism. There exists a homomorphism of groups $\varphi_F : \pi_1(F,\mathcal{T}) \longrightarrow \operatorname{colim}_{\mathcal{C}} F$ such that $\Phi_F = \varphi_F \circ i_F$.*

Proof. The functor φ_F maps the generator g to $\mathcal{B}\Phi_F(g)$ and the generator l^+ to the trivial element of the group $\operatorname{colim} F$. Clearly φ_F is well defined and $\Phi_F = \varphi_F \circ i_F$.

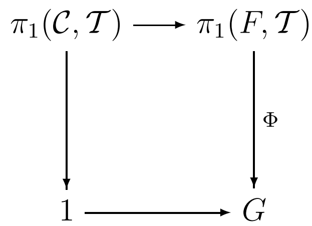
□

E.D. Farjoun has proved that the direct limit of the diagram of groups is a push-out of a certain diagram of groupoids (Corollary 5.4, [Fa]). The following Theorem was motivated by this observation.

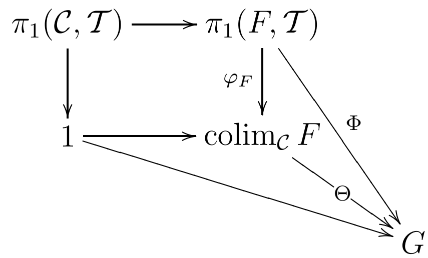
Theorem 4.2.13. *Let $F : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a diagram of groups. Then the group $\operatorname{colim} F$ is the push-out of the following diagram*

$$\begin{array}{ccc} \pi_1(\mathcal{C},\mathcal{T}) & \longrightarrow & \pi_1(F,\mathcal{T}) \\ \downarrow & & \downarrow{\scriptstyle \varphi_F} \\ 1 & \longrightarrow & \operatorname{colim}_{\mathcal{C}} F \end{array}$$

Proof. Consider a commutative diagram of groups

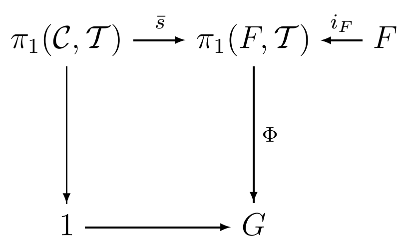


We will prove that there exists a unique homomorphism of groups $\Theta : \operatorname{colim}_{\mathcal{C}} F \longrightarrow G$ such that the following diagram commutes

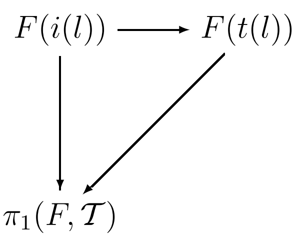


$(\star)$

Consider



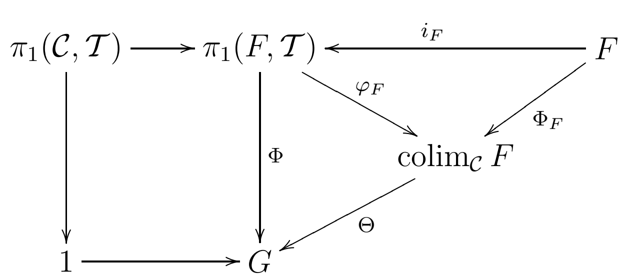
According ot 4.2.7 for each $l \in \operatorname{Mor}\mathcal{C}$ the corresponding diagram



commutes up to a conjugation with an element $l^+ \in \pi_1(F,\mathcal{T})$. Clearly $l^+ = \bar{s}(l^+)$, hence the composition $\Phi \circ i_F : F \longrightarrow G$ is a simple homomorphism. Then there exists a unique homomorphism

$$\Theta : \operatorname*{colim}_{\mathcal{C}} F \longrightarrow G$$

such that the following diagram is commutative



Therefore diagram $(\star)$ commutes which proves the Theorem.

$\square$

Corollary 4.2.14. *Let $F : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a diagram of groups. Then the fundamental group of F is isomorphic to the colimit of this diagram if and only if the geometric realization of the category $\mathcal{C}$ is simply connected.*

Chapter 5

Coverings of small categories and developable twisted diagrams of groups

The following Chapter starts with the theory of coverings of small categories. We say that a functor $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ is a covering if its geometric realization is a topological covering. A category $\mathrm{Cov}_{\mathcal{C}}$ of coverings of the given category $\mathcal{C}$ is a category whose objects are coverings of $\mathcal{C}$ and morphisms correspond to functors $F : \mathcal{C}_1' \longrightarrow \mathcal{C}_2'$ over the identity of $\mathcal{C}$. The geometric realization functor yields a functor $\mathrm{Cov}_{\mathcal{C}} \longrightarrow \mathrm{Cov}_{\mathrm{B}\,\mathcal{C}}$. We will prove that this functor is an equivalence of categories.

We say that a group G acts without inversion on a small category $\mathcal{D}$ if for each $g \in G$ and $d \in \mathrm{Ob}\,\mathcal{D}$ such that $gd = d$ we have $gk = k$ for each morphism $k \in \mathrm{Mor}\,\mathcal{D}$ such that $i(k) = d$. Given such an action one can define a quotient category $\mathcal{D}/G$ of this action and the induced projection $\mathcal{D} \longrightarrow \mathcal{D}/G$ is so called right covering. We will prove in Section 5.2 that the action without inversion yields an action of G on the geometric realization of $\mathcal{D}$ and $\mathrm{B}(\mathcal{D}/G) = (\mathrm{B}\,\mathcal{D})/G$. Assume that G acts freely on a small category $\mathcal{D}$, clearly a free action is an action without inversion. Then the geometric realization of $\mathcal{D} \longrightarrow \mathcal{D}/G$ is a G-covering.

Given an action without inversion of a group G on a small category $\mathcal{D}$, one can associate a twisted diagram of groups $\mathcal{G} : \mathcal{D}/G \longrightarrow \mathrm{Gr}$. Let $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ be the natural projection. Then for each object $c \in \mathrm{Ob}\,\mathcal{D}/G$ the group $\mathcal{G}(c)$ is isomorphic to the isotorpy subgroup of each $d \in p^{-1}(c) \subset \mathrm{Ob}\,\mathcal{D}$. If G acts freely on $\mathcal{D}$ then the isotropy subgroups are trivial, hence $\mathcal{BG} = \mathcal{D}/G$ and the functor $p : \mathcal{D} \longrightarrow \mathcal{BG} = \mathcal{D}/G$ is a G- covering. We will prove in Section 5.3 that for each twisted diagram of groups $\mathcal{G} : \mathcal{D}/G \longrightarrow \mathrm{Gr}$ associated to an action, there exists a functor $\mathcal{D} \longrightarrow \mathcal{BG}$ over $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ such that

- there exists a G-covering $\mathcal{E} \longrightarrow \mathcal{BG}$
- there exists an inclusion and an equivalence of categories $\mathcal{D} \hookrightarrow \mathcal{E}$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D} & \overset{\approx}{\hookrightarrow} & \mathcal{E} \\ & \searrow & \downarrow \\ & & \mathcal{BG} \end{array}$$

The action of the group G on the small category $\mathcal{D}$ yields a functor $S_{\mathcal{D}} : \mathcal{B}G \longrightarrow \mathrm{Cat}$. Let $\mathcal{E}G$ be the universal covering of the small category $\mathcal{B}G$. We will prove that the following diagram commutes

$$
\begin{array}{ccccc}
\mathcal{D} & \overset{\approx}{\hookrightarrow} & \mathcal{E} & \overset{\approx}{\hookrightarrow} & \mathcal{E}G \times \mathcal{D} \\
& \searrow & \downarrow{\scriptstyle /G} & & \downarrow{\scriptstyle /G} \\
& & \mathcal{B}\mathcal{G} & \overset{\approx}{\hookrightarrow} & \mathcal{B}S_{\mathcal{D}} \\
& & & & \downarrow \\
& & & & \mathcal{B}G
\end{array}
$$

Section 5.4 proves the developability theorems of Bridson and Haeflieger ([H1], [B-H]). Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and G a group. Given a homomorphism $\Phi : \mathcal{G} \longrightarrow G$ one can associate to it a small category $\mathcal{D}(\mathcal{G}, \Phi)$ with an action of the group G. Moreover a twisted diagram of groups $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ associated to this action is such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{G} & \overset{\varphi}{\longrightarrow} & \overline{\mathcal{G}} \\
& {\scriptstyle \Phi}\searrow & \downarrow{\scriptstyle \overline{\Phi}} \\
& & G
\end{array}
$$

where homomorphisms φ and $\overline{\Phi}$ are respectively surjective and injective on the local groups.

We say that a twisted diagram of groups is developable if it is equivalent to a twisted diagram of groups associated to an action. We will prove that it is the case when there exists a group G and a homomorphism $\Phi : \mathcal{G} \longrightarrow G$ which is injective on the local groups. Moreover, if $\mathcal{G}$ is developable then the functor $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$ is injective on the local groups and there exists a functor $\mathcal{D}(\mathcal{G}, \iota_{\mathcal{G}}) \longrightarrow \mathcal{D}(\mathcal{G}, \Phi)$. If the induced homomorphism $\Phi_* : \pi_1(\mathcal{G}, c_0) \longrightarrow G$ is surjective then this functor is a covering of small categories.

5.1 Coverings of small categories

A covering of a small category is a functor whose geometric realization is a toplogical covering. We will prove that for a given small category $\mathcal{C}$ the category of coverings of $\mathcal{C}$ is equivalent to the category of topological coverings of the geometric realization $\mathrm{B}\,\mathcal{C}$ of $\mathcal{C}$.

Definition 5.1.1. *Let $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ be a functor from a small category $\mathcal{C}'$ to a small connected category $\mathcal{C}$. We say that a functor ϕ is a covering if the geometric realization* $\mathrm{B}\,\phi : \mathrm{B}\,\mathcal{C}' \longrightarrow \mathrm{B}\,\mathcal{C}$ *is a topological covering.*

The category of coverings of $\mathcal{C}$ denoted $\mathrm{Cov}_{\mathcal{C}}$ *is a category whose objects are coverings $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ and morphism between two coverings $\phi_1 : \mathcal{C}'_1 \longrightarrow \mathcal{C}$, $\phi_2 : \mathcal{C}'_2 \longrightarrow \mathcal{C}$ is a functor $F : \mathcal{C}'_1 \longrightarrow \mathcal{C}'_2$ such that $\phi_1 F = \phi_2$.*

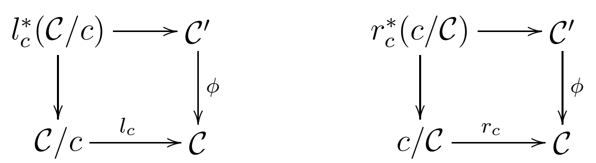
Let G be a group, we define a small category $\mathcal{E}G$ to be a category whose objects corresponds to elements of the group G and for each pair $g_1, g_2 \in G$ there exists the unique morphism $g_1 \overset{g_1^{-1}g_2}{\longrightarrow} g_2$. There exists an action of the group G on $\mathcal{E}G$ given by

$$g(g_1 \overset{g_1^{-1}g_2}{\longrightarrow} g_2) = (gg_1 \overset{g_1^{-1}g_2}{\longrightarrow} gg_2)$$

Example 5.1.2. The natural action of G on a small category $\mathcal{E}G$ yields a functor $\mathcal{E}G \longrightarrow \mathcal{B}G$. This functor is a covering because its geometric realization $\mathrm{E}\,G \longrightarrow \mathrm{B}\,G$ is the universal covering of $\mathrm{B}\,G$.

Theorem 5.1.3. *Let $\mathcal{C}$ be a connected category and $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ a functor. The following conditions are equivalent*

1. *$\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ is a covering*
2. *For each $c' \in \mathrm{Ob}\,\mathcal{C}'$ the induced functors $\phi/c' : \mathcal{C}'/c' \longrightarrow \mathcal{C}/\phi(c')$ and $c'/\phi : c'/\mathcal{C}' \longrightarrow \phi(c')/\mathcal{C}$ are bijections on the objects sets*
3. *Let $l_c : \mathcal{C}/c \longrightarrow \mathcal{C}$ and $r_c : c/\mathcal{C} \longrightarrow \mathcal{C}$ be the natural projections. For each $c \in \mathrm{Ob}\,\mathcal{C}$, the following pull-back categories*



are trivial coverings.

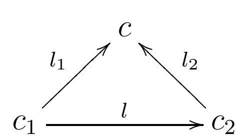
Proof. 1. $\Longrightarrow$ 2.

Assume $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ is a functor such that $\mathrm{B}\,\phi$ is a topological covering. Then the restriction of $\mathrm{B}\,\phi$ to the 1-skeletons $\mathrm{B}\,\phi^{(1)} : \mathrm{B}\,\mathcal{C}'^{(1)} \longrightarrow \mathrm{B}\,\mathcal{C}^{(1)}$ is also a covering. We will prove that for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ and for each $l \in \mathrm{Mor}\,\mathcal{C}$ such that $i(l) = \phi(c')$ (respectively $t(l) = \phi(c')$) there exists unique $l' \in \mathrm{Mor}\,\mathcal{C}'$ such that $i(l') = c'$ (respectively $t(l') = c'$) and $\phi(l') = l$.

For $l : \phi(c') \longrightarrow d$ a morphism in $\mathcal{C}$ its geometric realization $\omega = |l|$ is a path in $\mathrm{B}\,\mathcal{C}^{(1)}$. $\mathrm{B}\,\phi^{(1)}$ is a covering thus there exists a unique path ω' in $\mathrm{B}\,\mathcal{C}'^{(1)}$ such that $i(\omega') = c' \in \mathrm{B}\,\mathcal{C}'^{(0)}$ and $\mathrm{B}\,\phi^{(1)}(\omega') = \omega$. Since ϕ is a functor then ω' is an edge in $\mathrm{B}\,\mathcal{C}'$ corresponding to a morphism $l' \in \mathrm{Mor}\,\mathcal{C}'$ such that $i(l') = c'$ and $\phi(l') = l$. This implies that the restriction of ϕ to the subset of morphisms of $\mathcal{C}'$ that have c' as its initial object is a bijection onto the set of morphisms of $\mathcal{C}$ with initial object $\phi(c')$, hence $c'/\phi : c'/\mathcal{C}' \longrightarrow \phi(c')/\mathcal{C}$ is a bijection on the objects set. Choosing $l : d \longrightarrow \phi(c')$ we can prove the second part of the assertion.

2. $\Longrightarrow$ 3.

First note that if $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ satisfies assertion 2. then for each $c \in \mathrm{Ob}\,\mathcal{C}$ the preimage $\phi^{-1}(c)$ is a subset of $\mathrm{Ob}\,\mathcal{C}'$. The morphisms of $l_c^*(\mathcal{C}/c)$ are pairs (k, l') where $k \in \mathrm{Mor}\,\mathcal{C}/c$ and $l' \in \mathrm{Mor}\,\mathcal{C}'$ such that $l_c(k) = \phi(l')$. The morphism k is given by the diagram



and $l_c(k) = l = \phi(l')$. Assertion 2. implies that there exist the unique pair of morphisms l_1', l_2' and an object c' such that $\phi(l_1') = l_1$, $\phi(l_2') = l_2$ and $\phi(c') = c$ and morphisms l', l_1', l_2' form a diagram $k' \in \mathrm{Mor}\,\mathcal{C}'/c'$. This proves that $l_c^*(\mathcal{C}/c)$ is isomorphic to the disjoint union $\coprod_{c' \in \phi^{-1}(c)} \mathcal{C}'/c'$ hence its geometric realization is a trivial covering of

$\mathrm{B}(\mathcal{C}/c)$. Clearly $r_c^*(c/\mathcal{C}) \simeq \coprod_{c'\in\phi^{-1}(c)} c'/\mathcal{C}'$ and then its geometric realization is a trivial covering of $\mathrm{B}(c/\mathcal{C})$.

3. $\Longrightarrow$ 1.

Let $\mathrm{N}\,\phi : \mathrm{N}\,\mathcal{C}' \longrightarrow \mathrm{N}\,\mathcal{C}$ be a simplicial map of nerves induced by ϕ. A morphism $p : E \longrightarrow X$ of simplicial sets is said to be a covering if for each commutative diagram

$$\begin{array}{ccc} \Delta[0] & \xrightarrow{u} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta[n] & \xrightarrow{v} & X \end{array}$$

there is a unique morphism $s : \Delta[n] \longrightarrow E$ satisfying $p \circ s = v$, $s \circ i = u$.

We will prove that $\mathrm{N}\,\phi$ is a covering of simplicial sets. Let

$$\begin{array}{ccc} \Delta[0] & \xrightarrow{u} & \mathrm{N}\,\mathcal{C}' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathrm{N}\,\phi} \\ \Delta[n] & \xrightarrow{v} & \mathrm{N}\,\mathcal{C} \end{array}$$

be a commutative diagram such that $u(\Delta[0]) = c' \in (\mathrm{N}\,\mathcal{C}')^{(0)} = \mathrm{Ob}\,\mathcal{C}'$. The simplex $\Delta[n] \in \mathrm{N}\,\mathcal{C}$ corresponds to the lenght n sequence $c_n \xrightarrow{l_{n-1}} c_{n-1} \longrightarrow \dots \xrightarrow{l_0} c_0$ of nontrivial morphisms of $\mathcal{C}$. Therefore, there exists $0 \leq m \leq n$ such that $c_m = \phi(c')$. Thus we obtain sequences $c_n \xrightarrow{l_{n-1}} c_{n-1} \longrightarrow \dots \xrightarrow{l_m} \phi(c')$ and $\phi(c') \xrightarrow{l_{m-1}} c_{m-1} \longrightarrow \dots \xrightarrow{l_0} c_0$. These sequences yield the maps $\Delta[n-m] \longrightarrow \mathrm{N}(\mathcal{C}/\phi(c'))$ and $\Delta[m] \longrightarrow \mathrm{N}(\phi(c')/\mathcal{C})$. The functor ϕ satisfies 2. which implies that there exist $\Delta[n-m] \longrightarrow \mathrm{N}(\mathcal{C}'/c')$ and $\Delta[m] \longrightarrow \mathrm{N}(c'/\mathcal{C}')$ which are liftings of the latter maps. These maps yield $s : \Delta[n] \longrightarrow \mathrm{N}\,\mathcal{C}'$ satisfying $(\mathrm{N}\,\phi)\circ s = v$, $s\circ i = u$. The simplicial map $s : \Delta[n] \longrightarrow \mathrm{N}\,\mathcal{C}'$ satisfying the latter is unique because the liftings $\Delta[n-m] \longrightarrow \mathrm{N}(\mathcal{C}'/c')$ and $\Delta[m] \longrightarrow \mathrm{N}(c'/\mathcal{C}')$ are unique.

Thus $\mathrm{N}\,\phi$ is a covering of simplicial sets. According to Theroem 3.2. Appendix I [G-Z] the geometric realization of a covering of simplicial sets is a topological covering. Therefore $\mathrm{B}\,\phi : \mathrm{B}\,\mathcal{C}' \longrightarrow \mathrm{B}\,\mathcal{C}$ is a topological covering.

□

Corollary 5.1.4. *Let $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ be a covering of the connected category $\mathcal{C}$. Let $c \in \mathrm{Ob}\,\mathcal{C}$ and $c' \in \mathrm{Ob}\,\mathcal{C}'$ be such that $\phi(c') = c$. Any path in $\mathcal{C}$ starting at c can be lifted uniquely to a path in $\mathcal{C}'$ starting at c'. Moreover, if two paths starting from c' projects by ϕ to paths which are equal as morphisms in $\pi\mathcal{C}$, then these paths are equal in $\pi\mathcal{C}'$. Thus ϕ induces an injection $\pi_1(\mathcal{C}', c')$ into $\pi_1(\mathcal{C}, c)$.*

Let $\mathcal{D}$ be a connected category and fix $d \in \mathrm{Ob}\,\mathcal{D}$. Let $\phi_1, \phi_2 : \mathcal{D} \longrightarrow \mathcal{C}'$ be two functors such that $\phi \circ \phi_1 = \phi \circ \phi_2$ and $\phi_1(d) = \phi_2(d)$. Then $\phi_1 = \phi_2$.

Corollary 5.1.5. *Let $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ be a covering. Then the induced functor $\pi\phi : \pi\mathcal{C}' \longrightarrow \pi\mathcal{C}$ is a covering and the small category $\mathcal{C}'$ is the pull-back of the following diagram*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \pi\mathcal{C}' \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \pi\phi} \\ \mathcal{C} & \longrightarrow & \pi\mathcal{C} \end{array}$$

Proposition 5.1.6. *Let $\phi : (\mathcal{C}', c_0') \longrightarrow (\mathcal{C}, c_0)$ be a covering of small categories and $\lambda : (\mathcal{D}, d_0) \longrightarrow (\mathcal{C}, c_0)$ a functor, where $\mathcal{D}$ is connected small category. Then a lifting $\widetilde{\lambda} : (\mathcal{D}, d_0) \longrightarrow (\mathcal{C}', c_0')$ of λ exists if and only if $\lambda_*(\pi_1(\mathcal{D}, d_0)) \subset \phi_*(\pi_1(\mathcal{C}', c_0'))$.*

Proof. If a lifting $\widetilde{\lambda} : (\mathcal{D}, d_0) \longrightarrow (\mathcal{C}', c_0')$ of λ exists then obviousely
$\lambda_*(\pi_1(\mathcal{D}, d_0)) \subset \phi_*(\pi_1(\mathcal{C}', c_0'))$.

Assume that $\lambda_*(\pi_1(\mathcal{D}, d_0)) \subset \phi_*(\pi_1(\mathcal{C}', c_0'))$. For each $d \in \operatorname{Ob}\mathcal{D}$ we define $\widetilde{\lambda}(d)$ as follows. Choose any path γ in $\mathcal{D}$ joining d_0 to d, let $\widetilde{\gamma}$ be the unique lifting of the path $\lambda(\gamma)$ such that $i(\widetilde{\gamma}) = c_0'$. We define $\widetilde{\lambda}(d) := t(\widetilde{\gamma})$. Assume that γ' is another path joining d_0 to d, then the composition $\gamma^{-1} \circ \gamma'$ is a loop at d_0. Then according to the assumption, the lifting of a loop $\lambda(\gamma^{-1} \circ \gamma')$ is a loop (at c_0'). This lifting is unique thus $t(\widetilde{\gamma}) = t(\widetilde{\gamma'})$, and then $\widetilde{\lambda}(d)$ is well defined.

Let $l \in \operatorname{Mor}\mathcal{D}$, we define $\widetilde{\lambda}(l)$ to be the unique morphism $l' \in \operatorname{Mor}\mathcal{C}'$ such that $i(l') = \widetilde{\lambda}(i(l))$, $t(l') = \widetilde{\lambda}(t(l))$ and $\phi(l') = \lambda(l)$.

□

The following theorem was motivated by Quillen. He has proved in Proposition 1 of [Q] that the category of covering spaces of $\mathrm{B}\,\mathcal{C}$ is equivalent to the category of functors from the grupiod $\pi\mathcal{C}$ to the category of sets.

Theorem 5.1.7. *The category $\mathrm{Cov}_{\mathcal{C}}$ of covering categories of $\mathcal{C}$ is canonically equivalent to the category $\mathrm{Hom}(\pi\mathcal{C}, \textbf{Sets})$, where $\pi\mathcal{C}$ is the grupoid associated to the small category $\mathcal{C}$.*

Proof. Let $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ be a covering of small categories. Then ϕ satisfies assertion 2. of 5.1.3. The associated functor is defined as follows. The *fibre* of ϕ over $c \in \operatorname{Ob}\mathcal{C}$ is the set $\Lambda_c = \phi^{-1}(c) \subset \operatorname{Ob}\mathcal{C}'$. For $l \in \operatorname{Mor}\mathcal{C}$ with $i(l) = c$, let $\Lambda_l : \Lambda_{i(l)} \longrightarrow \Lambda_{t(l)}$ be the map associating to each $c' \in \Lambda_c$ the terminal object of the unique element $l' \in \operatorname{Mor}\mathcal{C}'$ such that $\phi(l') = l$ and $i(l') = c'$. This map is a bijection because ϕ is a covering. Moreover, for composable morphisms $l_1, l_2 \in \operatorname{Mor}\mathcal{C}$ we have $\Lambda_{l_1}\Lambda_{l_2} = \Lambda_{l_1 l_2}$, and $\Lambda_{\mathrm{id}_c} = \mathrm{id}_{\Lambda_c}$. In other words Λ can be considered as a functor from the category $\mathcal{C}$ to the category whose elements are bijections of sets. Therefore, according to 4.1.3 one can extend Λ to a functor $\widetilde{\Lambda} : \pi\mathcal{C} \longrightarrow \textbf{Sets}$.

Note that the small category $\mathcal{C}'$ is isomorphic to the Grothendieck category $\mathcal{B}\Lambda$, the isomorphism sends a morphism l' of $\mathcal{C}'$ to a pair $(l, \mathrm{id}_{t(l')}) \in \operatorname{Mor}\mathcal{B}\Lambda$. Moreover, $\pi\mathcal{C}' \simeq \mathcal{B}\widetilde{\Lambda}$.

Let $F : \mathcal{C}_1' \longrightarrow \mathcal{C}_2'$ over the identity of $\mathcal{C}$ be a morphism in $\mathrm{Cov}_{\mathcal{C}}$. Assume that $\Lambda_1,\ \Lambda_2 : \mathcal{C} \longrightarrow \textbf{Sets}$ are functors associated to the given coverings. We will define a natural transformation $\eta : \Lambda_1 \implies \Lambda_2$. For each $c \in \operatorname{Ob}\mathcal{C}$ the functor η_c is given by $\phi_1^{-1}(c) \xrightarrow{F} \phi_2^{-1}(c)$. Then for each morphism $l : c \longrightarrow d$ the following diagram commutes

$$\begin{array}{ccc} \Lambda_1(c) & \xrightarrow{\Lambda_1(l)} & \Lambda_1(d) \\ \Big\downarrow{\scriptstyle \eta_c} & & \Big\downarrow{\scriptstyle \eta_d} \\ \Lambda_2(c) & \xrightarrow{\Lambda_2(l)} & \Lambda_2(d) \end{array}$$

and then $\eta : \Lambda_1 \implies \Lambda_2$ is well defined. We can clearly extend η to a natural transformation $\widetilde{\eta} : \widetilde{\Lambda}_1 \implies \widetilde{\Lambda}_2$. Therefore, we have defined a functor

$$\Phi : \mathrm{Cov}_{\mathcal{C}} \longrightarrow \mathrm{Hom}(\pi\mathcal{C}, \textbf{Sets})$$

Assume that we have a morphism-inverting functor $\Lambda : \mathcal{C} \longrightarrow \mathbf{Sets}$. Then the natural projection $\mathcal{B}\Lambda \longrightarrow \mathcal{C}$ satisfies assertion 2. from 5.1.3 thus it is a covering. A natural transformation $\eta : \Lambda_1 \implies \Lambda_2$ gives a commutative diagram

$$\begin{array}{ccc} \mathcal{B}\Lambda_1 & \xrightarrow{\mathcal{B}\eta} & \mathcal{B}\Lambda_2 \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

which is a morphism in $\mathrm{Cov}_{\mathcal{C}}$. Therefore, we have

$$\Phi' : \mathrm{Hom}(\pi\mathcal{C}, \mathbf{Sets}) \longrightarrow \mathrm{Cov}_{\mathcal{C}}$$

Clearly $\Phi\Phi' = \mathrm{id}_{\mathrm{Hom}(\pi\mathcal{C}, \mathbf{Sets})}$ and $\Phi'\Phi \simeq \mathrm{id}_{\mathrm{Cov}_{\mathcal{C}}}$. This proves the Theorem. □

Remark 5.1.8. If $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ is a covering and $\Lambda : \mathcal{C} \longrightarrow \mathbf{Sets}$ the corresponding functor, then $\mathcal{C}' \simeq \mathcal{B}\Lambda$.

Corollary 5.1.9. *Assume that $\mathcal{D}'$ is the pull-back of the following diagram*

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{C}' \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathcal{D} & \xrightarrow{F} & \mathcal{C} \end{array}$$

*If ϕ is a covering and $\Lambda : \mathcal{C} \longrightarrow$ **Sets** the associated morphism inverting functor then ϕ' is a covering such that the associated morphism inverting functor is equal to $\Lambda \circ F$.*

The following Proposition is a corollary from 5.1.7.

Proposition 5.1.10. *The category $\mathrm{Cov}_{\mathcal{C}}$ of coverings of the small connected category $\mathcal{C}$ is equivalent to the category of $\pi_1(\mathcal{C}, c_0)$-sets.*

Proof. The small category $\pi\mathcal{C}$ is equivalent to the small category $\mathcal{B}\pi_1(\mathcal{C}, c_0)$. Then $\mathrm{Cov}_{\mathcal{C}} \approx \mathrm{Hom}(\pi\mathcal{C}, \mathbf{Sets}) \approx \mathrm{Hom}(\mathcal{B}\pi_1(\mathcal{C}, c_0), \mathbf{Sets})$ which proves the Theorem. □

Therefore we obtain a following

Theorem 5.1.11. *Let $\mathcal{C}$ be a small connected category. The category $\mathrm{Cov}_{\mathcal{C}}$ of coverings of the small category $\mathcal{C}$ is equivalent to the category $\mathrm{Cov}_{\mathrm{B}\mathcal{C}}$ of topological coverings of the topological space $\mathrm{B}\,\mathcal{C}$.*

The next observation will be usefull later.

Corollary 5.1.12. *Assume $\mathcal{C}'$, $\mathcal{C}$ are connected small categories and $\phi : (\mathcal{C}', c_0') \longrightarrow (\mathcal{C}, c_0)$ is a covering. Then for each $c \in \mathrm{Ob}\,\mathcal{C}$ the preimage $\phi^{-1}(c)$ is isomorphic to $\pi_1(\mathcal{C}, c_0)/\phi_*(\pi_1(\mathcal{C}', c_0'))$.*

Definition 5.1.13. *We say that a covering $\hat{\phi} : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ is a universal covering if for each covering $\phi : \mathcal{C}' \longrightarrow \mathcal{C}$ there exists a functor $\hat{\phi}' : \hat{\mathcal{C}} \longrightarrow \mathcal{C}'$ which is a morphism in $\mathrm{Cov}_{\mathcal{C}}$, i.e. the following diagram commutes*

$$\begin{array}{ccc} \hat{\mathcal{C}} & & \\ & \searrow^{\hat{\phi}'} & \\ {\scriptstyle \hat{\phi}}\downarrow & & \mathcal{C}' \\ & \swarrow_{\phi} & \\ \mathcal{C} & & \end{array}$$

Remark 5.1.14. Note $\hat{\phi}' : \hat{\mathcal{C}} \longrightarrow \hat{\phi}'(\hat{\mathcal{C}})$ satisfies assertion 2. of 5.1.3 hence is a covering.

Lemma 5.1.15. *Let $\mathcal{X}$ be a connected grupoid. There exists a "universal" functor $\hat{\Lambda} : \mathcal{X} \longrightarrow \mathrm{Gr}$ given by $x \longrightarrow \mathrm{Aut}_{\mathcal{X}}(x)$ such that for each functor $\Lambda : \mathcal{X} \longrightarrow$ **Sets** there exists a natural transformation $\eta : \hat{\Lambda} \implies \Lambda$.*

Proof. The functor $\hat{\Lambda} : \mathcal{X} \longrightarrow \mathrm{Gr}$ is given by $\hat{\Lambda}(x) = \mathrm{Aut}_{\mathcal{X}}(x)$ for each $x \in \mathrm{Ob}\,\mathcal{X}$ and for each $\gamma : x \longrightarrow x'$ we have $\hat{\Lambda}(\gamma) : \mathrm{Aut}_{\mathcal{X}}(x) \longrightarrow \mathrm{Aut}_{\mathcal{X}}(x')$ given by $g \longrightarrow \gamma g \gamma^{-1}$. Let $\Lambda : \mathcal{X} \longrightarrow$ **Sets** be any functor. Then the natural inclusion $\mathrm{Aut}_{\mathcal{X}}(x) \hookrightarrow \mathcal{X} \longrightarrow$ **Sets** defines an action of the group $\mathrm{Aut}_{\mathcal{X}}(x)$ on a set $\Lambda(x)$.

Our goal is to define a natural transformation $\eta : \hat{\Lambda} \implies \Lambda$. We choose $x_0 \in \mathrm{Ob}\,\mathcal{X}$, clearly $\Lambda(x_0)$ is an $\mathrm{Aut}_{\mathcal{X}}(x_0)$-set. Therefore there exsists a functor $\mathrm{Aut}_{\mathcal{X}}(x_0) \longrightarrow \Lambda(x_0)$ and we define $\eta_{x_0} : \hat{\Lambda}(x_0) \longrightarrow \Lambda(x_0)$ to be this functor. Note, η_{x_0} is onto if and only if $\Lambda(x_0)$ is a transitive $\mathrm{Aut}_{\mathcal{X}}(x_0)$-set. Moreover for each $g \in \mathrm{Aut}_{\mathcal{X}}(x_0)$ the following diagram commutes

$$\begin{array}{ccc} \Lambda(x_0) & \xrightarrow{\Lambda(g)} & \Lambda(x_0) \\ & \nwarrow \quad \nearrow & \\ & \mathrm{Aut}_{\mathcal{X}}(x_0) & \end{array}$$

For each $x \in \mathrm{Ob}\,\mathcal{X}$ we choose a morphism $\gamma_x : x_0 \longrightarrow x$ in $\mathcal{X}$. Then we define $\eta_x : \hat{\Lambda}(x) \longrightarrow \Lambda(x)$ to be $\mathrm{Aut}_{\mathcal{X}}(x) \xrightarrow{\hat{\Lambda}(\gamma_x)^{-1}} \mathrm{Aut}_{\mathcal{X}}(x_0) \xrightarrow{\eta_{x_0}} \Lambda(x_0) \xrightarrow{\Lambda(\gamma_x)} \Lambda(x)$, therefore the following diagram commutes

$$\begin{array}{ccc} \Lambda(x_0) & \xrightarrow{\Lambda(\gamma_x)} & \Lambda(x) \\ \eta_{x_0} \uparrow & & \uparrow \eta_x \\ \mathrm{Aut}_{\mathcal{X}}(x_0) & \xrightarrow{\mathrm{Ad}(\gamma_x)} & \mathrm{Aut}_{\mathcal{X}}(x) \end{array}$$

Clearly for each $\gamma : x \longrightarrow x'$ we have $\Lambda(\gamma) \circ \eta_x = \eta_{x'} \circ \hat{\Lambda}(\gamma)$, therefore η is a well defined natural transformation. □

Proposition 5.1.16. *Let $\mathcal{C}$ be a small connected category, $\hat{\phi} : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ a covering and $\hat{\Lambda} : \pi\mathcal{C} \longrightarrow$ **Sets** the associated functor. The following conditions are equivalent*

1. *$\hat{\phi} : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ is a universal covering*
2. *the fundamental group of the small category $\hat{\mathcal{C}}$ is trivial*
3. *$\hat{\Lambda} : \pi\mathcal{C} \longrightarrow$ **Sets** is isomorphic to the "universal" functor defined in 5.1.15*

Proof. 1. $\implies$ 2. Assume that $\hat{\phi} : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ is a universal covering and $\hat{\hat{\phi}} : \hat{\hat{\mathcal{C}}} \longrightarrow \mathcal{C}$ is a covering such that $\hat{\hat{\mathcal{C}}}$ is connected and the fundamental group of $\hat{\hat{\mathcal{C}}}$ is trivial. According to 5.1.6 there exists a functor $F : \hat{\hat{\mathcal{C}}} \longrightarrow \hat{\mathcal{C}}$ over the identity of $\mathcal{C}$. Clearly F is an isomorphism, therefore the fundamental group of $\hat{\mathcal{C}}$ is trivial.

2. $\implies$ 3. Due to 5.1.12 for each $c \in \mathrm{Ob}\,\mathcal{C}$ we have $\hat{\Lambda}(c) \simeq \pi_1(\mathcal{C}, c_0) \simeq \mathrm{Aut}_{\pi\mathcal{C}}(c)$.

3. $\implies$ 1. This implication follows directly from 5.1.15 and 5.1.7.

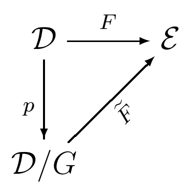
□

5.2 Action without inversion and G-coverings of small categories

Definition 5.2.1. *Let $\mathcal{D}$ be small category. An action without inversion of a group G on $\mathcal{D}$ is an action such that, if an element of G fixes an object $d \in \operatorname{Ob}\mathcal{D}$ then it acts trivially on the small category $d/\mathcal{D}$, i.e. fixes every morphism $l \in \operatorname{Mor}\mathcal{D}$ such that $i(l) = d$.*

Remark 5.2.2. An action of G on the category $\mathcal{D}$ induces an action of G on its geometric realization $\mathrm{B}\,\mathcal{D}$ in the obvious way. Geometrically, the above definition means that if g fixes a vertex d, then it fixes (pointwise) the union of the simplices corresponding to composable sequences $(l_1, \ldots, l_k)$ with $i(l_k) = d$.

Definition 5.2.3. *(Categorical quotient) Let $\mathcal{D}$ be a small category and G any group acting on it. Then the categorical quotient of the action of G on $\mathcal{D}$ is $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ such that for any small category $\mathcal{E}$ with the trivial action of G on it and the G-equivariant functor $F : \mathcal{D} \longrightarrow \mathcal{E}$, there is a G-equivariant functor $\widetilde{F} : \mathcal{D}/G \longrightarrow \mathcal{E}$ such that the following diagram is commutative*



Proposition 5.2.4. *Assume that G acts without inversion on a small category $\mathcal{D}$. Then the categorical quotient is isomorphic to the "naive" quotient of $\mathcal{D}$ by the action of G.*

Proof. Let $\mathcal{C}$ denote the "naive" quotient, i.e. $\operatorname{Mor}\mathcal{C} = \operatorname{Mor}\mathcal{D}/G$. We will prove that $\mathcal{C}$ is a small category. In order to do it we need to prove that for each pair of composable morphisms $l : c' \longrightarrow c$ and $l' : c'' \longrightarrow c'$ in $\mathcal{C}$ the composition $l'' : c'' \longrightarrow c$ is well defined.

First note that for each $\widetilde{l} \in \operatorname{Mor}\mathcal{D}$ we have an inclusion of isotropy subgroups

$$\operatorname{Stab}_G(i(\widetilde{l})) \subset \operatorname{Stab}_G(t(\widetilde{l}))$$

Let $p : \mathcal{D} \longrightarrow \mathcal{C}$ be a "naive" projection. Assume that we have two pairs of composable morphisms $(\widetilde{l}, \widetilde{l'})$ and $(g\widetilde{l}, g'\widetilde{l'})$ in $\mathcal{D}$ such that, $p(\widetilde{l}) = p(g\widetilde{l}) = l$, $p(\widetilde{l'}) = p(g'\widetilde{l'}) = l'$, and $p(\widetilde{l}\widetilde{l'}) = l''$. We will prove that $p(g\widetilde{l}g'\widetilde{l'}) = l''$.

We will denote $\widetilde{l''} := \widetilde{l}\widetilde{l'}$, $\widetilde{c''} := i(\widetilde{l'}) = i(\widetilde{l''})$, $\widetilde{c'} := t(\widetilde{l'}) = i(\widetilde{l})$, $\widetilde{c} := t(\widetilde{l}) = t(\widetilde{l''})$.

Then $g'\widetilde{c'} = t(g'\widetilde{l'}) = i(g\widetilde{l}) = g\widetilde{c'}$, which implies $g^{-1}g' \in \operatorname{Stab}(\widetilde{c'})$. The group G acts on $\mathcal{D}$ without inversion, and then $g^{-1}g' \in \operatorname{Stab}(\widetilde{l})$, $g'\widetilde{l} = g\widetilde{l}$.

Then $g\widetilde{l}g'\widetilde{l'} = g'\widetilde{l}g'\widetilde{l'}$. The action of the group G is functorial, so $g'\widetilde{l}g'\widetilde{l'} = g'(\widetilde{l}\widetilde{l'}) = g'\widetilde{l''}$. Then $p(g'\widetilde{l'}g\widetilde{l}) = p(g'\widetilde{l''}) = l''$.

□

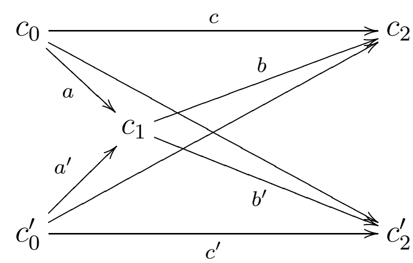
Proposition 5.2.5. *Assume that G acts without inversion on a small category $\mathcal{D}$ and $\mathcal{C}$ is the quotient. Then the induced action of the group G on $\mathrm{B}\,\mathcal{D}$ satisfies:*

$$\mathrm{B}\,\mathcal{C} = (\mathrm{B}\,\mathcal{D})/G$$

Proof. The topological space $\mathrm{B}\,\mathcal{D}$ is a geometric realization of a simplicial space $\mathrm{N}\,\mathcal{D}$. We have proved in 5.2.4 that $\mathrm{N}\,\mathcal{C}^{(0)} = (\mathrm{N}\,\mathcal{D})^{(0)}/G$ and $\mathrm{N}\,\mathcal{C}^{(1)} = (\mathrm{N}\,\mathcal{D})^{(1)}/G$. The n-simplex of the geometric realization of the small category corresponds to the sequence of n composable morphisms. Thus for each $n \in \mathbb{N}$, $\mathrm{N}\,\mathcal{C}^{(n)} = (\mathrm{N}\,\mathcal{D})^{(n)}/G$ and then $\mathrm{N}\,\mathcal{C} = (\mathrm{N}\,\mathcal{D})/G$. According to Theorem 3.1, Chapter III [G-Z] the geometric realization functor commutes with direct limits, therefore $\mathrm{B}\,\mathcal{C} = (\mathrm{B}\,\mathcal{D})/G$. □

The following example shows that if we drop the assumption of G acting without inversion, then the naive quotient fails to be a category.

Example 5.2.6. Let $\mathcal{D}$ be a category given by



we denote $ba' = d$ and $b'a = e$.

Assume that $\mathbb{Z}_2 = \langle g \rangle$ acts on $\mathcal{D}$ in the following way; $gc_0 = c_0'$, $gc_1 = c_1$, $gc_2 = c_2'$, $ga = a'$, $gb = b'$. Thus $gc = c'$ and $gd = e$.

Note, that the 'naive' quotient $\mathcal{D}//\mathbb{Z}_2$ is not a category. This quotient would have three objects $[c_0] = [c_0']$, $[c_1]$, $[c_2] = [c_2']$ and morphisms $[a] = [a']$, $[b] = [b']$, $[c] = [c']$ and $[d] = [e]$. Since $ba = c$, $b'a' = c'$, $ba' = d$, $b'a = e$, then $[b][a] = [c]$ and $[b][a] = [d]$ which is not possible, because $[c] \neq [d]$ in $\mathcal{D}//\mathbb{Z}_2$.

Let $\mathcal{D}/\mathbb{Z}_2$ be the categorical quotient. Let $F : \mathcal{D} \longrightarrow \mathcal{E}$ be any $\mathbb{Z}_2$-equivariant functor such that $\mathbb{Z}_2$ acts trivialy on $\mathcal{E}$. Then $F(gl) = F(l)$ for objects and morphisms of $\mathcal{D}$. Then we have $F(a) = F(a')$ and $F(b) = F(b')$ which implies $F(c) = F(c') = F(d) = F(e)$.

Then the category $\mathcal{D}/\mathbb{Z}_2$ has three objects and three morphisms $[a] : [c_0] \longrightarrow [c_1]$, $[b] : [c_1] \longrightarrow [c_2]$, $[c] : [c_0] \longrightarrow [c_2]$. Note that

$$\mathrm{B}(\mathcal{D}/\mathbb{Z}_2) \neq (\mathrm{B}\,\mathcal{D})/\mathbb{Z}_2$$

because $\mathrm{B}(\mathcal{D}/\mathbb{Z}_2)$ has one 2-simplex and $(\mathrm{B}\,\mathcal{D})/\mathbb{Z}_2$ two 2-simplices.

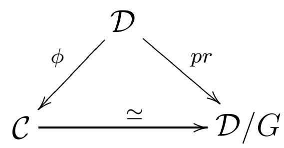
Definition 5.2.7. *We say that functor $\phi : \mathcal{D} \longrightarrow \mathcal{C}$ is a right covering if for each object d of $\mathcal{D}$ an induced functor $d/\phi : d/\mathcal{D} \longrightarrow \phi(d)/\mathcal{C}$ is a bijection on objects set.*

Proposition 5.2.8. *Assume that a group G acts without inversion on a small category $\mathcal{D}$. The natural projection $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ induced by the action of G is a right covering.*

Proof. Assume that k_1, k_2 are morphisms of $\mathcal{D}$ such that $i(k_1) = i(k_2) = d$ and $p(k_1) = p(k_2)$. This implies that there exists $g \in G$ such that $gk_1 = k_2$. Thus $gd = d$ and then $k_1 = k_2$. □

Assume that a group G acts freely on a small category $\mathcal{D}$. A free action of a group G is a special case of the action without inversion, thus there exists a quotient of that action $\mathcal{D}/G$ defined as in 5.2.4. According to 5.2.8 the natural projection $\mathcal{D} \longrightarrow \mathcal{D}/G$ is a right covering. Clearly it is also a left covering, therefore $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ is a covering.

Definition 5.2.9. *Let G be a group. We say that a covering of small categories $\phi : \mathcal{D} \longrightarrow \mathcal{C}$ is a G-covering if and only if the group G acts freely on the small category $\mathcal{D}$, the quotient $\mathcal{D}/G$ is isomorphic to the small category $\mathcal{C}$ and the following diagram commutes*



Proposition 5.2.10. *Let G be a group. Let $\phi : \mathcal{D} \longrightarrow \mathcal{C}$ be a G-covering and $\Lambda : \mathcal{C} \longrightarrow$ **Sets** the associated morphism inverting functor. Then for each $c \in \operatorname{Ob}\mathcal{C}$ $\Lambda(c) \simeq G$ and*

$$\mathcal{D} \simeq \mathcal{B}\Lambda$$

Proof. Follows directly form 5.1.8. ☐

We can generalize this observation in the following way. Assume that a group G acts without inversion on a small category $\mathcal{D}$, let $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ denote the natural projection induced by the action of G. For each object c of $\mathcal{D}/G$ consider $p^{-1}(c) \subset \operatorname{Ob}\mathcal{D}$. This preimage is a transitive G-set. Due to 5.2.8, for each $l : c \longrightarrow c'$ a morphism in $\mathcal{D}/G$ and $d \in p^{-1}(c)$ there exists unique morphism $k : d \longrightarrow d'$ such that $p(k) = l$. Thus each morphism $l \in \operatorname{Mor}\mathcal{C}$ defines a G-equivariant morphism $p^{-1}(c) \longrightarrow p^{-1}(c')$. Therefore we obtain a functor $L_{\mathcal{D}} : \mathcal{D}/G \longrightarrow (G - Sets)$.

Proposition 5.2.11. *Assume that a group G acts without inversion on a small category $\mathcal{D}$, let $L_{\mathcal{D}} : \mathcal{D}/G \longrightarrow$ **Sets** be a functor defined above. Then $\mathcal{B}L_{\mathcal{D}} \simeq \mathcal{D}$.*

Proof. A map sending a morphism $k : d \longrightarrow d'$ of $\mathcal{D}$ to a pair $(p(k), \mathrm{id}_{d'}) \in \operatorname{Mor}\mathcal{B}L_{\mathcal{D}}$ is an isomorphism.

☐

5.3 Twisted diagram of groups associated to an action

Remark 5.3.1. Assume that a group G acts without inversion on a small category $\mathcal{D}$. Let $L_{\mathcal{D}} : \mathcal{D}/G \longrightarrow (G - \mathbf{Sets})$ be a functor from 5.2.11. Assume $c \in \operatorname{Ob}\mathcal{D}/G$, then G-set $p^{-1}(c)$ defines a grupoid which is isomorphic to $G/\operatorname{Stab}_G(d)$ for each object $d \in p^{-1}(c)$. Therefore, the functor $L_{\mathcal{D}}$ induces a functor

$$F_{\mathcal{D}} : \mathcal{D}/G \longrightarrow \mathrm{Grp}$$

and a natural transformation $L_{\mathcal{D}} \implies F_{\mathcal{D}}$ given by $p^{-1}(c) \hookrightarrow \operatorname{Ob} F_{\mathcal{D}}(c)$. This natural transformation induces the inclusion

$$\mathcal{B}L_{\mathcal{D}} \hookrightarrow \mathcal{B}F_{\mathcal{D}}$$

Proposition 5.3.2. *Assume that a group G acts without inversion on a small category $\mathcal{D}$, we consider it as a functor $S_{\mathcal{D}} : \mathcal{B}G \longrightarrow$ Cat. Let $F_{\mathcal{D}} : \mathcal{D}/G \longrightarrow$ Grp be a functor defined in 5.3.1. Then the Grothendieck construction $\mathcal{B}S_{\mathcal{D}}$ is isomorphic to the Grothendieck construction $\mathcal{B}F_{\mathcal{D}}$.*

Proof. According to 5.2.11 the small category $\mathcal{D}$ is isomorphic to the Grothendieck construction $\mathcal{B}L_{\mathcal{D}}$. Thus we can assume that a morphism of $\mathcal{B}S_{\mathcal{D}}$ is a pair $(g,(l,\mathrm{id}_d))$ where $g \in G$ and $(l,\mathrm{id}_d) \in \mathrm{Mor}\,\mathcal{B}L_{\mathcal{D}}$. We define a functor $\mathcal{B}S_{\mathcal{D}} \longrightarrow \mathcal{B}F_{\mathcal{D}}$ to be an identity on the objets set and to be $(g,(l,\mathrm{id}_d)) \longrightarrow (g^{-1}l,(g,\mathrm{id}_d))$ on the set of morphisms. It is clearly an isomorphism which proves the Proposition. □

Let $\mathcal{E}G$ be the universal covering of the category $\mathcal{B}G$ defined in 5.1.2.

Proposition 5.3.3. *Consider the direct product $\mathcal{E}G \times \mathcal{D}$ and an action of the group G on it given by $g(h,d) = (gh,gd)$. Then this action is free and the quotient is isomorphic to $\mathcal{B}S_{\mathcal{D}}$.*

Proof. According to 5.2.11 the small category $\mathcal{D}$ is isomorphic to the Grothendieck category $\mathcal{B}L_{\mathcal{D}}$. Therefore $\mathcal{E}G \times \mathcal{D}$ is isomrophic to the Grothendieck category $\mathcal{B}\bar{L}_{\mathcal{D}}$, where $\bar{L}_{\mathcal{D}} : \mathcal{D}/G \longrightarrow \mathrm{Cat}$ is given by $\bar{L}_{\mathcal{D}}(c) = \mathcal{E}G \times L_{\mathcal{D}}(c)$ for each object $c \in \mathrm{Ob}\,\mathcal{D}/G$. $L_{\mathcal{D}}(c)$ is a transitive G-set, hence the action of the group G on $\mathcal{E}G \times \mathcal{D}$ gives an action of G on $\bar{L}_{\mathcal{D}}(c)$. Clearly $\bar{L}_{\mathcal{D}}(c)/G \simeq F_{\mathcal{D}}(c)$. Therefore $(\mathcal{E}G \times \mathcal{D})/G \simeq \mathcal{B}F_{\mathcal{D}} \simeq \mathcal{B}S_{\mathcal{D}}$. □

According to 5.2.5 the geometric realization $\mathrm{B}((\mathcal{E}G \times \mathcal{D})/G) = (\mathrm{E}\,G \times \mathrm{B}\,\mathcal{D})/G$, hence

Corollary 5.3.4. *The geometric realization of a sequence of functors $\mathcal{D} \hookrightarrow \mathcal{B}S_{\mathcal{D}} \longrightarrow \mathcal{B}G$ yields a topological fibration $\mathrm{B}\,\mathcal{D} \hookrightarrow \mathrm{E}\,G \times_G \mathrm{B}\,\mathcal{D} \longrightarrow \mathrm{B}\,G$. Therefore, the small category $\mathcal{B}S_{\mathcal{D}}$ is a categorical analogue of the Borel construction.*

Proposition 5.3.5. *Assume that the group G acts without inversion on the small category $\mathcal{D}$, let $\mathcal{C}$ denote the quotient of that action, and $F_{\mathcal{D}} : \mathcal{C} \longrightarrow \mathrm{Grp}$ be the functor constructed in 5.3.1. Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups associated to $F_{\mathcal{D}} : \mathcal{C} \longrightarrow \mathrm{Grp}$ (construction described in 4.1.22). Then for each $l : c \longrightarrow c'$ the corresponding homomorphism of groups $\mathcal{G}(l) : \mathcal{G}(c) \longrightarrow \mathcal{G}(c')$ is injective. We will call the twisted diagram of groups $\mathcal{G}$ a* twisted diagram of groups associated to an action *of the group G on the small category $\mathcal{D}$.*

Proof. For each object c of $\mathcal{C}$ the group $\mathcal{G}(c)$ is isomorphic to the group $\mathrm{Aut}_{F_{\mathcal{D}}(c)}(d) = \mathrm{Stab}_G(d) \subset G$. Let $k : d \longrightarrow d'$ be a morphism of $\mathcal{D}$. The group G acts without inversion thus $\mathrm{Stab}_G(d) \subset \mathrm{Stab}_G(d')$. Therefore the corresponding homomorphism is injective. □

Remark 5.3.6. According to 4.1.22, there exists a natural transformation $\mathcal{G} \implies F_{\mathcal{D}}$. Clearly it induces an equivalence of categories $\mathcal{B}\mathcal{G} \hookrightarrow \mathcal{B}F_{\mathcal{D}}$.

Corollary 5.3.7. *The composition of $\mathcal{B}\mathcal{G} \hookrightarrow \mathcal{B}F_{\mathcal{D}}$ with the isomorphism $\mathcal{B}F_{\mathcal{D}} \longrightarrow \mathcal{B}S_{\mathcal{D}}$ gives*

$$\mathcal{B}\mathcal{G} \overset{\approx}{\hookrightarrow} \mathcal{B}S_{\mathcal{D}}$$

Remark 5.3.8. Consider the composition $\mathcal{B}\mathcal{G} \hookrightarrow \mathcal{B}S_{\mathcal{D}} \longrightarrow \mathcal{B}G$. This functor defines a homomorphism of twisted diagrams of groups

$$\Phi : \mathcal{G} \longrightarrow G$$

Corollary 5.3.9. *The fundamental group of the category $\mathcal{D}$ is isomorphic to the kernel of the homomorphism of groups $\Phi_* : \pi_1(\mathcal{G}, c) \longrightarrow G$.*

Remark 5.3.10. If G acts freely on the small category $\mathcal{D}$ then the associated twisted diagram of groups is trivial hence $\mathcal{B}\mathcal{G} \simeq \mathcal{C}$. The associated homomorphism defined in 5.3.8 is given by $\Phi : \mathcal{I}_{\mathcal{C}} \longrightarrow G$.

G-covering of the classifying category $\mathcal{BG}$ If the group G acts freely on the small category $\mathcal{D}$ then the natural projection $p : \mathcal{D} \longrightarrow \mathcal{C} = \mathcal{D}/G$ is a G-covering. We can consider this projection as a homomomorphism of trivial twisted diagrams of groups $\mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{I}_{\mathcal{C}}$ over $p : \mathcal{D} \longrightarrow \mathcal{C}$. As we have observed in 5.3.10, $\mathcal{I}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a twisted diagram of groups associated to the free action of G on $\mathcal{D}$. We will prove that for each twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ associated to an action of G on $\mathcal{D}$ there exist a homomorphism $\mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{G}$ over $p : \mathcal{D} \longrightarrow \mathcal{C}$ and a small category $\mathcal{E}$ such that the associated functor $\mathcal{D} \longrightarrow \mathcal{BG}$ is equal to

$$\begin{array}{ccc} \mathcal{D} & \overset{\approx}{\hookrightarrow} & \mathcal{E} \\ & \searrow & \downarrow \\ & & \mathcal{BG} \end{array}$$

and $\mathcal{E} \longrightarrow \mathcal{BG}$ is a G-covering.

In order to prove it we need to construct a certain diagram of groups on the small category $\mathcal{D}$:

Remark 5.3.11. Assume that the group G acts without inversion on the small category $\mathcal{D}$. One can associate with this action a diagram of groups $\widetilde{G}_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Gr}$ given by

$$\widetilde{G}_{\mathcal{D}}(d) = \mathrm{Stab}_G(d)$$

for each object d of $\mathcal{D}$ and

$$\widetilde{G}_{\mathcal{D}}(k) : \mathrm{Stab}_G(d) \hookrightarrow \mathrm{Stab}_G(d')$$

for each morphism $k : d \longrightarrow d'$ of $\mathcal{D}$.

Proposition 5.3.12. *Let $\mathcal{B}\widetilde{G}_{\mathcal{D}}$ be the classifying category of the diagram of groups defined in 5.3.11. Then this category is isomorphic to the pull-back of the following diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{D}} & \xrightarrow{\tilde{p}} & \mathcal{BG} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{p} & \mathcal{C} \end{array}$$

Proof. Let $F_{\mathcal{D}} : \mathcal{C} \longrightarrow \mathrm{Grp}$ be a functor defined in 5.3.1. Consider a functor $F_{\mathcal{D}} \circ p : \mathcal{D} \longrightarrow \mathrm{Grp}$. For each object d of $\mathcal{D}$ the grupoid $(F_{\mathcal{D}} \circ p)(d)$ contains the set $p^{-1}(p(d)) \subset \mathrm{Ob}\,\mathcal{D}$. Clearly $d \in p^{-1}(p(d))$.

Consider the projection $\widetilde{\mathcal{D}} \longrightarrow \mathcal{D}$. According to 1.2.14 it is associated to the pull-back diagram of groups $\mathcal{G} \circ p : \mathcal{D} \longrightarrow \mathrm{Gr}$.

The (twisted) diagrams of groups $\mathcal{G} \circ p : \mathcal{D} \longrightarrow \mathrm{Gr}$ and $\widetilde{G}_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Gr}$ are associated to the functor $F_{\mathcal{D}} \circ p$ via the construction described in 4.1.22. Therefore these twisted digarams of groups are equivalent and then there exists an isomorphism $\mathcal{B}\widetilde{G}_{\mathcal{D}} \longrightarrow \widetilde{\mathcal{D}}$ over the identity of $\mathcal{D}$.

□

Corollary 5.3.13. *There exists an action without inversion of the group G on $\mathcal{B}\widetilde{G}_{\mathcal{D}}$ such that the following diagram*

$$\begin{array}{ccc} \mathcal{B}\widetilde{G}_{\mathcal{D}} & \xrightarrow{/G} & \mathcal{BG} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{/G} & \mathcal{C} \end{array}$$

commutes. This action is induced from the action of G on the pull-back $\widetilde{\mathcal{D}}$.

Remark 5.3.14. There exists a commutative diagram

$$\begin{array}{ccc} \mathcal{B}\widetilde{G}_{\mathcal{D}} & & \\ \uparrow & \searrow & \\ \mathcal{D} & \hookrightarrow & \mathcal{B}S_{\mathcal{D}} \end{array}$$

Given a diagram of groups one can assign to it a certain functor to the category of small categories.

Remark 5.3.15. Let $\widetilde{G}_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Gr}$ be a diagram of groups defined in (5.3.11). Consider a map $W : \mathcal{D} \longrightarrow \mathrm{Cat}$ given by

$$W(d) = \mathcal{E}\widetilde{G}_{\mathcal{D}}(d)$$

for each d an object of $\mathcal{D}$ and $W(k) : \mathcal{E}\widetilde{G}_{\mathcal{D}}(d) \longrightarrow \mathcal{E}\widetilde{G}_{\mathcal{D}}(d')$ induced by $\widetilde{G}_{\mathcal{D}}(k)$ for each $k : d \longrightarrow d'$ a morphism in $\mathcal{D}$. Then W is a functor and there exists a natural transformation $\eta : W \implies \widetilde{G}_{\mathcal{D}}$ such that for each object d the functor $\eta_d : \mathcal{E}\widetilde{G}_{\mathcal{D}}(d) \longrightarrow \mathcal{B}\widetilde{G}_{\mathcal{D}}(d)$ is the universal covering of $\mathcal{B}\widetilde{G}_{\mathcal{D}}(d)$.

Proposition 5.3.16. *Let $\mathcal{B}W$ be the Grothendieck construction of the functor W. There exists a free action of the group G on $\mathcal{B}W$ and the quotient is isomorphic to the small category $\mathcal{B}\mathcal{G}$.*

Proof. Let d be an object of $\mathcal{D}$. Consider the quotient $G/\operatorname{Stab}_G(d)$. In order to define the action of G we need to choose for each $\operatorname{Stab}_G(d)$-coset a representative in G. Then for each element g of G there exist a representative $g(d)$ and an element $h(g,d) \in \operatorname{Stab}_G(d)$ such that $g = g(d)h(g,d)$. If $g \in \operatorname{Stab}_G(d)$ then we assume that $g = h(g,d)$.

Let (d,h), where $d \in \operatorname{Ob}\mathcal{D}$ and $h \in \operatorname{Stab}_G(d) = \operatorname{Ob}\mathcal{E}\operatorname{Stab}_G(d)$, be an object of $\mathcal{B}W$. We define

$$g(d,h) = (gd, ghh(g,d)g^{-1})$$

Let (k,h), where $k \in \operatorname{Mor}\mathcal{D}$ and $(h_1 \xrightarrow{h} h_1 h) \in \operatorname{Mor}\mathcal{E}\operatorname{Stab}_G(t(k))$, be a morphism of $\mathcal{B}W$. The action of the group G is given by

$$g(k,h) = (gk, gh(g,i(k))^{-1}hh(g,t(k))g^{-1})$$

Note $g\big((k_1,h_1)(k_2,h_2)\big) = g(k_1,h_1) \circ g(k_2,h_2)$, which implies that the action is well defined. Assume that $g(d,h) = (d,h)$. Then $g \in \operatorname{Stab}_G(d)$ and $h(g,d) = g$. Thus $h = gh$ and this implies $g = 1$.

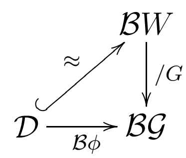
We will prove that the quotient category $(\mathcal{B}W)/G$ is isomorphic to the classifying category $\mathcal{B}\mathcal{G}$. First note that there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{B}W & \xrightarrow{\widetilde{r}} & \mathcal{D} \\ {\scriptstyle /G}\downarrow & & \downarrow{\scriptstyle p\,/G} \\ (\mathcal{B}W)/G & \xrightarrow{r} & \mathcal{D}/G \end{array}$$

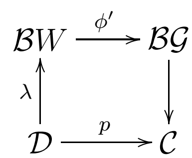
The functor r is clearly onto and it is a bijection on the objects set. For each object d of $\mathcal{D}$ the preimage $\widetilde{r}^{-1}(\mathrm{id}_d)$ is $\mathcal{E}\operatorname{Stab}_G(d)$ and then

$r^{-1}(\mathrm{id}_{p(d)}) \simeq \mathcal{E}\,\mathrm{Stab}_G(d)/\,\mathrm{Stab}_G(d) = \mathcal{B}\,\mathrm{Stab}_G(d)$. For each $k : d \longrightarrow d'$ a morphism in $\mathcal{D}$ the preimage $\widetilde{r}^{-1}(k)$ is isomorphic to the set of pairs (k, h) where $h \in \mathcal{E}\,\mathrm{Stab}_G(d')$. Thus the preimage $r^{-1}(p(k))$ is isomorphic to the set of pairs $(p(k), h)$ where $h \in \mathcal{E}\,\mathrm{Stab}_G(d')$. Thus $(\mathcal{B}W)/G \simeq \mathcal{B}\mathcal{G}$. □

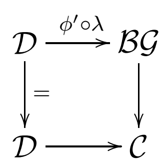
Theorem 5.3.17. *Let $\mathcal{I}_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Gr}$ be a trivial diagram of groups. There exists a homomorphism $\phi : \mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{G}$ over $p : \mathcal{D} \longrightarrow \mathcal{C}$ such that the corresponding homomorphism $\mathcal{B}\phi : \mathcal{D} \longrightarrow \mathcal{B}\mathcal{G}$ is equal to the composition*



Proof. Let $W : \mathcal{D} \longrightarrow \mathrm{Cat}$ be a functor defined in 5.3.15. There exists a functor $\lambda : \mathcal{D} \longrightarrow \mathcal{B}W$ given by $d \longrightarrow (d, 1)$. This functor is clearly inclusion and equivalence of categories. Consider a diagram



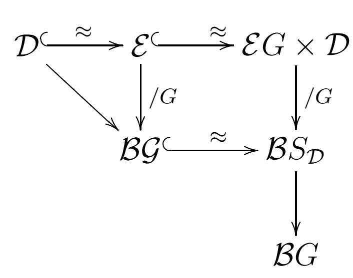
where ϕ' is a G covering from 5.3.16. The diagram



defines a homomorphism $\phi : \mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{G}$ over p. □

Let $\mathcal{E}$ denote the Grothendieck category $\mathcal{B}W$. Then Theorem 5.3.17 and Proposition 5.3.3 implies:

Corollary 5.3.18. *The following diagram commutes*



The following observations will be usefull in the next Chapter.

Proposition 5.3.19. *Let $\mathcal{B}\phi : \mathcal{D} \longrightarrow \mathcal{B}\mathcal{G}$ be a functor defined in 5.3.17. Note for each $c \in \mathrm{Ob}\,\mathcal{C}$*

$$\mathcal{G}(c) \cap \mathcal{B}\phi(\mathcal{D}) = \mathrm{id}_c$$

Let $\widetilde{l}$ be a morphism of $\mathcal{B}\mathcal{G}$ such that $\widetilde{l} \notin \mathcal{G}(c) \subset \mathrm{Mor}\,\mathcal{B}\mathcal{G}$ for each $c \in \mathrm{Ob}\,\mathcal{C}$. Then there exists a morphism $k \in \mathcal{D}$ such that $\mathcal{B}\phi(k) = \widetilde{l}$.

Proof. Let d be an object of $\mathcal{D}$ and $l \in \operatorname{Mor}\mathcal{C}$ such that $t(l) = p(d)$. Consider a subset Υ of $\operatorname{Mor}\mathcal{D}$ such that each $k \in \Upsilon$ satisfies $t(k) = d$ and $p(k) = l$. Let $g \in \operatorname{Stab}_G(d)$. We will prove that if $k \neq k'$ then $g\lambda(k) \neq \lambda(k')$.

For each $k \in \operatorname{Mor}\mathcal{D}$ $\lambda(k) = (k, 1) \in \operatorname{Mor}\mathcal{B}W$. If $g \in \operatorname{Stab}_G(d)$ and $k \in \Upsilon$ then $g(k, 1) = (gk, gh(g, i(k))^{-1})$ and if $k \neq gk$ then $gh(g, i(k))^{-1} \neq 1$. Therefore $\mathcal{B}\phi$ establishes a bijection between the set Υ and the set of morphism of $\mathcal{B}\mathcal{G}$ projecting on l. This proves the Proposition.

□

Corollary 5.3.20. *Let $\mathcal{B}\phi : \mathcal{D} \longrightarrow \mathcal{B}\mathcal{G}$ be a functor defined in 5.3.17. Let $\widetilde{l} \in \operatorname{Mor}\mathcal{B}\mathcal{G}$ be a morphism such that $\widetilde{l} \notin \mathcal{G}(c) \subset \operatorname{Mor}\mathcal{B}\mathcal{G}$ for each $c \in \operatorname{Ob}\mathcal{C}$. Then for each $d \in \operatorname{Ob}\mathcal{D}$ such that $p(d) = t(\widetilde{l})$ ($p(d) = i(\widetilde{l})$) there exists a unique morphism $k \in \operatorname{Mor}\mathcal{D}$ such that $d = t(k)$ ($d = i(k)$).*

5.4 Developable twisted diagrams of groups

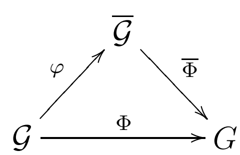
Definition 5.4.1. *A twisted diagram of groups equivalent to a twisted diagram of groups associated to an action of a group (5.3.5) is called developable.*

Example 5.4.2. Consider a short exact sequence $N \rightarrowtail \widetilde{G} \overset{\Theta}{\twoheadrightarrow} G$ and the twisted diagram of groups $\mathcal{F} : \mathcal{B}G \longrightarrow \mathrm{Gr}$ associated to it defined in 2.1.10. Consider an action of a group $\widetilde{G}$ on a small category $\mathcal{E}G$ given by

$$\widetilde{g}.g = \Theta(\widetilde{g})g$$

Note that $\mathcal{E}G/\widetilde{G} = \mathcal{B}G$ and the isotropy subgroup of each object of the category $\mathcal{E}G$ is isomorphic to the given group N. Moreover the twisted diagram of groups associated to this action is isomorphic to $\mathcal{F}$ thus it is developable.

Lemma 5.4.3. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $\Phi : \mathcal{G} \longrightarrow G$ be a homomorphism. There exists a twisted diagram of groups $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ such that homomorphism Φ is equal to the composition*



where $\varphi : \mathcal{G} \longrightarrow \overline{\mathcal{G}}$ is an epimorphism of twisted diagrams of groups and $\overline{\Phi} : \overline{\mathcal{G}} \longrightarrow G$ is injective on the local groups

Proof. We define a twisted diagram of groups as follows: $\overline{\mathcal{G}}(c) = \Phi_c(\mathcal{G}(c))$, $\overline{\mathcal{G}}(l) = \mathrm{Ad}(\Phi(l))$ and $\overline{g}_{l_1,l_2} = \Phi_{t(l_1)}(g_{l_1,l_2})$. Then the homomorphism $\overline{\Phi} = (\overline{\Phi}_c, \overline{\Phi}_l) : \overline{\mathcal{G}} \longrightarrow G$ is given by $\overline{\Phi}_c : \overline{\mathcal{G}}(c) \hookrightarrow G$ and $\overline{\Phi}(l) = \Phi(l)$ is injective on the local groups. The epimorphism $\varphi : \mathcal{G} \longrightarrow \overline{\mathcal{G}}$ is given by $\varphi_c : \mathcal{G}(c) \twoheadrightarrow \overline{\mathcal{G}}(c) = \operatorname{im}\Phi_c$. □

Lemma 5.4.4. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $\Phi : \mathcal{G} \longrightarrow G$ any homomorphism. One can associate to Φ a certain functor $L = L(\Phi) : \mathcal{C} \longrightarrow G -$ **Sets** satisfying*

$$L(c) = G/\Phi_c(\mathcal{G}(c))$$

for each $c \in \operatorname{Ob}\mathcal{C}$.

Proof. We define a functor $L : \mathcal{C} \longrightarrow (G - Sets)$ as follows: let $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups from 5.4.3. For each object c of $\mathcal{C}$ we put $L(c) = G/\overline{\mathcal{G}}(c)$ and for each morphism l a G-equivariant functor is given by $L(l)([g]) = [g\Phi(l)^{-1}]$. For $c_2 \xrightarrow{l_1} c_1 \xrightarrow{l_0} c_0$ the homomorphisms $\mathcal{G}(l_0 l_1)$ and $\mathcal{G}(l_0)\mathcal{G}(l_1)$ differ by a conjugation with an element of the group $\mathcal{G}(c_0)$. Therefore $L(l_0 l_1) = L(l_0)L(l_1)$. □

Theorem 5.4.5. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups, let G be a group and $\Phi : \mathcal{G} \longrightarrow G$ any homomorphism.*

1. *Canonically associated to each morphism $\Phi : \mathcal{G} \longrightarrow G$ there is an action of G on a small category $\mathcal{D} = \mathcal{D}(\mathcal{G}, \Phi)$ (called the development) with quotient $\mathcal{C}$. The twisted diagram of groups associated to this action is equivalent to $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$. If Φ is injective on the local groups, then $\mathcal{G}$ is equivalent to the twisted diagram of groups associated to this action.*

2. *If $\mathcal{G}$ is the twisted diagram of groups associated to an action of a group G on a small category $\mathcal{D}$ and if $\Phi : \mathcal{G} \longrightarrow G$ is the associated morphism, then there is a G-equivariant isomorphism $\mathcal{D}(\mathcal{G}, \Phi) \longrightarrow \mathcal{D}$ that projects to identity of $\mathcal{C}$.*

Proof. 1. Let $L : \mathcal{C} \longrightarrow G - \mathbf{Sets}$ be the functor defined in 5.4.4. We define a small category $\mathcal{D}(\mathcal{G}, \Phi)$ to be the Grothendieck construction $\mathcal{B}L$. The action of the group G on $\mathcal{D}$ is given by $g[h] = [gh]$ and it is an action without inversion. Clearly, the quotient $\mathcal{D}/G$ is isomorphic to $\mathcal{C}$ and the associated twisted diagram of groups is equivalent to $\overline{\mathcal{G}}$. If $\Phi : \mathcal{G} \longrightarrow G$ is injective on the local groups then $\mathcal{G}$ is equivalent to $\overline{\mathcal{G}}$.

The proof of 2. follows directly from 5.2.11. □

Corollary 5.4.6. *A twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is developable if and only if there exist a group G and a homomorphism $\Phi : \mathcal{G} \longrightarrow G$ which is injective on the local groups.*

Remark 5.4.7. Let $\mathcal{I}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a trivial twisted diagram of groups and $\Phi : \mathcal{I}_{\mathcal{C}} \longrightarrow G$ any homomorphism. Clearly Φ is injective on the local groups, and the natural projection $p : \mathcal{D}(\mathcal{I}_{\mathcal{C}}, \Phi) \longrightarrow \mathcal{C}$ is a G-covering.

Assume that $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a developable twisted diagram of groups and $\Phi : \mathcal{G} \longrightarrow G$ a homomorphism which is injective on the local groups. Let $\mathcal{D} = \mathcal{D}(\mathcal{G}, \Phi)$ be the associated development, and there exists an action of the group G on $\mathcal{D}$ such that the associated twisted diagram of groups is equivalent to $\mathcal{G}$. According to 5.3.17, there exists a homomorphism $\phi : \mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{G}$ and a G-covering $\phi' : \mathcal{E} \longrightarrow \mathcal{B}\mathcal{G}$ such that $\mathcal{B}\phi$ is equal to $\mathcal{D} \overset{\approx}{\hookrightarrow} \mathcal{E} \xrightarrow{\phi'} \mathcal{B}\mathcal{G}$. Moreover, due to 5.3.16 the following diagram commutes

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{/G} & \mathcal{B}\mathcal{G} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow[/G]{} & \mathcal{C} \end{array}$$

and the projection $\mathcal{E} \longrightarrow \mathcal{D}$ is G-equivariant.

The homomorphism $\Phi : \mathcal{G} \longrightarrow G$ yields a homomorphism $\bar{\Phi} : \mathcal{I}_{\mathcal{B}\mathcal{G}} \longrightarrow G$ such that $\mathcal{B}\Phi = \mathcal{B}\bar{\Phi}$. Then

Proposition 5.4.8. *The small category $\mathcal{E}$ is isomorphic to the development $\mathcal{D}(\mathcal{I}_{\mathcal{B}\mathcal{G}}, \bar{\Phi})$.*

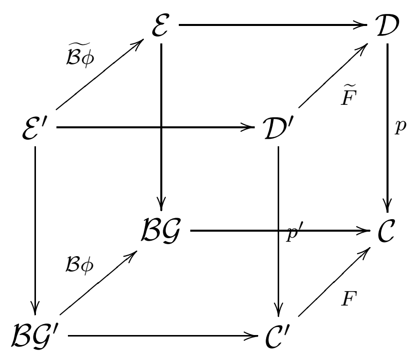
Proof. According to 5.4.7 the natural projection $\mathcal{D}(\mathcal{I}_{\mathcal{BG}}, \bar{\Phi}) \longrightarrow \mathcal{BG}$ is a G-covering of small categories. Due to 5.3.9 the fundamaental group of $\mathcal{D}(\mathcal{I}_{\mathcal{BG}}, \bar{\Phi})$ is isomorphic to the kernel of the homomorphism $\Phi_* : \pi_1(\mathcal{BG}, c_0) \longrightarrow G$. The equivalence of small categories $\mathcal{D} \overset{\approx}{\hookrightarrow} \mathcal{E}$ and 5.3.9 imply that the fundamental groups of $\mathcal{D}(\mathcal{I}_{\mathcal{BG}}, \bar{\Phi})$ and $\mathcal{E}$ are isomorphic. Therefore, there exists an isomorphism $\mathcal{E} \overset{\simeq}{\longrightarrow} \mathcal{D}(\mathcal{I}_{\mathcal{BG}}, \bar{\Phi})$ over the identity of $\mathcal{BG}$.

□

Proposition 5.4.9. *Assume that* $\mathcal{G}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$, $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ *are developable twisted diagrams of groups and* $\Phi' : \mathcal{G}' \longrightarrow G'$, $\Phi : \mathcal{G} \longrightarrow G$ *are the homomorphisms which are injective on the local groups. Assume that there exist a homomorphism* $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ *over* $F : \mathcal{C}' \longrightarrow \mathcal{C}$ *and a homomorphism of groups* $\Theta : G' \longrightarrow G$ *such that the following diagram*

$$\begin{array}{ccc} \mathcal{BG}' & \xrightarrow{\mathcal{B}\phi} & \mathcal{BG} \\ {\scriptstyle \mathcal{B}\Phi'}\downarrow & & \downarrow{\scriptstyle \mathcal{B}\Phi} \\ \mathcal{B}G' & \xrightarrow{\mathcal{B}\Theta} & \mathcal{B}G \end{array}$$

commutes up to a natural transformation $\eta : \mathcal{B}\Phi \circ \mathcal{B}\phi \Longrightarrow \mathcal{B}\Theta \circ \mathcal{B}\Phi'$. *Then there exist the* Θ*-eqivariant functors* $\widetilde{F} : \mathcal{D}' \longrightarrow \mathcal{D}$ *and* $\widetilde{\mathcal{B}\phi} : \mathcal{E}' \longrightarrow \mathcal{E}$ *of the developments (5.4.5) and coverings (5.4.8) associated to* Φ' *and* Φ *such that the following diagram commutes*



Proof. The natural transformation $\eta : \mathcal{B}\Phi \circ \mathcal{B}\phi \Longrightarrow \mathcal{B}\Theta \circ \mathcal{B}\Phi'$ is given by a family of elements $s_{c'} \in G$ indexed by $c' \in \mathrm{Ob}\,\mathcal{C}'$ such that

$$\Theta \circ \Phi'_{c'} = \mathrm{Ad}(s_{c'}) \circ \Phi_{(F(c'))} \circ \phi_{c'} \quad \text{and} \quad \mathcal{B}\Phi \circ \mathcal{B}\phi(\widetilde{l'}) = s_{t(\widetilde{l'})}\Theta(\Phi'(\widetilde{l'}))s^{-1}_{i(\widetilde{l'})} \quad \widetilde{l'} \in \mathrm{Mor}\,\mathcal{BG}'$$

Then a map $(\widetilde{l'}, g') \longrightarrow (\mathcal{B}\phi(\widetilde{l'}), \Theta(g')s_{t(\widetilde{l'})})$ defines a Θ-equivariant functor $\widetilde{\mathcal{B}\phi} : \mathcal{E}' \longrightarrow \mathcal{E}$ over $\mathcal{B}\phi : \mathcal{BG}' \longrightarrow \mathcal{BG}$.

We define a functor $\widetilde{F} : \mathcal{D}' \longrightarrow \mathcal{D}$ to be $\mathcal{D}' \overset{\approx}{\hookrightarrow} \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{D}$. Clearly $\widetilde{F} : \mathcal{D}' \longrightarrow \mathcal{D}$ is Θ-equivariant thus the following diagram

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{\widetilde{F}} & \mathcal{D} \\ {\scriptstyle /G'}\downarrow & & \downarrow{\scriptstyle /G} \\ \mathcal{C}' & \xrightarrow{F} & \mathcal{C} \end{array}$$

commutes, which proves the Proposition.

□

Proposition 5.4.10. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a developable twisted diagram of groups. Then the homomorphism $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$ from $\mathcal{G}$ to its fundamental group is injective on the local groups.*

Proof. According to 5.4.6 there exists a homomorphism $\Phi : \mathcal{G} \longrightarrow G$ which is injective on the local groups. The induced functor $\mathcal{B}\Phi : \mathcal{B}\mathcal{G} \longrightarrow \mathcal{B}G$ is equal to the composition

$$\begin{array}{ccc} & \pi\mathcal{B}\mathcal{G} & \\ {\scriptstyle \pi_{\mathcal{B}\mathcal{G}}}\nearrow & & \searrow{\scriptstyle \widetilde{\mathcal{B}\Phi}} \\ \mathcal{B}\mathcal{G} & \xrightarrow{\quad\mathcal{B}\Phi\quad} & \mathcal{B}G \end{array}$$

If Φ is injective on the local groups then for each c an object of $\mathcal{C}$ the composition $\mathcal{G}(c) \hookrightarrow \mathcal{B}\mathcal{G} \longrightarrow \mathcal{B}G$ is an inclusion. This implies that $\mathcal{G}(c) \longrightarrow \pi\mathcal{B}\mathcal{G}$ is an inclusion, hence $\mathcal{G}(c) \longrightarrow \pi\mathcal{B}\mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0) = \mathrm{Aut}_{\pi\mathcal{B}\mathcal{G}}(c_0)$ is an inclusion. Therefore $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$ is injective on the local groups.

□

Proposition 5.4.11. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a developable twisted diagram of groups and $\Phi : \mathcal{G} \longrightarrow G$ any homomorphism which is injective on the local groups. Let $\mathcal{D} = \mathcal{D}(\mathcal{G}, \Phi)$ be a dvelopment associated to Φ and $\hat{\mathcal{D}} = \mathcal{D}(\mathcal{G}, \iota_{\mathcal{G}})$ the development associated to the homomorphism $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$. Then there exists a functor $\hat{F} : \hat{\mathcal{D}} \longrightarrow \mathcal{D}$ and $\hat{\mathcal{D}} \longrightarrow \hat{F}(\hat{\mathcal{D}})$ is a covering of small categories. The functor $\hat{F}$ is onto if and only if $\Phi_* : \pi_1(\mathcal{G}, c_0) \longrightarrow G$ is onto.*

Proof. The homomorphism Φ induces a homomorphism of fundamental groups $\Phi_* : \pi_1(\mathcal{G}, c_0) \longrightarrow G$ given by a commutative diagram

$$\begin{array}{ccc} & \pi\mathcal{B}\mathcal{G} & \\ {\scriptstyle i}\nearrow & & \searrow{\scriptstyle \widetilde{\mathcal{B}\Phi}} \\ \mathrm{Aut}_{\pi\mathcal{B}\mathcal{G}}(c_0) & \xrightarrow{\quad\Phi_*\quad} & \mathcal{B}G \end{array}$$

Let $j : \pi\mathcal{B}\mathcal{G} \longrightarrow \mathrm{Aut}_{\pi\mathcal{B}\mathcal{G}}(c_0)$ be the inverse functor to the equivalence i. The homomorphism $\iota_{\mathcal{G}}$ is defined as the composition $j \circ \pi_{\mathcal{B}\mathcal{G}}$. We have $\widetilde{\mathcal{B}\Phi} \circ i = \Phi_*$ hence $\widetilde{\mathcal{B}\Phi} \circ i \circ j = \Phi_* \circ j$. This implies that there exists a natural isomorphism $\widetilde{\mathcal{B}\Phi} \implies \Phi_* \circ j$ and hence there exists a natural isomorphism $\alpha : \mathcal{B}\Phi \implies \Phi_* \circ \mathcal{B}\iota_{\mathcal{G}}$. According to 5.4.9 there exists a Φ_*-equivariant functor $\hat{F} : \hat{\mathcal{D}} \longrightarrow \mathcal{D}$.

For each $c \in \mathrm{Ob}\,\mathcal{C}$ the composition $\mathcal{G}(c) \longrightarrow \pi_1(\mathcal{G}, c_0) \xrightarrow{\Phi_*} G$ is injective, therefore $\mathcal{G}(c) \cap \ker \Phi_* = 1$. This implies that the group $\ker \Phi_*$ acts freely on $\hat{\mathcal{D}}$. Clearly the quotient of this action is isomorphic to the small category $\hat{F}(\hat{\mathcal{D}})$. □

Remark 5.4.12. According to Theorem 5.3.17 the development $\hat{\mathcal{D}}$ is equivalent to the universal covering of the small category $\mathcal{B}\mathcal{G}$, therefore $\hat{\mathcal{D}}$ is simply connected.

Chapter 6

Coverings of twisted diagrams of groups

We say that a twisted diagram of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is a generalized complex of groups if for each morphism $l \in \mathrm{Mor}\,\mathcal{C}$ the associated homomorphism of groups is injective. We will prove that a generalized complex of groups is locally developable, that is, for each object $c \in \mathrm{Ob}\,\mathcal{C}$ there exists a small category $\mathcal{D}_c$ (called the local development) with an action of a group $\mathcal{G}(c)$ such that the quotient of this action is isomorphic to the small category $\mathcal{C}/c$. Bridson and Haefliger in [B-H] defined a covering of complexes of groups to be a homomorphism $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ over a right covering $F : \mathcal{C}' \longrightarrow \mathcal{C}$ such that

- ϕ is injective on the local groups
- for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ the induced functor $\mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is an isomorphism (of the local developments)

They have proved in that $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ is a covering if and only if the associated functor $\mathcal{B}\phi$ is equal to the composition

$$\begin{array}{ccc} \mathcal{B}\mathcal{G}' & \overset{\approx}{\hookrightarrow} & \mathcal{E} \\ & {\scriptstyle \mathcal{B}\phi}\searrow & \downarrow \\ & & \mathcal{B}\mathcal{G} \end{array}$$

where $\mathcal{E} \longrightarrow \mathcal{B}\mathcal{G}$ is a covering of small categories. An arbitrary twisted diagram of groups does not have to be locally developable, hence the definition of a covering given in [B-H] no longer makes sense. Therefore we define a homomorphism of twisted diagrams of groups to be a covering if it satisfies the above property.

Section 6.1 presents some properties and examples of coverings of twisted diagrams of groups. Note that if ϕ is a covering of twisted diagrams of groups then it is injective on the local groups.

Section 6.2 is devoted to the proof of the theorem stated above, namely a homomorphism of generalized complexes of groups is a covering if it satisfies assertions 1. and 2. presented above.

We will prove in Section 6.3 that for each surjective homomorphism $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ there exists a "kernel" twisted diagram of groups. It will be a twisted diagram of groups $\mathcal{K} : \mathcal{D} \longrightarrow \mathrm{Gr}$ and a homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ such that the composition $\varphi \circ \phi$ is trivial on the local groups and the induced homomorphism $\varphi_* \circ \phi_*$ is trivial on the fundamental groups. Moreover ϕ turns out to be a covering of twisted diagrams of groups.

6.1 Coverings of twisted diagrams of groups

Definition 6.1.1. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a homomorphism of twisted diagrams of groups. We say that ϕ is a covering if and only if there exist a covering of small categories $\phi' : \mathcal{E} \longrightarrow \mathcal{BG}$ and an inclusion $\lambda : \mathcal{BG}' \longrightarrow \mathcal{E}$ which is an equivalence, such that $\mathcal{B}\phi = \phi'\lambda$.*

Corollary 6.1.2. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a covering of twisted diagrams of groups. The associated homomorphism of fundamental groups is injective.*

Proposition 6.1.3. *Assume that $\mathcal{G}' = H$ and $\mathcal{G} = G$ are twisted diagrams of groups on the category with one object and no morphisms and $\phi : H \longrightarrow G$ is a homomorphism. Then ϕ is a covering of twisted diagrams of groups if and only if ϕ is a monomorphism of groups.*

Proof. Assume that $\phi : H \longrightarrow G$ is a monomorphism. Then the corresponding functor $\mathcal{B}\phi$ is the inclusion of small categories $\mathcal{B}H \longrightarrow \mathcal{B}G$. We put $\mathcal{E} := \mathcal{E}G/H$, then $\mathcal{B}H \longrightarrow \mathcal{E}G/H$ is equivalence and inclusion of categories. The natural projection $\mathcal{E}G/H \longrightarrow \mathcal{B}G$ is a covering.

Assume that $\phi : H \longrightarrow G$ is a covering of twisted diagrams of groups. Then there exists a small category $\mathcal{E}$ such that $\mathcal{B}H \overset{\lambda}{\hookrightarrow} \mathcal{E} \overset{\phi'}{\longrightarrow} \mathcal{B}G$ The functor λ is an inclusion and an equivalence of small categories then for each $e \in \mathrm{Ob}\,\mathcal{E}$ the set $\mathrm{End}_{\mathcal{E}}(e) \simeq H$. The functor ϕ' is a covering, thus according to 5.1.3 we have an inclusion $\mathrm{End}_{\mathcal{E}}(e) \longrightarrow \mathcal{B}G$. Then ϕ is a monomorphism of groups. □

More generally:

Proposition 6.1.4. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a covering of twisted diagrams of groups over $F : \mathcal{C}' \longrightarrow \mathcal{C}$. Then ϕ is a monomorphism on the local groups, namely for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ the corresponding homomorphism of groups $\phi_{c'} : \mathcal{G}'(c') \longrightarrow \mathcal{G}(F(c'))$ is injective.*

Proof. Let $\mathcal{E}$ be a small category such that the composition $\mathcal{BG}' \overset{\lambda}{\hookrightarrow} \mathcal{E} \overset{\phi'}{\longrightarrow} \mathcal{BG}$ equals $\mathcal{B}\phi$. Then for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{G}'(c') & \hookrightarrow & \mathrm{End}_{\mathcal{E}}(\lambda(c')) \\ {\scriptstyle \phi_{c'}}\downarrow & & \downarrow{\scriptstyle \phi'_{|}} \\ \mathcal{G}(F(c')) & \hookrightarrow & \mathrm{End}_{\mathcal{BG}}(F(c')) \end{array}$$

The restriction $\phi'_{|}$ of ϕ' is an inclusion because ϕ' is a covering. Then $\phi_{c'}$ is injective. □

Note that a covering of trivial twisted diagrams of groups is not what one would expect:

Remark 6.1.5. Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a covering over $F : \mathcal{C}' \longrightarrow \mathcal{C}$ and assume that $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and $\mathcal{G}' : \mathcal{C}' \longrightarrow \mathrm{Gr}$ are trivial twisted diagrams of groups, i.e. such that the local groups are trivial. Then the related functor $\mathcal{B}\phi : \mathcal{BG}' \longrightarrow \mathcal{BG}$ equals $F : \mathcal{C}' \longrightarrow \mathcal{C}$. Note that F does not have to be a covering of small categories but the small category $\mathcal{C}'$ is equivalent to a covering category $\mathcal{E}$ of $\mathcal{C}$.

Proposition 6.1.6. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a covering over $F : \mathcal{C}' \longrightarrow \mathcal{C}$ of trivial twisted diagrams of groups. Assume that F is a right covering. Then $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is a covering of small categories.*

Proof. The functor F equals $\mathcal{C}' \overset{\lambda}{\hookrightarrow} \mathcal{E} \overset{\phi'}{\longrightarrow} \mathcal{C}$, where λ is an equivalence and ϕ' is a covering. We will prove that λ is an isomorphism. Choose any $e \in \mathrm{Ob}\,\mathcal{E}$. Then e is isomorphic to $\lambda(c')$ for some $c' \in \mathrm{Ob}\,\mathcal{C}'$. Let l denote the image of this isomorphism under ϕ'; $l : F(c') \longrightarrow \phi'(e)$. The functor F is the right covering thus there exists a unique morphism $l' : c' \longrightarrow c_1'$ such that $F(l') = l$. This implies $e = \lambda(c_1')$, thus λ is onto and then it is an isomorphism of small categories. Therefore F is a covering. □

Example 6.1.7. Assume that a group G acts without inversion on a small category $\mathcal{D}$ and $\mathcal{G} : \mathcal{D}/G \longrightarrow \mathrm{Gr}$ is an associated twisted diagram of groups. Then according to 5.3.17 there exists a homomorphism $\phi : \mathcal{I}_{\mathcal{D}} \longrightarrow \mathcal{G}$ over $p : \mathcal{D} \longrightarrow \mathcal{D}/G$ and it is a covering of twisted diagrams of groups.

6.2 Coverings of generalized complexes of groups

Definition 6.2.1. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups. We say that $\mathcal{G}$ is a generalized complex of groups if for each $l : c_1 \longrightarrow c_2$ in $\mathrm{Mor}\,\mathcal{C}$ the corresponding homomorphism of groups $\mathcal{G}(l) : \mathcal{G}(c_1) \longrightarrow \mathcal{G}(c_2)$ is injective.*

Remark 6.2.2. Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a developable twisted diagram of groups. Then $\mathcal{G}$ is a generalized complex of groups.

Local developability

Proposition 6.2.3. *Let $\mathcal{C}$ be a small category with the final object c and $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ a generalized complex of groups. Then $\mathcal{G}$ is developable and the universal covering of $\mathcal{G}$ is isomorphic to the small category $\mathcal{D}$ such that*

1. *$\mathcal{D}$ has a final object*
2. *let $p : \mathcal{D} \longrightarrow \mathcal{C}$ be the natural projection, for each $c' \in \mathrm{Ob}\,\mathcal{C}$ the preimage $p^{-1}(c')$ is a subset of $\mathrm{Ob}\,\mathcal{D}$ and is isomorphic to $\mathcal{G}(c)/\mathcal{G}(c')$*

Proof. Since $\mathcal{C}$ has a final object c and $\mathcal{G}$ is a generalized complex of groups then the fundamental group of $\mathcal{G}$ is isomorphic to $\mathcal{G}(c)$. The universal covering $\mathcal{D}$ is the development $\mathcal{D}(\mathcal{G}, \Phi)$ of the natural homomorphism $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathcal{G}(c)$ which is a monomorphism on the local groups. Thus the proof follows directly from 5.4.5. □

Corollary 6.2.4. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a generalized complex of groups. For each $c \in \mathrm{Ob}\,\mathcal{C}$ let $\mathcal{C}/c$ be a small category "over c" and $l_c : \mathcal{C}/c \longrightarrow \mathcal{C}$ the natural projection. Let $\mathcal{G}_c := l_c^*\mathcal{G}$ be the twisted diagram of groups induced by l_c. Then $\mathcal{G}_c$ is developable and equivalent to a diagram of subgroups of $\mathcal{G}(c)$.*

Remark 6.2.5. Let $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Cat}$ be a functor associated to $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ defined in 1.3.3. Then $\mathcal{B}\mathcal{G}_c = \overline{\mathcal{G}}(c)$.

Definition 6.2.6. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $\mathcal{G}_c$ the developable twisted diagram of groups defined above. Let $\mathcal{D}_c$ be the development associated to $\iota_{\mathcal{G}_c} : \mathcal{G}_c \longrightarrow \mathcal{G}(c)$. We will call it the local development of $\mathcal{G}$ at c.*

Proposition 6.2.7. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups associated to an action of a group G on a small category $\mathcal{D}$, let $p : \mathcal{D} \longrightarrow \mathcal{C}$ be the associated projection. Then for each $d \in \mathrm{Ob}\,\mathcal{D}$ the local devlopment at $p(d)$ is isomorphic to the category $\mathcal{D}/d$, namely*

$$\mathcal{D}_{p(d)} \simeq \mathcal{D}/d$$

Proof. Consider the action of the subgroup $\mathrm{Stab}_G(d)$ of the group G on the small category $\mathcal{D}$. This action yields an action of $\mathrm{Stab}_G(d)$ on the small category $\mathcal{D}/d$ with quotient $\mathcal{C}/p(d)$. The complex of groups associated to this action is isomorphic to $\mathcal{G}_{p(d)}$. Then according to 5.4.5 the small category $\mathcal{D}/d$ and the development of $\mathcal{G}_{p(d)}$ are isomorphic. □

Coverings of complexes of groups by [B-H]

Proposition 6.2.8. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a homomorphism of generalized complexes of groups over $F : \mathcal{C}' \longrightarrow \mathcal{C}$. For each $c' \in \mathrm{Ob}\,\mathcal{C}'$ there exists a homomorphism of the local complexes of groups $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ over $F/c' : \mathcal{C}'/c' \longrightarrow \mathcal{C}/F(c')$. This homomorphism yields a $\phi_{c'}$-equivariant functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ of local developments.*

Proof. Let $j : \mathcal{G} \longrightarrow \overline{\mathcal{G}}$ denote the natural transformation defined in 1.3.6. The homomorphism $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ is given by the natural transformation $\eta : \mathcal{G}' \longrightarrow F^*\mathcal{G}$, let $\overline{\eta} : \overline{\mathcal{G}'} \Longrightarrow F^*\overline{\mathcal{G}}$ be the natural transformation of functors $\overline{\mathcal{G}'}, F^*\overline{\mathcal{G}} : \mathcal{C}' \longrightarrow \mathrm{Cat}$ associated to η.

According to 1.3.6 we have

$$\begin{array}{ccc} \mathcal{G}' & \overset{\eta}{\Longrightarrow} & F^*\mathcal{G} \\ {\scriptstyle j'}\Downarrow & & \Downarrow{\scriptstyle F^*j} \\ \overline{\mathcal{G}'} & \underset{\overline{\eta}}{\Longrightarrow} & F^*\overline{\mathcal{G}} \end{array}$$

hence for each $c' \in \mathrm{Ob}\,\mathcal{C}'$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{G}'(c') & \xrightarrow{\phi_{c'}} & \mathcal{G}(F(c')) \\ {\scriptstyle j'_{c'}}\downarrow & & \downarrow{\scriptstyle j_{F(c')}} \\ \overline{\mathcal{G}'}(c') & \xrightarrow{\overline{\phi}_{c'}} & \overline{\mathcal{G}}(F(c')) \end{array}$$

Note, the functor $\overline{\phi}_{c'} : \overline{\mathcal{G}'}(c') \longrightarrow \overline{\mathcal{G}}(F(c'))$ defines a homomorphism $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ over $F/c' : \mathcal{C}'/c' \longrightarrow \mathcal{C}/F(c')$.

Due to 1.3.7 there exist $i' : \overline{\mathcal{G}'}(c') \longrightarrow \mathcal{G}'(c')$ and $i : \overline{\mathcal{G}}(F(c')) \longrightarrow \mathcal{G}(F(c'))$ such that we have the natural transformations $i' \circ j' \Longrightarrow \mathrm{id}_{\mathcal{G}'(c')}$, $i \circ j \Longrightarrow \mathrm{id}_{\mathcal{G}(F(c'))}$ and $\mathrm{id}_{\overline{\mathcal{G}'}(c')} \Longrightarrow j' \circ i'$, $\mathrm{id}_{\overline{\mathcal{G}}(F(c'))} \Longrightarrow j \circ i$. Therefore there exists a natural transformation $\alpha : i \circ \overline{\phi}_{c'} \Longrightarrow \phi_{c'} \circ i'$. Due to 5.4.9 the natural transformation α yields a $\phi_{c'}$-equivariant functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$. □

Assume that $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ is a covering of complexes of groups. Then the Defintion 6.1.1 becomes the Proposition A.24 from Chapter III.C [B-H].

Theorem 6.2.9. *Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ be a homomorphism of generalized complexes of groups over a functor $F : \mathcal{C}' \longrightarrow \mathcal{C}$ which is onto and is a right covering. Then ϕ is a covering if and only if it satisfies:*

1. *ϕ is a monomorphism on the local groups*

2. *for each $c' \in \operatorname{Ob}\mathcal{C}'$ the induced functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is an isomorphism*

Proof. $\Longrightarrow$ Assume that ϕ is a covering. Then according to 6.1.4 ϕ satisfies assertion 1. We will prove that ϕ satisfies assertion 2.

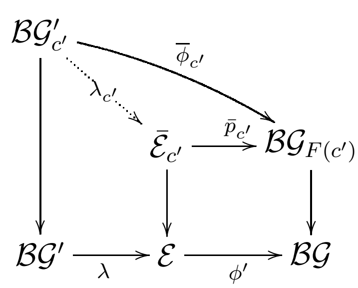
For each $c' \in \operatorname{Ob}\mathcal{C}'$ let $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ be the induced homomorphism of the local twisted diagrams of groups.

Lemma 6.2.10. *The homomorphism $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ over $F/c' : \mathcal{C}'/c' \longrightarrow \mathcal{C}/F(c')$ is a covering of twisted diagrams.*

Proof. The homomorphism ϕ is a covering then there exists a small category $\mathcal{E}$ such that $\mathcal{BG}' \overset{\lambda}{\hookrightarrow} \mathcal{E} \overset{\phi'}{\longrightarrow} \mathcal{BG}$. We define a small category $\bar{\mathcal{E}}_{c'}$ to be the pull back category of the diagram

$$\begin{array}{ccc} \bar{\mathcal{E}}_{c'} & \longrightarrow & \mathcal{BG}_{F(c')} \\ \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{BG} \end{array}$$

Then $\bar{p}_{c'} : \bar{\mathcal{E}}_{c'} \longrightarrow \mathcal{BG}_{F(c')}$ is a covering and the unique functor $\lambda_{c'} : \mathcal{BG}'_{c'} \longrightarrow \bar{\mathcal{E}}_{c'}$



is an inclusion. Let $\mathcal{E}_{c'}$ be a connected component of $\bar{\mathcal{E}}_{c'}$ containing $\lambda_{c'}(\mathcal{BG}'_{c'})$. Clearly $\lambda_{c'} : \mathcal{BG}'_{c'} \longrightarrow \mathcal{E}_{c'}$ is an equivalence (the proof is standard). Thus $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ is given by $\mathcal{BG}'_{c'} \overset{\approx}{\hookrightarrow} \mathcal{E}_{c'} \overset{p_{c'}}{\longrightarrow} \mathcal{BG}_{F(c')}$, hence is a covering. ☐

Lemma 6.2.11. *The $\phi_{c'}$-equivariant functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is an inclusion and an equivalence of categories.*

Proof. The equivalence $\mathcal{BG}'_{c'} \hookrightarrow \mathcal{E}_{c'}$ yields an equivalence of the universal coverings $\widehat{\mathcal{BG}'_{c'}} \hookrightarrow \widehat{\mathcal{E}_{c'}}$. The universal covering of the small category $\mathcal{E}_{c'}$ is isomorphic to the universal covering of the small category $\mathcal{BG}_{F(c')}$. This gives a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{BG}'_{c'}} & \overset{\approx}{\hookrightarrow} & \widehat{\mathcal{BG}_{F(c')}} \\ \downarrow & & \downarrow \\ \mathcal{BG}'_{c'} & \overset{\approx}{\hookrightarrow} & \mathcal{E}_{c'} \end{array}$$

According to 5.4.9 and 6.2.8 we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{B}\mathcal{G}'_{c'}} & \overset{\approx}{\hookrightarrow} & \widehat{\mathcal{B}\mathcal{G}_{F(c')}} \\ \downarrow & & \downarrow \\ \mathcal{D}'_{c'} & \xrightarrow{\widetilde{F}_{c'}} & \mathcal{D}_{F(c')} \end{array}$$

Clearly the following diagram

$$\begin{array}{ccc} \widehat{\mathcal{B}\mathcal{G}'_{c'}} & \overset{\approx}{\hookrightarrow} & \widehat{\mathcal{B}\mathcal{G}_{F(c')}} \\ \approx\uparrow & & \approx\uparrow \\ \mathcal{D}'_{c'} & \xrightarrow{\widetilde{F}_{c'}} & \mathcal{D}_{F(c')} \end{array}$$

commutes up to a natural isomorphism. Therefore the functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is an inclusion and equivalence of small categories. □

Lemma 6.2.12. *The functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is onto.*

Proof. By 6.2.11 it is enough to prove that $\widetilde{F}_{c'}$ is onto on the objects set. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{D}'_{c'} & \xrightarrow{\widetilde{F}_{c'}} & \mathcal{D}_{F(c')} \\ {\scriptstyle \pi_{c'}}\downarrow & & \downarrow{\scriptstyle \pi_{F(c')}} \\ \mathcal{C}'/c' & \xrightarrow{F/c'} & \mathcal{C}/F(c') \end{array}$$

The functors $\pi_{c'}$, F/c', $\pi_{F(c')}$ are right coverings and onto, therefore $\widetilde{F}_{c'}$ is a right covering. Choose $d \in \operatorname{Ob}\mathcal{D}_{F(c')}$. The functor $\widetilde{F}_{c'}$ is the equivalence therefore there exists an isomorphism $l : \widetilde{F}_{c'}(d') \longrightarrow d$ where $d' \in \operatorname{Ob}\mathcal{D}'_{c'}$. The functor $\widetilde{F}_{c'}$ is a left covering, hence there exists a morphism $l' : d' \longrightarrow d''$ in $\operatorname{Mor}\mathcal{D}'_{c'}$ such that $\widetilde{F}_{c'}(l') = l$. Therefore $\widetilde{F}_{c'}$ is onto on the objects set, hence in view of 6.2.11 is onto.

□

Due to 6.2.11 and 6.2.12 the functor $\widetilde{F}_{c'} : \mathcal{D}'_{c'} \longrightarrow \mathcal{D}_{F(c')}$ is an isomorphism. This proves assertion 2.

$\Longleftarrow$ Assume that $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ is a homomorphism of generalized complexes of groups which satisfies 1 and 2. We will prove that it is a covering of twisted diagrams of groups. In order to do this we will first prove that it is a covering locally.

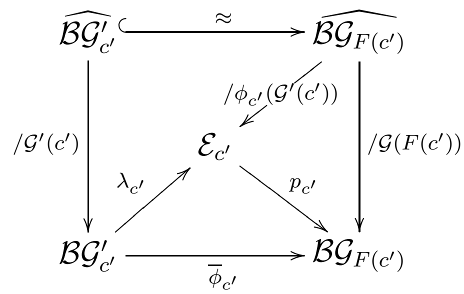
Lemma 6.2.13. *For each $c' \in \operatorname{Ob}\mathcal{C}'$ the homomorphism $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ is a covering of (generalized) complexes of groups.*

Proof. Let $W' : \mathcal{D}'_{c'} \longrightarrow \mathrm{Cat}$ and $W : \mathcal{D}_{F(c')} \longrightarrow \mathrm{Cat}$ be functors defined in 5.3.15. Then the $\phi_{c'}$-equivariant functor $\widetilde{F}_{c'}$ induces a functor $\mathcal{B}W' \longrightarrow \mathcal{B}W$ over $\widetilde{F}_{c'}$. The homomorphism ϕ is injective on the local groups, and $\widetilde{F}_{c'}$ is an isomorphism hence $\mathcal{B}W' \longrightarrow \mathcal{B}W$ is an inclusion. It is clearly an equivalence. Then the functor $\widehat{\mathcal{B}\mathcal{G}'_{c'}} \simeq \mathcal{B}W' \longrightarrow \mathcal{B}W \simeq \widehat{\mathcal{B}\mathcal{G}_{F(c')}}$ is an inclusion and an equivalence of categories.

Let $p_{c'} : \mathcal{E}_{c'} \longrightarrow \mathcal{B}\mathcal{G}_{F(c')}$ be a covering of small categories such that

$$(p_{c'})_*(\pi_1(\mathcal{E}_{c'}, e_{c'})) = \phi_{c'}(\mathcal{G}'(c')) \subset \mathcal{G}(F(c')) = \pi_1(\mathcal{B}\mathcal{G}_{F(c')}, \mathrm{id}_{F(c')})$$

Then according to 5.1.6 there exists a functor $\lambda_{c'} : \mathcal{B}\mathcal{G}'_{c'} \longrightarrow \mathcal{E}_{c'}$ such that $\overline{\phi}_{c'} = p_{c'}\lambda_{c'}$. Consider the following diagram



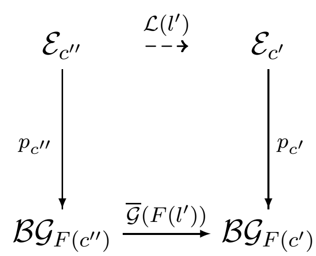
By assertion 1. the homomorphism of groups $\mathcal{G}'(c') \longrightarrow \phi_{c'}(\mathcal{G}'(c'))$ is an isomorphism. Moreover the functor $\widehat{\mathcal{B}\mathcal{G}'_{c'}} \hookrightarrow \widehat{\mathcal{B}\mathcal{G}_{F(c')}}$ is $\phi_{c'}$-equivariant, therefore $\lambda_{c'}$ is an inclusion and an equivalence of small categories. Then $\overline{\phi}(c') : \mathcal{G}'_{c'} \longrightarrow \mathcal{G}_{F(c')}$ is a covering of (generalized) complexes of groups.

□

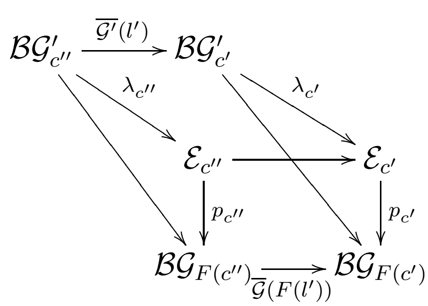
As we have proved ϕ is locally a covering. We will construct a global covering. In order to do this we will prove that

Lemma 6.2.14. *There exists a functor* $\mathcal{L} : \mathcal{C}' \longrightarrow \mathrm{Cat}$ *such that for each* $c' \in \mathrm{Ob}\,\mathcal{C}'$ $\mathcal{L}(c') = \mathcal{E}_{c'}$ *defined in 6.2.13. Moreover there exist the natural transformations* $\overline{\rho} : \mathcal{L} \Longrightarrow F^*\overline{\mathcal{G}}$, $\overline{\lambda} : \overline{\mathcal{G}'} \Longrightarrow \mathcal{L}$ *such that* $\overline{\rho}_{c'} = p_{c'}$ *is a covering and* $\overline{\lambda}_{c'} = \lambda_{c'}$ *is an equivalence of small categories.*

Proof. For each $c' \in \mathrm{Ob}\,\mathcal{C}'$ let $\mathcal{L}(c') = \mathcal{E}_{c'}$ defined in 6.2.13. We pick a base object $e_{c'}$ of $\mathcal{E}_{c'}$ such that $\lambda_{c'}(\mathrm{id}_{c'}) = e_{c'}$ and $p_{c'}(e_{c'}) = \mathrm{id}_{F(c')}$, where $\mathrm{id}_{c'}$ is a base object of $\overline{\mathcal{G}'}(c')$ and $\mathrm{id}_{F(c')}$ is a base object of $\overline{\mathcal{G}}(F(c'))$. We define $\overline{\rho}_{c'} = p_{c'}$ and $\overline{\lambda}_{c'} = \lambda_{c'}$. Let $l' \in \mathrm{Mor}\,\mathcal{C}'$ be any morphism $l' : c'' \longrightarrow c'$. We define a functor $\mathcal{L}(l')$



which is defined as a lifting of $\mathcal{E}_{c''} \longrightarrow \mathcal{B}\mathcal{G}_{F(c'')} \longrightarrow \mathcal{B}\mathcal{G}_{F(c')}$ such that $\mathcal{L}(l')(e_{c''}) = \lambda_{c'} \circ \overline{\mathcal{G}'}(l')(\mathrm{id}_{c''})$. Thus the following diagram commutes



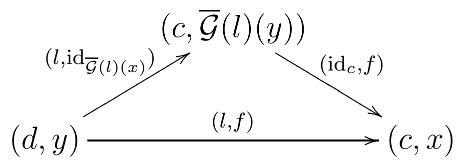
□

According to 1.2.9 the natural transformation $\overline{\lambda} : \overline{\mathcal{G}'} \Longrightarrow \mathcal{L}$ induces an equivalence (and inclusion) of small categories $\mathcal{B}\overline{\lambda} : \mathcal{B}\overline{\mathcal{G}'} \longrightarrow \mathcal{B}\mathcal{L}$.

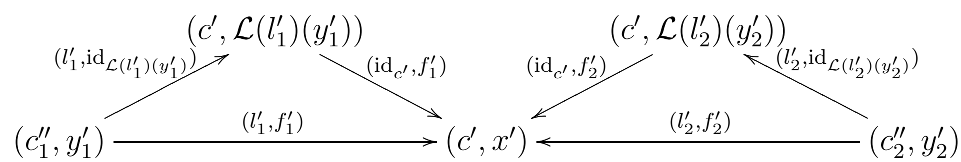
Lemma 6.2.15. *The functor* $\overline{p} : \mathcal{B}\mathcal{L} \longrightarrow \mathcal{B}\overline{\mathcal{G}}$ *induced by the natural transformation* $\overline{\rho} : \mathcal{L} \Longrightarrow F^*\overline{\mathcal{G}}$ *is a covering of small categories.*

Proof. We will prove that $\overline{p}$ satisfies assertion 2. from 5.1.3. Let $(c', x') \in \mathrm{Ob}\,\mathcal{B}\mathcal{L}$ and $(c, x) = \overline{p}(c', x') \in \mathrm{Ob}\,\mathcal{B}\overline{\mathcal{G}}$. Then the set of morphisms that have (c', x') as its initial object consits of pairs (l', f') such that $i(l') = c'$ and $i(f') = \mathcal{L}(l')(x')$. The set of morphisms of $\mathcal{C}'$ with the initial object c' is in bijection with the set of morphisms of $\mathcal{C}$ with the initial object c, because $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is a right covering. For each c' the functor $\overline{\rho}_{c'} = p_{c'}$ is a covering. Thus, according to 5.1.3, the restriction of $\overline{p}$ to the set of morphisms with the initial object (c', x') is a bijection onto the set of morphisms with the initial object (c, x).

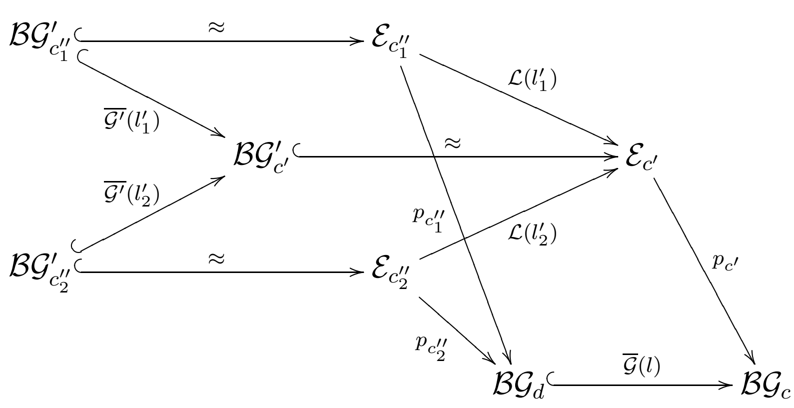
Let $(c, x) = \overline{p}(c', x') \in \mathrm{Ob}\,\mathcal{B}\overline{\mathcal{G}}$. Each morphism $(l, f) \in \mathrm{Mor}\,\mathcal{B}\overline{\mathcal{G}}$, $t(l, f) = (c, x)$ is equal to the composition



Assume that $(l_1', f_1'),\ (l_1', f_1') \in \mathrm{Mor}\,\mathcal{B}\mathcal{L}$ such that $\overline{p}((l_1', f_1')) = \overline{p}((l_2', f_2')) = (l, f)$. Then



project on (l, f). Since $p_{c'}$ is a covering then according to 5.1.3, $f_1' = f_2'$. Then $\mathcal{L}(l_1')(y_1') = \mathcal{L}(l_2')(y_2')$. Consider the following diagram



If $l_1' \neq l_2'$ then $\mathrm{Ob}\,\overline{\mathcal{G}'}(l_1')(\mathcal{B}\mathcal{G}'_{c_1''}) \cap \mathrm{Ob}\,\overline{\mathcal{G}'}(l_2')(\mathcal{B}\mathcal{G}'_{c_2''}) = \emptyset$ and then one can verify that $\mathrm{Ob}\,\mathcal{L}(l_1')(\mathcal{E}_{c_1''}) \cap \mathrm{Ob}\,\mathcal{L}(l_2')(\mathcal{E}_{c_2''}) = \emptyset$. Thus $l_1' = l_2'$ and then the restriction of $\overline{p}$ to the set of morphisms with the terminal object (c', x') is a bijection onto the set of morphisms with the terminal object (c, x). Thus the functor $\overline{p}$ is a covering of small categories.

□

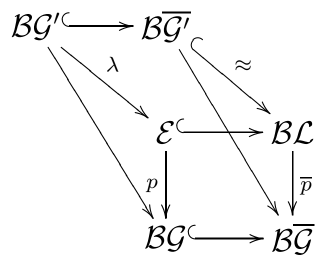
Let $\mathcal{E}$ be the pull-back category of the diagram

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{BL} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \overline{p}} \\ \mathcal{BG} & \hookrightarrow & \mathcal{B}\overline{\mathcal{G}} \end{array}$$

Then $p : \mathcal{E} \longrightarrow \mathcal{BG}$ is a covering and $\mathcal{E} \longrightarrow \mathcal{BL}$ is an inclusion of small categories. The small category $\mathcal{BG}'$ is the pull-back of the following diagram

$$\begin{array}{ccc} \mathcal{BG}' & \hookrightarrow & \mathcal{B}\overline{\mathcal{G}'} \\ \downarrow & & \downarrow \\ \mathcal{BG} & \hookrightarrow & \mathcal{B}\overline{\mathcal{G}} \end{array}$$

This implies that the unique functor $\lambda : \mathcal{BG}' \longrightarrow \mathcal{E}$



is an equivalence and inclusion of small categories (the proof is standard).

Thus the functor $\mathcal{B}\phi : \mathcal{BG}' \longrightarrow \mathcal{BG}$ equals $\mathcal{BG}' \overset{\approx}{\hookrightarrow} \mathcal{E} \overset{p}{\longrightarrow} \mathcal{BG}$ and then ϕ is a covering of twisted diagrams of groups.

□

6.3 G-coverings and extensions of twisted diagrams of groups

Definition 6.3.1. *We say that a covering $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ over $F : \mathcal{C}' \longrightarrow \mathcal{C}$ of twisted diagrams of groups is a G-covering if the associated covering $\phi' : \mathcal{E} \longrightarrow \mathcal{BG}$ is a G-covering.*

Note that the covering from Example 6.1.7 is a G-covering of twisted diagrams of groups.

Galois covering of a twisted diagram of groups

Theorem 6.3.2. *Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups and $\Phi : \mathcal{G} \longrightarrow G$ a homomorphism in the category of twisted diagrams of groups. Then there exists a G-covering $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ associated to Φ such that*

1. *$\mathcal{G}'$ is a twisted diagram of groups defined over the category $\mathcal{D} = \mathcal{D}(\mathcal{G}, \Phi)$ (development of Φ), let $p : \mathcal{D} \longrightarrow \mathcal{C}$ be the associated projection*

2. *$\mathcal{G}'(d) = \ker(\Phi_{p(d)} : \mathcal{G}(p(d)) \longrightarrow G)$*

3. $\pi_1(\mathcal{G}', d) = \ker(\pi_1(\mathcal{G}, p(d)) \longrightarrow G)$

Proof. Let $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a developable twisted diagram of groups defined in 5.4.3 and $\varphi : \mathcal{G} \twoheadrightarrow \overline{\mathcal{G}}$ the associated surjective homomorphism of twisted diagrams of groups. Then according to 3.4.4 there exists a twisted diagram of groups

$$\mathcal{F}_\varphi : \mathcal{B}\overline{\mathcal{G}} \longrightarrow \mathrm{Gr}$$

such that $\mathcal{F}_\varphi(c) = \ker(\varphi_c : \mathcal{G}(c) \longrightarrow \overline{\mathcal{G}}(c))$ and the classifying category of $\mathcal{F}_\varphi$ is isomorphic to the classifying category of $\mathcal{G}$.

Let $\mathcal{D} = \mathcal{D}(\mathcal{G}, \Phi)$ be the development of Φ and $\bar{\phi} : \mathcal{D} \longrightarrow \overline{\mathcal{G}}$ the associated covering of twisted diagrams of groups (6.1.7). Then homomorphism $\bar{\phi}$ gives a functor $\mathcal{B}\bar{\phi} : \mathcal{D} \longrightarrow \mathcal{B}\overline{\mathcal{G}}$. Note that the functor $p : \mathcal{D} \longrightarrow \mathcal{C}$ is the composition of $\mathcal{B}\bar{\phi}$ with the natural projection $\pi : \mathcal{B}\overline{\mathcal{G}} \longrightarrow \mathcal{C}$.

We define a twisted diagram of groups $\mathcal{G}'$ to be

$$\mathcal{G}' := (\mathcal{B}\bar{\phi})^* \mathcal{F}_\varphi : \mathcal{D} \longrightarrow \mathrm{Gr}$$

and a homomorphism $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ given by the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{B}\mathcal{G}' & \xrightarrow{=} & \mathcal{B}(\mathcal{B}\bar{\phi})^*\mathcal{F}_\varphi & \longrightarrow & \mathcal{B}\mathcal{F}_\varphi & \xrightarrow{\simeq} & \mathcal{B}\mathcal{G} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{D} & \xrightarrow{=} & \mathcal{D} & \xrightarrow{\mathcal{B}\bar{\phi}} & \mathcal{B}\overline{\mathcal{G}} & \longrightarrow & \mathcal{C}
\end{array}$$

Then $\mathcal{G}'(d) = \mathcal{F}_\varphi(\mathcal{B}\bar{\phi}(d)) = \ker(\mathcal{G}(\mathcal{B}\bar{\phi}(d)) \longrightarrow \overline{\mathcal{G}}(\mathcal{B}\bar{\phi}(d))) = \ker(\Phi_{p(d)} : \mathcal{G}(p(d)) \longrightarrow G)$.

The twisted diagram of groups $\overline{\mathcal{G}}$ is developable. Then the functor $\mathcal{B}\bar{\phi} : \mathcal{D} \longrightarrow \mathcal{B}\overline{\mathcal{G}}$ is equal to the composition

$$\mathcal{D} \xrightarrow{\lambda} \mathcal{E} \xrightarrow{\phi''} \mathcal{B}\overline{\mathcal{G}}$$

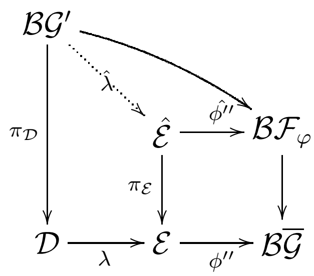
where λ is an inclusion and an equivalence of categories and ϕ'' is a G-covering of small categories. Let $\widehat{\mathcal{B}\bar{\phi}} : \mathcal{B}\mathcal{G}' \longrightarrow \mathcal{B}\mathcal{F}_\varphi$ be the functor from the above diagram which projects to $\mathcal{B}\bar{\phi}$. We will prove that $\widehat{\mathcal{B}\bar{\phi}}$ is equivalent to a G-covering of small categories.

Let $\hat{\mathcal{E}}$ be the pull-back category of the following diagram

$$\begin{array}{ccc}
\hat{\mathcal{E}} & \xrightarrow{\hat{\phi}''} & \mathcal{B}\mathcal{F}_\varphi \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{\phi''} & \mathcal{B}\overline{\mathcal{G}}
\end{array}$$

Then $\hat{\phi}'' : \hat{\mathcal{E}} \longrightarrow \mathcal{B}\mathcal{F}_\varphi$ is a G-covering. The action of the group G is induced from the action of G on $\mathcal{E}$ and $\hat{\phi}''$ is the natural projection induced by this action, thus $\hat{\phi}''$ is a G-covering.

Due to the universal property of the pull-back there exists a unique functor $\hat{\lambda} : \mathcal{BG}' \longrightarrow \hat{\mathcal{E}}$, such that the following diagram



commutes. We will prove that $\hat{\lambda}$ is an inclusion and an equivalence of categories.

First note that $\hat{\mathcal{E}} \longrightarrow \mathcal{E}$ is a functor associated to an induced twisted diagram of groups $(\phi'')^*\mathcal{F}_{\varphi}$ and the twisted diagram of groups $\mathcal{G}'$ is induced by λ. Then $\pi_{\mathcal{D}}$ and $\pi_{\mathcal{E}}$ satisfy properties from Theorem 2.2.9. This implies $\mathrm{Mor}_{\mathcal{BG}'}(d, d') \simeq \mathrm{Mor}_{\mathcal{D}}(d, d') \times \mathcal{G}'(d')$. But $\mathrm{Mor}_{\mathcal{D}}(d, d') \simeq \mathrm{Mor}_{\mathcal{E}}(\lambda(d), \lambda(d'))$ and $\mathrm{Mor}_{\mathcal{E}}(\lambda(d), \lambda(d')) \times \mathcal{G}'(d') \simeq \mathrm{Mor}_{\hat{\mathcal{E}}}(\hat{\lambda}(d), \hat{\lambda}(d'))$. Thus $\mathrm{Mor}_{\mathcal{BG}'}(d, d') \simeq \mathrm{Mor}_{\hat{\mathcal{E}}}(\hat{\lambda}(d), \hat{\lambda}(d'))$.
Choose $e \in \mathrm{Ob}\,\hat{\mathcal{E}} = \mathrm{Ob}\,\mathcal{E}$. There exists $d \in \mathrm{Ob}\,\mathcal{D} = \mathrm{Ob}\,\mathcal{BG}'$ such that $\lambda(d) \simeq e$. This implies $\hat{\lambda}(d) \simeq e$ in the category $\hat{\mathcal{E}}$. Thus $\hat{\lambda}$ is the the equivalence of categories.
The functor $\hat{\lambda}$ is inlusion on objects (because λ is) and equivalence, thus is inclusion of categories.

Thus $\mathcal{G}' \longrightarrow \mathcal{F}_{\varphi}$ is a G-covering of twisted diagrams which implies that $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ is a G-covering (because $\mathcal{BF}_{\varphi} \simeq \mathcal{BG}$). Note, $\hat{\mathcal{E}} \xrightarrow{\hat{\phi}''} \mathcal{BF}_{\varphi} \xrightarrow{\simeq} \mathcal{BG}$ is a G-covering associated to the homomorphism $\Phi : \mathcal{G} \longrightarrow G$. Then

1. $\mathcal{G}'$ is defined over the development of the homomorphism $\Phi : \mathcal{G} \longrightarrow G$
2. $\mathcal{G}'(d) = \mathcal{F}_{\varphi}(p(d)) = \ker \Phi_{p(d)}$
3. $\pi_1(\mathcal{G}', d) = \ker \Phi_* : \pi_1(\mathcal{G}, p(d)) \longrightarrow G$

□

Let $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a Haeflieger's complex of groups and $\Phi : \mathcal{G} \longrightarrow G$ any homomorphism. The Galois covering associated to Φ defined in 5.9 of Chapter III.C, [B-H] is equivalent to the G-covering associated to Φ defined in 6.3.2. Moreover the construction given in [B-H] carries over to the twisted diagrams of groups, namely:

Proposition 6.3.3. *Let $\Phi : \mathcal{G} \longrightarrow G$ be a homomorphism and $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ the associated G-covering (6.3.2). Then $\mathcal{G}' : \mathcal{D} \longrightarrow \mathrm{Gr}$ is equivalent to a certain twisted diagram of groups $\mathcal{G}'' : \mathcal{D}(\mathcal{G}, \Phi) \longrightarrow \mathrm{Gr}$ defined (for complexes of groups) in [B-H].*

Proof. We construct twisted diagram of groups $\mathcal{G}'' : \mathcal{D} \longrightarrow \mathrm{Gr}$ and the covering $\phi' : \mathcal{G}'' \longrightarrow \mathcal{G}$ as follows. For each $d \in p^{-1}(c) \subset \mathrm{Ob}\,\mathcal{D}$, the group $\mathcal{G}''(d)$ is the kernel of Φ_c and $\phi'_d : \mathcal{G}''(d) \longrightarrow \mathcal{G}(c)$ is the inclusion. Let $k \in \mathrm{Mor}\,\mathcal{D}$. In order to define $\mathcal{G}''(k)$ we need to choose for each $\Phi_c(\mathcal{G}(c))$-coset d a representative in G. We again denote this d, thus identyfying d to the coset $[d]_{\Phi_c(\mathcal{G}(c))}$. We also choose for each morphism $k \in p^{-1}(l) \subset \mathrm{Mor}\,\mathcal{D}$ an element $\phi'(k) \in \mathcal{G}(t(l))$ such that

$$i(k)\Phi(l)^{-1}\Phi_{t(l)}(\phi'(k)^{-1}) = t(k)$$

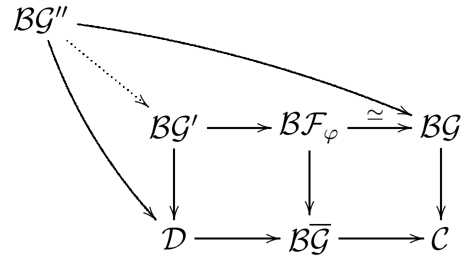
We then define $\mathcal{G}''(k) := \mathrm{Ad}(\phi'(k)) \circ \mathcal{G}(l)$.

For composable morphisms k_1, k_2 with $l_1 = p(k_1)$, $l_2 = p(k_2)$, we define

$$g''_{k_1,k_2} := \phi'(k_1)\mathcal{G}(l_1)(\phi'(k_2))g_{l_1,l_2}\phi'(k_1k_2)^{-1} \in \ker \Phi_{t(l_1)}$$

The homomorphism $\phi' : \mathcal{G}'' \longrightarrow \mathcal{G}$ over $p : \mathcal{D} \longrightarrow \mathcal{C}$ is given by the homomorphisms ϕ'_d and the elements $\phi'(k)$.

According to the universal property of the pull-back there exists unique functor $\mathcal{BG}'' \longrightarrow \mathcal{BG}'$ and a commutative diagram



This implies that there exists a morphism $\mathcal{G}'' \longrightarrow \mathcal{G}'$ which is the isomorphism on the local groups. Thus $\mathcal{G}'$ and $\mathcal{G}''$ are equivalent twisted diagrams.

□

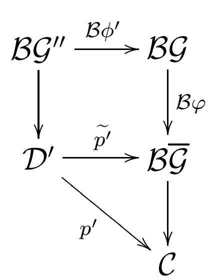
The universal property of the Galois covering Let $\phi : \mathcal{G}' \longrightarrow \mathcal{G}$ over $p : \mathcal{D} \longrightarrow \mathcal{C}$ is a G-covering associated to a homomorphism $\Phi : \mathcal{G} \longrightarrow G$. Then the composition $\Phi \circ \phi : \mathcal{G}' \longrightarrow G$ is trivial on the local groups and the fundamental group of the twisted diagram of groups $\mathcal{G}' : \mathcal{D} \longrightarrow \mathrm{Gr}$ is isomorphic to the kernel of $\Phi_* : \pi_1(\mathcal{G}, c_0) \longrightarrow G$, namely $\phi_*(\pi_1(\mathcal{G}', d_0)) = \ker(\pi_1(\mathcal{G}, p(d_0)) \longrightarrow G)$. We will prove that $\mathcal{G}'$ is the universal twisted diagram of groups satisfying these properties, namely

Proposition 6.3.4. *Assume that a homomorphism of twisted diagrams $\phi' : \mathcal{G}'' \longrightarrow \mathcal{G}$ over $p' : \mathcal{D}' \longrightarrow \mathcal{C}$ satisfies*

1. *$\Phi \circ \phi' : \mathcal{G}'' \longrightarrow G$ is trivial on the local groups*
2. *$(\Phi \circ \phi')_* : \pi_1(\mathcal{G}'', d'_0) \longrightarrow G$ is trivial*

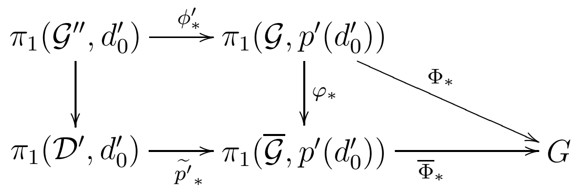
Then there exists a unique homomorphism $\bar{\phi'} : \mathcal{G}'' \longrightarrow \mathcal{G}'$ over $\bar{p'} : \mathcal{D}' \longrightarrow \mathcal{D}$ such that $\phi' = \phi \circ \overline{\phi'}$ and $p' = p \circ \overline{p'}$.

Proof. According to 5.4.3 the homomorphism $\Phi : \mathcal{G} \longrightarrow G$ is equal to the composition $\mathcal{G} \xrightarrow{\varphi} \overline{\mathcal{G}} \xrightarrow{\overline{\Phi}} G$, where φ is an epimorphism over the identity of $\mathcal{C}$ and $\overline{\Phi}$ is injective on the local groups. Then for each $d' \in \mathrm{Ob}\,\mathcal{D}'$ the composition $\mathcal{G}'' \xrightarrow{\phi'} \mathcal{G} \xrightarrow{\varphi} \overline{\mathcal{G}}$ maps a local group $\mathcal{G}''(d')$ to a trivial element of a group $\overline{\mathcal{G}}(p'(d')) \subset G$. Then according to 2.2.11 there exists a functor $\widetilde{p'} : \mathcal{D}' \longrightarrow \mathcal{B}\overline{\mathcal{G}}$ such that the following diagram commutes

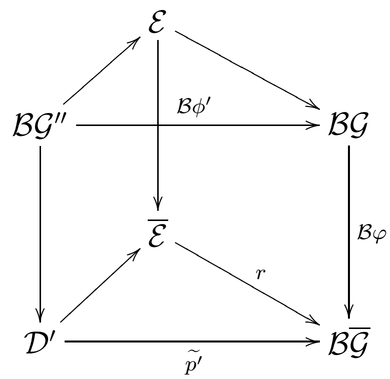


Clearly this diagram defines a homomorphism $\mathcal{G}'' \longrightarrow \mathcal{F}_\varphi$.

The latter diagram yields a commutative diagram



Due to 4.2.6 the vertical homomorphism are onto. Moreover, the composition $\Phi_* \circ \phi'_*$ is trivial, hence $\overline{\Phi}_* \circ \widetilde{p'}_*$ is trivial. Therefore the following diagram commutes



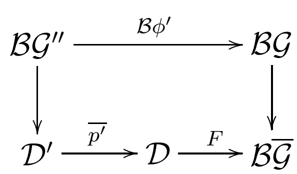
where $\mathcal{E} \longrightarrow \mathcal{B}\mathcal{G}$ and $\overline{\mathcal{E}} \longrightarrow \mathcal{B}\overline{\mathcal{G}}$ are corresponding G-coverings.

According to 5.3.17 there exists a functor $F : \mathcal{D} \longrightarrow \mathcal{B}\overline{\mathcal{G}}$ which is equal to the composition $\mathcal{D} \overset{\approx}{\hookrightarrow} \overline{\mathcal{E}} \overset{r}{\longrightarrow} \mathcal{B}\overline{\mathcal{G}}$. Due to 5.3.19 we have an inclusion $\widetilde{p'}(\mathcal{D}') \subset F(\mathcal{D})$.

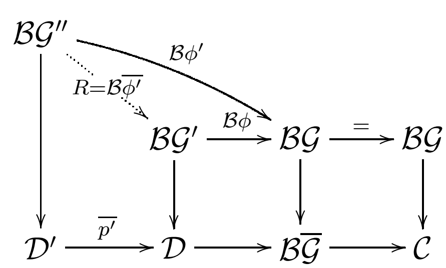
Lemma 6.3.5. *There exists a functor $\overline{p'} : \mathcal{D}' \longrightarrow \mathcal{D}$ such that $F \circ \overline{p'} = \widetilde{p'}$.*

Proof. The proof follows like the proof of 5.1.6. Use 5.3.20 and the fact that $\widetilde{p'}_*(\pi_1(\mathcal{D}', d_0')) \subset \ker \overline{\Phi}_* = F_*(\pi_1(\mathcal{D}, d_0))$. □

Therefore the following diagram commutes



Using the universal property of the pull-back there exists a unique functor $R : \mathcal{B}\mathcal{G}'' \longrightarrow \mathcal{B}\mathcal{G}'$ such that the following diagram commutes

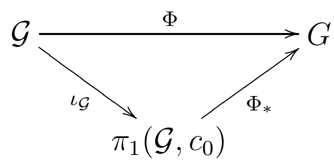


Therefore there exists a homomorphism $\overline{\phi'} : \mathcal{G}'' \longrightarrow \mathcal{G}'$ over $\overline{p'} : \mathcal{D}' \longrightarrow \mathcal{D}$ such that $R = \mathcal{B}\overline{\phi'}$ and $\phi' = \phi \circ \overline{\phi'}$ over $p' = p \circ \overline{p'}$. □

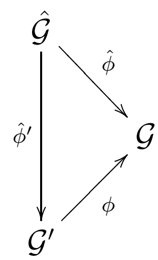
The universal covering of a twisted diagram of groups

Proposition 6.3.6. *Let* $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ *be a twisted diagram of groups and* $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$ *the associated homomorphism. Let* $\hat{\mathcal{G}} : \hat{\mathcal{D}} \longrightarrow \mathrm{Gr}$ *be the twisted diagram associated to* $\iota_{\mathcal{G}}$ *and* $\hat{\phi} : \hat{\mathcal{G}} \longrightarrow \mathcal{G}$ *over* $\hat{p} : \hat{\mathcal{D}} \longrightarrow \mathcal{C}$ *the associated covering (6.3.2). Then for each homomorphism* $\Phi : \mathcal{G} \longrightarrow G$ *there exists a homomorphism* $\hat{\phi}' : \hat{\mathcal{G}} \longrightarrow \mathcal{G}'$ *over* $\hat{p}' : \hat{\mathcal{D}} \longrightarrow \mathcal{D}$ *such that* $\phi \circ \hat{\phi}' = \hat{\phi}$ *and* $p \circ \hat{p}' = \hat{p}$. *This homomorphism is onto if and only if* $\Phi_* : \pi_1(\mathcal{G}, c_0) \longrightarrow G$ *is onto. Moreover* $\hat{\phi}' : \hat{\mathcal{G}} \longrightarrow \hat{\phi}'(\hat{\mathcal{G}})$ *is a covering.*

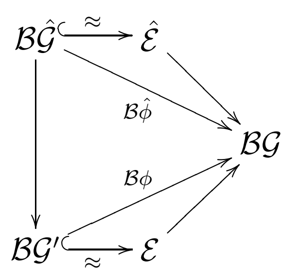
Proof. As we have observed in 5.4.11 the following diagram



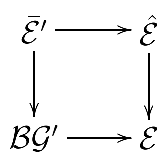
commutes up to a natural isomorphism. Therefore the composition $\hat{\mathcal{G}} \xrightarrow{\hat{\phi}} \mathcal{G} \xrightarrow{\Phi} G$ is trivial and $\pi_1(\hat{\mathcal{G}}, d_0) \xrightarrow{\hat{\phi}_*} \pi_1(\mathcal{G}, \hat{p}(d_0)) \xrightarrow{\Phi_*} G$ is trivial as well. Then according to 6.3.4 there exists a homomorphism $\hat{\phi}' : \hat{\mathcal{G}} \longrightarrow \mathcal{G}'$ over $\hat{p}' : \hat{\mathcal{D}} \longrightarrow \mathcal{D}$ such that the following diagram



commutes. The homomorphisms $\hat{\phi}$ and ϕ are coverings, hence the following diagram



where $\hat{\mathcal{E}} \longrightarrow \mathcal{B}\mathcal{G}$ is a universal covering and $\mathcal{E} \longrightarrow \mathcal{B}\mathcal{G}$ is a G-covering, commutes. Clearly there exists a functor $\hat{\mathcal{E}} \longrightarrow \mathcal{E}$, let $\bar{\mathcal{E}}'$ be a pull-back of the following diagram



Then there exists a functor $\lambda : \mathcal{B}\hat{\mathcal{G}} \longrightarrow \bar{\mathcal{E}}'$, let $\mathcal{E}'$ be a connected component of $\bar{\mathcal{E}}'$ such that $\lambda(\mathcal{B}\hat{\mathcal{G}}) \subset \mathcal{E}'$.

$$\begin{array}{ccccc} \mathcal{B}\hat{\mathcal{G}} & & \longrightarrow & & \hat{\mathcal{E}} \\ \downarrow & \searrow^{\lambda} & & \nearrow & \downarrow \\ & & \mathcal{E}' & & \\ \downarrow & \swarrow_{\rho} & & & \downarrow \\ \mathcal{B}\mathcal{G}' & & \longrightarrow & & \mathcal{E} \end{array}$$

Clearly $\lambda : \mathcal{B}\hat{\mathcal{G}} \longrightarrow \mathcal{E}'$ is an inclusion and an equivalence of small categories. Moreover $\rho : \mathcal{E}' \longrightarrow \rho(\mathcal{B}\mathcal{G}')$ is a covering of small categories. Therefore $\hat{\phi}' : \hat{\mathcal{G}} \longrightarrow \hat{\phi}'(\hat{\mathcal{G}})$ is a covering. The functor $\mathcal{B}\hat{\phi}'$ is onto if and only if $\hat{p}' : \hat{\mathcal{D}} \longrightarrow \mathcal{D}$ is onto. According to 5.4.11 this is the case when $\Phi_* : \pi_1(\mathcal{G}, c_0) \longrightarrow G$ is onto.

□

Extension of twisted diagrams of groups The following Theorem is a corollary from 6.3.2 and 6.3.4

Theorem 6.3.7. *Let $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ over $\mathcal{C}$ be an epimorphism of twisted diagrams of groups. Let $\hat{\mathcal{G}} : \hat{\mathcal{D}} \longrightarrow \mathrm{Gr}$ over $\hat{p} : \hat{\mathcal{D}} \longrightarrow \mathcal{C}$ be the universal covering of $\mathcal{G}$. Then there exists a twisted diagram of groups $\mathcal{K} : \mathcal{B}\hat{\mathcal{G}} \longrightarrow \mathrm{Gr}$ and a homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ over $F : \mathcal{B}\hat{\mathcal{G}} \longrightarrow \mathcal{C}$ satisfying*

1. *$\varphi \circ \phi : \mathcal{K} \longrightarrow \mathcal{G}$ is trivial on the local groups*
2. *$(\varphi \circ \phi)_* : \pi_1(\mathcal{K}, d_0) \longrightarrow \pi_1(\mathcal{G}, F(d_0))$ is trivial*
3. *the homomorphism $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ is a covering and it is final for homomorphisms $\phi' : \mathcal{K}' \longrightarrow \widetilde{\mathcal{G}}$ satisfying 1. and 2.*

Proof. We define a small category $\mathcal{B}\mathcal{K}$ to be the pull-back of the following diagram

$$\begin{array}{ccc} \mathcal{B}\mathcal{K} & \longrightarrow & \mathcal{B}\widetilde{\mathcal{G}} \\ \downarrow & & \downarrow \\ \mathcal{B}\hat{\mathcal{G}} & \longrightarrow & \mathcal{B}\mathcal{G} \end{array}$$

Clearly the associated homomorphism $\mathcal{K} \longrightarrow \mathcal{F}_\varphi$ is a covering, hence $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ given by

$$\begin{array}{ccccc} \mathcal{B}\mathcal{K} & \longrightarrow & \mathcal{B}\widetilde{\mathcal{G}} & \stackrel{=}{\longrightarrow} & \mathcal{B}\widetilde{\mathcal{G}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}\hat{\mathcal{G}} & \longrightarrow & \mathcal{B}\mathcal{G} & \longrightarrow & \mathcal{C} \end{array}$$

is a covering of twisted diagrams of groups. Then the composition $\varphi \circ \phi$ satisfies assertions 1. and 2.

If $\phi' : \mathcal{K}' \longrightarrow \widetilde{\mathcal{G}}$ satisfies 1. then the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}\mathcal{K}' & \longrightarrow & \mathcal{B}\widetilde{\mathcal{G}} \\ \downarrow & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{B}\mathcal{G} \end{array}$$

Let $\overline{\mathcal{G}} : \mathcal{C} \longrightarrow \mathrm{Gr}$ be a twisted diagram of groups such that $\iota_{\mathcal{G}} : \mathcal{G} \longrightarrow \pi_1(\mathcal{G}, c_0)$ is equal to $\mathcal{G} \longrightarrow \overline{\mathcal{G}} \longrightarrow \pi_1(\mathcal{G}, c_0)$ (5.4.3). If $\phi' : \mathcal{K}' \longrightarrow \widetilde{\mathcal{G}}$ satisfies 1. and 2. then as in proof of 6.3.4 we obtain commutative diagram

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{B}\mathcal{G} \\ \downarrow & & \downarrow \\ \hat{\mathcal{D}} & \longrightarrow & \mathcal{B}\overline{\mathcal{G}} \end{array}$$

The small category $\mathcal{B}\hat{\mathcal{G}}$ is defined as the pull-back of the latter diagram, hence there exists a functor $\mathcal{D}' \longrightarrow \mathcal{B}\hat{\mathcal{G}}$. Using the universal property of the pull-back category $\mathcal{B}\mathcal{K}$ we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{B}\mathcal{K}' & & & & \\ & \searrow & & \searrow & \\ \downarrow & & \mathcal{B}\mathcal{K} & \longrightarrow & \mathcal{B}\widetilde{\mathcal{G}} \\ & & \downarrow & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{B}\hat{\mathcal{G}} & \longrightarrow & \mathcal{B}\mathcal{G} \end{array}$$

Therefore there exists a homomorphism $\bar{\phi}' : \mathcal{K}' \longrightarrow \mathcal{K}$ such that $\phi \circ \bar{\phi}' = \phi'$ which proves the Theorem.

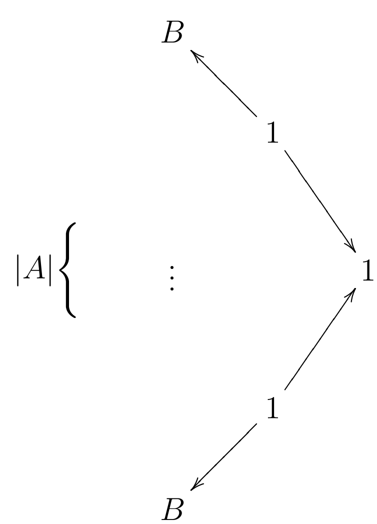
□

Example 6.3.8. Assume that $\varphi : \widetilde{G} \longrightarrow G$ is surjective homomorphism of groups. Then the twisted diagram of groups $\mathcal{K}$ is a group $N = \ker(\widetilde{G} \longrightarrow G)$.

Example 6.3.9. Consider an epimorphism of complexes of groups $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ given by

$$(B \longleftarrow 1 \longrightarrow A) \twoheadrightarrow (1 \longleftarrow 1 \longrightarrow A)$$

The complex of groups $\mathcal{G}$ is a graph of groups thus according to the theorem of Serre [S] it is developable, let $\hat{\mathcal{D}}$ denote its Bass-Serre tree. The complex of groups $\mathcal{K} : \hat{\mathcal{D}} \longrightarrow \mathrm{Gr}$ is defined as follows:



The covering $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ is given by the natural inclusions.

We obtain an "extension" of graphs of groups

$$\mathcal{K} \rightarrowtail \widetilde{\mathcal{G}} \twoheadrightarrow \mathcal{G}$$

which yields an exact sequence

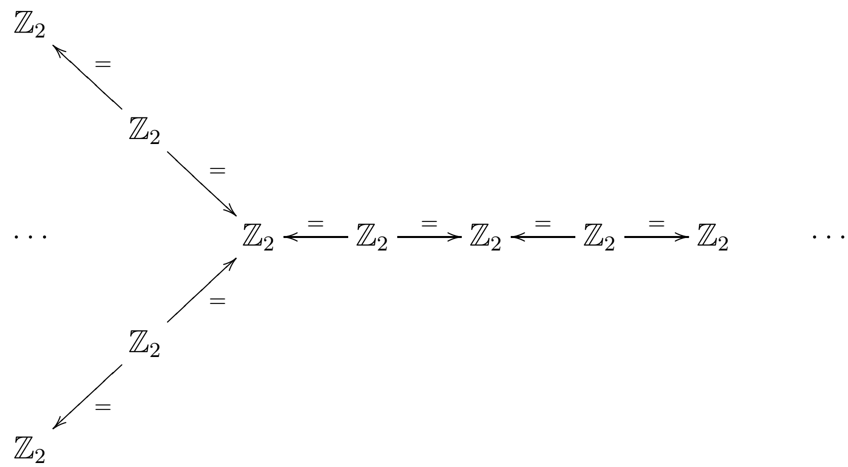
$$*_{|A|} B \rightarrowtail A * B \twoheadrightarrow A$$

of fundamental groups.

Example 6.3.10. Let $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ be an epimorphism of complexes of groups given by

$$(\mathbb{Z}_6 \longleftarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4) \twoheadrightarrow (\mathbb{Z}_3 \longleftarrow 1 \longrightarrow \mathbb{Z}_2)$$

The complex of groups $\mathcal{G}$ is developable. We define a complex of groups $\mathcal{K} : \hat{\mathcal{D}} \longrightarrow \mathrm{Gr}$ to be $\mathcal{K}(d) = \mathbb{Z}_2$ on objects and $\mathbb{Z}_2 \xrightarrow{=} \mathbb{Z}_2$ on morphisms:



The covering $\mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ is given by the natural inclusions.

The corresponding exact sequence of complexes of groups gives an exact sequence of its fundamental groups of the form

$$\mathbb{Z}_2 \rightarrowtail \mathrm{SL}_2\,\mathbb{Z} \twoheadrightarrow \mathrm{PSL}_2\,\mathbb{Z}$$

Example 6.3.11. Consider an epimorphism of complexes of groups $\varphi : \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ given by

$$(D_6 \longleftarrow \mathbb{Z}_2 \longrightarrow D_4) \twoheadrightarrow (\mathbb{Z}_2 \xleftarrow{=} \mathbb{Z}_2 \xrightarrow{=} \mathbb{Z}_2)$$

The universal covering of the complex of groups $\mathcal{G} : \mathcal{C} \longrightarrow \mathrm{Gr}$ is isomorphic to $\mathcal{C}$. Then the complex of groups $\mathcal{K} : \mathcal{C} \longrightarrow \mathrm{Gr}$ and the covering $\phi : \mathcal{K} \longrightarrow \widetilde{\mathcal{G}}$ are given by

$$(\mathbb{Z}_3 \longleftarrow 1 \longrightarrow \mathbb{Z}_2) \longrightarrow (D_6 \longleftarrow \mathbb{Z}_2 \longrightarrow D_4)$$

The exact sequence of fundamental groups equals

$$\mathrm{PSL}_2\,\mathbb{Z} \rightarrowtail \mathrm{PGL}_2\,\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$$

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